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A Note on Block Triangular Presentations of Rings and Finitistic Dimension.

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ABSTRACT - The connections between certain block triangular presentations of rings and their projectively defined finitistic dimensions are examined.

The «little» («big») finitistic dimension conjecture, asserting that over any left artinian ring there is a finitely generated (arbitrary) left module of maximal finite projective dimension, remains open. However, it has been verified for several classes of rings and algebras (see [11], [10] and [12], for example). Moreover, in some cases, such as [8], [9], [16] and [17], specific bounds on these dimensions have been presented.

In [4] and [7] we employed an equivalence relation on the indecomposable projective modules over a left artinian ring R that induces the «finest» presentation of R as a ring of lower block triangular matrices. The equivalence classes correspond to the simply connected components of the quiver of R and they contain information on such things as the Car-

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tan determinant and the finitistic dimensions of R. Our main purpose here is to show that such presentations provide bounds on the finitistic dimensions of certain rings and artin algebras that yield generalizations of some results of Platzeck and Coelho in [5] and [13]. In the process, we provide a simple proof of an old result from [6] and point out that a theorem of Smalø ([15]) provides a connection between the functorial finiteness in R-mod of the category $\mathcal{P}^{<\infty}(R)$ of modules of finite projective dimension, and the analogous categories over the irreducible components of R.

Let us assume for now that R is a ring which has a complete orthogonal set of primitive idempotents, $\mathfrak{J} = \{e_1, \ldots, e_n\}$. No further restrictions are imposed. It is possible to define an equivalence relation on $\mathfrak{J}$ as follows: We say that $e_j \lesssim e_i$ if there is a sequence of non-zero homomorphisms $\alpha_0 \colon Re_i \to Re_{i_1}$, $\alpha_1 \colon Re_{i_1} \to Re_{i_2}$, $\ldots$, $\alpha_l \colon Re_{i_l} \to Re_j$. This is a quasi-order relation and it defines an equivalence relation by $e_i \sim e_j$ if $e_i \lesssim e_j$ and $e_j \lesssim e_i$.

The quasi-order becomes a partial order, $\leqslant$, on the equivalence classes. Denote these classes by $\mathcal{C}_1, \ldots, \mathcal{C}_k$ and for each $j = 1, \ldots, k$, let $E_j = \sum_{e_i \in \mathcal{C}_j} e_i$. The classes $\mathcal{C}_1, \ldots, \mathcal{C}_k$ can be renumbered so that if $\mathcal{C}_i \leqslant \mathcal{C}_j$ then $i < j$. In terms of idempotents, this says, in particular, that $E_i R E_j = 0$ if $j > i$.

These data are enough to allow us to present R as a block lower triangular matrix ring:

$$(\clubsuit) \quad R \cong \begin{bmatrix} E_1RE_1 & 0 & \cdots & 0 \\ E_2RE_1 & E_2RE_2 & \cdots & 0 \\ & & \cdots & \\ E_kRE_1 & E_kRE_2 & \cdots & E_kRE_k \end{bmatrix}.$$

The subset $I = \sum_{i \neq j} E_i R E_j$ is, in fact, an ideal and $I^k = 0$.

In the case of a left artinian ring, the equivalence classes $\mathcal{C}_i$ can be identified with the strongly connected components of the ordinary quiver of R. See [7, pp 54] for a discussion of this case. Here, the rings $E_i R E_i$, $i = 1, \ldots, k$ are called the *irreducible components* of R, and $\{E_1, \ldots, E_k\}$ is a *complete set of irreducible idempotents* for R. In this left artinian case, the irreducible idempotents E_i are unique up to isomorphisms of the RE_i, and an argument like that of Platzeck [13, Lemma 1.3] can be used to show that I

is the intersection of the powers of the annihilators of the $RE_i/(\mathrm{Rad}\,R)E_i$, $i=1, \ldots, k$.

