Observability and forward-backward observability of discrete-time nonlinear systems

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Abstract

In this paper, we study the observability properties of nonlinear discrete time systems. Two types of contributions are given. First, we present observability criteria in terms of appropriate codistributions. For particular, but significant, classes of systems we provide criteria that require only a finite number of computations. Then, we consider invertible systems (which includes discrete-time models obtained by sampling of continuous-time systems) and prove that the weaker notion of forward-backward observability is equivalent to the stronger notion of (forward) observability.

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1 Introduction

This paper is a study on the observability of nonlinear discrete-time models. We only deal with the single input single output systems, since the general case involves only notational changes. More precisely, we consider systems of the form

$$\Sigma \qquad \begin{array}{lcl} x(t+1) & = & f(x(t), u(t)), \qquad t = 0, 1, 2, \ldots \\ y(t) & = & h(x(t)). \end{array} \tag{1}$$

In (1), we assume that $x(t) \in M$, $y(t) \in Y$ and $u(t) \in U$, with M and Y connected, second countable, Hausdorff, differentiable manifolds, of dimensions n and 1, respectively. Moreover, we assume that the control space U is an open interval of $\mathbb{R}$. A system is said to be of class C^k, if the manifolds M and Y are of class C^k, and the functions $f : M \times U \to M$ and $h : M \to Y$, are of class C^k. We will deal with the two cases $k = \infty$ and $k = \omega$.

This paper provides explicit criteria in terms of the functions f and h in (1) to decide if a state x_0 in the state space M is observable. This means that it is possible to distinguish x_0 from any other state (or possibly from any state in a neighborhood of x_0) by applying a suitable control sequence $u(1), u(2), \ldots$ and reading the corresponding output sequence

$h(x(0)), h(x(1)), \ldots$. Our approach is based on ideas from differential geometry. In particular, we characterize the observability properties of systems (1) using a sequence of codistributions generated by the output functions at different times.

The plan of the paper is as follows. In Section 2 we give the basic definitions concerning the observability of discrete-time nonlinear systems. In Section 3 we define a sequence of codistributions for the system (1) and using these codistributions we give a first general observability criterion in Theorem 3.1. Then Theorem 3.5 provides two independent sufficient conditions that guarantee that only a finite number of computations (in particular a number equal to the dimension n of the system) is sufficient to check observability. The special important case of invertible systems, namely systems of the form (1) with $f(\cdot, u)$ invertible for every u, is considered in Section 4. An important example of these models are discrete-time systems obtained from continuous-time systems after sampling. Stronger observability criteria are derived for these systems in Theorems 4.5 and 4.6. For invertible models of the form (1), we can define an inverse system which still has the form (1) but with f replaced by f^{-1}. We can ask the question of whether it is possible to distinguish two states by allowing the system to alternatively assume the *forward* form (1) or the *backward* form ((1) with f replaced by f^{-1}). This question of *forward-backward observability* is formalized in Section 4 and it is proven in sub-Section 4.2 that, under regularity assumptions, the weaker notion of *forward-backward observability* is equivalent to the stronger notion of *forward observability.* Conclusions are given in Section 5.

The paper [9] contains a previous study on the observability of discrete time nonlinear models, this paper deals with the case of systems without controls. In [8] differential geometric concepts of invariant distributions and codistributions were introduced in the discrete time setting. The classical paper [5] deals with questions of nonlinear observability in the continuous-time context. For a general treatment of this case see the books [6] and [10]. Paper [3] uses a similar approach to give necessary and sufficient conditions to transform a discrete-time nonlinear system into a state affine form. Related results can be found in [4].

Some results preliminary to the ones presented here were given in [1].

2 Basic Definitions

From now on, assume that a C^∞ system Σ of type (1) is given. For each $u \in U$, we denote by $f_u : M \to M$ the map $f_u(x) := f(x, u)$. Next we give definitions of *indistinguishability* and *observability* (This follows the classical definitions given in [10], [11]).

Definition 2.1 Two states $x_1, x_2 \in M$ are said to be *indistinguishable*, and we write $x_1 I x_2$, if, for each $j \geq 0$ and for each sequence of controls, $\{u_1, ..., u_j\} \in U^j$, we have:

$$h(f_{u_j} \circ \cdots \circ f_{u_1}(x_1)) = h(f_{u_j} \circ \cdots \circ f_{u_1}(x_2)).$$

Definition 2.2 A state $x_0 \in M$ is said to be *observable* if $x I x_0 \quad \Rightarrow \quad x = x_0, \forall\, x \in M$.

Next we give two definitions of *local observability.*

Definition 2.3 A state $x_0 \in M$ is said to be *locally weakly observable*, if there exists a neighborhood W of x_0, such that, for each $x_1 \in W$, $x_0 I x_1$ implies $x_0 = x_1$.

Definition 2.4 A state $x_0 \in M$ is said to be *locally strongly observable*, if there exists a neighborhood W of x_0, such that, for each $x_1, x_2 \in W$, $x_1 I x_2$ implies $x_1 = x_2$.

Clearly, observability of a point x_0 implies local weak observability of x_0. However it does not necessarily implies local strong observability. One simple example goes as follows: take $M = U = \mathbb{R}$ and $f(x,u) = x$, $h(x) = x^2$, then $x_0 = 0$ is observable but not locally strongly observable. One can also construct examples of points that are locally strongly observable but not observable. For example taking again $M = U = \mathbb{R}$, consider $h(x) = f(x,u) = x^2 - x$. The origin is indistiguishable from the point $x = 1$ (they both give the output sequence identically equal to zero). However every two points x_1 and x_2 in $(-\frac{1}{2}, \frac{1}{2})$ give a different values for the output. This discussion can be summarized by saying that both observability (*obs.*) and local strong observability (*l.s.o.*) imply local weak observability (*l.w.o.*) but they are not related to each other, so we have the following diagram.

$$obs. \rightarrow l.w.o. \leftarrow l.s.o. \tag{2}$$

Local strong observability is a more 'robust' oservability notion than local weak observability. If, in a given situation, the state of the system is known to be close (but not necessarily equal) to a nominal value, it is of interest to know whether by performing experiments on the system and reading the outputs we can detect a difference in the initial state.

