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Intersecting D-branes, Chern-kernels and the inflow mechanism

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Abstract

We analyse a system of arbitrarily intersecting D-branes in ten-dimensional supergravity. Chiral anomalies are supported on the intersection branes, called I-branes. For non-transversal intersections anomaly cancellation has been realized until now only cohomologically but not locally, due to short-distance singularities. In this paper we present a consistent local cancellation mechanism, writing the δ-like brane currents as differentials of the recently introduced Chern-kernels, $J = dK$. In particular, for the first time we achieve anomaly cancellation for dual pairs of D-branes. The Chern-kernel approach allows to construct an effective action for the RR-fields which is free from singularities and cancels the quantum anomalies on all D-branes and I-branes.

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1. Introduction and summary

Anomaly cancellation represents a basic quantum-consistency check for extended objects in M-theory, and constrains eventually the physically allowed excitations. A par-

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ticularly interesting case regards D-branes in ten dimensions. IIA-branes carry an odd-dimensional worldvolume and are trivially anomaly-free, while IIB-branes carry an even-dimensional worldvolume and are actually plagued by gravitational anomalies; the problem of their cancellation has been addressed for the first time in [1].

However, D-branes may also intersect with each other. The requirement of anomaly freedom for such configurations has indeed been used in [2] to deduce the anomalous (Wess–Zumino) couplings of the IIA and IIB RR-potentials to the fields on the branes: if the intersection manifold, called I-brane, is even-dimensional there are potential anomalies supported on it, which have to be cancelled by adding specific Wess–Zumino terms to the action and by modifying, correspondingly, the Bianchi-identities for RR-curvatures. Anomalies on I-branes represent the main topic of the present paper.

One has to distinguish two kinds of I-branes. In a D-dimensional spacetime two D-branes with i-dimensional worldvolume M_i and, respectively, j-dimensional worldvolume M_j may indeed intersect in two different ways, depending on the dimension of the I-brane manifold $M_{ij} = M_i \cap M_j$. In the first (generic) case we have

$$\dim(M_{ij}) = i + j - D,$$

and the intersection is called *transversal*; for such intersections the normal bundles of the two branes do not intersect, $N_{ij} \equiv N_i \cap N_j = \emptyset$. If $i + j - D < 0$ it is understood that $M_{ij} = \emptyset$. An example of a transversal intersection are two planes in three dimensions that intersect along a line.

In the second (exceptional) case the dimension of the I-brane satisfies

$$\dim(M_{ij}) > i + j - D,$$

and the intersection is called *non-transversal*. In this case $N_{ij} \neq \emptyset$ and $\dim(N_{ij}) = \dim(M_{ij}) + D - (i + j)$. If $i + j - D < 0$ it is understood that $M_{ij} \neq \emptyset$. Examples in three dimensions are two coinciding planes, or two lines which intersect in a point.

The anomaly cancellation mechanism presented in [2] applies to transversal I-branes, while the attempt of [1] was to include the case of non-transversal I-branes as well. Actually, the anomaly cancellation mechanism of [1] achieves only a "cohomological" cancellation, in a sense that we will specify more precisely in a moment. The main purpose of the present paper is to fill this gap, i.e., to present a cancellation mechanism for non-transversal intersections which works "locally", point-wise, as explained below.

The difference between these two kinds of I-branes can be translated in the language of differential forms as follows. Introduce the δ-function supported Poincaré dual forms for M_i and M_j, i.e., their "currents" J_i and J_j, of degree $D - i$ and $D - j$ respectively, and the current J_{ij} associated to M_{ij}, of degree $D - \dim(M_{ij})$. Then for transversal intersections the product $J_i J_j$ is well-defined and one has simply

$$J_i J_j = J_{ij}. \tag{1.1}$$

For non-transversal intersections the degree of the product $J_i J_j$ is greater then the degree of J_{ij} but, moreover, the product itself is ill defined. The reason is that since N_{ij} is non-empty, there exists at least one direction in N_{ij}, parametrized by a coordinate say u, such that J_i as well as J_j contain a factor $du\,\delta(u)$. The product $J_i J_j$ is therefore of the kind 0 $(du \wedge du)$ times ∞ $(\delta(u)\delta(0))$.

For what concerns anomalies on I-branes, the obstacle to their cancellation on non-transversal intersections arises as follows. Suppose first that the intersection is transversal. Then the anomaly polynomial due to chiral fermions on the I-brane M_{ij} is nonvanishing, and it amounts to a sum of factorized terms [1,2],

$$P_{ij} = 2\pi P_i P_j, \tag{1.2}$$

$P_i(P_j)$ being supported on $M_i(M_j)$. The anomaly is given by the descent[1]

$$\mathcal{A} = 2\pi \int_{M_{ij}} (P_i P_j)^{(1)}. \tag{1.3}$$

It is cancelled by a Wess–Zumino term in the action of the form

$$S_{\mathrm{WZ}} = 2\pi \int_{M_i} P_i \tilde{C}, \tag{1.4}$$

where the RR-potential $\tilde{C}$ entails an anomalous transformation supported on M_j,

$$\delta\tilde{C} = -P_j^{(1)} J_j. \tag{1.5}$$

The WZ-term varies according to

$$\delta S_{\mathrm{WZ}} = -2\pi \int_{M_i} P_j^{(1)} J_j P_i = -2\pi \int J_i J_j (P_i P_j)^{(1)}, \tag{1.6}$$

which cancels $\mathcal{A}$ thanks to (1.1).

For non-transversal intersections the addenda in the anomaly polynomial factorize only partially [1]

$$P_{ij} = 2\pi P_i P_j \chi_{ij},$$

due to the presence of the Euler-form $\chi_{ij} \equiv \chi(N_{ij})$ of the now no longer vanishing intersection of the normal bundles N_{ij}—a form of degree $\dim(M_{ij}) + D - (i + j)$. The anomaly reads then

$$\mathcal{A} = 2\pi \int_{M_{ij}} \chi_{ij} (P_i P_j)^{(1)}. \tag{1.7}$$

The Wess–Zumino term is still given by (1.4) but its *formal* variation leads to the now ill defined expression (1.6).

We can now make a more precise statement of the problem attacked in this paper and outline its solution. For non-transversal intersections the quantum anomaly (1.7) is still well-defined, what is ill defined is the Wess–Zumino term (1.4) itself, since its variation leads to an ill defined expression. The problem consists therefore in constructing a well-defined Wess–Zumino term and, afterwards, in checking whether its variation cancels (1.7) or not.

1 As usual for a generic polynomial we set $P = dP^{(0)}$, $\delta P^{(0)} = dP^{(1)}$.

The authors of [1] proposed a partial solution to this problem, maintaining the above Wess–Zumino term together with its variation (1.6), and trying to give a meaning to the product "$J_i J_j$". Clearly, to cancel the anomaly what one would need is the identification

$$J_i J_j \leftrightarrow J_{ij} \chi_{ij}. \tag{1.8}$$

As it stands this identification is rather contradictory since $J_i J_j$ is simply a product of δ-functions, while the r.h.s. contains, apart from δ-functions, the gravitational curvatures present in χ_{ij}. Moreover, the l.h.s. is ill defined. The authors of [1] proposed first to substitute say J_i by a smooth *cohomological* representative $\hat{J}_i$. Then they showed that the product $\hat{J}_i J_j$ is *cohomologically equivalent to* $J_{ij}\chi_{ij}$, in the sense of de Rham. Although this is clearly not enough to realize a local anomaly cancellation mechanism, the above identification bears convincingly the correct idea.

The main lines of our solution are, indeed, as follows. The expression (1.4) itself looks canonical and rigid: the unique feature one can try to change is the definition of the RR-potential $\tilde{C}$. Above it is indeed (implicitly) assumed that the RR-field strength is given in terms of $\tilde{C}$ as[2]

$$dR = P_j J_j \leftrightarrow R = d\tilde{C} + P_j^{(0)} J_j, \tag{1.9}$$

which obliges $\tilde{C}$ to the transformation law (1.5), carrying a δ-like singularity on M_j, meaning that $\tilde{C}$ itself is singular on M_j, and therefore that (1.4) is ill defined. Our strategy instead consists in keeping (1.4), while introducing a RR-potential that is regular on M_j, actually on all branes. A key step in this direction is to search for a convenient antiderivative K_j of the current,

$$J_j = dK_j. \tag{1.10}$$

Then one can solve the Bianchi identity for R alternatively in terms of a different potential

$$R = dC + P_j K_j, \qquad \tilde{C} = C - P_j^{(0)} K_j, \tag{1.11}$$

where C is invariant and, for a convenient choice of K_j, regular on M_j. This time $\delta\tilde{C} = -dP_j^{(1)} K_j$ and the variation of the Wess–Zumino term (1.4) amounts to

$$\delta S_{\mathrm{WZ}} = -2\pi \int_{M_i} d(P_i P_j)^{(1)} K_j = -2\pi \int d(J_i K_j)(P_i P_j)^{(1)}.$$

The difference w.r.t. (1.6) is that for a convenient choice of K_j *the product* $J_i K_j$ *may be well-defined together with its differential*—contrary to what happens to $J_i J_j$. The apparent paradox is solved by the fact that, due to the singularities present, one is not allowed to use Leibnitz's rule when computing $d(J_i K_j)$.

