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Higher-order parabolic equations without conditions at infinity

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1. Introduction

In this note we study the well-posedness of the following Cauchy problem:

$$\begin{cases} \rho \dfrac{\partial u}{\partial t} = \displaystyle\sum_{k=0}^{m} (-1)^{k+1} \dfrac{\partial^k}{\partial x^k}\left(a_k \dfrac{\partial^k u}{\partial x^k}\right) - c_0 |u|^{p-1} u & \text{in } S := \mathbb{R} \times (0, T), \\ u = u_0 & \text{in } \mathbb{R} \times \{0\}. \end{cases} \tag{1.1}$$

Here $p > 1$, $m \in \mathbb{N}$ while the coefficients $\rho = \rho(x,t)$, $a_k = a_k(x,t)$, and $c_0 = c_0(x,t)$ are positive functions defined in S, which satisfy among others the following growth conditions:

(P_1) there exist $K > 0$, $\alpha \in \mathbb{R}$ such that

$$\rho(x,t) \geqslant \frac{1}{K}(1 + |x|)^{\alpha}, \qquad \frac{1}{K} \leqslant c_0(x,t) \leqslant K;$$

(P_2) for any $k = 1, \dots, m$ there exist $M_k > 0$, $\alpha_k \in \mathbb{R}$ such that

$$a_k(x,t) \leqslant M_k (1 + |x|)^{\alpha_k}$$

for any $(x,t) \in S$.

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The motivation of our study comes from the recent paper [1], where the same question was addressed for the Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\gamma \dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^2 u}{\partial x^2} + u - u^3 & \text{in } S, \\ u = u_0 & \text{in } \mathbb{R} \times \{0\} \end{cases} \tag{1.2}$$

($\gamma > 0$), which is a particular case of problem (1.1) (see Section 2). In particular, uniqueness of solutions to problem (1.2) was proved in a class of functions satisfying the growth condition

$$|u(x,t)| \leqslant c_1 \exp\bigl\{\beta_1 |x|^{4/3}\bigr\} \quad \text{as } |x| \to \infty \quad (t \in [0,T]) \tag{1.3}$$

for some $c_1, \beta_1 > 0$.

Let us recall that, even in the linear case $c_0 \equiv 0$, uniqueness of solutions to problem (1.1) holds in classes of functions which "do not grow too rapidly at infinity," depending on the behaviour of the coefficients of the first equation as $|x| \to \infty$. For instance, for the general second-order parabolic Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \displaystyle\sum_{i,j=1}^{N} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i x_j} + \sum_{i=1}^{N} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u & \text{in } \mathbb{R}^N \times (0,T), \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\} \end{cases} \tag{1.4}$$

uniqueness holds in the class of bounded solutions if there exists $M > 0$ such that

$$|a_{ij}(x,t)| \leqslant M\bigl(1+|x|^2\bigr), \qquad |b_i(x,t)| \leqslant M\bigl(1+|x|^2\bigr)^{1/2},$$
$$c(x,t) \leqslant M \quad (i,j = 1,\dots,N).$$

On the other hand, if

$$|a_{ij}(x,t)| \leqslant M, \qquad |b_i(x,t)| \leqslant M, \qquad c(x,t) \leqslant M$$
$$(i,j = 1,\dots,N)$$

(in particular, in the case of constant coefficients), there exists at most one solution to problem (1.4) such that

$$|u(x,t)| \leqslant c_2 \exp\bigl\{\beta_2 |x|^2\bigr\} \quad \text{as } |x| \to \infty \quad (c_2, \beta_2 > 0,\ t \in [0,T]) \tag{1.5}$$

(see [2,3]; see also [4] for a related problem). As is well known, condition (1.5) is essential for uniqueness; in fact, a celebrated counterexample proves the existence of a nontrivial solution to the Cauchy problem

$$\begin{cases} u_t = \Delta u & \text{in } \mathbb{R}^N \times (0,T), \\ u = 0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases}$$

which grows like $\exp\{\beta_2 |x|^{2+\epsilon}\}$ for some $\epsilon > 0$ (see [5,6]).

Concerning parabolic equations of arbitrary order $2m$ $(m \geqslant 1)$, analogous results are given in [7] for the case of constant coefficients, respectively in [8] for bounded coefficients (always in the linear case $c_0 \equiv 0$; see also [9] for general results concerning parabolic systems). In all these cases uniqueness holds in the class of functions that do not grow faster than $\exp\{\beta_3|x|^{2m/(2m-1)}\}$ $(\beta_3 > 0)$; observe that this condition reduces to the growth condition (1.5) when $m = 1$, respectively to (1.3) when $m = 2$.

In light of the previous remarks, the above-mentioned uniqueness result in [1] appears as a nontrivial extension to the *semilinear* problem (1.2) of the growth condition (1.3), already known for the related *linear* case. However, as we shall see below, we can take advantage of the nonlinear term $-u^3$ in the differential equation to prove uniqueness in a wider class of locally integrable solutions of problem (1.2), regardless of their behaviour as $|x| \to \infty$. More generally, Theorem 2.1 below gives sufficient conditions—depending on the growth of the coefficients—for the uniqueness of locally integrable solutions to problem (1.1) (see Definition 2.1), if the coefficient c_0 satisfies assumption (P_1).

The above uniqueness result (which is due to the effect of the nonlinear term on the right-hand side of the first equation of (1.1)) is not surprising, for it partly generalizes to the present situation previous results obtained by Brezis for second-order (elliptic and parabolic) problems (see [10]) and by Bernis for a class of higher-order problems with constant coefficients ([11]; see also [12,13] for some generalizations). In this connection, let us observe that well-posedness results analogous to those for problem (1.1) can be proved for the elliptic equation

$$\sum_{k=0}^{m}(-1)^{k+1}\frac{d^k}{dx^k}\left[a_k(x)\frac{d^k u}{dx^k}\right] - c_0(x)|u|^{p-1}u = f \quad \text{in } \mathbb{R} \tag{1.6}$$

without prescribing any growth condition at infinity of the data f (see Theorem 2.2).