Definition 2.5 A system Σ is *(locally weakly) (locally strongly) observable*, if each state $x \in M$ enjoys this property.

In the following, we say that x_e is an equilibrium point, if there exists $u_e \in U$ such that $f(x_e, u_e) = x_e$. We say that a subset of M is *generic* if its complement is contained in a proper analytic subset of M. Given a set L of C^∞ functions, defined on M, we shall denote by dL the codistribution spanned by all the differentials of these functions. By definition, these are exact differentials.

3 Observability Criteria in terms of Codistributions

We first define some sets of functions which will be used to obtain observability criteria. We let, for each $k \geq 1$:

$$\begin{aligned} \Theta_1 &= \{h(\cdot)\} \\ \Theta_k &= \{h(f_{u_j} \circ \cdots \circ f_{u_1}(\cdot)) \mid \forall\, i = 1, \cdots, j,\ u_i \in U, \quad \text{and} \quad 1 \leq j \leq k-1\}, \end{aligned}$$

and $\Theta = \cup_{k \geq 1} \Theta_k$. The following theorem states a first criterion where local observability is related to the full dimensionality of the codistribution $d\Theta$. For completeness we give its proof, which can also be found in [1].

Theorem 3.1 *Let Σ be a system for type (1) and fix a state $x_0 \in M$.*

(a) If dim $d\Theta(x_0) = n$, then x_0 is a locally strongly observable state for Σ.

(b) If Σ is locally weakly observable, then there exists an open set A, with $A \subset M$, such that $\dim d\Theta(x) = n$ *for all $x \in A$. If, in addition, the system is analytic then A can be chosen to be a generic subset of M.*

Proof. (a) If *dim* $d\Theta(x_0) = n$, then there exist n functions in Θ, $H_i(\cdot) := h(f_{u^i_{j^i}} \circ \cdots \circ f_{u^i_1}(\cdot))$, $i = 1, ..., n$, whose differentials are linearly independent at x_0. By continuity, they remain independent in a neighborhood $W \subset M$ of x_0. Therefore, $H_i(\cdot)$, $i = 1, ..., n$, define a smooth mapping from M to Y^n, which, restricted to W, is injective. Let $x_1, x_2 \in W$, if $x_1 I x_2$, in particular, for all $i = 1, ..., n$, it must hold $H_i(x_1) = H_i(x_2)$. By the injectivity of $H_i(\cdot)$ $i = 1, ..., n$, it follows that $x_1 = x_2$. Thus x_0 is a locally strongly observable state.

(b) We prove this part by the way of contradiction. Assume that Σ is locally weakly observable, but it does not exist an open subset of M where $dim\, d\Theta(x) = n$, which is equivalent to saying that $dim\, d\Theta(x) < n$ for all $x \in M$. Let $l = \max_{x \in M} dim\, d\Theta(x) (< n)$, and choose $x_0 \in M$, such that $dim\, d\Theta(x_0) = l$. By continuity and maximality of l, there exists an open neighborhood W of x_0, such that $dim\, d\Theta(x) = l$, for all $x \in W$. Thus there exist $H_1(\cdot), ..., H_l(\cdot)$, in Θ, whose differentials in W are linearly independent. It is thus possible to take these functions, $H_1(\cdot), ..., H_l(\cdot)$, along with a set of complementary independent functions, as partial coordinates in W. Since every function in Θ only depends on the first $l < n$ coordinates, points in W, which differ only in the last $n - l$ coordinates, cannot be distinguished. This contradicts the hypothesis of the local observability of the system. Thus there must exists an open subset A in M, such that $dimd\Theta(x) = n$ for all $x \in A$.

To prove the last sentence of the theorem it is enough to observe that if the system is analytic then $\{x \in M \mid dim\, d\Theta(x) < n\}$ is the set of zeros of an analytic function, namely an analytic set. Thus, since M is connected, A is a generic subset of M. ∎

The converse of statement (a) in Theorem 3.1 is false as the following example shows:

Example 3.2 Let $M = \mathbb{R}^2$, $U = \mathbb{R}$,

$$\begin{cases} x_1(t+1) &= u x_2^3, \\ x_2(t+1) &= x_2. \end{cases}$$

and $y(t) = x_1(t)$.

It is easy to see that this model is observable and the origin is locally strong observable. In fact, two different states can be distinguished at the first step if their x_1 coordinates are different or at the second step if their second coordinates are different (set $u = 1$). However, $\dim d\Theta(x_1, x_2) = 2$ for all $(x_1, x_2) \neq (0, 0)$, while $\dim d\Theta(0, 0) = 1$

A stronger result about tests of local observability can be derived if we assume that $d\Theta$ is constant dimensional in a neighborhood of x_0. More precisely, on gets:

Proposition 3.3 Let Σ be a system for type (1) and fix a state $x_0 \in M$. Assume that there exists a neighborhood W of x_0 where $d\Theta$ is constant dimensional. Then the following are equivalent:

i) x_0 is locally strongly observable,

ii) x_0 is locally weakly observable,

iii) dim $d\Theta(x_0) = n$.

Proof. The implications $iii) \to i) \to ii)$ follow from part a) of Theorem 3.1 and (2). Now assume that x_0 is locally weakly oservable but dim $d\Theta(x_0) = l < n$. By the constant dimensionality assumption for $d\Theta$, dim $d\Theta(x) = l < n$ for every x in a neighborhood W of x_0. As in the proof of Theorem 3.1 choose l functions $H_1(\cdot), ..., H_l(\cdot)$ whose differentials are linearly independent in W. Taking these functions along with a set of complementary independent functions as partial coordinates in W, since every function in Θ (the set of all possible outputs) only depends on the first $l < n$ coordinates, states which differ only in the last $n - l$ coordinates cannot be distinguished. This contradicts the observability assumption. ∎

Dealing with the whole set of functions Θ may be difficult in some cases because we have to compute an infinite number of iterations of the map f_u. Therefore, in the following Theorem 3.5, we look for observability conditions expressed in terms of the codistribution $d\Theta_n$, where n is the dimension of the system (i.e. $\dim M = n$ where M is the state space). Define

$$l_k = \max_{x \in M} \dim d\Theta_k(x), \qquad \text{and} \qquad l = \max_{x \in M} \dim d\Theta(x).$$

Lemma 3.4 Let Σ be a model of type (1), assume that there exists a $\bar{k} > 0$ such that

$$l_{\bar{k}} = l_{\bar{k}+1}. \tag{3}$$

Moreover assume that at least one of the following two conditions is verified:

(a) Σ is an analytic system and, for each $u \in U$, the map f_u is open;

(b) For each $k \geq 1$, the dimension of $d\Theta_k(x)$ does not depend on x, i.e. $\dim d\Theta_k(x) = l_k$ for all $x \in M$.