The key observation of the present paper is that if one chooses for K_j a Chern-kernel [3] then the product is not only well-defined, but one also has the new fundamental identity

$$d(J_i K_j) = J_{ij} \chi_{ij}, \tag{1.12}$$

[2] We focus here only on the above anomaly; the complete Bianchi identity for a RR-curvature is more complicated, especially for the presence of the B_2-field, and it is given in the text.

realizing in some sense the "identification" (1.8), which is precisely what is needed to cancel the anomaly. To be precise, this formula holds whenever $M_i \not\subset M_j$. The extremal case $M_i \subset M_j$ needs a slight adaptation that is given in the text. For previous applications of the Chern-kernel approach to anomaly cancellation see [4–7].

A special case of non-transversal I-branes is represented by a couple of electromagnetically dual branes,

$$i + j = D - 2,$$

e.g., a D1- and a D5-brane, which have a non-empty intersection manifold M_{ij}. Then the intersection of the normal bundles has dimension $\dim(N_{ij}) = \dim(M_{ij}) + 2$, and if it is even it has a non-vanishing Euler-form of the same degree. In this case the anomaly polynomial on M_{ij} is given simply by the Euler-form itself [1],

$$P_{ij} = 2\pi\, \chi_{ij},$$

but its cancellation has not yet been achieved, not even "cohomologically"; for the solution of a particular example in $D = 11$ see however [7]. As stated in [1], the cancellation of these anomalies requires "a more powerful approach": Chern-kernels provide actually such an approach. Indeed, in this case the relevant contribution in the Bianchi-identity, realizing the minimal coupling but ignored in [1], is

$$dR = J_j \leftrightarrow R = dC + K_j, \tag{1.13}$$

and the Wess–Zumino term is conveniently written as the integral over an eleven-dimensional manifold, with spacetime R^{10} as boundary, of a closed eleven-form:

$$S_{\mathrm{WZ}} = 2\pi \int_{M_{11}} \left(R J_i - J_{ij}\, \chi_{ij}^{(0)}\right). \tag{1.14}$$

The eleven-form is closed thanks to (1.12), and

$$\delta S_{\mathrm{WZ}} = -2\pi \int_{M_{11}} d\left(J_{ij}\, \chi_{ij}^{(1)}\right) = -2\pi \int_{R^{10}} J_{ij}\, \chi_{ij}^{(1)} = -2\pi \int_{M_{ij}} \chi_{ij}^{(1)},$$

which cancels the anomaly. Again, as we will see the definition (1.13) leads to a potential C that is regular on M_j.

A third case regards the anomalies on the (even-dimensional) D-branes of IIB-supergravity. Formally the anomalies supported on a D-brane can be interpreted as anomalies on the I-brane of two copies of the same D-brane (self-intersection). In light of this interpretation these anomalies are just a special case of anomalies on I-branes (their cancellation has been discussed in [1], again from a cohomological point of view). So the Chern-kernel approach furnishes automatically a consistent local cancellation mechanism also for IIB D-branes.

For concreteness in this paper we consider a system of arbitrarily interacting and intersecting Abelian IIB-branes (one for each woldvolume dimension), the case of Abelian IIA-branes requiring only a straightforward adaptation. Actually, IIB-branes have a richer anomaly structure because, being even-dimensional, they carry anomalies even in the absence of intersections. The generalization to non-Abelian branes is exposed briefly in the conclusions.

Usually a magnetic equation of the kind $dR = J_j$ requires the introduction of a Dirac-brane, as antiderivative of J_j, whose unobservability is guaranteed by charge quantization. The consistency of the employment of Chern-kernels as antiderivatives, instead of Dirac-branes, has been proven in [4].

In Section 2 we recall the definition of odd (IIB) and even (IIA) Chern-kernels, and review in a self-contained way their basic properties. In Section 3 we present the basis for the fundamental identity (1.12). In Section 4 we show how one arrives at formula (1.14) for a pair of dual branes, explaining the interplay between Dirac-branes and Chern-kernels. In Section 5 we recall the specific form of the anomalies produced by chiral fermions on D-branes and I-branes, we give the complete set of Bianchi-identities/equations of motion for the RR-field strengths of IIB-Sugra in presence of branes, and present their solutions in terms of Chern-kernels and regular potentials. This section is based on a systematic application and elaboration of our proposals for the introduction of regular potentials, made in (1.11) and (1.13). We take also a non-vanishing NS B_2-field into account, whose consistent inclusion is not completely trivial. In this section we write eventually the action, in particular the Wess–Zumino term, producing the correct equations of motion (for the "basic" potentials C_0, C_2 and C_4), verifying that it is well-defined and that it cancels all anomalies. Section 6 is more technical, in that there we write a manifestly duality-invariant (physically equivalent) action, in which the RR-potentials C_0, C_2, C_4 and their duals C_6, C_8 appear on the same footing. In this form the distinctive features of our action with respect to previous results emerge more clearly. Section 7 is devoted to generalizations and conclusions.

We remark briefly on our conventions and framework. We will assume that there are no topological obstructions in spacetime, in particular closed forms in the bulk are then always exact. Since we are in presence of δ-like currents, for consistency differential forms are intended as distribution-valued, and the differential calculus is performed in the sense of distributions. This implies that our differential operator d is *always* nilpotent, $d^2 = 0$. With our conventions it acts from the right rather than from the left.

2. Chern-kernels: definition and properties

In this section we review briefly the definition of Chern-kernels and recall their main properties [4]. Since we will treat in detail only IIB-branes, that have an even-dimensional worldvolume, we concentrate mainly on odd Chern-kernels, but for completeness and comparison we report also shortly on even kernels. For more details we refer the reader to the above reference.

2.1. Odd kernels

Let M be a closed $(D-n)$-dimensional brane worldvolume in a D-dimensional spacetime, and introduce a set of normal coordinates y^a, $(a = 1, \dots, n)$ associated to M; the brane stays at $y^a = 0$. Then locally one can write the current associated to M as

$$J = \frac{1}{n!}\varepsilon^{a_1\ldots a_n}\, dy^{a_1}\cdots dy^{a_n}\,\delta^n(y). \tag{2.1}$$

One can also introduce an $SO(n)$-connection A^{ab} and its curvature $F = dA + AA$ (both are target-space forms), which are only constrained to reduce, if restricted to M, respectively to the $SO(n)$-normal-bundle connection and curvature, defined intrinsically on M.

For odd rank Chern-kernels (even currents, IIB-branes) n is even, $n = 2m$. Then one can define the Euler n-form[3] associated to F and its Chern–Simons form,

$$\chi = \frac{1}{m!(4\pi)^m}\varepsilon^{a_1\ldots a_n} F^{a_1a_2}\cdots F^{a_{n-1}a_n} = d\chi^{(0)}.$$

Its anomaly descent is indicated as usual by $\delta\chi^{(0)} = d\chi^{(1)}$. Notice that the rank of the Euler-form equals the rank of the current. The Chern-kernel K associated to the even current J is written as the sum

$$K = \Omega + \chi^{(0)}, \tag{2.2}$$

$$dK = J, \tag{2.3}$$

where Ω is an $SO(n)$-*invariant* $(n-1)$-*form with inverse-power-like singularities on* M, polynomial in F and $D\hat{y} = d\hat{y} - A\hat{y}$, where $\hat{y}^a = y^a/|y|$. For the expression of Ω for a generic n see [4]; for example for $n = 2$ and $n = 4$ respectively, the formula reported there gives

$$\Omega = -\frac{1}{2\pi}\varepsilon^{a_1a_2}\hat{y}^{a_1}D\hat{y}^{a_2}, \tag{2.4}$$

$$\Omega = -\frac{1}{2(4\pi)^2}\varepsilon^{a_1\ldots a_4}\hat{y}^{a_1}D\hat{y}^{a_2}\left(4F^{a_3a_4} + \frac{8}{3}D\hat{y}^{a_3}D\hat{y}^{a_4}\right). \tag{2.5}$$

Chern-kernels are not unique due to the arbitrariness of normal coordinates and of A *away from the brane*, i.e., in the bulk, and due to the presence of the non-invariant Chern–Simons form $\chi^{(0)}$. But since for a different kernel one has in any case $dK' = J$, one obtains

$$K' = K + dQ, \tag{2.6}$$

for some target-space form Q. What matters eventually is the behaviour of Q on M. Since Ω has a singular but *invariant* behaviour near the brane, it is only $\chi^{(0)}$ that induces an *anomalous but finite* change on M,

$$Q|_M = \chi^{(1)}\big|_M. \tag{2.7}$$

The transformation (2.6) has been called Q-transformation in [4] and it is in some sense the analogous of a change of Dirac-brane. From (2.7) one sees that on M a Q-transformation reduces to a normal-bundle $SO(n)$-transformation, and it can give rise to anomalies supported on M. On the contrary, we demand a theory to be Q-*invariant in the bulk*. As we will see below, also the RR-potentials have to transform under Q-transformations, and so Q-invariance and anomalies are intimately related.