2. Mathematical framework and results

Following [11] we shall work in an (H^{-m}, H^m) framework. We denote as usual by $H^k(Q)$, $H^{-k}(Q)$ the Sobolev spaces $W^{k,2}(Q)$, $W^{-k,2}(Q)$, respectively $(Q \subseteq \mathbb{R},\ k \leqslant m)$. We set also

$$H_c^k(\mathbb{R}) := \left\{u \in H^k(\mathbb{R}) \mid \operatorname{supp} u \text{ is compact}\right\},$$
$$H_{\mathrm{loc}}^k(\mathbb{R}) := \left\{u \in L_{\mathrm{loc}}^2(\mathbb{R}) \mid u|_{B_R} \in H^k(B_R) \text{ for any } R > 0\right\},$$
$$H_{\mathrm{loc}}^{-k} := \left\{f \in \mathcal{D}'(\mathbb{R}) \mid f|_{B_R} \in H^{-k}(B_R) \text{ for any } R > 0\right\},$$

where $B_R := \{x \in \mathbb{R} \mid |x| < R\}$ and $\mathcal{D}'$ denotes the space of distributions.

Let us recall that $f \in H^{-k}_{\mathrm{loc}}(\mathbb{R})$ if and only if there exists $g \in L^2_{\mathrm{loc}}(\mathbb{R})$ whose kth distributional derivative is equal to f (see [11]). For any $f \in H^{-k}_{\mathrm{loc}}(\mathbb{R})$, $u \in H^k_c(\mathbb{R})$ the following duality product is defined:

$$\langle f, u\rangle_{-k,k} := (-1)^k \int_{\mathbb{R}} g \frac{d^k u}{dx^k}.$$

Concerning the coefficients ρ, a_k, besides the growth conditions (P_1), (P_2) we always assume that

(P_3) $\rho \in C(\bar{S}) \cap C^{0,1}_{x,t}(S)$, $a_k \in C(\bar{S}) \cap C^{1,0}_{x,t}(S)$ $(k = 0, \ldots, m)$;
(P_4) $\partial\rho/\partial t \leqslant 0$, $a_k > 0$ $(k = 0, \ldots, m)$;
(P_5) for any $k = 1, \ldots, m$ there exists $\bar{M}_k > 0$ such that

$$\left|\frac{\partial a_k(x,t)}{\partial x}\right| \leqslant \frac{\bar{M}_k}{1+|x|} a_k(x,t)$$

for any $(x,t) \in S$.

Let us make the following definition.

Definition 2.1. Let $u_0 \in \mathbb{L}^2_{\mathrm{loc}}(\mathbb{R})$. By a solution to problem (1.1) we mean any function $u \in C([0,T]; \mathbb{L}^2_{\mathrm{loc}}(\mathbb{R})) \cap \mathbb{L}^2((0,T); H^m_{\mathrm{loc}}(\mathbb{R})) \cap \mathbb{L}^{p+1}_{\mathrm{loc}}(S)$ such that

$$\rho \frac{\partial u}{\partial t} = \sum_{k=0}^{m} (-1)^{k+1} \frac{\partial^k}{\partial x^k}\left(a_k \frac{\partial^k u}{\partial x^k}\right) - c_0|u|^{p-1}u$$

in $\mathcal{D}'(S)$ and, moreover,

$$u = u_0 \quad \text{a.e. in } \mathbb{R}.$$

We can now state the following result.

Theorem 2.1. *Let assumptions* (P_1)–(P_5) *be satisfied*; *let* $u_0 \in \mathbb{L}^2_{\mathrm{loc}}(\mathbb{R})$. *Assume that*

$$\alpha_k - 2k < \max\left\{\alpha, -\frac{p-1}{p+1}\right\} \tag{2.1}$$

for any $k = 1, \ldots, m$. *Then there exists exactly one solution to problem* (1.1).

The proof of the above result relies on local estimates of the solution (see [10,11]); the presence of variable coefficients and of lower-order terms requires some nontrivial adaptation of the method (see Section 3).

Let us observe that by the standard transformation $v := \exp(-\lambda t)u$ ($\lambda \geqslant 1$) problem (1.2) reads

$$\begin{cases} \dfrac{\partial v}{\partial t} = -\gamma \dfrac{\partial^4 v}{\partial x^4} + \dfrac{\partial^2 v}{\partial x^2} - (\lambda - 1)v - \exp(2\lambda t)v^3 & \text{in } S, \\ v = u_0 & \text{in } \mathbb{R} \times \{0\}. \end{cases}$$

It is easily seen that Theorem 2.1 applies in this case; in particular, inequality (2.1) is satisfied since $\alpha = \alpha_1 = \alpha_2 = 0$. Then there exists a unique solution (in the sense of Definition 2.1) to problem (1.2). Similarly, the well-posedness result in [11, Theorem 9.1] follows from Theorem 2.1 in the case of one space dimension.

Concerning the elliptic equation (1.6), the following result can be proved.

Theorem 2.2. *Let the following assumptions be satisfied*:

(E_1) *there exists* $K > 0$ *such that*

$$\frac{1}{K} \leqslant c_0(x) \leqslant K;$$

(E_2) *for any* $k = 1, \dots, m$ *there exist* $M_k > 0$, $\alpha_k \in \mathbb{R}$ *such that*

$$a_k(x) \leqslant M_k(1 + |x|)^{\alpha_k}$$

for any $x \in \mathbb{R}$;

(E_3) $a_k \in C^1(\mathbb{R})$, $a_k > 0$ ($k = 0, \dots, m$);

(E_4) *for any* $k = 1, \dots, m$ *there exists* $\bar{M}_k > 0$ *such that*

$$\left| \frac{\partial a_k(x)}{\partial x} \right| \leqslant \frac{\bar{M}_k}{1 + |x|} a_k(x)$$

for any $x \in \mathbb{R}$. *Moreover, let*

$$\alpha_k - 2k < -\frac{p-1}{p+1} \tag{2.2}$$

for any $k = 1, \dots, m$.

