

Tri-bimaximal neutrino mixing, A_4 and the modular symmetry

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Abstract

We formulate and discuss a 4-dimensional SUSY version of an A_4 model for tri-bimaximal neutrino mixing which is completely natural. We also study the next-to-the-leading corrections and show that they are small, once the ratios of A_4 breaking VEVs to the cutoff are fixed in a specified interval. We also point out an interesting way of presenting the A_4 group starting from the modular group. In this approach, which could be interesting in itself as an indication on a possible origin of A_4 , the Lagrangian basis where the symmetry is formulated coincides with the basis where the charged leptons are diagonal. If the same classification structure in A_4 is extended from leptons to quarks, the CKM matrix coincides with the unit matrix in leading order and a study of non-leading corrections shows that the departures from unity of the CKM matrix are far too small to accommodate the observed mixing angles.

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1. Introduction

It is an experimental fact [1] that within measurement errors the observed form of the neutrino mixing matrix is compatible with the so-called tri-bimaximal form, discussed by Harrison, Perkins and Scott (HPS) [2], which, apart from phase redefinitions is given by:

$$U = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix}. \quad (1)$$

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It is an interesting challenge to formulate dynamical principles that can lead to this specific mixing pattern in a completely natural way as a first approximation, with small corrections determined by higher order terms in a well-defined expansion. In a series of papers [3–5] it has been pointed out that a broken flavour symmetry based on the discrete group A_4 appears to be particularly fit for this purpose. Other solutions based on continuous flavour groups like $SU(3)$ or $SO(3)$ have also been recently presented [6,7], but the A_4 models have a very simple (for example, in terms of field content) and attractive structure. In a recent paper [8] we have constructed an explicit A_4 model where the problem as stated above is solved. A crucial feature of all HPS models is the mechanism used to guarantee the necessary VEV alignment of the flavon field φ_T which determines the charged lepton mass matrix with respect to the direction in flavour space chosen by the flavon φ_S that gives the neutrino mass matrix. In Ref. [8] we adopted an extra-dimensional framework, with φ_T and φ_S on different branes so that the minimization of the respective potentials is kept to a large extent independent. The advantage of this approach is that the HPS mixing is reproduced quite naturally in a comparatively simple way. A moderately hierarchical neutrino mass spectrum is obtained. The correction terms from all possible higher dimensionality operators allowed by the symmetries of the model are shown to be small for a wide range of values for the cut-off scales. Finally, the observed hierarchy of charged lepton masses can be reproduced without fine tuning by enlarging the flavour symmetry with an extra $U(1)$.

In the present article we address a number of questions which are left open. First, we give an alternative formulation of the A_4 model in 4 dimensions with supersymmetry (SUSY) which shows that the connection of A_4 with the HPS matrix is robust, in the sense that it can be obtained in different ways and does not necessarily require extra dimensions. The two versions differ in the set of additional fields and in the pattern of non leading corrections, so that experimental tests are in principle possible. The advantage of SUSY is to considerably simplify the problem of obtaining the right vacuum alignment from the minimization of the relevant potential. We present a detailed discussion of the pattern of non leading corrections in the new version of the model. We also address the important problem of trying to understand the dynamical origin of A_4 . To this end we reformulate the 12 elements of A_4 as products of two matrices S and T with $S^2 = (ST)^3 = T^3 = 1$. In this formulation the Lagrangian basis directly coincides with that where the charged leptons are diagonal. The main virtue of this formulation is that A_4 is seen as a subgroup of the modular group of transformations which often plays a role in the formalism of string theories, for example in the context of duality transformations [9]. We then discuss the extension to quarks. We show that a direct extrapolation to quarks of the classification scheme adopted in A_4 for leptons immediately leads in lowest approximation to a diagonal CKM mixing matrix. This fact had been already observed in a similar A_4 context in Refs. [3,4]. We study the higher order corrections in our specific framework and show that a direct extension to the quark sector of the A_4 classification leads, apart from negligible terms, to the same contributions for up and down mass matrices, so that the CKM matrix remains diagonal. Thus new sources of A_4 breaking are needed in the quark sector.

2. A_4 revisited

A_4 can also be defined as the group generated by the two elements S and T obeying the relations [10] (a “presentation” of the group):

$$S^2 = (ST)^3 = T^3 = 1. \quad (2)$$

It is immediate to see that one-dimensional unitary representations are given by:

$$\begin{aligned} 1 \quad S = 1, \quad T = 1, \\ 1' \quad S = 1, \quad T = e^{i4\pi/3} \equiv \omega^2, \\ 1'' \quad S = 1, \quad T = e^{i2\pi/3} \equiv \omega. \end{aligned} \tag{3}$$

It is simple to check that a two-dimensional unitary representation does not exist (only $\det(T^3) = -1$ is in this case compatible with $S^2 = (ST)^3 = 1$). The three-dimensional unitary representation, in a basis where the element T is diagonal, is given by:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}. \tag{4}$$

The elements of A_4 can be represented by the 3×3 matrices M_i ($i = 1, 2, \dots, 12$) given by: $1, S, T, ST, TS, T^2, ST^2, STS, TST, T^2S, TST^2, T^2ST$. Indeed if one performs the unitary transformation $M_i \rightarrow M'_i = V^\dagger M_i V$ with

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \tag{5}$$

one obtains the 12 matrices listed in Eqs. (5)–(8) of Ref. [3] which span the three-dimensional representation of A_4 .

Starting from the explicit expressions of S and T we can build the multiplication rules for triplet representations. Consider the two triplets:

$$a = (a_1, a_2, a_3), \quad b = (b_1, b_2, b_3). \tag{6}$$

The combination $xa_1b_1 + ya_2b_3 + za_3b_2$ is invariant under T . If we also enforce invariance under S it is easy to see that we get the invariant singlet

$$1 \equiv (ab) = (a_1b_1 + a_2b_3 + a_3b_2). \tag{7}$$

Similarly, the singlets $1'$ and $1''$ are obtained from the combinations $x'a_3b_3 + y'a_1b_2 + z'a_2b_1$ and $x''a_2b_2 + y''a_3b_1 + z''a_1b_3$, by imposing invariance under S :

$$\begin{aligned} 1' \equiv (ab)' &= (a_3b_3 + a_1b_2 + a_2b_1), \\ 1'' \equiv (ab)'' &= (a_2b_2 + a_1b_3 + a_3b_1). \end{aligned} \tag{8}$$

With a little algebra it is possible to see that the remaining 6 independent combinations fill two triplets, a symmetric one and an antisymmetric one:

$$\begin{aligned} 3 \equiv (ab)_S &= \frac{1}{3}(2a_1b_1 - a_2b_3 - a_3b_2, 2a_3b_3 - a_1b_2 - a_2b_1, 2a_2b_2 - a_1b_3 - a_3b_1), \\ 3' \equiv (ab)_A &= \frac{1}{2}(a_2b_3 - a_3b_2, a_1b_2 - a_2b_1, a_1b_3 - a_3b_1). \end{aligned} \tag{9}$$

Moreover, if c, c' and c'' are singlets of the type $1, 1'$ and $1''$, and $a = (a_1, a_2, a_3)$ is a triplet, then the products ac, ac' and ac'' are triplets explicitly given by $(a_1c, a_2c, a_3c), (a_3c', a_1c', a_2c')$ and (a_2c'', a_3c'', a_1c'') , respectively.

The group A_4 has two obvious subgroups: G_S , which is a reflection subgroup generated by S and G_T , which is the group generated by T , isomorphic to Z_3 . If the flavour symmetry associated

to A_4 is broken by the VEV of a triplet $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ of scalar fields, there are two interesting breaking pattern. The VEV

$$\langle \varphi \rangle = (1, 1, 1) \quad (10)$$

breaks A_4 down to G_S , while

$$\langle \varphi \rangle = (1, 0, 0) \quad (11)$$

breaks A_4 down to G_T . As we will see, G_S and G_T are the relevant low-energy symmetries of the neutrino and the charged-lepton sectors, respectively. Note that the vectors (10) and (11) are interchanged under the transformation in Eq. (5).