Then

$$l_k = l_{\bar{k}}, \qquad \forall\, k \geq \bar{k}. \tag{4}$$

Proof. We will prove that the assumption (3) along with either one of the conditions (a) and (b) implies

$$l_{\bar{k}+2} = l_{\bar{k}} \tag{5}$$

From (5), and using an induction argument, (4) follows. We will establish (5) by contradiction. Assume there exists $\bar{x} \in M$ such that $\dim d\Theta_{\bar{k}+2}(\bar{x}) > l_{\bar{k}}$. Then, there exists a function

$$H(\cdot) = h\left(f_{u_{\bar{k}+1}} \circ \ldots \circ f_{u_1}\right)(\cdot) \in \Theta_{\bar{k}+2} \setminus \Theta_{\bar{k}+1}, \tag{6}$$

such that $dH(\bar{x})$ is linearly independent from $d\Theta_{\bar{k}+1}(\bar{x})$. Let

$$\tilde{H}(\cdot) = h\left(f_{u_{\bar{k}+1}} \circ \ldots \circ f_{u_2}\right)(\cdot) \in \Theta_{\bar{k}+1},$$

where the controls $u_2, ..., u_{\bar{k}+2}$ are the same as in (6). Then, by continuity, there exists an open neighborhood W of $\bar{x}$ such that if $x \in W$, then $dH(x)$ is still linearly independent from $d\Theta_{\bar{k}+1}(x)$. Let $V = f_{u_1}(W)$.

Now, we prove that in both cases (a) and (b), there exists $\tilde{y} \in V$ and $\tilde{x} \in W$ such that $\tilde{y} = f_{u_1}(\tilde{x})$ and

$$\dim d\Theta_{\bar{k}}(\tilde{y}) = l_{\bar{k}} = \dim d\Theta_{\bar{k}+1}(\tilde{y}). \tag{7}$$

- If (a) holds then V is open since f_{u_1} is an open map, thus $\tilde{y} \in V$ satisfying (7) exists by analyticity. Then one chooses any $\tilde{x} \in W$ such that $\tilde{y} = f_{u_1}(\tilde{x})$.
- If (b) holds then we may choose $\tilde{x} = \bar{x}$ and $\tilde{y} = f_{u_1}(\bar{x})$.

Let $\tilde{H}_1, \ldots, \tilde{H}_{l_{\bar{k}}} \in \Theta_{\bar{k}}$ be such that $d\tilde{H}_1(\tilde{y}), \ldots, d\tilde{H}_{l_{\bar{k}}}(\tilde{y})$ is a basis for $d\Theta_{\bar{k}}(\tilde{y})$. Then for each $i = 1, \ldots, l_{\bar{k}}$, there exists $u_1^i, \ldots, u_{j_i}^i \in U$, with $j_i \leq \bar{k} - 1$, such that

$$\tilde{H}_i(\cdot) = h\left(f_{u_{j_i}^i} \circ \ldots \circ f_{u_1^i}\right)(\cdot) \in \Theta_{\bar{k}}. \tag{8}$$

From (7) we get that there exists $\alpha_1, \ldots, \alpha_{l_{\bar{k}}} \in \mathbb{R}$, such that $d\tilde{H}(\tilde{y}) = \sum_{i=1}^{l_{\bar{k}}} \alpha_i d\tilde{H}_i(\tilde{y})$. On the other hand, we have:

$$dH(\tilde{x}) \;=\; d\tilde{H}(\tilde{y}) \cdot \frac{\partial f_{u_1}}{\partial x}(\tilde{x}) = \sum_{i=1}^{l_{\bar{k}}} \alpha_i d\tilde{H}_i(\tilde{y}) \cdot \frac{\partial f_{u_1}}{\partial x}(\tilde{x}) = \sum_{i=1}^{l_{\bar{k}}} \alpha_i dH_i(\tilde{x}),$$

Where we have used the notation (see (8))

$$H_i(\cdot) \;=\; h\left(f_{u^i_{j_i}} \circ \ldots \circ f_{u^i_1} \circ f_{u_1}\right)(\cdot) \in \Theta_{\bar{k}+1}.$$

Thus $dH(\tilde{x})$ is linearly dependent from $d\Theta_{\bar{k}+1}(\tilde{x})$, which gives the desired contradiction. ∎

Assumption (a) of Lemma 3.4 holds for example when the map f_u is an analytic diffeomorphism of the state space. This fact will be used later in Section 4.

The following theorem shows that, under appropriate assumptions, it is sufficient, in testing observability, to check the dimension of $d\Theta_n$ (as opposed to the dimension of the whole co-distribution $d\Theta$ as in Proposition 3.3). Thus we have an "a priori" bound (given by the dimension of the state space) on the iterations of the maps f_u we need to take into account to test observability.

Theorem 3.5 *Let Σ be a model of type (1), assume that we are in one of the following cases:*

(a) Σ is an analytic system and, for each $u \in U$, the map f_u is open;

(b) For each $k \geq 1$, the dimension of $d\Theta_k(x)$ does not depend on x, i.e. $\dim d\Theta_k(x) = l_k$ for all $x \in M$.