Then for any $f \in \mathbb{L}^{(p+1)/p}_{\mathrm{loc}}(\mathbb{R})$ *there exists exactly one* $u \in H^m_{\mathrm{loc}}(\mathbb{R}) \cap L^{p+1}_{\mathrm{loc}}(\mathbb{R})$ *satisfying Eq.* (1.6) *in* $\mathcal{D}'(\mathbb{R})$.

3. A useful inequality

It is expedient for further purposes to consider the following family of test functions

$$\zeta_R(x) := \begin{cases} \left[R\left(1 - \dfrac{x^{2m}}{R^{2m}}\right)\right]^s & \text{if } x \in B_R, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

where $R > 0$, $m \in \mathbb{N}$, and $s > 2m$.

In this connection it is useful to introduce another family of functions. Let $k = 1, \ldots, m$ be fixed; define for any $j = 0, \ldots, k-1$

$$\psi_j(x) \equiv \psi_{j,k,R}(x) := \begin{cases} \left[R\left(1 - \dfrac{x^{2m}}{R^{2m}}\right)\right]^{s-2k+2j} \left(\dfrac{x}{R}\right)^{2k+2j} & \text{if } x \in B_R, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

It is the purpose of this section to prove the following result.

Proposition 3.1. *There exists $M > 0$ (only depending on s, k, and m) such that for any $u \in H^m_{\mathrm{loc}}(\mathbb{R})$ and $t \in (0, T)$ there holds*

$$\sum_{k=0}^{m} (-1)^k \left\langle \frac{\partial^k}{\partial x^k}\left[a_k(\cdot, t)\frac{d^k u}{dx^k}\right], u\zeta_R \right\rangle_{-k,k} \geqslant \frac{1}{2}\sum_{k=0}^{m} \int_{B_R} a_k(\cdot, t)\left(\frac{d^k u}{dx^k}\right)^2 \zeta_R - M \sum_{k=1}^{m} \int_{B_R} a_k(\cdot, t) u^2 \psi_0. \tag{3.3}$$

Let us first prove some preliminary results concerning the functions ζ_R, ψ_j.

Lemma 3.1. *For any $k \geqslant 1$ there holds*

$$\frac{d^k \zeta_R(x)}{dx^k} = \begin{cases} \left[R\left(1 - \dfrac{x^{2m}}{R^{2m}}\right)\right]^{s-k} \left(\dfrac{x}{R}\right)^{2m-k} P_k\left[\dfrac{x^{2m}}{R^{2m}}\right] & \textit{if } x \in B_R, \\ 0 & \textit{otherwise}; \end{cases} \tag{3.4}$$

here P_k is a polynomial of degree $k-1$, whose coefficients depend only on s, k, and m.

Proof. Let us proceed by induction. It is immediately seen that equality (3.4) holds for $k = 1$ with $P_1(z) := -2ms$. On the other hand, if equality (3.4) holds for $1 \leqslant l \leqslant k$, then it holds for $l = k+1$ with

$$P_{k+1}(z) := -2m(s-k)zP_k(z) + (2m-k)(1-z)P_k(z) + 2m(1-z)zP_k'(z),$$

as is easily checked. Then the conclusion follows. □

It follows from equality (3.4) that $\zeta_R \in C^k_0(\mathbb{R})$ for any $k < s$ $(k = 1, \ldots, m)$.

Observe that by definition $\psi_k \leqslant \zeta_R$ in B_R. Another link between the functions ζ_R and ψ_j is given by the following lemma.

Lemma 3.2. *For any $j = 0, \dots, k-1$ there exists a constant $\kappa_j > 0$ (only depending on s, k, and m) such that in B_R there holds*

$$\frac{1}{\zeta_R}\left(\frac{d^{k-j}\zeta_R}{dx^{k-j}}\right)^2 \leqslant \kappa_j \psi_j. \tag{3.5}$$

Proof. It follows from equality (3.4) that in B_R

$$\frac{1}{\zeta_R}\left(\frac{d^{k-j}\zeta_R}{dx^{k-j}}\right)^2 = \left[R\left(1 - \frac{x^{2m}}{R^{2m}}\right)\right]^{s-2k+2j}\left(\frac{x}{R}\right)^{4m-2k+2j} \times \left[P_{k-j}\left(\frac{x^{2m}}{R^{2m}}\right)\right]^2.$$

Since $|x|/R \leqslant 1$ the conclusion easily follows. □

Lemma 3.3. *For any $j = 1, \dots, k-1$:*

(i) *there holds*

$$\frac{{\psi_j}^2}{\psi_{j+1}} = \psi_{j-1};$$

(ii) *there exists $\lambda_j > 0$ (only depending on s, k, and m) such that*

$$\frac{1}{\psi_j}\left(\frac{d\psi_j}{dx}\right)^2 \leqslant \lambda_j \psi_{j-1}.$$

Proof. (i) Due to definition (3.2) there holds

$$\psi_j(x) = x^2\left(1 - \frac{x^{2m}}{R^{2m}}\right)^2 \psi_{j-1}(x),$$

whence the claim follows.

(ii) It is easily checked that

$$\frac{1}{\psi_j}\left(\frac{d\psi_j}{dx}\right)^2 = \left[R\left(1 - \frac{x^{2m}}{R^{2m}}\right)\right]^{s-2k+2j-2}\left(\frac{x}{R}\right)^{2k+2j-2} \times \left[-2m(s-2k+2j)\left(\frac{x^{2m}}{R^{2m}}\right) + 2(k+j)\left(1 - \frac{x^{2m}}{R^{2m}}\right)\right]^2.$$

Then the conclusion follows. □

Now we can prove the following

Lemma 3.4. *For any $k = 1, \dots, m$, $j = 0, \dots, k-1$ and $\epsilon > 0$ sufficiently small there exists $\mu_j = \mu_j(\epsilon) > 0$ (only depending on ϵ, s, k, and m) such that*

$$\int_{B_R} a_k(\cdot,t)\left(\frac{d^j u}{dx^j}\right)^2 \psi_j \leqslant \epsilon \int_{B_R} a_k(\cdot,t)\left(\frac{d^{j+1} u}{dx^{j+1}}\right)^2 \psi_{j+1} + \mu_j(\epsilon)\int_{B_R} a_k(\cdot,t)u^2\psi_0 \tag{3.6}$$

for any $u \in H^m_{\mathrm{loc}}(\mathbb{R})$ *and* $t \in (0,T)$.