3. Basic structure of the model

We discuss here general properties of the model based on the A_4 realization discussed above which are independent of the particular mechanism adopted to guarantee the required VEV alignment which will be specified in the next section. Following Ref. [8] we assigns leptons to the four inequivalent representations of the group A_4 : left-handed lepton doublets l transform as a triplet 3, while the right-handed charged leptons e^c , μ^c and τ^c transform as 1, $1''$ and $1'$, respectively. The flavour symmetry is broken by two triplets φ_S and φ_T and by a singlet ξ . Actually we may need more singlets and indeed in the next section we will introduce two of them. But we can always choose a basis in the space of these singlets such that ξ denotes the field with a non-vanishing VEV, whereas all the other ones have a zero VEV and do not contribute to the neutrino mass matrices. So in this section we only keep the terms with ξ for simplicity. All these fields are gauge singlets. Two Higgs doublets $h_{u,d}$, invariant under A_4 , are also introduced. We assume that some mechanism produces and maintains the hierarchy $\langle h_{u,d} \rangle = v_{u,d} \ll \Lambda$ where Λ is the cut-off scale of the theory (for example, by adopting a supersymmetric version of the model, as in the next section). The Yukawa interactions in the lepton sector read:

$$\mathcal{L}_l = y_e e^c (\varphi_T l) + y_\mu \mu^c (\varphi_T l)' + y_\tau \tau^c (\varphi_T l)'' + x_a \xi (ll) + x_b (\varphi_S ll) + \text{h.c.} + \dots \quad (12)$$

To keep our formulae compact, we use a two-component notation for the fermion fields and omit to write the Higgs fields $h_{u,d}$ and the cut-off scale Λ . For instance $y_e e^c (\varphi_T l)$ stands for $y_e e^c (\varphi_T l) h_d / \Lambda$, $x_a \xi (ll)$ stands for $x_a \xi (l h_u l h_u) / \Lambda^2$ and so on. The Lagrangian \mathcal{L}_l contains the lowest order operators in an expansion in powers of $1/\Lambda$. Dots stand for higher-dimensional operators that will be discussed later on. Some terms allowed by the flavour symmetry, such as the terms obtained by the exchange $\varphi_T \leftrightarrow \varphi_S$, or the term (ll) are missing in \mathcal{L}_l . Their absence is crucial and will be motivated later on. As we will demonstrate, the fields φ_T , φ_S and ξ develop a VEV along the directions:

$$\langle \varphi_T \rangle = (v_T, 0, 0), \quad \langle \varphi_S \rangle = (v_S, v_S, v_S), \quad \langle \xi \rangle = u. \quad (13)$$

At the leading order in $1/\Lambda$ and after the breaking of the flavour and electroweak symmetries, the mass terms from the Lagrangian (12) are

$$\begin{aligned} \mathcal{L}_l = & v_d \frac{v_T}{\Lambda} (y_e e^c e + y_\mu \mu^c \mu + y_\tau \tau^c \tau) \\ & + x_a v_u^2 \frac{u}{\Lambda^2} (v_e v_e + 2v_\mu v_\tau) \\ & + x_b v_u^2 \frac{2v_S}{3\Lambda^2} (v_e v_e + v_\mu v_\mu + v_\tau v_\tau - v_e v_\mu - v_\mu v_\tau - v_\tau v_e) + \text{h.c.} + \dots \end{aligned} \quad (14)$$

Here we have made use of Eqs. (7)–(9). In the charged lepton sector the flavour symmetry A_4 is broken by $\langle \varphi_T \rangle$ down to G_T . Actually the above mass terms for charged leptons are the most general allowed by the symmetry G_T . At leading order in $1/\Lambda$, charged lepton masses are diagonal simply because there is a low-energy G_T symmetry. In the neutrino sector A_4 is broken down to G_S , though neutrino masses in this model are not the most general ones allowed by G_S . From Eq. (14), at the leading order of the $1/\Lambda$ expansion, we read the mass matrices m_l and m_ν for charged leptons and neutrinos:

$$m_l = v_d \frac{v_T}{\Lambda} \begin{pmatrix} y_e & 0 & 0 \\ 0 & y_\mu & 0 \\ 0 & 0 & y_\tau \end{pmatrix}, \quad (15)$$

$$m_\nu = \frac{v_u^2}{\Lambda} \begin{pmatrix} a + 2b/3 & -b/3 & -b/3 \\ -b/3 & 2b/3 & a - b/3 \\ -b/3 & a - b/3 & 2b/3 \end{pmatrix}, \quad (16)$$

where

$$a \equiv 2x_a \frac{u}{\Lambda}, \quad b \equiv 2x_b \frac{v_S}{\Lambda}. \quad (17)$$

Charged fermion masses are given by:

$$m_e = y_e v_d \frac{v_T}{\Lambda}, \quad m_\mu = y_\mu v_d \frac{v_T}{\Lambda}, \quad m_\tau = y_\tau v_d \frac{v_T}{\Lambda}. \quad (18)$$

We can easily obtain a natural hierarchy among m_e , m_μ and m_τ by introducing an additional $U(1)_F$ flavour symmetry under which only the right-handed lepton sector is charged. We write the F-charge values in this model as 0, q and $2q$ for τ^c , μ^c and e^c , respectively. By assuming that a flavon θ , carrying a negative unit of F, acquires a VEV $\langle \theta \rangle / \Lambda \equiv \lambda < 1$, the Yukawa couplings become field dependent quantities $y_{e,\mu,\tau} = y_{e,\mu,\tau}(\theta)$ and we have

$$y_\tau \approx O(1), \quad y_\mu \approx O(\lambda^q), \quad y_e \approx O(\lambda^{2q}). \quad (19)$$

The neutrino mass matrix is diagonalized by the transformation:

$$U^T m_\nu U = \frac{v_u^2}{\Lambda} \text{diag}(m_1 = a + b, m_2 = a, m_3 = -a + b), \quad (20)$$

with

$$U = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & +1/\sqrt{2} \end{pmatrix}. \quad (21)$$

Thus the HPS mixing matrix is obtained in the leading approximation. The constraints on the parameters and on the scales of the model in order to obtain a realistic neutrino mass spectrum are exactly as in our previous paper [8]. In fact the formulation of A_4 adopted in this paper is such that the Lagrangian basis where the symmetry is specified coincides with the basis where the charged leptons are diagonal. This makes the role of A_4 in producing the HPS mixing more transparent but, at the leading level, the old and the new versions are related by a unitary change of basis. Thus the present version of A_4 model is completely natural, as much as our previous version with an extra dimension, both needing only a moderate amount of fine tuning to reproduce the small value of $r = \Delta m_{\text{sun}}^2 / \Delta m_{\text{atm}}^2$. The difference between the extra-dimensional version and the present SUSY version is however relevant at the level of subleading

terms. We recall the expected range for the parameters in the symmetry breaking sector. By assuming that all the VEVs breaking A_4 have approximately the same value, in Ref. [8] we found that

$$0.004 < \frac{v_S}{\Lambda} \approx \frac{v_T}{\Lambda} \approx \frac{u}{\Lambda} < 1 \tag{22}$$

and that the cutoff Λ should be limited between 10^{13} GeV and 2×10^{15} GeV. In particular, the lower bound in (22) comes from requiring that the tau Yukawa coupling is within a perturbative regime. In the supersymmetric version of the model discussed in the next section, the tau Yukawa coupling is given by $y_\tau = (m_\tau \Lambda)/(v \cos \beta v_T)$ where $v \approx 174$ GeV and $\tan \beta = v_u/v_d$. By asking $y_\tau < 4\pi$ we find $v_T/\Lambda > 0.0022(0.024)$ when $\tan \beta = 2.5(30)$. In what follows we take as lower limit $v_T/\Lambda > 0.0022$.

In conclusion, if one can construct a natural mechanism to guarantee the necessary alignment of the VEVs φ_S and φ_T , one obtains a first approximation where the neutrino mixing is of the HPS form. In the next section we will present a supersymmetric version of the model in four dimensions and later we will discuss the non-leading corrections in this context.