[3] In the following odd-dimensional Euler forms are taken to be zero *by definition*.

2.2. Even kernels

For odd currents (IIA-branes) n is odd, and the kernel is even. In this case it is only constructed from an *$SO(n)$-invariant $(n-1)$-form with inverse-power-like singularities on M*, $(K = \Omega)$

$$K = \frac{\Gamma(n/2)}{2\pi^{n/2}(n-1)!}\varepsilon^{a_1\cdots a_n}\hat{y}^{a_1}\mathcal{F}^{a_2a_3}\cdots\mathcal{F}^{a_{n-1}a_n}, \tag{2.8}$$

$$dK = J, \tag{2.9}$$

with $\mathcal{F}^{ab} \equiv F^{ab} + D\hat{y}^a D\hat{y}^b$. The reason is that for an odd normal bundle the Euler form is vanishing.

Also this kernel is defined modulo Q-transformations, $K' = K + dQ$, but since Ω is invariant this time we have

$$Q|_M = 0.$$

We can thus write in general $K = \Omega + \chi^{(0)}$, with the convention that for even kernels the Euler Chern–Simons form is set to zero.

3. A new identity

In this section we illustrate the new identity

$$d(J_i K_j) = J_{ij}\chi_{ij}, \quad M_i \not\subset M_j, \tag{3.1}$$

where it is understood that the Euler-form of an empty normal bundle is unity, $\chi(\emptyset) = 1$. Its proof is worked out in Appendix A. When $i + j < D$ (3.1) is an identity between forms whose degree exceeds D. In that case M_i and M_j have to be extended to worldvolumes in a larger spacetime, keeping the degrees of the K's and the J's unchanged; see, e.g., [4].

The "non-extremality" condition $M_i \not\subset M_j$ is needed to guarantee that the product $J_i K_j$ is well-defined, implying that also its differential is so. In Appendix A it is then shown that $d(J_i K_j)$ is (1) closed, (2) invariant, (3) supported on M_{ij} and (4) constructed from the curvatures of N_{ij}. The above identity follows then essentially for uniqueness reasons.

To simplify some formulae of the following sections, and motivated by (3.1), we *define* for arbitrary intersections

$$(J_i J_j)_{\mathrm{reg}} \equiv J_{ij}\chi_{ij}.$$

Consider now an extremal intersection $M_i \subset M_j$, where the product $J_i K_j$ is ill defined. Fortunately, as we will see, in the dynamics of intersecting D-branes such a product will never show up. However, for notational convenience it will be useful to *define*

$$(J_i K_j)_{\mathrm{reg}} \equiv \begin{cases} J_i K_j & \text{if } M_i \not\subset M_j, \\ J_i \chi_j^{(0)} & \text{if } M_i \subset M_j,\ K_j \text{ odd}, \\ 0 & \text{if } M_i \subset M_j,\ K_j \text{ even}. \end{cases} \tag{3.2}$$

This definition is motivated as follows. If $M_i \subset M_j$ then for the normal bundles we have $N_j = N_{ij}$, and therefore for the Euler-forms $\chi_j = \chi_{ij}$. If K_j is of odd rank, then J_j is even and $\chi_j \neq 0$; if K_j is even, then J_j is odd and $\chi_j = 0$. This implies that with the above definitions we have *in any case*

$$d(J_i K_j)_{\mathrm{reg}} = (J_i J_j)_{\mathrm{reg}}.$$

We conclude this section presenting an alternative, but equivalent, way of writing the information contained in (3.1). We may rewrite its l.h.s. in terms of the restriction of K_j to M_i, $J_i K_j = J_i(K_j|_{M_i})$. Since this restriction is well-defined, in this form we can apply Leibnitz's rule to get

$$J_i d(K_j|_{M_i}) = J_{ij}\chi_{ij}.$$

Denoting the δ-function supported Poincaré-dual of M_{ij} w.r.t. M_i with $\mathcal{J}_{ij}$—this is a form on M_i and not on target-space—we have $J_{ij} = J_i \mathcal{J}_{ij}$. The target-space relation (3.1) is then equivalent to the relation on M_i,

$$d(K_j|_{M_i}) = \mathcal{J}_{ij}\chi_{ij}. \tag{3.3}$$

We can go one step further and observe that, if the intersection is effectively non-transversal, i.e., $\chi_{ij} \neq 1$, then the above relation is equivalent to the existence of a form $\mathcal{L}_{ij}$ on M_i such that

$$K_j|_{M_i} - \mathcal{J}_{ij}\chi_{ij}^{(0)} = d\mathcal{L}_{ij}, \tag{3.4}$$

transforming under Q-transformations of K_j and under normal bundle transformations of N_{ij} respectively as

$$\delta\mathcal{L}_{ij} = Q_j|_{M_i}, \qquad \delta\mathcal{L}_{ij} = -\mathcal{J}_{ij}\chi_{ij}^{(1)},$$

apart from closed forms.

If the intersection is extremal, $K_j|_{M_i}$ is not defined and according to above one would rather consider the expression $\chi_j^{(0)}|_{M_i} - \mathcal{J}_{ij}\chi_{ij}^{(0)}$, which vanishes identically since $\mathcal{J}_{ij} = 1$. This suggests to *define*, for $M_i \subset M_j$, $\mathcal{L}_{ij} = 0$.

4. Basic applications

We present here two basic applications of the above identity, to dual pairs of branes and to self-dual branes. These cases enter as main building blocks in the construction of the action for arbitrary intersections, given in the next section. These two examples illustrate the role played by Q-invariance, which is fundamental also in the general case. For the sake of clarity we ignore here all other couplings but the minimal ones. We restore now the brane charges g_i and Newton's constant G, taken until now as $g_i = 1$ and $G = 1/2\pi$.

4.1. Electromagnetically dual pairs of branes

Suppose that a RR-field strength satisfies the Bianchi identity and equation of motion

$$dR = g_j J_j, \tag{4.1}$$

$$d * R = g_i J_i, \tag{4.2}$$

where J_j (J_i) is the current on the high-dimensional (low-dimensional) magnetic (electric) D-brane with worldvolume M_j, (M_i) and charge g_j (g_i). M_i and M_j form an electromagnetically dual pair whose dimensionalities satisfy $i + j = 8$. For the self-dual D3-brane the equation of motion has to be replaced by $R = *R$, but we do not consider this case in the present section.

To write an action for such a system one must first solve (4.1), introducing a potential for R, and to do this one must search for an antiderivative of J_j or of J_i (or of both).

There are two candidates for such antiderivatives. The standard one is a Dirac-brane M_{j+1}, i.e., a brane whose boundary is M_j, with δ-function supported Poincaré dual, say W_j. Then one has

$$J_j = dW_j.$$

The same can be done for J_i. The $(9 - j)$-form W_j carries by construction δ-like singularities on M_{j+1} and hence also on M_j. Since the Dirac-brane is unphysical one must eventually ensure that it is unobservable. In this case one would solve the Bianchi identity (4.1) through $R = dC + g_j W_j$, and C would carry the known singularities along the Dirac-brane and on M_j.

The second candidate for an antiderivative is a Chern-kernel K_j, which carries *invariant inverse-power-like singularities on* M_j,

$$J_j = dK_j,$$

and due to this fact the potential C introduced according to $R = dC + g_j K_j$ is *regular* on M_j, because all singularities are contained in K_j. Since the Chern-kernel is defined modulo Q-transformations, one must eventually ensure that the theory is Q-invariant, i.e., independent of the particular kernel one has chosen.

We recall now the recipe developed in [4] for writing an action for the system (4.1), (4.2) if M_i and M_j have a transversal intersection. Then we will present its adaptation to a non-transversal one. (Remember that for a dual pair of branes a transversal intersection amounts to no intersection at all, while a non-transversal one means simply $M_{ij} \neq \emptyset$.) The recipe goes as follows. Introduce a Chern-kernel for the magnetic brane $J_j = dK_j$, and a Dirac-brane for the electric brane $J_i = dW_i$. Then solve the Bianchi-identity (4.1) according to

$$R = dC + g_j K_j. \tag{4.3}$$

Under Q-transformations of K_j the potential must now also transform,

$$C' = C - g_j Q_j, \qquad K_j' = K_j + dQ_j, \tag{4.4}$$

to keep the field-strength R invariant. For the restrictions on M_j (2.7) implies then, for an odd kernel,

$$\delta C|_{M_j} = -g_j \chi_j^{(1)}, \tag{4.5}$$

while for an even kernel $C|_{M_j}$ is invariant. From these transformations one sees that the potential C is a field regular on M_j, because $\delta C|_{M_j}$ is finite. Equivalently, all singularities of R are contained in the Chern-kernel K_j, more precisely in the invariant form Ω_j.