Proof. (i) Let us first prove the following claim: For any $k = 2,\ldots,m$, $j = 1,\ldots,k-1$ and $\theta > 0$ there exists $\nu_j = \nu_j(\theta) > 0$ (only depending on θ, s, k, and m) such that

$$\int_{B_R} a_k\left(\frac{d^j u}{dx^j}\right)^2 \psi_j \leqslant \theta \int_{B_R} a_k\left(\frac{d^{j+1} u}{dx^{j+1}}\right)^2 \psi_{j+1} + \nu_j(\theta)\int_{B_R} a_k\left(\frac{d^{j-1} u}{dx^{j-1}}\right)^2 \psi_{j-1} \tag{3.7}$$

for any $u \in H^m_{\mathrm{loc}}(\mathbb{R})$ and $t \in (0,T)$ (we set $a_k \equiv a_k(\cdot,t)$ for brevity).

Integrating by parts gives

$$\int_{B_R} a_k\left(\frac{d^j u}{dx^j}\right)^2 \psi_j = -\int_{B_R} a_k \frac{d^{j+1}u}{dx^{j+1}}\frac{d^{j-1}u}{dx^{j-1}}\psi_j - \int_{B_R} \frac{\partial a_k}{\partial x}\frac{d^j u}{dx^j}\frac{d^{j-1}u}{dx^{j-1}}\psi_j - \int_{B_R} a_k \frac{d^j u}{dx^j}\frac{d^{j-1}u}{dx^{j-1}}\frac{d\psi_j}{dx} =: I_1 + I_2 + I_3.$$

By Young inequality, for any $\eta > 0$ there holds

$$|I_1| \leqslant \eta \int_{B_R} a_k\left(\frac{d^{j+1}u}{dx^{j+1}}\right)^2 \psi_{j+1} + \frac{1}{4\eta}\int_{B_R} a_k\left(\frac{d^{j-1}u}{dx^{j-1}}\right)^2 \frac{\psi_j^2}{\psi_{j+1}},$$

$$|I_2| \leqslant \frac{\eta}{4}\int_{B_R} a_k\left(\frac{d^j u}{dx^j}\right)^2 \psi_j + \frac{1}{\eta}\int_{B_R} \frac{1}{a_k}\left(\frac{\partial a_k}{\partial x}\right)^2\left(\frac{d^{j-1}u}{dx^{j-1}}\right)^2 \psi_j,$$

$$|I_3| \leqslant \frac{\eta}{4}\int_{B_R} a_k\left(\frac{d^j u}{dx^j}\right)^2 \psi_j + \frac{1}{\eta}\int_{B_R} a_k\left(\frac{d^{j-1}u}{dx^{j-1}}\right)^2 \frac{1}{\psi_j}\left(\frac{d\psi_j}{dx}\right)^2.$$

Due to Lemma 3.3 and assumption (P_5), from the above inequalities we obtain easily

$$\int_{B_R} a_k\left(\frac{d^j u}{dx^j}\right)^2 \psi_j \leqslant \frac{2\eta}{(2-\eta)}\int_{B_R} a_k\left(\frac{d^{j+1}u}{dx^{j+1}}\right)^2 \psi_{j+1}$$

$$+\frac{2}{\eta(2-\eta)}\left(\frac{1}{4}+\bar{M}_k^2+\lambda_j\right)\int\limits_{B_R} a_k\left(\frac{d^{j-1}u}{dx^{j-1}}\right)^2\psi_{j-1}$$

for any $\eta < 2$, where λ_j denotes the constant in Lemma 3.3(ii). Choosing $\eta = 2\theta/(2+\theta)$, then defining

$$\nu_j := \frac{(2+\theta)^2}{4\theta}\left(\frac{1}{4}+\bar{M}_k^2+\lambda_j\right), \tag{3.8}$$

we obtain inequality (3.7); hence the claim follows.

(ii) For $k = 1,\dots,m$, $j = 0$ inequality (3.6) is clearly satisfied. If $k = 2,\dots,m$ and $j = 1,\dots,k-1$ we proceed by induction. For $j = 1$ the statement follows from inequality (3.7) with $\epsilon = \theta$ and $\mu_1 := \nu_1$. Further suppose the statement to be true for $1 \leqslant l \leqslant j-1$; then by inequality (3.7) we obtain easily

$$\begin{aligned}\big[1-\theta^2\nu_j(\theta)\big]\int\limits_{B_R} a_k\left(\frac{d^ju}{dx^j}\right)^2\psi_j \\ \leqslant \theta\int\limits_{B_R} a_k\left(\frac{d^{j+1}u}{dx^{j+1}}\right)^2\psi_{j+1} + \mu_{j-1}\big(\theta^2\big)\nu_j(\theta)\int\limits_{B_R} a_k u^2\psi_0.\end{aligned}$$

For any $\epsilon > 0$ sufficiently small choose $\bar{\theta} = \bar{\theta}(\epsilon) > 0$ such that $\bar{\theta}/(1-\bar{\theta}^2\nu_j(\bar{\theta})) = \epsilon$ (this is possible since by definition (3.8) there holds $\theta^2\nu_j(\theta) \to 0$ as $\theta \to 0$). Defining recursively

$$\mu_j(\epsilon) := \frac{\nu_j(\bar{\theta})}{1-\bar{\theta}^2\nu_j(\bar{\theta})}\mu_{j-1}\big(\bar{\theta}^2\big)$$

we obtain inequality (3.6); then the conclusion follows. □

Lemma 3.5. *For any $k = 1,\dots,m$, $j = 0,\dots,k-1$ and $\epsilon > 0$ sufficiently small there holds*

$$\begin{aligned}\int\limits_{B_R} a_k(\cdot,t)\left(\frac{d^ju}{dx^j}\right)^2\psi_j \\ \leqslant \epsilon^{k-j}\int\limits_{B_R} a_k(\cdot,t)\left(\frac{d^ku}{dx^k}\right)^2\zeta_R + \sum_{i=0}^{k-j-1}\epsilon^i\mu_{j+i}(\epsilon)\int\limits_{B_R} a_k(\cdot,t)u^2\psi_0\end{aligned}$$

for any $u \in H^m_{\mathrm{loc}}(\mathbb{R})$ and $t \in (0,T)$ (the constants μ_j being the same as in inequality (3.6)).