4. Vacuum alignment

Here we discuss a supersymmetric solution to the vacuum alignment problem. In a SUSY context, the right-hand side of Eq. (12) should be interpreted as the superpotential w_l of the theory, in the lepton sector:

$$w_l = y_e e^c (\varphi_T l) + y_\mu \mu^c (\varphi_T l)' + y_\tau \tau^c (\varphi_T l)'' + (x_a \xi + \tilde{x}_a \tilde{\xi})(ll) + x_b (\varphi_S ll) + \text{h.c.} + \dots, \tag{23}$$

where dots stand for higher-dimensional operators that will be discussed in the next section and where we have also added an additional A_4 -invariant singlet $\tilde{\xi}$. Such a singlet does not modify the structure of the mass matrices discussed previously, but plays an important role in the vacuum alignment mechanism. A key observation is that the superpotential w_l is invariant not only with respect to the gauge symmetry $SU(2) \times U(1)$ and the flavour symmetry $U(1)_F \times A_4$, but also under a discrete Z_3 symmetry and a continuous $U(1)_R$ symmetry under which the fields transform as shown in Table 1.

We see that the Z_3 symmetry explains the absence of the term (ll) in w_l : such a term transforms as ω^2 under Z_3 and need to be compensated by the field ξ in our construction. At the same time Z_3 does not allow the interchange between φ_T and φ_S , which transform differently under Z_3 . The singlets ξ and $\tilde{\xi}$ have the same transformation properties under all symmetries and, as we shall see, in a finite range of parameters, the VEV of $\tilde{\xi}$ vanishes and does not contribute to neutrino masses. Charged leptons and neutrinos acquire masses from two independent sets of fields. If the two sets of fields develop VEVs according to the alignment described in Eq. (13), then the desired mass matrices follow.

Table 1

Field	1	e^c	μ^c	τ^c	$h_{u,d}$	φ_T	φ_S	ξ	$\tilde{\xi}$	φ_0^T	φ_0^S	ξ_0
A_4	3	1	1'	1''	1	3	3	1	1	3	3	1
Z_3	ω	ω^2	ω^2	ω^2	1	1	ω	ω	ω	1	ω	ω
$U(1)_R$	1	1	1	1	0	0	0	0	0	2	2	2

Finally, there is a continuous $U(1)_R$ symmetry that contains the usual R -parity as a subgroup. Suitably extended to the quark sector, this symmetry forbids the unwanted dimension two and three terms in the superpotential that violate baryon and lepton number at the renormalizable level. The $U(1)_R$ symmetry allows us to classify fields into three sectors. There are “matter fields” such as the leptons l , e^c , μ^c and τ^c , which occur in the superpotential through bilinear combinations. There is a “symmetry breaking sector” including the Higgs doublets $h_{u,d}$ and the flavons φ_T , φ_S , $(\xi, \tilde{\xi})$. As we will see these fields acquire non-vanishing VEVs and break the symmetries of the model. Finally, there are “driving fields” such as φ_0^T , φ_0^S and ξ_0 that allows to build a non-trivial scalar potential in the symmetry breaking sector. Since driving fields have R -charge equal to two, the superpotential is linear in these fields.

The full superpotential of the model is

$$w = w_l + w_d, \quad (24)$$

where, at leading order in a $1/\Lambda$ expansion, w_l is given by the right-hand side of Eq. (12) and the “driving” term w_d reads:

$$w_d = M(\varphi_0^T \varphi_T) + g(\varphi_0^T \varphi_T \varphi_T) + g_1(\varphi_0^S \varphi_S \varphi_S) + g_2 \tilde{\xi}(\varphi_0^S \varphi_S) + g_3 \xi_0(\varphi_S \varphi_S) + g_4 \xi_0 \xi^2 + g_5 \xi_0 \xi \tilde{\xi} + g_6 \xi_0 \tilde{\xi}^2. \quad (25)$$

At this level there is no fundamental distinction between the singlets ξ and $\tilde{\xi}$. Thus we are free to define $\tilde{\xi}$ as the combination that couples to $(\varphi_0^S \varphi_S)$ in the superpotential w_d . We notice that at the leading order there are no terms involving the Higgs fields $h_{u,d}$. We assume that the electroweak symmetry is broken by some mechanism, such as radiative effects when SUSY is broken. It is interesting that at the leading order the electroweak scale does not mix with the potentially large scales u , v and v' . The scalar potential is given by:

$$V = \sum_i \left| \frac{\partial w}{\partial \phi_i} \right|^2 + m_i^2 |\phi_i|^2 + \dots, \quad (26)$$

where ϕ_i denote collectively all the scalar fields of the theory, m_i^2 are soft masses and dots stand for D-terms for the fields charged under the gauge group and possible additional soft breaking terms. Since m_i are expected to be much smaller than the mass scales involved in w_d , it makes sense to minimize V in the supersymmetric limit and to account for soft breaking effects subsequently. From the driving sector we have:

$$\begin{aligned} \frac{\partial w}{\partial \varphi_{01}^T} &= M \varphi_{T1} + \frac{2g}{3} (\varphi_{T1}^2 - \varphi_{T2} \varphi_{T3}) = 0, \\ \frac{\partial w}{\partial \varphi_{02}^T} &= M \varphi_{T3} + \frac{2g}{3} (\varphi_{T2}^2 - \varphi_{T1} \varphi_{T3}) = 0, \\ \frac{\partial w}{\partial \varphi_{03}^T} &= M \varphi_{T2} + \frac{2g}{3} (\varphi_{T3}^2 - \varphi_{T1} \varphi_{T2}) = 0, \\ \frac{\partial w}{\partial \varphi_{01}^S} &= g_2 \tilde{\xi} \varphi_{S1} + \frac{2g_1}{3} (\varphi_{S1}^2 - \varphi_{S2} \varphi_{S3}) = 0, \\ \frac{\partial w}{\partial \varphi_{02}^S} &= g_2 \tilde{\xi} \varphi_{S3} + \frac{2g_1}{3} (\varphi_{S2}^2 - \varphi_{S1} \varphi_{S3}) = 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial \varphi_{03}^2} &= g_2 \tilde{\xi} \varphi_{S2} + \frac{2g_1}{3} (\varphi_{S3}^2 - \varphi_{S1} \varphi_{S2}) = 0, \\ \frac{\partial w}{\partial \xi_0} &= g_4 \xi^2 + g_5 \xi \tilde{\xi} + g_6 \tilde{\xi}^2 + g_3 (\varphi_{S1}^2 + 2\varphi_{S2} \varphi_{S3}) = 0. \end{aligned} \tag{27}$$

A solution to the first three equations is:

$$\varphi_T = (v_T, 0, 0), \quad v_T = -\frac{3M}{2g}. \tag{28}$$

This VEV breaks A_4 down to G_T .¹ The need of an additional singlet can be understood by looking at the remaining equations. Indeed, if a unique singlet were present, which can be realized by setting $\xi = 0$, then the only solution to these equations would be that with all vanishing VEVs for φ_S and $\tilde{\xi}$. The additional singlet is therefore essential to recover a non-trivial solution.² In particular, if we choose $m_\xi^2 > 0$, thus enforcing $\langle \tilde{\xi} \rangle = 0$, in a finite portion of the parameter space we find the solution

$$\begin{aligned} \tilde{\xi} &= 0, \\ \xi &= u, \\ \varphi_S &= (v_S, v_S, v_S), \quad v_S^2 = -\frac{g_4}{3g_3} u^2 \end{aligned} \tag{29}$$

with u undetermined.³ By choosing $m_{\varphi_0}^2, m_{\varphi'_0}^2, m_{\xi_0}^2 > 0$, the driving fields φ_0, φ'_0 and ξ_0 vanish at the minimum. Moreover, if $m_{\varphi_S}^2, m_\xi^2 < 0$, then u slides to a large scale, which we assume to be eventually stabilized by one-loop radiative corrections.