We shall denote the projective dimension of a left module ${}_RM$ by $\mathrm{pdim}_R M$, and the «little» and «big» projectively defined (left) finitistic dimensions of R by fin dim R and Fin dim R, respectively. We wish to obtain upper bounds on these dimensions in terms of the corresponding dimensions of the E_iRE_i in the case where $\mathrm{pdim}_R I < \infty$. A formula of Fossum, Griffith and Reiten, [6, Corollary 4.21(iii)], can be applied to do so for the «big» finitistic dimension. Employing methods in the same spirit as those in [8] we present a module theoretic proof of this result (Proposition 2) which shows that analogous bounds also work for the «little» finitistic dimension. For this, and later uses, the following lemma will be helpful.

LEMMA 1. *Let R be a ring with idempotents $e \neq 0, 1$ and $f = 1 - e$ such that $eRf = 0$. Then for each module ${}_RM$ there is an exact sequence of R-modules*

$$0 \to fM \to M \to eM \to 0$$

in which

$$\mathrm{pdim}_R\, fM = \mathrm{pdim}_{fRf}\, fM$$

and

$$\mathrm{pdim}_R\, eM \leq \mathrm{pdim}_{eRe}\, eM + \mathrm{pdim}_{fRf}\, fRe + 1\,.$$

Moreover, if $\mathrm{pdim}_{fRf}\, fRe < \infty$ then ${}_RM$ has finite projective dimension if and only if so do ${}_{fRf}fM$ and ${}_{eRe}eM$.

PROOF. We note first that $fR = RfR$ is an ideal of R and that $eRe \cong R/fR$ as rings. Moreover, $M/fM \cong (R/fR)\otimes_R M \cong eM$, both as R and as eRe-modules. The existence of the exact sequence then follows. Since $eRf = 0$, fM has a projective resolution whose terms all belong to $\mathrm{Add}\,(Rf) = \mathrm{Add}\,(fRf)$. Thus, $\mathrm{pdim}_R\, fM = \mathrm{pdim}_{fRf}\, fM$.

Now suppose that $\mathrm{pdim}_{fRf}\, fRe = l < \infty$. Then we have $\mathrm{pdim}_R\, fR = \mathrm{pdim}_{fRf}(fRe \oplus fRf) = l$. Since $R/fR \cong eRe$, each projective eRe-module ${}_{eRe}Q$ has $\mathrm{pdim}_R\, Q \leq l+1$. It follows that if eM has an eRe-projective resolution

$$0 \to Q_k \xrightarrow{d_k} \ldots \to Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} eM \to 0\,,$$

then an inductive argument employing the exact sequences

$$0 = \mathrm{Ext}_R^{k+l+1}(\mathrm{Im}\, d_1, L) \to \mathrm{Ext}_R^{k+l+2}(eM, L) \to \mathrm{Ext}_R^{k+l+2}(Q_0, L) = 0$$

shows that $\mathrm{pdim}_R\, eM \leqslant k + l + 1$. Thus we have verified the first statement of the lemma, and the necessity part of the last statement follows.

For the sufficiency we observe that if $P \oplus P' = R^{(X)}$ then $eP \oplus eP' = eR^{(X)} = eRe^{(X)}$ and $fP \oplus fP' = (fRe \oplus fRf)^{(X)}$. Thus, if ${}_RP$ is projective then so is ${}_{eRe}eP$ and $\mathrm{pdim}_{fRf} fP \leqslant l$. Since multiplication by an idempotent (amounting to tensoring by a projective) is an exact functor, $\mathrm{pdim}_R M = p < \infty$ implies $\mathrm{pdim}_{eRe}\, eM \leqslant p$ and (see [14, Exercise 9.9]) $\mathrm{pdim}_{fRf}\, fM < \infty$. ■

PROPOSITION 2 ([6, Cf. Corollary 4.21(iii)]). *Let R be a ring with idempotents $e \neq 0, 1$ and $f = 1 - e$ so that $eRf = 0$. Suppose that* $\mathrm{pdim}_{fRf}\, fRe = l < \infty$. *Then*

(1) fin dim $R \leqslant \max\{\text{fin dim } eRe + l + 1, \text{fin dim } fRf\}$;

(2) Fin dim $R \leqslant \max\{\text{Fin dim } eRe + l + 1, \text{Fin dim } fRf\}$.