Then the following statements are equivalent:

- *(i) There exists a generic set $A_1 \subseteq M$ such that $\dim d\Theta_n(x) = n$ for all $x \in A_1$.*
- *(ii) There exists a generic set $A_2 \subseteq M$ such that all $x \in A_2$ are locally strongly observable.*
- *(iii) There exists a generic set $A_3 \subseteq M$ such that all $x \in A_3$ are locally weakly observable.*

Proof. The fact that (ii) implies (iii) is obvious, while $(i) \Rightarrow (ii)$ is given by part (a) of Theorem 3.1. Thus we only need to prove that $(iii) \Rightarrow (i)$. Assume (iii), and recall that we have defined $l = \max_{x\in M} \dim d\Theta(x)$ and $l_k = \max_{x\in M} \dim d\Theta_k(x)$. Notice that to obtain (i), it is enough to show, both for cases (a) and (b), that $l_n = n$. Clearly $0 \leq l_1 \leq l_2 \leq \ldots l_k \leq \ldots \leq l \leq n$.

First, notice that $l_1 > 0$, in fact, if $l_1 = 0$ then the output function h would be constant, and so no point in M would be locally weakly observable. Moreover, we have that necessarily $l = n$. In fact, let x_0 be locally weakly observable and such that $\dim d\Theta(x_0) = l$. Then, for both cases (a) and (b), $\dim d\Theta$ is constant in a neighborhood of x_0, thus by Proposition 3.3, we may conclude $l = n$.

Assume, by contradiction, that $l_n < n$, then necessarily there exists $\bar{k} < n$ such that $l_{\bar{k}} = l_{\bar{k}+1}$. Thus, by Lemma 3.4, we have that $l_k = l_{\bar{k}}\ \forall k \geq \bar{k}$, which, in turn, implies $l = l_{\bar{k}} < n$. Thus, $l_n = n$, as desired. ∎

Remark 3.6

- It is interesting to notice that the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ hold pointwise, thus we have:
$$A_1 \subseteq A_2 \subseteq A_3,$$
where both inclusions may be proper. Notice that since the set of points locally (strongly or weakly) observable is a superset of the set of points where $\dim d\,\Theta_n(x) = n$, the latter condition at some point x implies local observability at x.

- We may rephrase the conclusion of the previous theorem as follows. Assume we have given a model Σ which satisfies the assumptions of Theorem 3.5 then the following statements are equivalent:

  1. Σ is locally (strongly or weakly) unobservable;
  2. $\dim d\Theta_n(x) < n$ for all $x \in M$.

  Thus the maximal dimension of $d\Theta_n$, l_n, provides good information on observability. If $l_n < n$ we can conclude that Σ is unobservable, on the other hand if $l_n = n$ then local observability holds in a generic subset of M.

Remark 3.7 A class of models where assumption (b) of Theorem 3.5 holds is the one in which $M = \mathbb{R}^n$ and whose both dynamics and output function are linear in x, i.e. $f(x,u) = g(u)x + d(u)$ and $h(x) = C^T x$. For this class it is not difficult to show that all the codistributions $d\Theta_k$ are constant dimensional and the notion of local and global observability are equivalent.

4 Invertible Systems

In this section, we study a particular class of discrete-time models.

Definition 4.1 A system Σ of type (1) is said to be *invertible*, if for all $u \in U$, the map f_u is a diffeomorphism (we denote by f_u^{-1} the inverse function of f_u).

The invertibility assumption holds, for example, for discrete-time systems obtained by sampling continuous-time dynamics for which the corresponding vector fields are complete. For further motivations to the study of this class of models we refer to [7].

To an invertible model Σ described by (1), it is possible to associate an *inverse* system Σ^- as follows. Σ^- will have the same state, control and output spaces as Σ, while its dynamics will be described by

$$\Sigma^- \qquad \begin{array}{lcl} x(t+1) & = & f^{-1}(x(t), u(t)), \qquad t = 0, 1, 2, \ldots \\ y(t) & = & h(x(t)). \end{array}$$

Using this system it is possible to define *backward* indistinguishability and observability, following the same lines as in Definitions 2.1-2.4 and 2.5. Moreover, these definitions extend to *forward-backward* indistinguishability and observability in an obvious manner. For example, we will say that $x_1 \in M$ is forward-backward indistinguishable from $x_2 \in M$ if for all $k \geq 0$, for all sequences $u_1, \ldots, u_k$ and $\epsilon_1, \ldots, \epsilon_k$, with $u_j \in U$ and $\epsilon_j = \pm 1$ it holds:

$$h\left(f_{u_k}^{\epsilon_k} \circ \ldots \circ f_{u_1}^{\epsilon_1}\right)(x_1) \;=\; h\left(f_{u_k}^{\epsilon_k} \circ \ldots \circ f_{u_1}^{\epsilon_1}\right)(x_2).$$

In general, the two notions of forward observability and forward-backward observability are not equivalent (see Example 4.2 below). In Section 4.2, we will look for sufficient conditions for these two notions to coincide. This is a question of fundamental interest and from a practical point of view, it may be more convenient, in some cases, to check the weaker notion of forward-backward observability rather than forward observability (see Example 4.12 in Section 4.2).

Example 4.2 Let's consider the following system: $M = \mathbb{R}^2$, $U = (-1, 1)$,

$$\begin{cases} x_1(t+1) &= x_1(t) + 1 + \frac{u}{2} g(x_1)\sin(x_2), \\ x_2(t+1) &= x_2(t) + u. \end{cases}$$

and $h(x_1, x_2) = x_1$, where

$$g(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0, \\ 1 & x = 0. \end{cases}$$

It is easy to verify that $g(x) = 0$ if and only if $x \in \mathbb{Z} \setminus \{0\}$ and $|g'(x)| \leq 1$. From the fact that $|g'(x)| \leq 1$ one gets that this model is locally invertible. However, it is not difficult to check that, for each control $u \in U$, the map f_u is one-to-one and onto, thus the model is invertible.

Consider the state $x_0 = (1, 0)$. Notice that, for any state of the type (n, a) with $n \in \mathbb{N}$, $n \geq 1$, and $a \in \mathbb{R}$, independently of the control u, we always reach the state $(n+1, a+u)$. Thus, the state x_0 is not weakly local forward observable, since in any neighborhood of x_0 there is a state of the type $x_1 = (1, \epsilon)$ which can not be distinguished from x_0.