As for the $U(1)_F$ field θ it is easy to see that, in the unbroken SUSY limit, its VEV remains undetermined as a consequence of the vanishing of the charged lepton field VEVs. When SUSY is broken the $m_\theta^2 |\theta|^2$ term in the potential would drive the θ VEV to zero (or to ∞ if $m_\theta^2 < 0$). However, in the presence of a renormalizable coupling of θ to additional field(s), like, for example, $g_\sigma \theta \sigma^2$, one-loop radiative corrections typically bring back the θ VEV near the cutoff. We implicitly assume that such field(s) σ, \dots , which are completely neutral except for the appropriate $U(1)_F$ and $U(1)_R$ charges, are included in our model. Thus the value of the ratio $\langle \theta \rangle / \Lambda$ can be taken as a free parameter that, together with the charge value q , fixes the charged lepton mass ratios as given in Eq. (19).

5. Higher-order corrections

The results of the previous section hold to first approximation. Higher-dimensional operators, suppressed by additional powers of the cut-off Λ , can be added to the leading terms in the La-

¹ More precisely, since the solutions lie in an orbit of the group A_4 , the non-trivial solutions are (28) and those generated by acting on (28) by the elements of A_4 : $\varphi_T = (M/2g)(1, -2, -2)$, $\varphi_T = (M/2g)(1, -2\omega^2, -2\omega)$ and $\varphi_T = (M/2g)(1, -2\omega, -2\omega^2)$. Each of these vacua leaves unbroken a Z_3 subgroup of A_4 . It is not restrictive to choose the vacuum $\varphi_T = -(3M/2g)(1, 0, 0)$. The trivial solution $\varphi_T = (0, 0, 0)$ can be eliminated by choosing $m_{\varphi_T}^2 < 0$.

² We are indebted to G.G. Ross for pointing out this possibility to us.

³ Also in this case we find other degenerate solutions, obtained by acting on (29) with the elements of A_4 : $\varphi_S = v_S(1, \omega, \omega^2)$ and $\varphi_S = v_S(1, \omega^2, \omega)$. Any of these solutions produces the same neutrino mass matrix. For instance $\varphi_S = v_S(1, \omega, \omega^2)$ is equivalent to $\varphi_S = v_S(1, 1, 1)$, the two being related by the local field transformations $\nu_e \rightarrow \nu_e, \nu_\mu \rightarrow \omega^2 \nu_\mu, \nu_\tau \rightarrow \omega \nu_\tau$.

grangian. Here we will classify these non-leading terms and analyze their physical effects. In particular we will show that these corrections are completely under control in our model and that they can be made negligibly small without any fine-tuning. We can classify higher-order operators into three groups.

5.1. Corrections to m_l

The leading operators giving rise to m_l are of order $1/\Lambda$ (see Eqs. (14), (15)). At order $(1/\Lambda)^2$ there are no new structures contributing to m_l . Indeed the only invariant operator:

$$\frac{1}{\Lambda^2}(f^c l \varphi_T \varphi_T) h_d \quad (f^c = e^c, \mu^c, \tau^c) \tag{30}$$

replicates the leading-order pattern, as can be seen from the fact that the symmetric triplet

$$(\varphi_T \varphi_T)_S = \frac{2}{3}(\varphi_{T1}^2 - \varphi_{T2} \varphi_{T3}, \varphi_{T3}^2 - \varphi_{T1} \varphi_{T2}, \varphi_{T2}^2 - \varphi_{T3} \varphi_{T1}) \tag{31}$$

has a VEV in the same direction as φ_T . Thus, in the charged lepton sector, the first corrections arise at relative order $1/\Lambda^2$.

5.2. Corrections to m_ν

The leading operators contributing to m_ν are of order $1/\Lambda^2$ (see Eqs. (14), (16)). At the next order we have three operators, whose contribution to m_ν , after symmetry breaking, cannot be absorbed by a redefinition of the parameters $x_{a,b}$:

$$\frac{x_c}{\Lambda^3}(\varphi_T \varphi_S)'(ll)'' h_u h_u, \quad \frac{x_d}{\Lambda^3}(\varphi_T \varphi_S)''(ll)' h_u h_u, \quad \frac{x_e}{\Lambda^3} \xi(\varphi_T ll) h_u h_u. \tag{32}$$

The corrections from these operators will be taken into account in Section 5.4.

5.3. Corrections to the vacuum alignment

In Appendix B, the operators of higher dimension contributing to the superpotential w_d introduced in Eq. (25) are listed and the procedure of minimization is repeated. The leading corrections to the VEVs are of relative order $1/\Lambda$ and affect all the flavon fields. The correction to the VEV of φ_T , apart from a shift of v_T , is proportional to the VEV of φ_S . In turn, the VEV of φ_S is shifted in a generic direction, the VEV of ξ , which was vanishing at leading order, acquires a small component and that of ξ remains undetermined:

$$\begin{aligned} \langle \varphi_T \rangle &\rightarrow (v'_T + \delta v_T, \delta v_T, \delta v_T), \\ \langle \varphi_S \rangle &\rightarrow (v_S + \delta v_1, v_S + \delta v_2, v_S + \delta v_3), \\ \langle \xi \rangle &\rightarrow u, \\ \langle \tilde{\xi} \rangle &\rightarrow \delta u', \end{aligned} \tag{33}$$

where u is undetermined, $v'_T - v_T, \delta v_T, \delta v_i$ and $\delta u'$ are suppressed with respect to v_T and v_S by a factor $1/\Lambda$.

5.4. Modified masses and mixing angles

The new vacuum in Eq. (33) modifies the leading order mass matrix m_l in Eq. (15) into $m'_l = m_l(v_T \rightarrow v'_T) + \delta m_l$

$$\delta m_l = v_d \frac{\delta v_T}{\Lambda} \begin{pmatrix} y_e & y_e & y_e \\ y_\mu & y_\mu & y_\mu \\ y_\tau & y_\tau & y_\tau \end{pmatrix}. \tag{34}$$

The matrix $m_l'^{\dagger} m'_l$ is diagonalized by sending l into $U_e l$ where, by neglecting terms of relative order $(y_e/y_\tau)^2$ and $(y_\mu/y_\tau)^2$, U_e is given by:

$$U_e = \begin{pmatrix} 1 & \frac{\delta v_T}{v_T} & \frac{\delta v_T}{v_T} \\ -\frac{\delta v_T}{v_T} & 1 & \frac{\delta v_T}{v_T} \\ -\frac{\delta v_T}{v_T} & -\frac{\delta v_T}{v_T} & 1 \end{pmatrix}. \tag{35}$$

Charged lepton masses are modified by an overall factor $(v'_T + \delta v_T)/v_T$.

The neutrino mass matrix is modified by both the new vacuum, Eq. (33), and by the new operators (32). The non-vanishing VEV of $\tilde{\xi}$ can be absorbed into a redefinition of the parameter a of Eq. (17). The remaining effects change m_ν of Eq. (16) into $m'_\nu = m_\nu + \delta m_\nu$

$$\delta m_\nu = \begin{pmatrix} 2\delta z_1/3 & -\delta z_3/3 + \delta z_5 & -\delta z_2/3 + \delta z_4 \\ -\delta z_3/3 + \delta z_5 & 2\delta z_2/3 + \delta z_4 & -\delta z_1/3 \\ -\delta z_2/3 + \delta z_4 & -\delta z_1/3 & 2\delta z_3/3 + \delta z_5 \end{pmatrix} \frac{v_u^2}{\Lambda}, \tag{36}$$

where

$$\delta z_1 \equiv 2 \left(\frac{x_b}{\Lambda} \delta v_1 + x_e \frac{u v_T}{\Lambda^2} \right), \quad \delta z_2 \equiv 2 \frac{x_b}{\Lambda} \delta v_2, \quad \delta z_3 \equiv 2 \frac{x_b}{\Lambda} \delta v_3, \tag{37}$$

$$\delta z_4 \equiv 2 x_c \frac{v_S v_T}{\Lambda^2}, \quad \delta z_5 \equiv 2 x_d \frac{v_S v_T}{\Lambda^2}. \tag{38}$$