Proof. It suffices to apply inequality (3.6) $k-j$ times. □

Now we can prove Proposition 3.1.

Proof of Proposition 3.1. Since $u \in H^m_{\text{loc}}(\mathbb{R})$, by assumption (P$_3$) there holds

$$\frac{\partial^k}{\partial x^k}\left(a_k \frac{d^k u}{dx^k}\right) \in H^{-m}_{\text{loc}}(\mathbb{R}) \quad \text{for any } k \leqslant m.$$

On the other hand, by Lemma 3.1 we have $u\zeta_R \in H^m_c(\mathbb{R})$. It follows that for any $k \leqslant m$ the equality

$$\left\langle \frac{\partial^k}{\partial x^k}\left(a_k \frac{d^k u}{dx^k}\right), u\zeta_R \right\rangle_{-k,k} = (-1)^k \int_{B_R} a_k \frac{d^k u}{dx^k} \frac{d^k (u\zeta_R)}{dx^k}$$

is well defined. Then there holds

$$\begin{aligned} &\sum_{k=0}^{m} (-1)^k \left\langle \frac{\partial^k}{\partial x^k}\left(a_k \frac{d^k u}{dx^k}\right), u\zeta_R \right\rangle_{-k,k} \\ &\quad = \sum_{k=0}^{m} \int_{B_R} a_k \left(\frac{d^k u}{dx^k}\right)^2 \zeta_R + \sum_{k=1}^{m} \sum_{j=0}^{k-1} \binom{k}{j} \int_{B_R} a_k \frac{d^k u}{dx^k} \frac{d^j u}{dx^j} \frac{d^{k-j}\zeta_R}{dx^{k-j}}. \end{aligned} \tag{3.9}$$

Due to Lemmas 3.2 and 3.5, by Young's inequality we obtain for any $k = 1, \ldots, m$, $j = 0, \ldots, k-1$ and $\epsilon > 0$ small enough

$$\begin{aligned} \left| \int_{B_R} a_k \frac{d^k u}{dx^k} \frac{d^j u}{dx^j} \frac{d^{k-j}\zeta_R}{dx^{k-j}} \right| &\leqslant \sqrt{\epsilon} \int_{B_R} a_k \left(\frac{d^k u}{dx^k}\right)^2 \zeta_R + \frac{\kappa_j}{4\sqrt{\epsilon}} \int_{B_R} a_k \left(\frac{d^j u}{dx^j}\right)^2 \psi_j \\ &\leqslant \sqrt{\epsilon}\left(1 + \frac{\kappa_j}{4}\epsilon^{k-j-1}\right) \int_{B_R} a_k \left(\frac{d^k u}{dx^k}\right)^2 \zeta_R \\ &\quad + \frac{\kappa_j}{4\sqrt{\epsilon}} \sum_{i=0}^{k-j-1} \epsilon^i \mu_{j+i}(\epsilon) \int_{B_R} a_k u^2 \psi_0. \end{aligned}$$

From the above inequality and equality (3.9) it follows that

$$\begin{aligned} &\sum_{k=0}^{m} (-1)^k \left\langle \frac{\partial^k}{\partial x^k}\left(a_k \frac{d^k u}{dx^k}\right), u\zeta_R \right\rangle_{-k,k} \\ &\quad \geqslant \int_{B_R} a_0 u^2 \zeta_R + \sum_{k=1}^{m} \left[1 - \sqrt{\epsilon} \sum_{j=0}^{k-1} \binom{k}{j}\left(1 + \frac{\kappa_j}{4}\right)\right] \int_{B_R} a_k \left(\frac{d^k u}{dx^k}\right)^2 \zeta_R \\ &\qquad - \frac{1}{4} \sum_{k=1}^{m} \sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} \binom{k}{j} \kappa_j \epsilon^{i-1/2} \mu_{j+i}(\epsilon) \int_{B_R} a_k u^2 \psi_0. \end{aligned}$$

Set

$$H_0 := \max_{k=1,\dots,m} \left\{ \sum_{j=0}^{k-1} \binom{k}{j} \left(1 + \frac{\kappa_j}{4}\right) \right\},$$

then choose $\epsilon = \epsilon_0 < 1/4H_0^2$. Set also

$$M := \frac{1}{4} \max_{k=1,\dots,m} \left\{ \sum_{j=0}^{k-1} \sum_{i=0}^{k-j-1} \binom{k}{j} \kappa_j \epsilon_0^{i-1/2} \mu_{j+i}(\epsilon_0) \right\}.$$

Then the latter inequality implies (3.3); hence the conclusion follows. □

4. Proof of the main results

In order to prove Theorem 2.1 the following local estimate of solutions of problem (1.1) is important.

Proposition 4.1. *Let u be any solution to problem* (1.1). *Then for any $R > 0$, $R_1 > R$, and s sufficiently large there exists $N > 0$* (*only depending on m, s, p, T, R, R_1*) *such that*

$$\sup_{0 \leqslant t \leqslant T} \int_{B_R} u^2(x,t)\,dx + \sum_{k=0}^{m} \iint_{S_R} a_k \left(\frac{d^k u}{dx^k}\right)^2 + \iint_{S_R} |u|^{p+1} \leqslant N\left(1 + \int_{B_{R_1}} u_0^2(x)\,dx\right), \tag{4.1}$$

where $S_R := B_R \times (0,T)$.