PROOF. The proofs of the two parts proceed in the same way so we present only that of (1). Suppose fin dim $eRe = t$ and fin dim $fRf = r$. Then, according to Lemma 1, if ${}_RM$ is (finitely generated and) of finite projective dimension, so are ${}_{eRe}eM$ and ${}_{fRf}fM$. Moreover, in the exact sequence

$$0 \to fM \to M \to eM \to 0$$

$\mathrm{pdim}_R\, eM \leqslant t + l + 1$; and if $\mathrm{pdim}_R M = q > \mathrm{pdim}_R\, eM$, then from the exact sequence

$$0 = \mathrm{Ext}_R^q(eM, L) \to \mathrm{Ext}_R^q(M, L) \to \mathrm{Ext}_R^q(fM, L) \to 0$$

we see that $\mathrm{pdim}_R M \leqslant \mathrm{pdim}_R\, fM \leqslant \mathrm{pdim}_{fRf}\, fM \leqslant r$. ■

As an application of these formulas we obtain the following result from which we recover those of Platzeck, [13, Theorem 2.5], on left artinian rings whose idempotent ideals are projective (IIP rings), and Coelho and Platzeck, [5, Theorem 2.4], where $\dim_R I < \infty$ but R is *weakly triangular* in the sense that its irreducible components are primary.

THEOREM 3. *Suppose that R is a ring with block lower triangular matrix presentation* (♣). *Assume that* $\operatorname{pdim}_R I = l < \infty$ *where* $I = \sum_{i \neq j} E_i R E_j$. *Then*

(i) $\text{fin dim} R \leqslant \max\left\{ \max_{1 \leqslant i \leqslant k-1} \{\text{fin dim} E_i R E_i + l + 1\}, \text{fin dim} E_k R E_k \right\}$,

and

(ii) $\text{Fin dim} R \leqslant \max\left\{ \max_{1 \leqslant i \leqslant k-1} \{\text{Fin dim} E_i R E_i + l + 1\}, \text{Fin dim} E_k R E_k \right\}$.

PROOF. We prove (i), (ii) being quite similar. Proposition 2 gives the necessary induction step. When $k = 1$ the result is obvious. The inequality holds when $k = 2$, by the proposition. Now suppose $k \geqslant 2$ and that R has $k + 1$ blocks on its diagonal and that the theorem holds for rings with k blocks. In the proposition, put $e = E_1$ and $f = 1 - E_1$, and we see that the induction assumption applies to fRf since its ideal $K = \sum_{2 \leqslant i \neq j \leqslant k} E_i R E_j$ is a summand of I and $\operatorname{pdim}_{fRf} K = \operatorname{pdim}_R K$. Hence, fRf has «little» finitistic dimension, bounded by

$$\max\left\{ \max_{2 \leqslant i \leqslant k-1} \{\text{fin dim } E_i R E_i + l + 1\}, \text{fin dim } E_k R E_k \right\}.$$

The proposition now says that $\text{fin dim} R \leqslant \max\{\text{fin dim } E_1 R E_1 + l + 1,$ $\text{fin dim } fRf\}$, which is the predicted formula. ∎

By [6] (see also [8, Corollary 1.6]), if, in a ring R of the form (♣), the finitistic dimension of each $E_i R E_i$ is finite then $\text{fin dim } R < \infty$, without any assumptions on I: in fact, $\text{fin dim } R \leqslant \sum_{i=1}^{k} \text{fin dim } E_i R E_i + k - 1$.