The action, generating the equation of motion (4.2), is given by

$$S = \frac{1}{G} \int \left(\frac{1}{2} R * R + g_i R W_i \right). \tag{4.6}$$

Under a change of the (unphysical) electric Dirac-brane, $W_i \to W_i + dZ_i$, this action changes by an integer multiple of 2π, if the quantization condition

$$\frac{g_i g_j}{G} = 2\pi n$$

holds, see, e.g., [10]. Moreover, the action S is trivially free from gravitational anomalies.

Two comments are in order. First, the equation of motion (4.2) could also be obtained more simply from the action $\frac{1}{G} \int (\frac{1}{2} R * R + g_i \, dC \, W_i) = \frac{1}{2G} \int R * R + \frac{g_i}{G} \int_{M_i} C$. However, this action is inconsistent in that it breaks Q-invariance in the bulk, because C is not Q-invariant. Second, introducing an eleven-dimensional manifold which bounds target-space, the above WZ-term can be rewritten equivalently as

$$\frac{g_i}{G} \int R W_i = \frac{g_i}{G} \int_{M_{11}} (R J_i - g_j J_j W_i) = \frac{g_i}{G} \int_{M_{11}} R J_i \quad \mod 2\pi, \tag{4.7}$$

where in the last step we used that $\int J_j W_i$ is integer [10]. The eleven-form $\frac{g_i}{G} R J_i$ is indeed closed modulo a well-defined integer form,[4] because we are assuming that M_i and M_j are not intersecting: $d(\frac{g_i}{G} R J_i) = 2\pi n J_j J_i$.

Let us now adapt this recipe to a dual pair with a non-empty intersection manifold M_{ij}, starting from (4.7). In this case, due to the identity (3.1), the form $\frac{g_i}{G} R J_i$ is no longer closed, not even modulo integer forms, and there is the additional problem that for extremal intersections the product $K_j J_i$ is ill defined. But, in turn, thanks to this identity we know how to amend the WZ-term. Replace

$$\frac{g_i}{G} \int_{M_{11}} R J_i \to S_{\mathrm{WZ}} \equiv \frac{g_i}{G} \int_{M_{11}} \left((R J_i)_{\mathrm{reg}} - g_j J_{ij} \chi_{ij}^{(0)} \right), \tag{4.8}$$

where the subscript reg refers to the product $K_j J_i$, defined in (3.2): the integrand is now again a *well-defined closed* eleven-form. This is the WZ-term anticipated in the introduction, see (1.14), holding now for extremal intersections as well.

[4] "Integer forms" are by definition forms that integrate over an arbitrary manifold (closed or open) to an integer. It can be shown that all such forms are necessarily δ-functions on some manifold M. In particular our currents J are integer forms, and a product of integer forms, whenever it is well-defined, is an integer form as well.

For an extremal intersection, $M_i \subset M_j$, due to $\chi_j = \chi_{ij}$ and $J_i = J_{ij}$, the above WZ simplifies to

$$S_{\mathrm{WZ}} = \frac{g_i}{G} \int_{M_{11}} dC\, J_i = \frac{g_i}{G} \int_{M_i} C.$$

Despite the formal cancellation of the "anomalous" term $\chi_{ij}^{(0)}$, the anomaly itself has not disappeared. Indeed, the potential C transforms under a Q-transformation, and thanks to (4.5), since $M_{ij} = M_i \subset M_j$, we have

$$\delta C|_{M_i} = -g_j \chi_j^{(1)} = -g_j \chi_{ij}^{(1)}.$$

The anomaly polynomial supported on M_{ij} is then $-2\pi n \chi_{ij}$ also for extremal intersections, and the action remains Q-invariant in the bulk.

As we observed, for non-empty (non-extremal) intersections it is Q-invariance that forces to put in (4.8) the Q-invariant combination RJ_i instead of the closed form $dC\, J_i$. So it is eventually Q-invariance in the bulk that requires the presence of the anomalous term $J_{ij}\chi_{ij}^{(0)}$, needed to get back a closed eleven-form in the WZ. This interplay between Q-invariance and anomalies will be a guiding principle also in our construction of an anomaly-free effective action for an arbitrary system of intersecting D-branes in the next section.

We may then summarize the properties of S_{WZ} in (4.8) as follows. (1) It is written in terms of a potential C that carries an anomalous transformation law but that is regular on M_j. (2) It gives rise to the equation of motion (4.2). (3) It exhibits no singularities. (4) For transversal intersections ($J_{ij} = 0$) it is well-defined modulo 2π, Q-invariant and anomaly free. (5) For non-transversal intersections it is well-defined, it carries the gravitational anomaly $-2\pi n \chi_{ij}$ supported on M_{ij}, and it is Q-invariant in the bulk. (6) For empty intersections the second term drops, because J_{ij} is vanishing, and one gets back the WZ-term for transversal intersections.

We conclude and summarize this subsection, giving a ten-dimensional representation of (4.8). For transversal intersections one introduces an electric Dirac-brane for M_i, $J_i = dW_i$, and uses that (4.8) reduces to (4.7), while for non-transversal ones one uses (3.4). The results are:

$$S_{\mathrm{WZ}} = \frac{g_i}{G} \int_{M_i} C + \begin{cases} \frac{g_i g_j}{G} \int K_j W_i & \text{for transversal intersections,} \\ \frac{g_i g_j}{G} \int_{M_i} \mathcal{L}_{ij} & \text{for non-transversal intersections.} \end{cases} \tag{4.9}$$

We recall that for extremal intersections $\mathcal{L}_{ij}$ is zero by definition. The eleven-dimensional representation (4.8) for S_{WZ} is, however, *universal* in that it holds for arbitrary intersections, transversal or not.

4.2. Self-dual branes

As second example we consider a self-dual brane, i.e., a brane M_j with $2j = D - 2$, which is coupled to a field-strength satisfying a self-duality condition rather than a

Maxwell equation,

$$dR = g_j J_j, \quad R = *R.$$

The brane we will be interested in is clearly the self-dual D3-brane in IIB-Sugra. Strictly speaking, this case can be treated without using the identity (3.1), so we recall simply the results of [4]. One solves the Bianchi identity as above, $R = dC + g_j K_j$, and using the covariant PST-approach [8] to deal with the self-duality condition, one can write the action as

$$S = \frac{1}{4G}\int (R * R + f_4 f_4) + \frac{g_j}{2G}\int_{M_j} C.$$

The additional (overall) factor of $1/2$ will be explained in the next section. A part from this the WZ appearing here coincides with the one of the extremal intersection of a dual pair discussed above; a self-dual brane amounts in some sense indeed to $M_i = M_j$, a special case of $M_i \subset M_j$, $i + j = D - 2$. The potential C transforms under Q according to (4.5). This means that S entails the anomaly polynomial

$$-\frac{g_j^2}{2G}\chi_j = -\frac{2\pi n}{2}\chi_j,$$

supported on M_j.

Due to the presence of the NS B-field and of the gauge-fields on the branes, the technical details of the anomaly cancellation mechanism of a generic system of intersecting D-branes in IIB-Sugra appear slightly involved. For this reason we presented the new ingredients of the Chern-kernel approach, which are crucial for the entire construction, separately and in some detail in these first four sections.

We conclude this more general part by stressing again that the Chern-kernel approach, as we saw, allows to write well-defined actions which entail no ambiguities or singularities, despite the dangerous short-distance configurations represented by branes intersecting non-transversally. Furthermore, this approach does not require any regularization (or smoothing) of the currents, a procedure that would immediately run into troubles with the unobservability of the Dirac-brane, see [5,10].

5. Anomaly cancellation for intersecting IIB D-branes

In this section we consider the full interacting system of IIB supergravity and Abelian D-branes (one for each dimensionality) with arbitrary intersections. We recall first the quantum anomalies of the system, including from the beginning the NS two-form B in the dynamics. Then we give the full set of Bianchi identities and equations of motion and solve them in terms of regular potentials, applying systematically the proposals of Eqs. (1.11) and (1.13). Next we present the action, check it is well defined and show it cancels all the anomalies. Eventually we discuss Born–Infeld actions and equations of motion on the branes.

5.1. Anomalies and the B-field

On each D-brane with worldvolume M_i the pullback of the NS 2-form B couples to the Abelian gauge field A_i living on M_i through the invariant field strength

$$h_i = B - 2\pi\alpha' F_i, \qquad F_i = dA_i. \tag{5.1}$$

Under a gauge transformation $\delta B = d\Lambda$, the $U(1)$ potential transforms as

$$\delta A_i = \frac{1}{2\pi\alpha'}\Lambda, \tag{5.2}$$

where pull backs are understood.