However, using the fact that $g(0) \neq 0$, one get that x_0 is locally strongly forward-backward observable. In fact, let V be any open ball centered in x_0 with radius less then 1 and take any two points $\bar{x}_1, \bar{x}_2 \in V$. If the first coordinates of $\bar{x}_1$ and $\bar{x}_2$ are different then the two states are, clearly, distinguishable. Assume that $\bar{x}_1 = (a, b_1)$ and $\bar{x}_2 = (a, b_2)$ with $b_1 \neq b_2$. Fix any control $\bar{u} \neq 0$, and let:

$$f_{\bar{u}}^{-1}(a, b_1) = (\alpha_1, b_1 - \bar{u}), \quad f_{\bar{u}}^{-1}(a, b_2) = (\alpha_2, b_2 - \bar{u}).$$

Then, since

$$\alpha_1 + 1 + \frac{\bar{u}}{2} g(\alpha_1)\sin(b_1 - \bar{u}) = a = \alpha_2 + 1 + \frac{\bar{u}}{2} g(\alpha_2)\sin(b_2 - \bar{u}), \tag{9}$$

and $\sin(b_1 - \bar{u}) \neq \sin(b_2 - \bar{u})$, we may conclude $\alpha_1 \neq \alpha_2$, unless $g(\alpha_1) = 0$ (or $g(\alpha_2) = 0$). This last situation cannot occur since it would imply $\alpha_1 = a - 1$ (or $\alpha_2 = a - 1$), and so $|\alpha_1| < 1$ (or $|\alpha_2| < 1$), but the map g has no zeros with absolute value less then 1. Thus, x_0 is locally strongly forward-backward observable

4.1 Observability criteria for invertible systems

This subsection is devoted to study how the forward observability criteria of the previous section can be strengthened in the case of invertible systems.

The next lemma will be useful to prove sufficient conditions for constant dimensionality of the distribution $d\Theta$ for an invertible system.

Lemma 4.3 Let Σ be an invertible system. Assume we are given two states $x_1,\ x_2 \in M$ and a sequence of control values $u_1, \ldots, u_k$ such that:

$$x_2 \;=\; f_{u_k} \circ \ldots \circ f_{u_1}(x_1).$$

For each $j \geq 1$, if $\dim d\Theta_j(x_2) \;=\; r$, then $\dim d\Theta_{j+k}(x_1) \;\geq\; r$.

Proof. Let $g_1, \ldots, g_r \in \Theta_j$ such that $dg_1(x_2), \ldots, dg_r(x_2)$, form a basis for $d\Theta_j(x_2)$. Consider the functions $\tilde{g}_i = g_i \circ f_{u_k} \circ \ldots \circ f_{u_1}$, $i = 1, \ldots, r$. Then, for all $i = 1, \ldots, r$, $\tilde{g}_i \in \Theta_{j+k}$. Moreover

$$d\tilde{g}_i(x_1) \;=\; d\left(g_i \circ f_{u_k} \circ \ldots \circ f_{u_1}\right)(x_1) \;=\; dg_i(x_2) \frac{\partial}{\partial x} \left(f_{u_k} \circ \ldots \circ f_{u_1}\right)(x_1).$$

By the invertibility assumption of $f_{u_k} \circ \ldots \circ f_{u_1}$, $\frac{\partial}{\partial x}\left(f_{u_k} \circ \ldots \circ f_{u_1}\right)(x_1)$ is nonsingular, and therefore $d\tilde{g}_i(x_1)$, for $i = 1, \ldots, r$, are linearly independent. ∎

Now we recall some definitions about controllability.

Definition 4.4 Let Σ be a system of type (1), then:

- $x_0 \in M$ is said to be *forward accessible* (resp. *backward accessible*) if there exists an open set $A \subset M$ such that, for each $x \in A$, it is possible to find a sequence of control values $u_1, \ldots, u_k$ such that

$$x \;=\; f_{u_k} \circ \ldots \circ f_{u_1}(x_0) \quad \left(\text{resp.}\;\; x \;=\; f_{u_k}^{-1} \circ \ldots \circ f_{u_1}^{-1}(x_0)\right).$$

- $x_0 \in M$ is said to be *transitive* if there exists an open set $A \subset M$ such that, for each $x \in A$, it is possible to find a sequence of control values $u_1, \ldots, u_k$ and a sequence $\epsilon_1, \ldots, \epsilon_k$, with $\epsilon_j = \pm 1$, and such that

$$x \;=\; f_{u_k}^{\epsilon_k} \circ \ldots \circ f_{u_1}^{\epsilon_1}(x_0).$$

Theorem 4.5 *Consider an invertible analytic system Σ and let $\bar{x} \in M$ be a forward accessible point. Then the dimension of $d\Theta$ is maximum at $\bar{x}$, i.e. if $l = \max_{x \in M} \dim d\Theta(x)$, we have*

$$\dim d\Theta(\bar{x}) \;=\; l.$$

Thus, $\bar{x}$ is locally strongly observable if and only if $\bar{x}$ is locally weakly observable if and only if $\dim d\Theta(\bar{x}) \;=\; n$.

Proof. Since $\bar{x}$ is forward accessible, there exists an open set $A \subseteq M$ such that each point in A is reachable from $\bar{x}$. By analyticity, there exists $\tilde{x} \in A$ such that $\dim d\Theta_j(\tilde{x}) = l$, for some $j \geq 1$. Since $\tilde{x} \in A$, there exists a sequence of control values $u_1, \ldots, u_k$ such that:

$$\tilde{x} \;=\; f_{u_k} \circ \ldots \circ f_{u_1}(\bar{x}).$$

From Lemma 4.3 we have:

$$\dim d\Theta_{j+k}(\bar{x}) \geq \dim d\Theta_j(\tilde{x}) = l,$$

which implies $\dim d\Theta(\bar{x}) = l$, as desired.

The second statement follows from Proposition 3.3.

∎

The next result shows that for invertible models which have a compact state space, it is possible to give an a-priori bound on the number k such that $d\Theta_k$ has maximal dimension.