To first order in δz_i neutrino masses are given by:

$$\begin{aligned} m_1 &= \frac{v_u^2}{\Lambda} \left[a + b + \frac{1}{3}(\delta z_1 + \delta z_2 + \delta z_3) - \frac{1}{2}(\delta z_4 + \delta z_5) \right], \\ m_2 &= \frac{v_u^2}{\Lambda} [a + \delta z_4 + \delta z_5], \\ m_3 &= \frac{v_u^2}{\Lambda} \left[-a + b + \frac{1}{3}(\delta z_1 + \delta z_2 + \delta z_3) + \frac{1}{2}(\delta z_4 + \delta z_5) \right]. \end{aligned} \tag{39}$$

By combining the first order corrections to neutrino and charged lepton masses we find the modified parameters of the lepton mixing matrix (a bar on a letter indicates complex conjugation):

$$\begin{aligned} |U_{e3}| &= \frac{1}{\sqrt{2}} \left| \left[\frac{1}{3(\bar{a}b + \bar{b}a - |b|^2)} (\bar{a}(\delta z_2 - \delta z_3) + (b - a)(\delta \bar{z}_2 - \delta \bar{z}_3)) \right. \right. \\ &\quad \left. \left. - \frac{1}{2(\bar{a}b + \bar{b}a)} ((\bar{a} + \bar{b})(\delta z_4 - \delta z_5) - (a - b)(\delta \bar{z}_4 - \delta \bar{z}_5)) \right] \right|, \end{aligned} \tag{40}$$

$$\tan^2 \theta_{12} = \frac{1}{2} - 3 \frac{\delta v_T}{v_T} + \frac{1}{4(\bar{a}b + \bar{b}a + |b|^2)} [(2\bar{a} + \bar{b})(-2\delta z_1 + \delta z_2 + \delta z_3) + \text{c.c.}], \tag{41}$$

$$\tan^2 \theta_{23} = 1 + 4 \frac{\delta v_T}{v_T} - \frac{2}{3(\bar{a}b + \bar{b}a - |b|^2)} [\bar{b}(\delta z_2 - \delta z_3) + \text{c.c.}] - \frac{1}{(\bar{a}b + \bar{b}a)} [\bar{b}(\delta z_4 - \delta z_5) + \text{c.c.}]. \tag{42}$$

Given the expected range for the VEVs for v_T , v_S and u , Eq. (22), we see that all the corrections can be kept small, below the percent level. We see that deviations from the leading order predictions are obtained for all measurable quantities at approximately the same level. If we require that the subleading terms do not spoil the leading order picture, these deviations should not be larger than about 0.05. This can be inferred by the agreement of both $\Delta m_{\text{sol}}^2 / \Delta m_{\text{atm}}^2$ and $\tan^2 \theta_{12}$ with the experimental values. We then go back to Eq. (22) and include this constraint:

$$0.0022 < \frac{v_S}{\Lambda} \approx \frac{v_T}{\Lambda} \approx \frac{u}{\Lambda} < 0.05. \tag{43}$$

We recall that the lower bound 0.0022 was derived from the requirement that the Yukawa coupling y_τ is small enough to justify a truncated perturbative expansion. Note that the ratio $\langle \theta \rangle / \Lambda$ is not limited by the constraint in Eq. (43).

6. See-saw realization

We can easily modify the previous model to accommodate the see-saw mechanism. We introduce conjugate right-handed neutrino fields ν^c transforming as a triplet of A_4 and we modify the transformation law of the other fields according to Table 2.

The superpotential becomes

$$w = w_l + w_d, \tag{44}$$

where the ‘driving’ part is unchanged, whereas w_l is now given by:

$$w_l = y_e e^c (\varphi_T l) + y_\mu \mu^c (\varphi_T l)' + y_\tau \tau^c (\varphi_T l)'' + y(\nu^c l) + (x_A \xi + \tilde{x}_A \tilde{\xi})(\nu^c \nu^c) + x_B (\varphi_S \nu^c \nu^c) + \text{h.c.} + \dots \tag{45}$$

dots denoting higher-order contributions. The vacuum alignment proceeds exactly as discussed in Section 4 and also the charged lepton sector is unaffected by the modifications. In the neutrino sector, after electroweak and A_4 symmetry breaking we have Dirac and Majorana masses:

$$m_\nu^D = y \nu u \mathbf{1}, \quad M = \begin{pmatrix} A + 2B/3 & -B/3 & -B/3 \\ -B/3 & 2B/3 & A - B/3 \\ -B/3 & A - B/3 & 2B/3 \end{pmatrix} u, \tag{46}$$

where $\mathbf{1}$ is the unit 3×3 matrix and

$$A \equiv 2x_A, \quad B \equiv 2x_B \frac{v_S}{u}. \tag{47}$$

Table 2

Field	ν^c	φ_S	ξ	$\tilde{\xi}$	φ_0^S	ξ_0
A_4	3	3	1	1	3	1
Z_3	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
$U(1)_R$	1	0	0	0	2	2

The mass matrix for light neutrinos is $m_\nu = (m_\nu^D)^T M^{-1} m_\nu^D$ with eigenvalues

$$m_1 = \frac{y^2}{A+B} \frac{v_u^2}{u}, \quad m_2 = \frac{y^2}{A} \frac{v_u^2}{u}, \quad m_3 = \frac{y^2}{-A+B} \frac{v_u^2}{u}. \quad (48)$$

The mixing matrix is the HPS one, Eq. (21). In the presence of a see-saw mechanism both normal and inverted hierarchies in the neutrino mass spectrum can be realized. If we call Φ the relative phase between the complex number A and B , then $\cos \Phi > -|B|/2|A|$ is required to have $|m_2| > |m_1|$. In the interval $-|B|/2|A| < \cos \Phi \leq 0$, the spectrum is of inverted hierarchy type, whereas in $|B|/2|A| \leq \cos \Phi \leq 1$ the neutrino hierarchy is of normal type. The quantity $|B|/2|A|$ cannot be too large, otherwise the ratio r cannot be reproduced. When $|B| \ll |A|$ the spectrum is quasi degenerate. When $|B| \approx |A|$ we obtain the strongest hierarchy. For instance, if $B = -2A + z$ ($|z| \ll |A|, |B|$), we find the following spectrum:

$$|m_1|^2 \approx \Delta m_{\text{atm}}^2 \left(\frac{9}{8} + \frac{1}{12} r \right), \quad |m_2|^2 \approx \Delta m_{\text{atm}}^2 \left(\frac{9}{8} + \frac{13}{12} r \right), \\ |m_3|^2 \approx \Delta m_{\text{atm}}^2 \left(\frac{1}{8} + \frac{1}{12} r \right). \quad (49)$$

When $B = A + z$ ($|z| \ll |A|, |B|$), we obtain:

$$|m_1|^2 \approx \Delta m_{\text{atm}}^2 \left(\frac{1}{3} r \right), \quad |m_2|^2 \approx \Delta m_{\text{atm}}^2 \left(\frac{4}{3} r \right), \quad |m_3|^2 \approx \Delta m_{\text{atm}}^2 \left(1 - \frac{1}{3} r \right). \quad (50)$$

These results can be affected by higher-order corrections induced by non-renormalizable operators. As before, charged lepton masses and mixing angles are unaffected at first order. Dirac and Majorana neutrino mass terms are instead corrected at first order, through the insertion of φ_T/A and also the VEVs receive a corrections. It is interesting to note that the contribution to the light neutrino masses coming directly from local operators of the type $(llh_u h_u \dots)$ is highly suppressed compared to the see-saw contribution. The latter is of order $1/\text{VEV}$, whereas the former, due to the Z_3 assignment, is of order VEV^2/Λ^3 . In conclusion, the symmetry structure of our model is fully compatible with the see-saw mechanism.