Let us now look at the cases considered by Platzeck and Coelho in [13] and [5]. It is assumed in [13] that R is left artinian and that all idempotent ideals are projective (an IIP ring). For such rings, Platzeck shows that the equivalence classes $\mathcal{C}_i$ are the isomorphism classes of primitive idempotents and so the irreducible components of R are *primary* rings (i.e., matrix rings over local rings), and, hence, of finitistic dimension 0. The ideal I is projective ([13, Propositions 1.2]) and $\text{fin dim } R \leqslant 1$ ([13, Theorem 2.5]). In [5] a left artinian ring is called *weakly triangular* if the irreducible components are all primary rings. In the case where all the idempotent ideals have left projective dimension $\leqslant \operatorname{pdim}_R I$, the authors obtain the bound $\text{fin dim } R \leqslant \operatorname{pdim}_R I + 1$. These bounds are special cases of those found in Theorem 3.

COROLLARY 4 ([13, Theorem 2.5] and [5, Theorem 2.4]). *Assume that R is a left artinian ring. (i) Suppose all the idempotent ideals of R are (left) projective. Then* fin dim $R \leqslant 1$. *(ii) Suppose that all idempotent ideals of R are of (left) projective dimension* $\leqslant l$ *and that the irreducible components of R are rings of matrices over local rings («weakly triangular» in* [5]). *Then* fin dim $\leqslant l+1$. *There are analogous bounds for* Fin dim R.

PROOF. By [13, Lemma 2.1(b)] a ring as in (i) is weakly triangular (i.e., the equivalence classes $\mathcal{C}_i$ are the isomorphism classes of primitive idempotents) and so (i) is a special case of (ii) with $l=0$. Under the conditions of (ii), the ideal we have called I is $\bigoplus_{i=1}^{k}\left(\sum_{j=i+1}^{k} E_j R E_i\right)$, each term of which is $R(1-E_i)RE_i$, a direct summand of the idempotent ideal $R(1-E_i)R$. Hence, pdim $I \leqslant l$, and the result follows from the theorem since, for each i, fin dim $E_i R E_i = 0$. ■

The last statement of Lemma 1 was presented, in slightly different terminology, as Lemma 2.4 of [15]. In the main theorem of that paper Smalø also proved the following: Let R be an artin algebra with idempotents e and f satisfying the hypotheses of Lemma 1; if $\mathcal{S}$ and $\mathcal{T}$ are full subcategories of eRe-mod and fRf-mod, respectively, then the full subcategory of R-modules M such that $eM \in \mathcal{S}$ and $fM \in \mathcal{T}$ is functorially (covariantly) (contravariantly) finite if $\mathcal{S}$ and $\mathcal{T}$ are. Thus arguing inductively we see that Smalø's results, together with the fact ([3, Proposition 4.2]) that any subcategory of R-mod containing only finitely many indecomposable modules is functorially finite, we readily get the following.

PROPOSITION 5. *Let R be an artin algebra such that* $I = \sum_{i \neq j} E_i R E_j$ *has finite projective dimension, where* $E_1, \ldots, E_k$ *are the irreducible idempotents. Then* $\mathcal{P}^{<\infty}(R)$ *is functorially (covariantly) (contravariantly) finite, if and only if so is each* $\mathcal{P}^{<\infty}(E_i R E_i)$. *In particular (cf.,* [5, *Theorem* 5.2]) *if each irreducible component has finitistic dimension* 0 *(e.g., when R is weakly triangular), then* $\mathcal{P}^{<\infty}(R)$ *is functorially finite.*

It can also be remarked, using the proof of [15, Theorem 2.1] and induction on k , that, under the hypotheses of the proposition, when $\mathcal{P}^{<\infty}(R)$ is contravariantly finite, the $\mathcal{P}^{<\infty}(R)$-cover (left approximation) of a simple ${}_R S$ with $E_i S = S$ is its $\mathcal{P}^{<\infty}(E_i R E_i)$-cover, $f\colon Y_S \to S$. To see this, put $E = E_i + \ldots + E_k$ and note that the induction hypothesis would say that the $\mathcal{P}^{<\infty}(ERE)$-cover of S would be $f: Y_S \to S$. We would need, for *any* $M \in \mathcal{P}^{<\infty}(R)$, that

$$\mathrm{Hom}(M, Y_S) \xrightarrow{\mathrm{Hom}(M, f)} \mathrm{Hom}(M, S)$$

be surjective. However, $EM \cong M/(1-E)M$ and any $g: M \to S$ factors through EM. This allows us to use the induction hypothesis. The modules Y_S serve as building blocks for $\mathcal{P}^{<\infty}(R)$ ([2, Proposition 3.8]).