The anomaly polynomial that describes the anomalies of the system has been derived in [1] and is given by the 12-form

$$P_{12} = -\pi \sum_{i,j} (-1)^{\frac{i}{2}} e^{\gamma(h_j - h_i)} \sqrt{\frac{\hat{A}(T_i)}{\hat{A}(N_i)}} \sqrt{\frac{\hat{A}(T_j)}{\hat{A}(N_j)}} J_{ij}\chi_{ij}, \tag{5.3}$$

where $\gamma \equiv \frac{1}{4\pi^2\alpha'}$, $\hat{A}$ is the roof genus and N_i and T_i are the normal and tangent bundles respectively. Since $h_j - h_i = 2\pi\alpha'(F_i - F_j)$ the polynomial P_{12} is independent of B and invariant under (5.2), as it should. The term in the sum is symmetric under exchange of i with j, so that on single branes (which can be considered as self-intersection) a factor of π appears since summation is on diagonal terms $i = j$, while for intersecting branes one recovers a factor of 2π. (5.3) reproduces indeed the anomaly polynomial for both single and intersecting D-branes. Notice that the anomaly on the (self intersecting) D(−1)- and D1-brane ($i = 0, 2$) vanishes since the degree $(10 - i)$ of the Euler form χ_{ii} exceeds $i + 2$, while the anomalies on self intersecting D3-, D5-, D7-branes are different from zero. The case of the D9 is special in that the worldvolume theory is the super-Yang–Mills part of type I string theory [9] and its anomaly is cancelled by the Green–Schwarz mechanism. We do not include this contribution in our discussion since we consider it well known.

The anomalies that have not been dealt with both in [2] and [1] are those for pairs of dual branes, D(−1)–D7, D1–D5, D3–D3, where the integrand in (5.3) for dimensional reasons reduces entirely to the Euler form. In this case it was not understood how to perform an inflow of charge and, as mentioned in [1], "a more powerful approach is needed". This can be achieved using Chern-kernels as we have seen in the previous sections.

The charge of the D-brane with worldvolume M_i is given by [9]

$$g_i = \sqrt{2\pi G}\,\gamma^{\frac{i-4}{2}}, \qquad g_i g_{8-i} = 2\pi G. \tag{5.4}$$

As noticed in [1,2] a crucial step towards anomaly cancellation is to notice that the anomaly polynomial (5.3) is partially "factorized". On each D-brane one can introduce the closed and invariant polynomial

$$Y_i = e^{-\frac{F_i}{2\pi}} \sqrt{\frac{\hat{A}(T_i)}{\hat{A}(N_i)}} = dY_i^{(0)} + 1. \tag{5.5}$$

The forms Y_i are those appearing in the Bianchi identities and equations of motion of [1,2], where B was kept zero. For a non-vanishing B-field one notices that (5.5) is not invariant under (5.2), so what should appear in the Bianchi identities and equations of motion is rather the invariant expression

$$\bar{Y}_i = e^{\gamma h_i}\sqrt{\frac{\hat{A}(T_i)}{\hat{A}(N_i)}} = e^{\gamma B_i} Y_i. \tag{5.6}$$

These $\bar{Y}_i$ are, however, no longer closed and satisfy

$$d\bar{Y}_i = \gamma \bar{Y}_i H, \quad H = dB. \tag{5.7}$$

In terms of these forms one can define the following forms of degree $n = 2, \ldots, 10$,

$$\Delta_n = g_{10-n} \sum_i (-1)^{\frac{i}{2}} J_i \bar{Y}_i, \tag{5.8}$$

where it is understood that on the r.h.s. one has to extract the n-form part. (5.4) implies then the important relation

$$d\Delta_n = \Delta_{n-2} H. \tag{5.9}$$

These forms are crucial since they allow eventually to factorize the anomaly polynomial completely. Indeed, it is not difficult to show that one can rewrite (5.3) as

$$P_{12} = -\frac{1}{2G} \sum_n (-1)^{n/2} (\Delta_n \Delta_{12-n})_{\mathrm{reg}}, \tag{5.10}$$

where the "regularized" product of two currents has been defined in the previous section. This formula holds in the case of non-transversal intersections for all couples of dual branes. In case these intersections are all transversal one has to add the term $-2\pi(J_6 J_2 - J_8 J_0)$ which subtracts the same term from the above expression. Notice however that, since for transversal intersections the form $J_6 J_2 - J_8 J_0$ is integer, it gives rise through the descent formalism to an anomaly which is a well-defined integer multiple of 2π, and can be disregarded.

Notice also that (5.10) is independent on B—despite its explicit appearance—as is obvious by construction. This can also be checked explicitly noticing that under a generic variation of B one has

$$\delta\Delta_n = \Delta_{n-2}\delta B. \tag{5.11}$$

δP_{12} vanishes under such a variation thanks to the alternating signs $(-1)^{n/2}$ in Eq. (5.10). This explains also the appearance of those signs.

Eventually, since P_{12} is invariant under (5.2) there exist a Chern–Simons form and a second descent which respect this symmetry, $P_{12} = dP_{11}$, $\delta P_{11} = dP_{10}$. The resulting anomaly

$$\mathcal{A} = \int P_{10} \tag{5.12}$$

can therefore always be chosen to respect this symmetry, too.

5.2. Bianchi identities and equations of motion

In this section we describe the theory of IIB supergravity in presence of arbitrary D-branes and with a non-zero B field turned on. We give new Bianchi identities and equations of motion that apply in the presence of D-branes. Using the techniques developed in the previous sections we solve Bianchi identities in terms of regular potentials, and then proceed to write down a classical action that gives the equations of motion. Q-invariance of this classical action generates an anomalous transformation law that exactly cancels the quantum anomaly.

Start defining the formal sum of generalized currents as

$$\Delta = \sum_n \Delta_n, \tag{5.13}$$

where Δ_n is defined in (5.8). The full set of Bianchi identities and equations of motion is given by the compact formula

$$dR = RH + \Delta, \tag{5.14}$$

where $R_9 = *R_1$, $R_7 = *R_3$ and $R_5 = *R_5$. Three comments are in order. First, in the limit where each brane charge g_i is set to zero, (5.14) reduces to the Bianchi identities and equations of motion of free IIB supergravity. Secondly, (5.14) is well defined since the right-hand side is a closed form, as can be seen using (5.9). Third point to mention is electro–magnetic duality. While an equation of the form $dR = J$ is electro–magnetically symmetric, the full equation (5.14) is not, because of the RH and $J\bar{Y}$ terms. These latter can be thought of as currents associated to smeared branes.

The compact formula describing solutions of the Bianchi identities is

$$R = dC - CH + f, \tag{5.15}$$

where f_n, $n = 1, 3, 5$, is the n-form

$$f_n = g_{9-n} \sum_i (-1)^{\frac{i}{2}} K_i \bar{Y}_i, \tag{5.16}$$

and it satisfies

$$df = fH + \Delta. \tag{5.17}$$

Notice that under Q transformations $\delta K_i = dQ_i$ and the RR-curvatures are invariant provided the potentials transform as

$$\delta C_n = -g_{8-n} \sum_i (-1)^{i/2} Q_i \bar{Y}_i, \tag{5.18}$$

and therefore the pullback of the potentials on the branes is regular. Such pullback amounts to an anomalous transformation of the potential, which plays an important role in cancellation of anomalies.

As a specific example, the corrected form of the Bianchi identities for R_1 and R_3 is

$$dR_1 = g_8 J_8, \tag{5.19}$$

$$dR_3 = R_1 H + g_6(-J_6 + J_8 \bar{Y}_{8,2}), \tag{5.20}$$

(by $\bar{Y}_{8,2}$ we mean the degree 2 part of the $\bar{Y}$ form defined on the D7-brane). Now we introduce Chern-kernels K_8, K_6 (and K_4 for R_5) associated to the D7-, D5- (and D3-) branes that appear in the Bianchi identities. Further sources appearing in the equations of motion, J_2 and J_0, have instead to be treated using Dirac branes W_2 and W_0, as explained in [4]. The solution we propose for the Bianchi identities is

$$R_1 = dC_0 + g_8 K_8, \tag{5.21}$$

$$R_3 = dC_2 - C_0 H + g_6(-K_6 + K_8 \bar{Y}_{8,2}), \tag{5.22}$$

and similarly for R_5. It is straightforward to check that these definitions ensure that the Bianchi identities are satisfied, using (5.7), (5.4). Notice the fact that this solution requires the $\bar{Y}$ forms to be extended to target space forms. One could object that a more standard way to solve Bianchi identities that involves only Y forms evaluated on branes would rather be

$$R_1 = d\tilde{C}_0 + g_8 K_8, \tag{5.23}$$

$$R_3 = d\tilde{C}_2 - \tilde{C}_0 H + g_6\big(-K_6 + J_8 Y^{(0)}_{8,1} + K_8 B\big), \tag{5.24}$$

and similarly for R_5. This is the approach used in [1], and we remark that there the solution of Bianchi identities is incomplete, in that it misses the terms were K appears without any Y form. However, this definition leads to singular potentials and therefore to inconsistencies. Consider for example an anomalous gauge transformation of $Y^{(0)}_{8,1}$ in (5.24), $\delta Y^{(0)}_{8,1} = dY^{(1)}_{8,0}$. Since the curvature R_3 is invariant the potential $\tilde{C}_2$ transforms accordingly as

$$\delta\tilde{C}_2 = -g_6 J_8 Y^{(1)}_{8,0}. \tag{5.25}$$

The variation of $\tilde{C}_2$ is always singular on the D7-brane so that $\tilde{C}_2$ itself is singular. Similarly, one shows that an analogous $\tilde{C}_4$ would be singular on the D7- and D5-branes.