Proof. Let $0 < R < R_1$. By Definition 2.1 we have $u \in \mathbb{L}^2((0,T); H^m_{\mathrm{loc}}(\mathbb{R})) \cap \mathbb{L}^{p+1}_{\mathrm{loc}}(S)$. Since $\zeta_{R_1} \in C_0^m(\mathbb{R})$, there also holds $u\zeta_{R_1} \in \mathbb{L}^2((0,T); H^m_c(\mathbb{R})) \cap \mathbb{L}^{p+1}_{\mathrm{loc}}(S)$. Then by Definition 2.1

$$\int_0^T \left\langle \rho \frac{\partial u}{\partial t} + \sum_{k=0}^{m} (-1)^k \frac{\partial^k}{\partial x^k}\left(a_k \frac{\partial^k u}{\partial x^k}\right) + c_0 |u|^{p-1} u, u\zeta_{R_1} \right\rangle_{-m,m} = 0. \tag{4.2}$$

On the other hand, $\rho(\partial u/\partial t) \in \mathbb{L}^2((0,T); H^{-m}_{\mathrm{loc}}(\mathbb{R})) + \mathbb{L}^{(p+1)/p}_{\mathrm{loc}}(S)$. Then by assumption ($P_4$) we obtain

$$\int_0^T \left\langle \rho \frac{\partial u}{\partial t}, u\zeta_{R_1} \right\rangle_{-m,m} = \iint_{S_{R_1}} \rho \frac{\partial u}{\partial t} u \zeta_{R_1}$$

$$\geqslant \frac{1}{2}\int\limits_{B_{R_1}} \rho(x,T)u^2(x,T)\zeta_{R_1}(x)\,dx - \frac{1}{2}\int\limits_{B_{R_1}} \rho(x,0)u_0^2(x)\zeta_{R_1}(x)\,dx. \tag{4.3}$$

From equality (4.2), inequalities (3.3) and (4.3), and assumption (P_1) we obtain easily

$$\frac{1}{2}\min_{S_{R_1}}\rho \sup_{0\leqslant t\leqslant T}\int\limits_{B_{R_1}} u^2(x,t)\zeta_{R_1} + \frac{1}{2}\sum_{k=0}^{m}\iint\limits_{S_{R_1}} a_k\left(\frac{\partial^k u}{\partial x^k}\right)^2\zeta_{R_1} + \frac{1}{K}\iint\limits_{S_{R_1}} |u|^{p+1}\zeta_{R_1} \leqslant 2M\sum_{k=1}^{m}\iint\limits_{S_{R_1}} a_k u^2\psi_0 + \int\limits_{B_{R_1}} \rho(x,0)u_0^2(x)\zeta_{R_1}, \tag{4.4}$$

where K is the constant in assumption (P_1) and $\psi_0 \equiv \psi_{0,k,R_1}$. By Hölder and Young inequalities, for any $k = 1,\dots,m$ and $\epsilon > 0$ there holds

$$\iint\limits_{S_{R_1}} a_k u^2\psi_0 \leqslant \left[\iint\limits_{S_{R_1}} \left(u^2(\epsilon\zeta_{R_1})^{2/(p+1)}\right)^{(p+1)/2}\right]^{\frac{2}{p+1}} \times \left[\iint\limits_{S_{R_1}} \left(\frac{a_k\psi_0}{(\epsilon\zeta_{R_1})^{2/(p+1)}}\right)^{\frac{p+1}{p-1}}\right]^{\frac{p-1}{p+1}}$$
$$\leqslant \frac{2\epsilon}{p+1}\iint\limits_{S_{R_1}} |u|^{p+1}\zeta_{R_1} + \frac{p-1}{p+1}\epsilon^{-2/(p-1)}\iint\limits_{S_{R_1}} \frac{a_k^{(p+1)/(p-1)}{\psi_0}^{(p+1)/(p-1)}}{{\zeta_{R_1}}^{2/(p-1)}}.$$

Observe that for any $k = 1,\dots,m$

$$I_k := \iint\limits_{S_{R_1}} \frac{a_k^{(p+1)/(p-1)}{\psi_0}^{(p+1)/(p-1)}}{{\zeta_{R_1}}^{2/(p-1)}} < \infty$$

if $s > 2m(p+1)/(p-1) - 1$, as is easily checked (here use of assumption (P_2) is made). Set

$$\epsilon = \epsilon_1 := \frac{p+1}{8mKM}, \qquad H_2 := 2\max\left\{\frac{1}{\min_{S_{R_1}}\rho}, K, 1\right\}.$$

Then from (4.4) and the above inequality we obtain

$$\sup_{0\leqslant t\leqslant T}\int_{B_R} u^2(x,t)\,dx+\sum_{k=0}^{m}\iint_{S_R} a_k\left(\frac{\partial^k u}{\partial x^k}\right)^2+\iint_{S_R}|u|^{p+1}$$
$$\leqslant \frac{H_2}{\zeta_{R_1}(R)}\left\{2M\frac{p-1}{p+1}\epsilon_1^{-2/(p-1)}\sum_{k=1}^{m} I_k+R_1^s\max_{x\in B_{R_1}}\rho(x,0)\int_{B_{R_1}} u_0^2(x)\,dx\right\}. \tag{4.5}$$

Then by a proper definition of the constant N the conclusion follows. □

Now we can prove Theorem 2.1.

Proof of Theorem 2.1. Existence of solutions can be proved by a standard procedure, due to estimate (4.1); we refer the reader to [11] for details.