Theorem 4.6 *Consider an invertible analytic system Σ. Assume that M is a compact manifold and that any $x \in M$ is transitive. Then, there exists a $\bar{k} \geq 1$ such that:*

$$\dim d\Theta_{\bar{k}}(x) \;=\; l, \qquad \forall\, x \in M.$$

Proof. It is known (see [2]) that for invertible models whose state space is compact, transitivity implies that any $x \in M$ is forward accessible. Thus, using Theorem 4.5, we have that $\dim d\Theta(x) = l$ for each $x \in M$. Let $k_x \geq 1$ be the minimum k such that $\dim d\Theta_k(x) = l$. By continuity, for each $x \in M$, there exists O_x neighborhood of x such that $\dim d\Theta_{k_x}(y) = l$ for all $y \in O_x$. Then, the $\cup_{x\in M} O_x$ is an open covering of M. By compactness there exists a finite covering $\cup_{i=1}^{p} O_{x_i}$. Let:

$$\bar{k} = \max \left\{ k_{x_1}, \ldots, k_{x_p} \right\},$$

then $\dim d\Theta_{\bar{k}}(x) = l$ for all $x \in M$. ∎

4.2 Relation between forward-backward observability and forward observability

We look now at the relation between the two notions of local (strong or weak) forward observability and local (strong or weak) forward-backward observability. We will prove that, for analytic invertible systems, these two notions are equivalent in a generic set, and, pointwise for forward accessible points and for transitive equilibrium points.

Before stating and proving these results, we introduce some notations. We let:

$$\Theta_1^+ = \Theta_1^- = \Theta_1^{+,-} = \{h\},$$

and

$$\Theta_k^+ = \Theta_k,$$

$$\Theta_k^- = \{h(f_{u_j}^{-1} \circ \cdots \circ f_{u_1}^{-1}(\cdot)) \mid u_i \in U,\, i = 1, \cdots, j, \text{ and } 1 \leq j \leq k-1\},$$

$$\Theta_k^{+,-} = \{h(f_{u_j}^{\epsilon_j} \circ \cdots \circ f_{u_1}^{\epsilon_1}(\cdot)) \mid u_i \in U,\, \epsilon_i = \pm 1,\, i = 1, \cdots, j, \text{ and } 1 \leq j \leq k-1\}.$$

Moreover we let:

$$\Theta^+ = \Theta, \quad \Theta^- = \cup_{k\geq 1} \Theta_k^-, \quad \Theta^{+,-} = \cup_{k\geq 1} \Theta_k^{+,-},$$

and $l_k^\alpha = \max_{x\in M} \dim d\Theta_k^\alpha(x)$, $l^\alpha = \max_{x\in M} \dim d\Theta^\alpha(x)$, where $\alpha = +, -$, or $+, -$.

All of the previous results can be re-written with the $+$ for Σ, with the $-$ for Σ^-, and with $+, -$ for the system where we allow both dynamics.

Theorem 4.7 *Let Σ be an analytic invertible system. The following statements are equivalent:*

i) There exists a generic set $A_1 \subseteq M$ such that all $x \in A_1$ are locally weakly forward observable.

ii) There exists a generic set $A_2 \subseteq M$ such that all $x \in A_2$ are locally strongly forward observable.

iii) There exists a generic set $A_3 \subseteq M$ such that all $x \in A_3$ are locally weakly forward-backward observable.

iv) There exists a generic set $A_4 \subseteq M$ such that all $x \in A_4$ are locally strongly forward-backward observable.

Remark 4.8 Notice that, if we set: $A = A_1 \cap A_2 \cap A_3 \cap A_4$, then clearly $A \subseteq M$ is still a generic set. Thus we made rephrase the statement of Theorem 4.7 by saying that, if Σ is an analytic invertible system, then there exists a generic subset of the state space M in which the four different notions of local observability are equivalent.

Proof. Since strong implies weak and forward observability implies forward-backward observability, we only need to prove that *iii*) implies *ii*). We will establish this implication by contradiction.

Assume that almost every point is locally weakly forward-backward observable but not strongly locally forward observable. Since Σ is analytic, this implies that no state in M is strongly locally forward observable. Thus $l^+ < n$. Fix any $\bar{x} \in M$ such that $\dim d\Theta^+(\bar{x}) = l^+$. Let $H_1, \ldots, H_{l^+} \in \Theta^+$ be such that $dH_1(\bar{x}), \ldots, dH_{l^+}(\bar{x})$ is a basis for $d\Theta^+(\bar{x})$. Choose any set of complementary independent functions $g_1, \ldots, g_r$ ($r = n - l^+$), and let:

$$\tilde{A} = \left\{ x \in M \;\middle|\; \begin{array}{l} dH_1(x), \ldots, dH_{l^+}(x) \\ dg_1(x), \ldots, dg_r(x) \end{array} \quad \text{are linear. indep.} \right\}.$$

Since $\bar{x} \in \tilde{A}$, by analyticity, $\tilde{A}$ is an open and dense subset of M. Let

$$A = \left\{ x \in M \,|\, x \in \tilde{A} \;\text{ and }\; \dim d\Theta_n^{+,-}(x) = n \right\}.$$

Since Σ is analytic and locally weakly forward-backward observable from almost every point, then $\dim d\Theta_n^{+,-}(x) = n$ for almost every $x \in M$, thus A is still an open and dense subset of M. This follows from Theorem 3.5 for forward-backward observability, since the maps f_u are analytic and, being diffeomorphism, they are also open.
Fact. It is possible to define a sequence of states $\bar{x}_i$, for $i = -n, \ldots, n$ such that:

$$\left\{ \begin{array}{ll} \bar{x}_i \in A & \forall\, i \\ \bar{x}_{i+1} = f_0(\bar{x}_i) & i = -n, \ldots, n-1. \end{array} \right. \tag{10}$$

To define these states one may proceed as follows. Let:

$$\left\{ \begin{array}{lcl} A_{-n} & = & A \\ A_i & = & f_0\left(A_{i-1}\right) \cap A \end{array} \right.$$

Then one may choose any $\bar{x}_n \in A_n$ and let $\bar{x}_i = f_0^{-1}(\bar{x}_{i+1})$, $i = n-1, \ldots, -n$. Clearly this sequence of states satisfies both properties of equation (10).