7. Quarks

There are several possibilities to include quarks. At first sight the most appealing one is to adopt for quarks the same classification scheme under A_4 that we have used for leptons. Thus we tentatively assume that left-handed quark doublets q transform as a triplet 3, while the right-handed quarks (u^c, d^c) , (c^c, s^c) and (t^c, b^c) transform as 1, $1''$ and $1'$, respectively. We can similarly extend to quarks the transformations of Z_3 and $U(1)_R$ given for leptons in the table of Section 4. The superpotential for quarks reads:

$$w_q = y_d d^c (\varphi_T q) + y_s s^c (\varphi_T q)' + y_b b^c (\varphi_T q)'' \\ + y_u u^c (\varphi_T q) + y_c c^c (\varphi_T q)' + y_t t^c (\varphi_T q)'' + \text{h.c.} + \dots \quad (51)$$

It is interesting to note that such an extrapolation to quarks leads to a diagonal CKM mixing matrix in first approximation [3,4]. In fact, starting from Eq. (51) and proceeding as described in detail for the lepton sector, we see that the up quark and down quark mass matrices are separately diagonal with mass eigenvalues which are left unspecified by A_4 and with a hierarchy that could

be accommodated by a suitable $U(1)_F$ set of charge assignments for quarks. Thus the V_{CKM} matrix is the identity in leading order, providing a good first order approximation.

The problems come when we discuss non-leading corrections. As seen in Section 5, first-order corrections to the lepton sector should be typically below 0.05, approximately the square of the Cabibbo angle. Therefore it seems difficult to reproduce the quark mixing between the first two generations in this scheme, without introducing new ingredients in the symmetry breaking sector. Also, by inspecting these corrections more closely, we see that, exactly as in the case of charged leptons, the quark mass matrices are not modified to first order by higher-dimensional Yukawa operators. The only possible first order changes could only come from the new vacuum, Eq. (33). Unfortunately, up to very small terms of order $y_{u(d)}^2/y_{t(b)}^2$ and $y_{c(s)}^2/y_{t(b)}^2$, these corrections are the same in the up and down sectors—see Eq. (35)—and therefore they almost exactly cancel in the mixing matrix V_{CKM} . We conclude that, if one insists in adopting for quarks the same flavour properties as for leptons, than new sources of A_4 breaking are needed in order to produce an acceptable V_{CKM} .

An other point of view is to regard A_4 as a special feature of the lepton sector, and to provide an independent description for quarks. For instance one could take quarks as invariant under A_4 and charged only with respect to the $U(1)$ part of the flavour group which controls the mass hierarchy. Masses and mixing angles for quarks would emerge from the symmetry breaking of an Abelian continuous flavour symmetry, as in many models of fermion masses [11]. This possibility has the obvious disadvantage of preventing a unified description of both quarks and lepton masses, as expected for instance in grand unified theories.

8. Relation with the modular group

There is an interesting relation between the A_4 model considered so far and the modular group. The modular group is the group of linear fractional transformations acting on a complex variable z :

$$z \rightarrow \frac{az + b}{cz + d}, \quad ad - bc = 1, \tag{52}$$

where a, b, c, d are integers. These transformations can be represented by the matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{53}$$

with integer coefficients and determinant 1, belonging to the group $SL(2, Z)$. Since however a matrix and its opposite in $SL(2, Z)$ define the same linear fractional transformation, the modular group Γ coincides with $PSL(2, Z)$, the matrices in $SL(2, Z)$ up to an overall sign. There are infinite elements in Γ , but all of them can be generated by the two transformations:

$$s: z \rightarrow -\frac{1}{z}, \quad t: z \rightarrow z + 1, \tag{54}$$

represented by the matrices⁴:

$$m_s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad m_t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{55}$$

⁴ Notice that m_t is not unitary and also that $m_s^2 = -1$ while $m_t^n \neq 1$ for all integer $n > 1$.

In string theory the transformations (54) operate in many different contexts. For instance the role of the complex variable z can be played by a field, whose VEV can be related to a physical quantity like a compactification radius or a coupling constant. In that case s in Eq. (54) represents a duality transformation and t in Eq. (54) represent the transformation associated to an “axionic” symmetry.

The transformations s and t in (54) satisfy the relations

$$s^2 = (st)^3 = 1 \tag{56}$$

and, conversely, these relations provide an abstract characterization of the modular group. Since the relations (2) are a particular case of the more general constraint (56), it is clear that the representations given in (3) and (4) are also representations of the modular group. Then the natural questions is: how much special are (3) and (4) from the point of view of Γ ? To answer this we should inspect the unitary irreducible representations of Γ . To this purpose it is sufficient to look at linear unitary irreducible representations of the two elements s and t , from which it is possible to reconstruct the representatives of any other element in Γ .

The singlet representations act simply as multiplications by a phase factor:

$$s = e^{i2\pi\alpha}, \quad t = e^{i2\pi\beta}. \tag{57}$$

It is immediate to see that there are six inequivalent choices for α and β that satisfy (56):

$$\begin{aligned} 1 & \quad s = 1, \quad t = 1, \\ 1^I & \quad s = 1, \quad t = e^{i4\pi/3}, \\ 1^{II} & \quad s = 1, \quad t = e^{i2\pi/3}, \\ 1^{III} & \quad s = -1, \quad t = e^{i5\pi/3}, \\ 1^{IV} & \quad s = -1, \quad t = e^{i3\pi/3}, \\ 1^V & \quad s = -1, \quad t = e^{i\pi/3}. \end{aligned} \tag{58}$$

The representation 1 is the invariant representation. If we have two fields a and b transforming according two representations of the above list, then their product ab also transforms according to a singlet representation and the multiplication table can be easily deduced by Eqs. (58):

	1	1 ^I	1 ^{II}	1 ^{III}	1 ^{IV}	1 ^V
1	1	1 ^I	1 ^{II}	1 ^{III}	1 ^{IV}	1 ^V
1 ^I	1 ^I	1 ^{II}	1	1 ^{IV}	1 ^V	1 ^{III}
1 ^{II}	1 ^{II}	1	1 ^I	1 ^V	1 ^{III}	1 ^{IV}
1 ^{III}	1 ^{III}	1 ^{IV}	1 ^V	1 ^I	1 ^{II}	1
1 ^{IV}	1 ^{IV}	1 ^V	1 ^{III}	1 ^{II}	1	1 ^I
1 ^V	1 ^V	1 ^{III}	1 ^{IV}	1	1 ^I	1 ^{II}

(59)

In particular we see that, beyond the invariant representation, there is an interesting subset of representations closed under the product. It is given by 1, 1^I, 1^{II}. These representations corresponds to the representations 1, 1' and 1'' used in our model.

The relations $s^2 = (st)^3 = 1$ lead to a quantization of α and β in Eq. (57) and, in the unidimensional case, there is a finite number of unitary representations. This is no-longer true when going to higher-dimensional representations [12]. We are particularly interested in the three-dimensional case which is discussed in detail in Appendix A. The result is that we have unitary

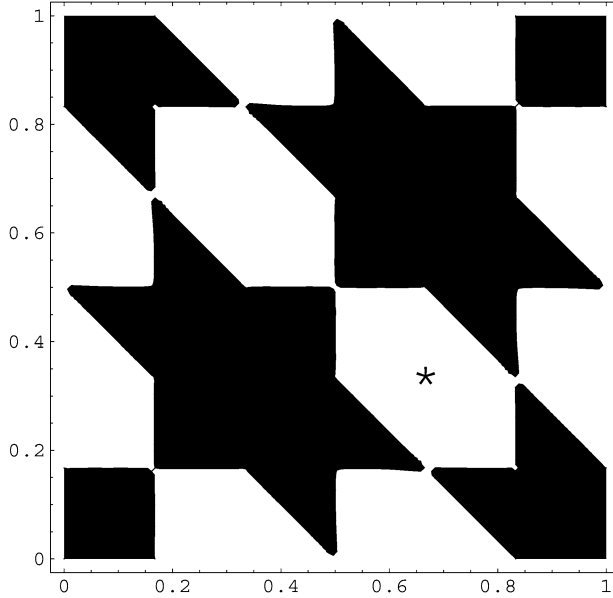


Fig. 1. Allowed (white) region in the parameter space (β_2, β_3) . The conditions (A.6), (A.7) of Appendix A are saturated along the lines $\beta_2 = 1/6, 3/6, 5/6$, $\beta_3 = 1/6, 3/6, 5/6$, $\beta_3 = -\beta_2 + 3/6$, $\beta_3 = -\beta_2 + 5/6$, $\beta_3 = -\beta_2 + 7/6$, $\beta_3 = -\beta_2 + 9/6$. Accounting for the periodicity in $\beta_{2,3}$, we have six allowed regions with the same shape and area, differing only by the location of their center. These six regions are related by permutations of β_1 , β_2 and β_3 and the inequivalent representations can be labelled by the points of one of them. The irreducible representation used in our model is marked by a star.