We now show that the RR-curvatures are independent of the extension of the $\bar{Y}$ forms. Consider target space forms $Y_i = dY^{(0)}_i + 1$. Under a change of extension they transform as

$$\delta Y_i = dX_i, \tag{5.26}$$

where X_i is an arbitrary target space form such that

$$J_i X_i = J_i X_i|_{M_i} = 0, \tag{5.27}$$

since Y_i is well defined on its D-brane. From here on we will refer to these as X-transformations. Here we explicitly consider the R_3 curvature, but similar formulae apply to all the potentials C. Under (5.26) R_3 is invariant if C_2 shifts by

$$\delta C_2 = -g_6 K_8 X_{8,1}. \tag{5.28}$$

Such X-transformations of the potentials always exist due to consistency of the Bianchi identities. Even though K_8 does not admit limit close to the D7-brane, it remains finite and the product $K_8 X_{8,1}$ is well behaved and goes to zero since $X_{8,1}$ goes to zero. This proves that the RR-potentials as defined by (5.15) are completely regular close to the branes, and that the curvatures do not depend on the arbitrary extensions of the Y forms. In the next section we will present the action for the system and see that it does not depend on such extensions.

5.3. Action and anomaly cancellation

In this section we present an action that gives rise to the equations of motion (5.14). There are two ways to discuss the Wess–Zumino part of the action. One possibility is write it as the integral of a closed 11-form. The advantage of this formulation is that it is the most clear one: it immediately displays Q- and X-invariances of the theory and how the gravitational anomaly is cancelled. However, even if the procedure is rigorous and does not depend on the arbitrary extra dimension, nevertheless it is a natural expectation to ask for the existence of a well-defined ten-dimensional action. Our approach to the problem will be that of presenting at first the Wess–Zumino part of the action as the integral of an 11-form and discuss its properties. A well-defined 10D action will be given in the end of the section, and its relationship with the former discussed in Appendix C.

We define the total effective action of the theory as the sum of a classical and quantum part

$$\Gamma = \frac{1}{G}(S_{\mathrm{kin}} + S_{\mathrm{WZ}}) + \Gamma_{\mathrm{quant}}, \tag{5.29}$$

where Γ_{quant} generates the anomalies described in Section 5.1. S_{kin} is given by

$$\begin{aligned} S_{\mathrm{kin}} = {} & \int d^{10}x\sqrt{-g}e^{-2\phi}R + \frac{1}{2}\int e^{-2\phi}(8H_1 * H_1 + H_3 * H_3) \\ & + \frac{1}{2}\int\left(R_1 * R_1 - R_3 * R_3 + \frac{1}{2}R_5 * R_5 - \frac{1}{2}f_4 * f_4\right), \end{aligned} \tag{5.30}$$

where the field $f_4 = \iota_v(R_5 - *R_5)$ appears as part of the PST approach [8], enforcing the self-duality equation, and v^m is the unit vector

$$v_m \equiv \frac{\partial_m a}{\sqrt{\partial a \partial a}}, \qquad v_m v^m = 1, \tag{5.31}$$

where a is an arbitrary scalar field. The PST formulation then guarantees that a is a non-propagating, auxiliary field.

To describe the Wess–Zumino term instead take an eleven-dimensional manifold M_{11} whose boundary is the spacetime M_{10} of the theory, $\partial M_{11} = M_{10}$. Assume no topological obstruction, and take the extra dimension to be parallel to the D-branes so that the degree of J forms, which counts the number of normal directions to a brane, is not changed. Then one can write the Wess–Zumino as

$$S_{\mathrm{WZ}} = \int_{M_{11}} L_{11}, \tag{5.32}$$

$$dL_{11} = 0, \tag{5.33}$$

and L_{11} is given by

$$L_{11} = \left[-\frac{1}{2} R_5 (R_3 H + \Delta_6) - R_3 \Delta_8 - R_1 \Delta_{10} \right]_{\text{reg}} - G P_{11}. \tag{5.34}$$

P_{11} is the Chern–Simons form of the anomaly and is given by

$$P_{11} = \pi \sum_{i,j} (-1)^{i/2+1} P_{ij}^{(0)} J_{ij} \chi_{ij} + 2\pi \left(J_{2-6} \chi_{2-6}^{(0)} - J_{0-8} \chi_{0-8}^{(0)} \right) - \pi J_4 \chi_4^{(0)}, \tag{5.35}$$

and $P_{ij}^{(0)}$, in turn, is a Chern–Simons form of[5]

$$P_{ij} = e^{\frac{(F_i - F_j)}{2\pi}} \sqrt{\frac{\hat{A}(T_i)}{\hat{A}(N_i)}} \sqrt{\frac{\hat{A}(T_j)}{\hat{A}(N_j)}}. \tag{5.36}$$

Some comments are in order. A direct check shows that the classical part of the action gives the correct equations of motion. Moreover, Q- and X-invariances in this picture are immediately displayed, since only RR invariant curvatures appear.

Anomaly cancellation is guaranteed since the only term in the action which is not invariant under anomalous transformations is $-\int P_{11}$, whose variation exactly cancels the quantum anomaly. Lastly, notice the factor of $1/2$ in the minimal coupling $R_5 \Delta_6$ of Eq. (5.34). On one side it is an artifact of the PST formalism (see the kinetic part of the action), and should not misunderstood as a novel feature. On the other side, it combines exactly with $\pi J_4 \chi_4^{(0)}$ leaving only a C_4 potential term, that cancels the anomaly on the self-dual D3-brane according to what explained in Section 4.2.

We now write down the Wess–Zumino term in a ten-dimensional fashion. Here we simply display it as it is, leaving the proof of its equivalence with (5.34) to Appendix C. We split the Wess–Zumino into a term depending on the potentials and a remainder:

$$S_{\text{WZ}} = \int \left(L_{10}^C + L_{10}^{\text{rem}} \right). \tag{5.37}$$

The part depending on the potentials is

$$L_{10}^C = -\frac{1}{2} C_4 (R_3 H + \Delta_6) - \frac{1}{2} C_2 H f_5 - C_2 \Delta_8 - C_0 \Delta_{10}. \tag{5.38}$$

The remainder can be written in two ways. In the case of transversal intersection for dual branes it is given by

$$L_{10}^{\text{rem}} = \frac{1}{2} \left[f_3 f_7 - f_1 f_9 + 2\pi G \sum_{i,j} (-1)^{i/2} P_{ij}^{(0)} K_i J_j + 4\pi G (K_6 W_2 - K_8 W_0) \right], \tag{5.39}$$

where the forms f_7, f_9 are formally defined as in (5.16), but using Dirac branes W_2 and W_0 instead of K_2 and K_0. For non-transversal intersection the two last $K_i W_j$ terms have to be modified according to Eq. (4.9).

[5] Notice that $P_{ij} = d P_{ij}^{(0)} + 1$, which explains the form of (5.35).

In this 10D picture, Q- and X-invariances are hidden and have to be checked one by one. Anomaly cancellation arises for non-dual branes from the terms $P^{(0)}_{ij} J_i K_j$. For dual branes with non extremal intersection it arises from the terms of the kind $\mathcal{L}_{ij} J_i$. If the intersection is extremal then $\mathcal{L}_{ij} = 0$ and the anomaly is cancelled by the anomalous transformation of the potentials. This is always true for the D3-brane. All the other Q-variations instead are cancelled between terms in L^C_{10} and in L^{rem}_{11}.

5.4. Born–Infeld actions and equations of motion on the branes

In this section we describe the dynamics of $U(1)$ fields on each D-brane and of the NS form B. The action (5.29) describes all the dynamics of RR-fields but, as it stands, is not complete. One has to add to it Born–Infeld terms for the $U(1)$ fields on each brane

$$\Gamma = \frac{1}{G}(S_{\mathrm{kin}} + S_{\mathrm{WZ}} + S_{\mathrm{BI}}) + \Gamma_{\mathrm{quant}}, \tag{5.40}$$

with

$$S_{\mathrm{BI}} = \sum_i g_i I^i_{\mathrm{BI}}, \tag{5.41}$$

$$I^i_{\mathrm{BI}} = -\int_{Di} dx^i\, e^{-\phi} \sqrt{-\det\bigl(g^i_{mn} + B_{mn} - 2\pi\alpha' F^i_{mn}\bigr)}. \tag{5.42}$$

From such Born–Infeld term on can define generalized field strenghts

$$\tilde{h}^i_{mn} = \frac{2}{\sqrt{-\det g^i_{mn}}} \frac{\delta I^i_{\mathrm{BI}}}{\delta B_{mn}}, \tag{5.43}$$

justified by the fact that under a variation of the field B one has $\delta I^i_{\mathrm{BI}} = \int_{Di} \delta B * \tilde{h}^i$. In terms of these one can obtain, after a straightforward but lenghty calculation, the equation of motion of B:

$$d * H = R_3 R_5 - R_1 R_7 + \sum_i g_i J_i * \tilde{h}^i, \tag{5.44}$$

and the equations of motion for the $U(1)$ field strenghts, that we report in Appendix B. These new equations of motion have three important properties. First of all, they are invariant under Q- and P-transformations. This is required by consistency since the action we wrote down is invariant in first instance. Q-invariance happens because a direct check shows that such equations display no dependence on Chern-kernels K_i. P-invariance is evident since the equations are expressed in terms of $\bar{Y}$ forms pulled back on the appropriate branes and of RR-curvatures. Second point is that the equations are explicitly invariant under gauge transformations of B

$$\delta B = d\Lambda, \tag{5.45}$$

$$\delta A^i = \frac{1}{2\pi\alpha'} \Lambda|_{Di}, \tag{5.46}$$

again as it should, by consistency. Third property is that, as expected, the $U(1)$ theory on the branes is anomalous. If one writes the equations of motion in the form

$$d * \tilde{h}^i = \tilde{j}_i, \tag{5.47}$$

then an explicit check shows that

$$d\tilde{j}_i \neq 0. \tag{5.48}$$

However the equation of motion for B (5.44), even though it contains the anomalous field strenghts $\tilde{h}^i$, is non-anomalous.