Let us prove the statement concerning uniqueness. Let u and v be two solutions to problem (1.1); then the function $w:=u-v$ satisfies

$$\int_0^T\left\langle\rho\frac{\partial w}{\partial t}+\sum_{k=0}^{m}(-1)^k\frac{\partial^k}{\partial x^k}\left(a_k\frac{\partial^k w}{\partial x^k}\right)\right.$$
$$\left.+c_0\big(|u|^{p-1}u-|v|^{p-1}v\big),w\zeta_R\right\rangle_{-m,m}dt=0.$$

Due to assumption (P_4), to inequality (3.3) and to the elementary inequality

$$\big(|s|^{p-1}s-|t|^{p-1}t\big)(s-t)\geqslant 2^{1-p}|s-t|^{p+1}\quad (s,t\in\mathbb{R},\ p>1)$$

we obtain as in the proof of inequality (4.4)

$$\frac{1}{2}\sup_{0\leqslant t\leqslant T}\int_{B_R}\rho(x,t)w^2(x,t)\zeta_R(x)\,dx+\frac{1}{2}\sum_{k=0}^{m}\iint_{S_R} a_k\left(\frac{\partial^k w}{\partial x^k}\right)^2\zeta_R$$
$$+2^{1-p}\iint_{S_R} c_0|w|^{p+1}\zeta_R\leqslant 2M\sum_{k=1}^{m}\iint_{S_R} a_k w^2\psi_0.$$

Since

$$\sup_{0\leqslant t\leqslant T}\int_{B_R}\rho(x,t)w^2(x,t)\zeta_R(x)\,dx\geqslant\frac{1}{T}\iint_{S_R}\rho w^2\zeta_R,$$

from the previous inequality and assumption (P_1) we obtain

$$\iint\limits_{S_R} \rho w^2 \zeta_R + \iint\limits_{S_R} |w|^{p+1} \zeta_R \leqslant C \sum_{k=1}^{m} \iint\limits_{S_R} a_k w^2 \psi_0, \tag{4.6}$$

where $C := M \max\{2T, K/2^{1-p}\}$.

Using Young and Hölder inequalities, we obtain for any $r \in (1, \infty)$ the following estimate from below the left-hand side of inequality (4.6)

$$\begin{aligned} &\iint\limits_{S_R} \rho w^2 \zeta_R + \iint\limits_{S_R} |w|^{p+1} \zeta_R \\ &\quad \geqslant \left[\iint\limits_{S_R} \rho w^2 \zeta_R \right]^{\frac{1}{r}} \left[\iint\limits_{S_R} |w|^{p+1} \zeta_R \right]^{\frac{r-1}{r}} \geqslant \iint\limits_{S_R} \rho^{1/r} |w|^{\beta} \zeta_R, \end{aligned} \tag{4.7}$$

where

$$\beta = \beta(r) := \frac{1}{r}\big[2 + (p+1)(r-1)\big]. \tag{4.8}$$

Now observe that by definitions (3.1), (3.2) there holds

$$\psi_0 = \zeta_R^{2/\beta} \zeta_R^{(\beta-2)/\beta - 2k/s} \left(\frac{x}{R}\right)^{2k}$$

for any $k = 1, \dots, m$. Using the previous equality and Hölder inequality with conjugate exponents $r' = \beta/2$, $s' = \beta/(\beta-2)$ (which is feasible since $\beta > 2$; see (4.8)) we find for any $k = 1, \dots, m$ and $r \in (1, \infty)$

$$\begin{aligned} \iint\limits_{S_R} a_k w^2 \psi_0 &\leqslant \left[\iint\limits_{S_R} \rho^{1/r} |w|^{\beta} \zeta_R \right]^{\frac{2}{\beta}} \\ &\quad \times \left[\iint\limits_{S_R} \frac{a_k^{\beta/(\beta-2)} \zeta_R^{1-2k\beta/s(\beta-2)}}{\rho^{2/r(\beta-2)}} \left(\frac{x}{R}\right)^{\frac{2k\beta}{\beta-2}} \right]^{\frac{\beta-2}{\beta}}. \end{aligned} \tag{4.9}$$

From inequalities (4.6), (4.7), and (4.9) we obtain

$$\begin{aligned} &\left[\iint\limits_{S_R} \rho^{1/r} |w|^{\beta} \zeta_R \right]^{\frac{\beta-2}{\beta}} \\ &\quad \leqslant C \sum_{k=1}^{m} \left[\iint\limits_{S_R} \frac{a_k^{\beta/(\beta-2)} \zeta_R^{1-2k\beta/s(\beta-2)}}{\rho^{2/r(\beta-2)}} \left(\frac{x}{R}\right)^{\frac{2k\beta}{\beta-2}} \right]^{\frac{\beta-2}{\beta}}. \end{aligned} \tag{4.10}$$

Now observe that

$$\iint\limits_{S_R} \rho^{1/r}|w|^\beta \zeta_R \geqslant \left(\frac{4^m-1}{4^m}\right)^s R^s \iint\limits_{S_{R/2}} \rho^{1/r}|w|^\beta; \tag{4.11}$$

moreover, by assumptions (P_1), (P_2) and the definition of ζ_R (see (3.1)) there holds

$$\begin{aligned}
&\iint\limits_{S_R} \frac{a_k^{\beta/(\beta-2)}\zeta_R^{1-2k\beta/s(\beta-2)}}{\rho^{2/r(\beta-2)}}\left(\frac{x}{R}\right)^{\frac{2k\beta}{\beta-2}} \\
&\quad\leqslant \left(M_k K^{2/r\beta}\right)^{\beta/(\beta-2)} R^{s-2k\beta/(\beta-2)} \iint\limits_{S_R} \left(1+|x|\right)^{[1/(\beta-2)](\alpha_k\beta-2\alpha/r)} \\
&\quad\leqslant T\left(M_k K^{2/r\beta}\right)^{\beta/(\beta-2)} R^{s+1-2k\beta/(\beta-2)} \\
&\qquad\times \int\limits_{B_1} \left(1+|R\xi|\right)^{[1/(\beta-2)](\alpha_k\beta-2\alpha/r)}\,d\xi \\
&\quad\leqslant 2^{1+[1/(\beta-2)][\alpha_k\beta-2\alpha/r]_+} T\left(M_k K^{2/r\beta}\right)^{\beta/(\beta-2)} \\
&\qquad\times R^{s+1-2k\beta/(\beta-2)+[1/(\beta-2)][\alpha_k\beta-2\alpha/r]_+}
\end{aligned} \tag{4.12}$$