inequivalent representations described by

$$t = \begin{pmatrix} e^{i2\pi\beta_1} & 0 & 0 \\ 0 & e^{i2\pi\beta_2} & 0 \\ 0 & 0 & e^{i2\pi\beta_3} \end{pmatrix}, \quad (60)$$

and a matrix s also determined as function of β_i once the following relations are satisfied: $\beta_1 + \beta_2 + \beta_3 = 0 \pmod{1}$ and (β_2, β_3) in the region $3/6 \leq \beta_2 \leq 5/6$, $1/6 \leq \beta_3 \leq 3/6$, $-\beta_2 + 5/6 \leq \beta_3 \leq -\beta_2 + 7/6$. Therefore, there is a double infinity of possible three-dimensional representations which in the (β_2, β_3) plane appears as in Fig. 1. The representation (4) adopted in the construction of the model falls just in the center of the region allowed to (β_2, β_3) . Perhaps in the underlying theory there is a dynamical principle that selects this particularly symmetric point.

9. Conclusion

The A_4 discrete group appears to be particularly suitable to economically reproduce the precise relations among neutrino mass matrix elements which are needed to obtain the HPS mixing matrix. We have presented here a 4-dimensional SUSY version of an A_4 model which is completely natural, in the sense that no arbitrary tuning of parameters is necessary to lead to the HPS mixing angles, once the ratios of VEVs to the cutoff are fixed in a given interval. A moderate fine tuning is only present in the neutrino mass spectrum, in order to reproduce the observed

small value of $r = \Delta m_{\text{sun}}^2 / \Delta m_{\text{atm}}^2$. In a previous paper we had presented a model which was equally natural, actually in a more extended interval for the cut-off, but assumed extra dimensions, thus making the model more exotic. We then pointed out an interesting way of presenting the A_4 group as a particular set of transformations of the modular group. This approach has the immediate advantage that the Lagrangian basis where the symmetry is formulated coincides with the basis where the charged leptons are diagonal. But this connection could possibly lead to an insight on the possible origin of the A_4 symmetry within the context of a more fundamental theory. Finally, if the same structure of left-handed and right-handed field classification in A_4 is extended from leptons to quarks, then, in leading order, the CKM matrix coincides with the unit matrix. A study of non-leading corrections in this model shows that the departures from unity of the CKM matrix are far too small to reproduce the observed mixing angles. Thus the quark mixing angles, in this picture, should arise from additional effects specific of the quark sector.

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Appendix A

We explicitly classify and build the three-dimensional, irreducible unitary representations of the modular group. It is not restrictive to go to a basis where T is diagonal:

$$T = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \equiv \begin{pmatrix} e^{i2\pi\beta_1} & 0 & 0 \\ 0 & e^{i2\pi\beta_2} & 0 \\ 0 & 0 & e^{i2\pi\beta_3} \end{pmatrix}. \quad (\text{A.1})$$

In this basis and without loss of generality, the most general 3 by 3 unitary matrix S satisfying $S^2 = 1$ can be parametrized as:

$$S = \eta \begin{pmatrix} -\cos \chi & \sin \chi \sin \varphi & \sin \chi \cos \varphi \\ \sin \chi \sin \varphi & \cos \chi \sin^2 \varphi - \cos^2 \varphi & (1 + \cos \chi) \cos \varphi \sin \varphi \\ \sin \chi \cos \varphi & (1 + \cos \chi) \cos \varphi \sin \varphi & \cos \chi \cos^2 \varphi - \sin^2 \varphi \end{pmatrix}, \quad (\text{A.2})$$

where $\eta = \pm 1$. We can take χ between 0 and π , and φ between 0 and $\pi/2$.⁵ We first discuss $\eta = 1$. The condition $(ST)^3 = 1$, written in the form $TST = ST^{-1}S$, gives rise to the equations:

$$\begin{aligned} -\frac{\cos^2 \chi}{t_1} - \cos \chi t_1^2 - \frac{\sin^2 \chi}{t_2 t_3} (\cos^2 \varphi t_2 + \sin^2 \varphi t_3) &= 0, \\ -\frac{\sin^2 \chi \sin^2 \varphi}{t_1} - \frac{(\cos^2 \varphi - \cos \chi \sin^2 \varphi)^2}{t_2} + (\cos \chi \sin^2 \varphi - \cos^2 \varphi) t_2^2 \\ - \frac{(1 + \cos \chi)^2 \cos^2 \varphi \sin^2 \varphi}{t_3} &= 0, \end{aligned}$$

⁵ It is not restrictive to assume the non-diagonal elements of S real and of the same sign. If this is not the case, we can make them real and of the same sign by means of a unitary diagonal transformation that leaves T invariant and does not affect the diagonal elements of S .

$$\begin{aligned}
 & -\frac{\sin^2 \chi \cos^2 \varphi}{t_1} - \frac{(\sin^2 \varphi - \cos \chi \cos^2 \varphi)^2}{t_3} + (\cos \chi \cos^2 \varphi - \sin^2 \varphi)t_3^2 \\
 & - \frac{(1 + \cos \chi)^2 \cos^2 \varphi \sin^2 \varphi}{t_2} = 0, \\
 & t_1 t_2 + \cos^2 \varphi \left(\frac{1}{t_2} - \frac{1}{t_3} \right) + \cos \chi \left(\frac{1}{t_1} - \frac{\sin^2 \varphi}{t_2} - \frac{\cos^2 \varphi}{t_3} \right) = 0, \\
 & t_1 t_3 + \sin^2 \varphi \left(-\frac{1}{t_2} + \frac{1}{t_3} \right) + \cos \chi \left(\frac{1}{t_1} - \frac{\sin^2 \varphi}{t_2} - \frac{\cos^2 \varphi}{t_3} \right) = 0, \\
 & -\frac{\sin^2 \chi}{t_1} + \frac{(1 + \cos \chi)}{t_2 t_3} (\sin^2 \varphi t_2 + \cos^2 \varphi t_3 + t_2^2 t_3^2 - \cos \chi (\cos^2 \varphi t_2 + \sin^2 \varphi t_3)) = 0.
 \end{aligned}
 \tag{A.3}$$

By combining the fourth and the fifth equations above we obtain:

$$t_1 t_2 t_3 = 1, \tag{A.4}$$

which can be solved, for instance, to express t_1 in terms of t_2 and t_3 . By ignoring solutions that lead to reducible representations, the remaining independent conditions in (A.3) are solved by:

$$\begin{aligned}
 \cos \chi &= -\frac{t_2 t_3 (t_2 + t_3)}{1 - t_2^2 t_3 - t_2 t_3^2 + t_2^3 t_3^3}, \\
 \cos \varphi &= \sqrt{\frac{t_2 (1 + t_3^3) (1 - t_2^2 t_3)}{(t_2 - t_3) (1 + t_2^3 t_3^3 - 2 t_2 t_3 (t_2 + t_3))}}.
 \end{aligned}
 \tag{A.5}$$

It is easy to see that $\cos \chi$ is automatically real and that the condition $|\cos \chi| \leq 1$ reads:

$$t_2^3 t_3^3 + \frac{1}{t_2^2 t_3^3} - 2 t_2 t_3 (t_2 + t_3) - \frac{2}{t_2 t_3} \left(\frac{1}{t_2} + \frac{1}{t_3} \right) + 2 \geq 0. \tag{A.6}$$