The action constructed so far only involves C_0, C_2, C_4 potentials. Often in the literature the action is written in term of all the RR-potentials, and hence in the next section we rewrite our results in a duality-invariant language.

6. PST duality-invariant formulation

In order to make contact with the formulation of [1] in this section we construct the action for the same system but using all the possible RR-potentials C_i, $i = 0, 2, \dots, 8$, instead of the minimal ones C_0, C_2, C_4. A duality-invariant formulation may also be useful for the purpose of analysing the flux quantization of dual potentials, or for dimensional reductions involving dual branes and dual potentials.

Let us then introduce the new RR-potentials C_6, C_8 and define the forms f_7 and f_9 as in (5.16), but this time using proper Chern-kernels K_2 and K_0 instead of Dirac branes. Introduce RR-curvature R_7 and R_9 using the same recipe of (5.15). In order to deal with C_6 and C_8 one has to exploit the PST formalism. Introduce an arbitrary scalar field a and construct the unit vector v^m as in (5.31). In term of v^m construct the forms

$$r_0 \equiv \iota_v(R_1 - *R_9), \tag{6.1}$$

$$r_2 \equiv \iota_v(R_3 - *R_7). \tag{6.2}$$

Then the PST duality-invariant action is

$$S_{\text{dual}} = S_{\text{kin}} + S_{\text{WZ}} + S_{\text{BI}} + \frac{1}{2}(r_2 * r_2 - r_0 * r_0). \tag{6.3}$$

Such action has all the PST symmetries necessary to prove that a is an auxiliary field and that the conditions $R_9 = *R_1$, $R_7 = *R_3$ are enforced (see [4,8]). In order to make contact with the usual formulations one can use the following identities:

$$R_3 * R_3 - r_2 * r_2 = (R_3, R_7)P(v)\begin{pmatrix} R_3 \\ R_7 \end{pmatrix} - R_3 R_7, \tag{6.4}$$

$$R_1 * R_1 - r_0 * r_0 = (R_1, R_9)P(v)\begin{pmatrix} R_1 \\ R_9 \end{pmatrix} + R_1 R_9, \tag{6.5}$$

where $P(v)$ is the operator valued matrix

$$\begin{pmatrix} -v\iota_v * & v\iota_v \\ v\iota_v & -v\iota_v * \end{pmatrix}. \tag{6.6}$$

Substituting this into the action (6.3) gives $S_{\mathrm{dual}} = S_{\mathrm{kin,dual}} + S_{\mathrm{WZ,dual}} + S_{\mathrm{BI}}$, where $S_{\mathrm{kin,dual}}$ is a kinetic term for all the RR-potentials given by

$$S_{\mathrm{kin,dual}} = \frac{1}{2}\int\left[(R_1, R_9)P(v)\begin{pmatrix} R_1 \\ R_9 \end{pmatrix} - (R_3, R_7)P(v)\begin{pmatrix} R_3 \\ R_7 \end{pmatrix} + \frac{1}{2}R_5 * R_5 - \frac{1}{2}f_4 * f_4\right], \tag{6.7}$$

where there is symmetry under $R_1 \leftrightarrow R_9$, $R_3 \leftrightarrow R_7$, see [10]. $S_{\mathrm{WZ,dual}} = \int L_{\mathrm{dual}}$ is a modified Wess–Zumino term whose 11D version reads

$$L_{11,\mathrm{dual}} = -\frac{1}{2}\sum_n R_{n+1}\Delta_{10-n} - GP_{11}, \tag{6.8}$$

while the 10D one is

$$\begin{aligned} L_{10,\mathrm{dual}} &= -\frac{1}{2}\sum_n C_n \Delta_{10-nn} \\ &\quad + \frac{2\pi}{2}G\left[\sum_{i,j}(-1)^{i/2}P^{(0)}_{ij}(K_iJ_j)_{\mathrm{reg}} + 2(K_6W_2 - K_8W_0)\right] \\ &= -\frac{1}{2}\sum_n R_{n+1}f_{9-n} \\ &\quad + \frac{2\pi}{2}G\left[\sum_{i,j}(-1)^{i/2}P^{(0)}_{ij}(K_iJ_j)_{\mathrm{reg}} + 2(K_6W_2 - K_8W_0)\right]. \end{aligned} \tag{6.9}$$

Again the usual remark for non-transversal intersections of dual branes applies, where Eq. (4.9) should be used. Now we can try to make contact with Eq. (2.11) of [1]. In our notation it says that on each brane the Wess–Zumino term goes like

$$-\frac{1}{2}\int_{M_i}\left(\tilde{C}_i + RY^{(0)}_i\right). \tag{6.10}$$

From Eq. (5.16) one can decompose the forms f_n, in the limit $B \equiv 0$, as

$$f_n = g_{9-n}\sum_i (-1)^{\frac{i}{2}}\left[\left(J_iY^{(0)}_i + K_i\right) + d\left(K_iY^{(0)}_i\right)\right]_n, \tag{6.11}$$

and plug them into the second line of (6.9). Consider the first three terms in (6.11). f_n on its own has only inverse power singularities near each brane, but the first and third term in the decomposition individually display δ-like singularities. Therefore, in (6.9) it is not allowed to multiply each single term times a RR-curvature, but only the whole sum. Suppose however we want to formally forget about this difficulty. Then we can see that the first term in (6.11) reproduces the second term of (6.10). The second term has two effects. Part of it is multiplied in Eq. (6.9) times the potential part in R. Joint with some of the $P^{(0)}_{ij}(K_iJ_j)_{\mathrm{reg}}$ terms, it reconstructs the $\tilde{C}_i$ part of (6.10). Remember that the passage from C_i to $\tilde{C}_i$ is purely formal since the latter is ill defined. The remaining part of the second term in (6.11), together with the third term $d(K_i Y_i^{(0)})$, are dependent on the Chern-kernels K_i and cancel completely with the rest of the $P_{ij}^{(0)}(K_i J_j)_{\text{reg}}$ terms of (6.9). This is guaranteed since the (ill defined) potentials $\tilde{C}_i$ of [1] are Q-invariant and so the K-dependent terms have to disappear.

In conclusion, in the formal approximation when one can forget δ-like divergencies, in the limit $B \equiv 0$ and assuming it is possible to use the $\tilde{C}_i$ potentials, one exactly recovers the Wess–Zumino term of [1], plus the extra terms that cancel the anomaly for dual branes. The anomaly cancellation for the self-dual D3-brane is given by transformations of C_4 which are present only in the Chern-kernel formulation and cannot be reproduced in the context of [1].

7. Conclusions and outlooks

We conclude by summarizing our results and commenting on their extension.

We have considered the system IIB supergravity interacting with all possible combinations of single D-branes, with arbitrary intersections as long as there are no topological obstructions. We have constructed a regular action that gives the equations of motion, which is written in terms of potentials that are everywhere well defined.

We have provided a correct understanding of the mechanism of charge inflow using Chern-kernel techniques. In particular, we have shown that, for pairs of dual branes which had proven to be intractable before, charge inflow is not produced by curvatures but either by potentials, in the case of the self-dual D3-brane and extremal intersections of dual branes, or by the $\mathcal{L}$ forms for non-extremal, non-transversal intersections of dual branes. Another important part of the understanding of charge inflow, that is used in order to implement a Chern-kernel analysis, is the new fundamental identity (3.1).

The Wess–Zumino term we obtained differs from other expressions that appeared in the literature, like those of [1,2,11] and in particular it contains extra terms which are related to anomaly cancellation for pairs of dual branes. Moreover, it contains all the corrections due to the presence of the NS form B.

We have obtained the full, corrected equations of motion for B interacting with RR fields, gravity and $U(1)$ Yang–Mills fields, and for the $U(1)$ fields themselves. The classical $U(1)$ theory on the branes is anomalous but quantum corrections restore the full symmetry.

We insist on remarking that Chern-kernel techniques have wide application to all theories with extended objects, and not only D-branes of supergravity. They can be used for example to deal with orientifolds, like in problems considered in [12,13], with O-planes [14], with non-BPS branes [15]. Another possible system to which apply these techniques is IIA supergravity in presence of D-branes. There, branes have odd-dimensional worldvolume but they still admit anomalies on their intersections.