Since A is open, by continuity, there exist $V \subset U$ (the set of possible inputs) neighborhood of 0 and, for each $i = -n, \ldots, n$, $W_i \subset A$ neighborhoods of $\bar{x}_i$, such that:

$$f_v\left(W_i\right) \subseteq W_{i+1}, \qquad \forall\, v \in V,\; i = -n, \ldots, n-1. \tag{11}$$

Since $W_i \subset A$, for each $i = -n, \ldots, n-1$, we may perform a local change of coordinates on W_i using $H_1, \ldots, H_{l^+}$ and $g_1, \ldots, g_r$ as new coordinates. We write $\hat{h}$, $\hat{f}$, $\hat{H}_j$, and $\hat{g}_j$ for h, f, H_j, and g_j respectively, in the new coordinates. In particular, if we denote by z the new coordinate, we may assume, without loss of generality, that $z_j = \hat{H}_j$, for $j = 1, \ldots, l^+$. Notice that the same change of coordinates can be performed on every W_i.

We claim that for $(x, u) \in W_i \times V$, the system Σ written in the new coordinates $z = (z_1, z_2)$, where z_1 represents the block of the first l^+ components, reads as:

$$\begin{cases} z_1(t+1) &= \hat{f}\,(z_1(t), u(t))\,, \\ z_2(t+1) &= \hat{f}\,(z_1(t), z_2(t), u(t))\,, \\ y(t) &= \hat{h}\,(z_1(t))\,. \end{cases} \tag{12}$$

Since $dh \in d\Theta$, clearly $\hat{h}$ depends only on the first l^+ coordinates, thus $\hat{h}(z) = \hat{h}\,(z_1(t))$. Moreover, since $d\,(H_i(f_u)) \in d\Theta$, we also have:

$$0 = \frac{\partial}{\partial z_j}\hat{H}_i(f_u) = \frac{\partial}{\partial z_j}\left(\hat{f}_u\right)_i, \qquad \forall\, j = l^+ + 1, \ldots, n, \;\; \forall\, i = 1, \ldots, l^+.$$

Thus Σ, in the z-coordinates reads as in (12). For $i = -n+1, \ldots, n$, let

$$\tilde{W}_i = \{\, f(x, v) \,|\, x \in W_{i-1}, \;\; v \in V \,\}\,.$$

It is clear that we can choose subsets $\hat{W}_i \subseteq \tilde{W}_i$ neighborhoods of $\bar{x}_i$, such that:

$$f_v^{-1}(x) \in W_{i-1}, \qquad \forall\, v \in V, \; \forall\, x \in \hat{W}_i.$$

Moreover, it holds that, by using the same coordinates z as before, the inverse system Σ^- for $(x, v) \in \hat{W}_i \times V$ $(i = -n+1, \ldots, n)$ reads as:

$$\begin{cases} z_1(t+1) &= \hat{f}^{-1}\,(z_1(t), u(t))\,, \\ z_2(t+1) &= \hat{f}^{-1}\,(z_1(t), z_2(t), u(t))\,, \\ y(t) &= \hat{h}\,(z_1(t))\,. \end{cases} \tag{13}$$

By continuity, there exists $\delta_0 > 0$ and $N_0 \subseteq \hat{W}_0$ neighborhood of $\bar{x}_0$ such that for all $x_0 \in N_0$, for all $1 \le k \le n$, for all $u_i \in U$, with $|u_i| < \delta_0$ $(i = 1, \ldots, k)$, and for all $\epsilon_i = \pm 1$ $(i = 1, \ldots, k)$, it holds that

$$x_k = f_{u_k}^{\epsilon_k} \circ \cdots \circ f_{u_1}^{\epsilon_1}(x_0) \in \hat{W}_{\epsilon_1 + \ldots + \epsilon_k}, \tag{14}$$

with $\hat{W}_{-n} = \{f_v^{-1}(x) \,|\, x \in \hat{W}_{-n+1}, \;\; v \in V\}$.

Since $\dim d\Theta_n^{+,-}(\bar{x}_0) = n$, there exists a neighborhood N of $\bar{x}_0$ such that each $x \in N$ is forward-backward distinguishable from $\bar{x}_0$ with a sequence of input values of length at most n.

Choose any $\tilde{x}_0 \in N \cap N_0$ such that in the z-coordinates the first l^+ components of $\bar{x}_0$ and of $\tilde{x}_0$ are equal. Then there exists $k \le n$, $u_1, \ldots, u_k \in U$, and $\epsilon_1, \ldots, \epsilon_k \in \{+1, -1\}$, such that:

$$h\left(f_{u_k}^{\epsilon_k} \circ \cdots \circ f_{u_1}^{\epsilon_1}\right)(\bar{x}_0) \neq h\left(f_{u_k}^{\epsilon_k} \circ \cdots \circ f_{u_1}^{\epsilon_1}\right)(\tilde{x}_0)$$

By analyticity, there exists $v_1, \ldots, v_k \in U$, with $|v_i| < \delta_0$ for all $i = 1, \ldots, k$, such that

$$h\left(f_{v_k}^{\epsilon_k} \circ \cdots \circ f_{v_1}^{\epsilon_1}\right)(\bar{x}_0) \neq h\left(f_{v_k}^{\epsilon_k} \circ \cdots \circ f_{v_1}^{\epsilon_1}\right)(\tilde{x}_0). \tag{15}$$

On the other hand, since $\tilde{x}_0, \bar{x}_0 \in N_0$, $|v_i| < \delta_0$ for all $i = 1, \ldots, k$, and $k \le n$, from equation (14) we have that:

$$f_{v_i}^{\epsilon_i} \circ \cdots \circ f_{v_1}^{\epsilon_1}(\tilde{x}_0) \in \hat{W}_{\epsilon_1 + \ldots + \epsilon_i},$$

and

$$f_{v_i}^{\epsilon_i} \circ \cdots \circ f_{v_1}^{\epsilon_1}(\bar{x}_0) \in \hat{W}_{\epsilon_1 + \ldots + \epsilon_i},$$

for all $i = 1, \ldots, k$. Thus, by the triangular form of Σ and Σ^- in $\hat{W}_i$, $i = -n, \ldots, n$, in the z-coordinates, the first l^+ components of $\tilde{x}_i$ and of $\bar{x}_i$ remain equal. In particular, this implies:

$$h(f_{v_i}^{\epsilon_i} \circ \cdots \circ f_{v_1}^{\epsilon_1}(\tilde{x}_0)) = h(f_{v_i}^{\epsilon_i} \circ \cdots \circ f_{v_1}^{\epsilon_1}(\bar{x}_0)),$$

which contradicts equation (15). ∎

From the previous Theorem, we can derive some pointwise versions of the equivalence between the two notions of observability.