for any $R > 1$; here $[r]_+ := \max\{r, 0\}$ ($r \in \mathbb{R}$). From inequalities (4.10)–(4.12) it follows that

$$\left[\iint\limits_{S_{R/2}} \rho^{1/r}|w|^\beta\right]^{\frac{\beta-2}{\beta}} \leqslant \sum_{k=1}^m C_k R^{\eta_k}, \tag{4.13}$$

where for any $k = 1, \dots, m$

$$\begin{aligned}
C_k = C_k(r) &:= 2^{(1/\beta)\{\beta-2+[\alpha_k\beta-2\alpha/r]_+\}} C\left(\frac{4^m}{4^m-1}\right)^{s\frac{\beta-2}{\beta}} \\
&\quad\times T^{(\beta-2)/\beta} M_k K^{2/r\beta}, \\
\eta_k = \eta_k(r) &:= -2k + \frac{1}{\beta}\left\{\beta - 2 + \left[\alpha_k\beta - \frac{2\alpha}{r}\right]_+\right\}.
\end{aligned} \tag{4.14}$$

Then the statement concerning uniqueness will follow, if there exists some $r \in (1, \infty)$ such that $\eta_k < 0$ for any $k = 1, \dots, m$. This is easily seen to be the case if condition (2.1) is satisfied, since by definitions (4.8) and (4.14) we have

$$\begin{aligned}
\eta_k(r) = \frac{1}{2+(p+1)(r-1)} &\Big\{\big[(p-1) - 2k(p+1)\big](r-1) - 4k \\
&+ \big[2(\alpha_k - \alpha) + (p+1)(r-1)\alpha_k\big]_+\Big\}.
\end{aligned}$$

In fact,

(i) if $\alpha_k - \alpha < 0$, there holds $\eta_k \to -2k < 0$ as $r \to 1^+$;

(ii) if $\alpha_k - \alpha \geqslant 0$ and $\alpha_k < 0$, there holds

$$\eta_k \to \frac{(p-1) - 2k(p+1)}{p+1} < 0$$

as $r \to \infty$;

(iii) if both $\alpha_k - \alpha \geqslant 0$ and $\alpha_k \geqslant 0$, there holds

$$\eta_k(r) = \frac{1}{2 + (p+1)(r-1)} \times \left\{ 2(\alpha_k - 2k - \alpha) + (p-1)(r-1)\left[1 + (\alpha_k - 2k)\frac{p+1}{p-1}\right]\right\}.$$

In this case

$$\eta_k \to \alpha_k - 2k - \alpha$$

as $r \to 1^+$, respectively

$$\eta_k \to \alpha_k - 2k + \frac{p-1}{p+1}$$

as $r \to \infty$.

Due to condition (2.1) the conclusion follows. □

Let us finally prove Theorem 2.2. First we state the following result, which is the counterpart of Proposition 4.1; the proof is omitted.

Proposition 4.2. *Let $u \in H^m_{\mathrm{loc}}(\mathbb{R}) \cap L^{p+1}_{\mathrm{loc}}(\mathbb{R})$ be any solution to Eq.* (1.6) *in $\mathcal{D}'(\mathbb{R})$. Then for any $R > 0$, $R_1 > R$, and s sufficiently large there exists $\bar{N} > 0$ (only depending on m, s, p, R, R_1) such that*

$$\sum_{k=0}^{m} \int_{B_R} a_k \left(\frac{d^k u}{dx^k}\right)^2 + \int_{B_R} |u|^{p+1} \leqslant \bar{N}\left(1 + \int_{B_{R_1}} |f|^{(p+1)/p}\right). \tag{4.15}$$

Proof of Theorem 2.2. Existence of solutions can be proved by a variational argument using estimate (4.15). The proof is the same as in [11], thus we omit it.

Concerning uniqueness, let u and v be two solutions to Eq. (1.6); then the function $w := u - v$ satisfies

$$\left\langle \sum_{k=0}^{m} (-1)^k \frac{d^k}{dx^k}\left(a_k \frac{d^k w}{dx^k}\right) + c_0\big(|u|^{p-1}u - |v|^{p-1}v\big), w\zeta_R \right\rangle_{-m,m} = 0.$$

As in the proof of Theorem 2.1 it follows that

$$\frac{1}{2}\sum_{k=0}^{m} \int_{B_R} a_k \left(\frac{d^k w}{dx^k}\right)^2 \zeta_R + 2^{1-p} \int_{B_R} c_0 |w|^{p+1} \zeta_R \leqslant M \sum_{k=1}^{m} \int_{B_R} a_k w^2 \psi_0.$$

Arguing as in the proof of Proposition 4.1 we obtain the following inequality, analogous to (4.5),

$$\sum_{k=0}^{m} \int_{B_R} a_k \left(\frac{d^k w}{dx^k}\right)^2 + \int_{B_R} |w|^{p+1} \leqslant \frac{H_3}{R^s} \sum_{k=1}^{m} \int_{B_R} \frac{a_k^{(p+1)/(p-1)} {\psi_0}^{(p+1)/(p-1)}}{{\zeta_R}^{2/(p-1)}},$$

where $H_3 > 0$ is a constant not depending on R. If $s > 2m(p+1)/(p-1) - 1$, each integral on the right-hand side of the above inequality is finite. As in the proof of Theorem 2.1, for any $R > 1$ from the above inequality we obtain

$$\int_{B_R} |w|^{p+1} \leqslant H_4 \sum_{k=1}^{m} R^{1+([\alpha_k]_+ - 2k)(p+1)/(p-1)},$$

for some constant $H_4 > 0$ which does not depend on R. Then the conclusion easily follows. $\square$

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