We conclude with a family of examples suggested by [5, Example 1.2]. Consider a finite dimensional algebra whose quiver is

$$1 \rightleftarrows 2 \to 3 \rightleftarrows 4 \to 5 \rightleftarrows 6\,.$$

The irreducible components are each serial with two distinct simple modules. Thus, for each i, Fin dim $E_iRE_i = 0$, 1 or 2. Assuming various paths as relations and assuring that $\mathrm{pdim}_R I < \infty$, one can use this quiver to create a family of monomial algebras to which these results apply.

REFERENCES

[1] F. W. Anderson - K. R. Fuller, *Rings and Categories of Modules, Second Edition*, Springer-Verlag, New York, 1992.

[2] M. Auslander -I. Reiten, *Applications of contravariantly finite subcategories*, Adv. in Math., **86** (1991), pp. 111-152.

[3] M. Auslander - S. O. Smalø, *Preprojective modules over artin algebras*, J. Algebra, **66** (1980), pp. 61-122.

[4] W. D. Burgess, *The graded Cartan matrix and global dimension of 0-relations algebras*, Proc. Edinburgh Math. Soc., **30** (1987), pp. 351-362.

[5] F. U. Coelho - M. I. Platzeck, *On artin rings whose idempotent ideals have finite projective dimension*, preprint, 1999.

[6] R. M. Fossum - P. A. Griffith - I. Reiten, *Trivial extensions of abelian categories*, Lecture Notes in Math., **456**, Springer-Verlag (1975).

[7] K. R. Fuller, *The Cartan determinant and global dimension of Artinian rings, Azumaya algebras, actions, and modules (Bloomington, IN, 1998)*, pp. 51-72, Contemp. Math., **124**, Amer. Math. Soc, Providence, RI, 1992.

[8] K. R. Fuller - M. Saorín, *On the finitistic dimension conjecture for artinian rings*, Manuscripta Math., **74** (1992), pp. 117-132.

[9] K. R. Goodearl - B. Huisgen-Zimmermann, *Repetitive resolutions over classical orders and finite-dimensional algebras, Algebras and modules, II (Geiranger,* 1996), pp. 205-225, CMS Conf. Proc., **24**, Amer. Math. Soc., Providence, RI, 1998.

[10] E. L. Green - B. Huisgen-Zimmermann, *Finitistic dimension of artinian rings with vanishing radical cube*, Math. Z., **206** (1991), pp. 505-526.

[11] E. L. Green - E. Kirkman - J. Kuzmanovich, *Finitistic dimension of finite dimensional monomial algebras*, J. Algebra, **136** (1991), pp. 37-50.

[12] K. Igusa - D. Zacharia, *Syzygy pairs in monomial algebras*, Proc. Amer. Math. Soc., **108** (1990), pp. 601-604.

[13] M. I. Platzeck, *Artin rings with all idempotent ideals projective*, Comm. Algebra, **24** (1996), pp. 2515-2533.

[14] J. R. Rotman, *An Introduction to Homological Algebra*, Academic Press, Orlando, 1979.

[15] S. O. Smalø, *Functorially finite subcategories over triangular matrix rings*, Proc. Amer. Math. Soc., **111** (1991), pp. 651-656.

[16] Y. Wang, *Finitistic dimensions of semiprimary rings*, Manuscripta Math., **81** (1993), pp. 79-87.

[17] Y. Wang, *Finitistic dimensions of serial-like rings*, J. Pure Appl. Algebra, **96** (1994), pp. 81-96.

[18] B. Zimmermann-Huisgen, *Syzygies and homological dimensions over left serial rings.* Methods in module theory (Colorado Springs, CO, 1991), pp. 161-174, Lecture Notes in Pure and Appl. Math., 140, Dekker, New York, 1993.

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