Another possible generalization is to couple our system of D-branes to an NS5-brane, which is interesting since in that case the NS curvature would not be closed. Its treatment should go along the lines of [6] and we expect it to be straightforward to implement.

Lastly, we discuss generalization to the $U(N)$ case. In this case, it is reasonable to argue that, in constructing physical $U(N)$ fields on each brane, the colourless NS form B will be coupled to some $U(1)$ subgroup on $U(N)$. Let $\mathcal{F}$ then be the full $U(N)$ curvature, and decompose it into a $U(1)$ part F, that couples to B as in (5.1), and an $SU(N)$ part with curvature $\tilde{F}$. Since the $U(1)$ part commutes with the rest, it is easy to see that the non-Abelian Chern character that enters in the anomaly has to be generalized to

$$\mathrm{ch}\left(\frac{\mathcal{F}}{2\pi}\right) \to e^{-\gamma h}\,\mathrm{ch}\left(\frac{\tilde{F}}{2\pi}\right). \tag{7.1}$$

This would be the ingredient necessary to form the new $\bar{Y}$ forms. Since B enters in the $U(1)$ part the identity (5.7) continues to hold and from this one is able to impose again Bianchi identities, equations of motion, and to find a Wess–Zumino term for the action from which they come from. The only limitation is that one does not have a Born–Infeld action that is uniquely fixed so far, and therefore for the time being it is not possible to find equations of motion for B and the Yang–Mills fields on the brane.

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Appendix A. Proof of the new identity

The following proof holds for arbitrary Chern-kernels, even or odd. We begin by considering the properties of $J_i K_j$. As we saw, $K_j = \Omega_j + \chi_j^{(0)}$ is singular on M_j because Ω_j involves $\hat{y}^a = y^a/|y|$, and M_j stays at $y^a = 0$. But since $M_i \not\subset M_j$ the product

$$J_i K_j = J_i (K_j|_{M_i})$$

is well defined and, therefore, in the sense of distributions also its differential is so. The subtle point is only that one cannot apply Leibnitz's rule to evaluate it, because the product has (inverse-power and δ-like) singularities on M_{ij}. Away from M_{ij} one can apply Leibnitz and there the result is $d(J_i K_j) = 0$. This means that $d(J_i K_j)$ is supported on M_{ij} and hence proportional to J_{ij},

$$d(J_i K_j) = J_{ij}\Phi,$$

for some form Φ defined on M_{ij}. Furthermore, since the l.h.s. is closed also Φ must be a closed form. Moreover, Φ must be a completely invariant form as is the l.h.s., because J_i is intrinsically defined and K_j transforms as $K_j' = K_j + dQ_j$, where Q_j is *regular* on M_j. This means that one can apply Leibnitz and $d(J_i\, dQ_j) = 0$.

Furthermore, Φ can depend only on the curvature components of the intersection of the normal bundles N_{ij}. This can be seen as follows. Since K_j is made out only of gravitational curvatures belonging to N_j, also Φ is a polynomial made out only of (a subset of) those curvatures. If (in addition to $M_i \not\subset M_j$) we have also $M_j \not\subset M_i$, we can apply Leibnitz to $d(K_i K_j) = K_i J_j \pm J_i K_j$, giving $d(J_i K_j) = \pm d(J_j K_i)$. This implies that Φ depends, moreover, only on the curvatures of N_i, and hence only on those of N_{ij}. If on the contrary $M_j \subset M_i$ then, using, e.g., the regularizations of [4], one can show that $d(K_i K_j) = \chi_i^{(0)} J_j \pm J_i K_j$. Applying the differential to this one gets directly (3.1), since in this case $J_j = J_{ij}$ and $\chi_i = \chi_{ij}$.

Eventually, Φ is a form of degree $\dim(M_{ij}) + D - (i + j)$ and it is odd under parity. Φ shares all these properties uniquely with the Euler-form of N_{ij} and we conclude therefore that it is proportional to it.

A cohomological argument can finally be used to fix the proportionality coefficient to one. Perform a regularization $K_j \to K_j^\varepsilon$, $J_j \to J_j^\varepsilon$, $J_j^\varepsilon = dK_j^\varepsilon$, as for example the one given in the appendix of [4], where J_j^ε is cohomologically equivalent to J_j and regular on M_j. Then $d(J_i K_j) = \lim_{\varepsilon\to 0} d(J_i K_j^\varepsilon) = \lim_{\varepsilon\to 0} J_i J_j^\varepsilon$, and as shown in [1], $J_i J_j^\varepsilon$ is cohomologically equivalent to $J_{ij}\chi_{ij}$ for every ε. This proves that the l.h.s. of (3.1) is *cohomologically* equivalent to $J_{ij}\chi_{ij}$, and it fixes the proportionality coefficient of our *local* i.e., point–wise derivation to unity.

Appendix B. Equations of motion for the $U(1)$ theory on the D-branes

The equations of motion obtained by the action (5.40) for the $U(1)$ fields are:

$$d * \tilde{h}_2 = -R_1, \tag{B.1}$$

$$d * \tilde{h}_4 = +R_3 - \frac{1}{\gamma} R_1 \bar{Y}_{4,2} - \frac{g_6}{2} J_8 \frac{A_4 - A_8}{2\pi}, \tag{B.2}$$

$$d * \tilde{h}_6 = -R_5 + \frac{1}{\gamma} R_3 \bar{Y}_{6,2} - \frac{1}{\gamma^2} R_1 \bar{Y}_{6,4} + \frac{g_4}{3} J_8 \frac{A_6 - A_8}{2\pi} \frac{F_6 - F_8}{2\pi}, \tag{B.3}$$

$$\begin{aligned} d * \tilde{h}_8 = {}& +R_7 - \frac{1}{\gamma} R_5 \bar{Y}_{8,2} + \frac{1}{\gamma^2} R_3 \bar{Y}_{8,4} - \frac{1}{\gamma^3} R_1 \bar{Y}_{8,6} \\ & - g_2 \left[\frac{1}{2} J_4 \frac{A_8 - A_4}{2\pi} - \frac{1}{3} J_6 \frac{A_8 - A_6}{2\pi} \frac{F_8 - F_6}{2\pi} \right]. \end{aligned} \tag{B.4}$$

On the right-hand side the regularized products of currents and Chern-kernels are always understood.

Appendix C. The Wess–Zumino term

Here we make contact between the Wess–Zumino written as the integral of an 11-form (5.34) and the one written in usual ten-dimensional notation, (5.38) and (5.39). The procedure one realizes in practice is the following: first of all construct (5.38), that is completely fixed by equations of motion as showed in Section 5.3. Then, the remaining part (5.39) is completely fixed by asking invariance of the action under Q- and P-transformations. The actual calculations are lenghty, though straightforward, and we do not include them here. Once the ten-dimensional Wess–Zumino is fixed, one can take its differential and get the much simpler form (5.34), which displays all invariances and anomaly cancellations at a first sight. What we do here instead is to proceed in the opposite direction, that is to show how to transform (5.34) into the ten-dimensional Wess–Zumino.

As a first step, consider (5.34) and extract all the terms dependent on the potentials. After some algebra and integration by part one shows that this amounts to

$$d\left[-\frac{1}{2}C_4(R_3H+\Delta_6)-\frac{1}{2}C_2Hf_5-C_2\Delta_8-C_0\Delta_{10}\right]=dL_{10}^C. \tag{C.1}$$

Now, take the remainder in (5.34). This is equal to

$$\begin{aligned}
&\frac{1}{2}f_5(f_3H-\Delta_6)+f_3\Delta_8-f_1\Delta_{10}-GP_{11}\\
&\quad=\frac{1}{2}d(f_3f_7-f_1f_9)+\frac{1}{2}(-\Delta_2f_9+\Delta_4f_7-\Delta_6f_5+\Delta_8f_3-\Delta_{10}f_1)-GP_{11}\\
&\quad=\frac{1}{2}d(f_3f_7-f_1f_9)+\frac{1}{2}\sum_n(-1)^{\frac{n}{2}+1}\Delta_{10-n}f_{n+1}-GP_{11}\\
&\quad=\frac{1}{2}d(f_3f_7-f_1f_9)+\frac{1}{2}\sum_n(-1)^{\frac{n}{2}+1}\Delta_{10-n}f_{n+1}\bigg|_{B_2=0}-GP_{11},
\end{aligned} \tag{C.2}$$

where in the last passage independence from B_2 depends crucially on the alternating sign and can be checked using (5.11) and an analogous variation for f. Given this, one shows with some algebra that

$$\sum_n(-1)^{\frac{n}{2}+1}\Delta_{10-n}f_{n+1}\bigg|_{B_2=0}=2\pi G\sum_{i,j}(-1)^{i/2+1}P_{ij}K_iJ_j. \tag{C.3}$$

Putting together Eqs. (C.2), (C.3) and the expression (5.35) for P_{11} one gets that the reminder is exactly given by (5.39).

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