Theorem 4.9 *Let Σ be an analytic invertible system, and $\bar{x} \in M$ be a forward accessible point. Then, $\bar{x}$ is locally (strong or weak) forward observable if and only if it is locally (strong or weak) forward-backward observable.*

Proof. Assume that $\bar{x}$ is a local weak forward-backward observable point. Since Σ is analytic, then we have that $l^{+,-} = n$. Thus, using the proof of Theorem 4.7, it must hold that $l^+ = n$. Since $\bar{x}$ is forward accessible, from Theorem 4.5 we know that $\dim d\Theta^+(\bar{x}) = l^+ = n$, which implies that $\bar{x}$ is also locally strongly forward observable.

Being all the other implications obvious, the statement is proved. ∎

In [2] it is proved that for analytic model the following two statements hold:

- if a state is an equilibrium point and it is transitive then it is also forward accessible,
- if the state space M is compact then forward accessibility and transitivity are equivalent.

From these facts and the Theorem 4.9, we get:

Corollary 4.10 Let Σ be an analytic invertible system, and $\bar{x} \in M$ be a transitive equilibrium point. Then, $\bar{x}$ is locally (strong or weak) forward observable if and only if it is locally (strong or weak) forward-backward observable.

Corollary 4.11 Let Σ be an analytic invertible system, and assume that M is a compact manifold and that the model Σ is transitive. Then, the two notions of local (strong or weak) forward observability and locally (strong or weak) forward-backward observability are equivalent.

The next example shows a system for which checking forward-backward observability requires a much smaller number of operations then checking forward observability.

Example 4.12 Fix any positive integer $N \geq 2$, and let g be an analytic function of one real variable such that:

(a) g is periodic of period $4N$,

(b) $g(x) = 0$ if and only if $x \in \mathbb{Z}$ and $|x| \leq N$,

We shall construct a model Σ on the bidimensional torus. Since g is periodic and analytic, also g' is periodic and analytic, thus there exists $L > 0$ such that $|g'(x)| \leq L$ for all $x \in \mathbb{R}$. Let $U = \left(-\frac{1}{2L}, \frac{1}{2L}\right)$ and parametrize the torus with coordinates in $[-2N, 2N) \times [-2N, 2N)$ where $2N$ and $-2N$ are identified. In these coordinates, the dynamics is described by

$$\left\{ \begin{array}{lcll} x_1(t+1) & = & x_1(t) + 1 + ug(x_1)\sin\left(\frac{\pi}{2N}x_2\right) & \text{mod. } 4N, \\ x_2(t+1) & = & x_2(t) + u & \text{mod. } 4N, \end{array} \right. \tag{16}$$

and $h(x_1, x_2) = x_1$.

We claim that this model is analytic and invertible. Analyticity is obvious. Notice that, the evolution of the second component is independent of the first one and clearly invertible. Thus, since $|\sin\left(\frac{\pi}{2N}x\right)| \leq 1$, to show that the whole map f_u is invertible, it is enough to prove that the map $d(x) := x + 1 + vg(x)$ mod. $4N$, for a given fixed $v \in U$ and x in the one dimensional torus, is one-to-one and onto. Since $d'(x) = 1 + vg'(x) > 0$, the map d without the *mod* operation is invertible as a map from $\mathbb{R}$ to $\mathbb{R}$. We first show that d is still onto on the torus. Fix $y \in [-2N, 2N)$, then there exists $x \in \mathbb{R}$ such that $x + 1 + ug(x) = y$. Given x, there exists $\bar{k} \in \mathbb{Z}$ such that $\bar{x} = x + 4N\bar{k} \in [-2N, 2N)$. Since g is periodic of period $4N$, we have $g(x) = g(\bar{x})$, which implies $\bar{x} + 1 + vg(\bar{x}) = y$ mod. $4N$, thus the map d is onto on the torus. We now prove injectivity by contradiction. Assume that d is not injective. This implies that there exist two points $y_1,\ y_2 \in [-2N, 2N)$ such that:

$$y_1 + 1 + vg(y_1) = y_2 + 1 + vg(y_2) + 4Nk,$$

for some integer k. Let $y_3 = y_2 + 4Nk$. Since $g(y_2) = g(y_3)$, we can rewrite the previous equation as:

$$y_1 + vg(y_1) = y_3 + vg(y_3).$$

This means:

$$|y_1 - y_3| = |u(g(y_1) - g(y_3))| \leq |u| \left| \sup_{y \in \mathbb{R}} g'(y) \right| |y_1 - y_3| \leq \frac{1}{2}|y_1 - y_3| < |y_1 - y_3|,$$

which gives the desired contradiction.

This model is easily seen to be transitive. In fact, in at most N steps, using Σ or the inverse system Σ^-, from any initial conditions we reach an open subset of M. Being M compact and Σ transitive, we know that forward observability and forward-backward observability are equivalent.

Notice that if we take $x_0 = (-N, 0)$, then to check forward observability we need to perform $2N + 1$ steps, in fact $\dim d\Theta_k^+(x_0) = 1$ for all $k \leq 2N$, while to check forward-backward observability we only need 1 step, since $\dim d\Theta_2(x_0) = 2$.

5 Conclusions

This paper has presented a number of results concerning the observability of discrete time nonlinear systems. Criteria to check observability, have been provided in terms of appropriate codistributions. For relevant classes of systems, such as invertible models, these criteria have been strengthened and more practical criteria can be given. Moreover, for invertible systems, we have define the notion of forward-backward observability, which means that it is possible to distinguish two states by evolving the system both forward and backward in time. Since observability implies forward-backward observability, it is natural to ask the question of wheather the converse is also true. We have given an example of a system which is forward-backward observable but not observable, as well as a number of sufficient conditions for the two notions to be equivalent. We also showed a simple example where the equivalence of the two notions can be used to render the test of observability more efficient.

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