The argument of the squared root in Eq. (A.5) is always real and the conditions to obtain $\cos \varphi$ real and bounded between 0 and 1 are, respectively:

$$\begin{aligned}
 & -\frac{2(1 + t_3^3)}{t_2^3 t_3^4} (t_2 - t_3 + 2 t_2 t_3^3 + t_2^2 t_3^2 - 3 t_2^3 t_3 - 3 t_2^3 t_3^4 + t_2^4 t_3^3 + 2 t_2^5 t_3^2 + t_2^5 t_3^5 - t_2^6 t_3^4) \geq 0, \\
 & -\frac{2(1 + t_2^3)}{t_2^4 t_3^3} (t_3 - t_2 + 2 t_3 t_2^3 + t_3^2 t_2^2 - 3 t_3^3 t_2 - 3 t_3^3 t_2^4 + t_3^4 t_2^3 + 2 t_3^5 t_2^2 + t_3^5 t_2^5 - t_3^6 t_2^4) \geq 0.
 \end{aligned}
 \tag{A.7}$$

Recalling that $t_{2,3} = e^{i2\pi\beta_{2,3}}$, we find that in the fundamental region $0 \leq \beta_{2,3} \leq 1$ the values of $\beta_{2,3}$ that are compatible with the conditions (A.6), (A.7) are those displayed in Fig. 1. Then t_1 is given by the constraint (A.4). The representation chosen in our model, corresponds to one of the centers of the allowed regions, the point $(\beta_2, \beta_3) = (2/3, 1/3)$. Then Eq. (A.5) gives $\cos \chi = 1/3$, $\cos \varphi = 1/\sqrt{2}$ and $\sin \chi = 2\sqrt{2}/3$, $\sin \varphi = 1/\sqrt{2}$.

Finally, the irreducible representations corresponding to $\eta = -1$, which flips the sign of S , are obtained from those given above by sending (S, T) into $(-S, -T)$.

Appendix B

Here we discuss how the vacuum alignment achieved at the leading order is modified by the inclusion of higher dimensionality operators. The part of the superpotential depending on the driving fields φ_0^T , φ_0^S and ξ_0 is modified into

$$w_d + \Delta w_d. \tag{B.1}$$

Here w_d is the leading order contribution:

$$w_d = M(\varphi_0^T \varphi_T) + g(\varphi_0^T \varphi_T \varphi_T) + g_1(\varphi_0^S \varphi_S \varphi_S) + g_2 \tilde{\xi}(\varphi_0^S \varphi_S) + 3\tilde{g}_3^2 \xi_0(\varphi_S \varphi_S) - \tilde{g}_4^2 \xi_0 \xi^2 + g_5 \xi_0 \xi \tilde{\xi} + g_6 \xi_0 \tilde{\xi}^2, \tag{B.2}$$

where, for convenience, we have redefined $g_3 \equiv 3\tilde{g}_3^2$ and $g_4 \equiv -\tilde{g}_4^2$. We recall that w_d gives rise to the minimum:

$$\begin{aligned} \varphi_T &= (v_T, 0, 0), & v_T &= -\frac{3M}{2g}, \\ \varphi_S &= (v_S, v_S, v_S), & v_S &= \frac{\tilde{g}_4}{3\tilde{g}_3} u, \\ \xi &= u, \\ \tilde{\xi} &= 0 \end{aligned} \tag{B.3}$$

with u undetermined. The remaining term, Δw_d is the most general quartic, A_4 -invariant polynomial linear in the driving fields:

$$\Delta w_d = \frac{1}{\Lambda} \left(\sum_{k=3}^{13} t_k I_k^T + \sum_{k=1}^{12} s_k I_k^S + \sum_{k=1}^3 x_k I_k^X \right) \tag{B.4}$$

where t_k , s_k and x_k are coefficients and $\{I_k^T, I_k^S, I_k^X\}$ represent a basis of independent quartic invariants:

$$\begin{aligned} I_3^T &= (\varphi_0^T \varphi_T)(\varphi_T \varphi_T), & I_9^T &= (\varphi_0^T (\varphi_S \varphi_S)_S) \xi, \\ I_4^T &= (\varphi_0^T \varphi_T)'(\varphi_T \varphi_T)'', & I_{10}^T &= (\varphi_0^T (\varphi_S \varphi_S)_S) \tilde{\xi}, \\ I_5^T &= (\varphi_0^T \varphi_T)''(\varphi_T \varphi_T)', & I_{11}^T &= (\varphi_0^T \varphi_S) \xi^2, \\ I_6^T &= (\varphi_0^T \varphi_S)(\varphi_S \varphi_S), & I_{12}^T &= (\varphi_0^T \varphi_S) \xi \tilde{\xi}, \\ I_7^T &= (\varphi_0^T \varphi_S)'(\varphi_S \varphi_S)'', & I_{13}^T &= (\varphi_0^T \varphi_S) \tilde{\xi}^2, \\ I_8^T &= (\varphi_0^T \varphi_S)''(\varphi_S \varphi_S)', \end{aligned} \tag{B.5}$$

$$\begin{aligned} I_1^S &= ((\varphi_0^S \varphi_T)_S)(\varphi_S \varphi_S)_S, & I_7^S &= (\varphi_0^S (\varphi_T \varphi_S)_S) \tilde{\xi}, \\ I_2^S &= ((\varphi_0^S \varphi_T)_A)(\varphi_S \varphi_S)_S, & I_8^S &= (\varphi_0^S (\varphi_T \varphi_S)_A) \xi, \\ I_3^S &= (\varphi_0^S \varphi_T)(\varphi_S \varphi_S), & I_9^S &= (\varphi_0^S (\varphi_T \varphi_S)_A) \tilde{\xi}, \\ I_4^S &= (\varphi_0^S \varphi_T)'(\varphi_S \varphi_S)'', & I_{10}^S &= (\varphi_0^S \varphi_T) \xi^2, \\ I_5^S &= (\varphi_0^S \varphi_T)''(\varphi_S \varphi_S)', & I_{11}^S &= (\varphi_0^S \varphi_T) \xi \tilde{\xi}, \\ I_6^S &= (\varphi_0^S (\varphi_T \varphi_S)_S) \xi, & I_{12}^S &= (\varphi_0^S \varphi_T) \tilde{\xi}^2, \end{aligned} \tag{B.6}$$

$$\begin{aligned}
 I_1^X &= \xi_0(\varphi_T(\varphi_S\varphi_S)S), & I_3^X &= \xi_0(\varphi_S\varphi_S)\tilde{\xi}, \\
 I_2^X &= \xi_0(\varphi_S\varphi_S)\tilde{\xi}.
 \end{aligned}
 \tag{B.7}$$

The new minimum for φ_S , φ_T , ξ and $\tilde{\xi}$ is obtained by searching for the zeros of the F terms, the first derivatives of $w_d + \Delta w_d$, associated to the driving fields φ_0^S , φ_0^T and ξ_0 . We look for a solution that perturbs (B.3) to first order in the $1/\Lambda$ expansion:

$$\begin{aligned}
 \varphi_T &= (v_T + \delta v_1^T, \delta v_2^T, \delta v_3^T), & v_T &= -\frac{3M}{2g}, \\
 \varphi_S &= (v_S + \delta v_1, v_S + \delta v_2, v_S + \delta v_3), & v_S &= \frac{\tilde{g}_4}{3\tilde{g}_3}u, \\
 \xi &= u, \\
 \tilde{\xi} &= \delta u'.
 \end{aligned}
 \tag{B.8}$$

The minimum conditions become equations in the unknown δv_i^T , δv_i , u and $\delta u'$ which can be expanded in $1/\Lambda$. By keeping only the first order in the expansion, we get:

$$\begin{aligned}
 -\delta v_1^T - \frac{27t_3}{8g^3} \frac{M^2}{\Lambda} + \frac{\tilde{g}_4}{3\tilde{g}_3} \left(t_{11} + \frac{\tilde{g}_4^2}{3\tilde{g}_3^2}(t_6 + t_7 + t_8) \right) \frac{u^3}{M\Lambda} &= 0, \\
 2\delta v_3^T + \frac{\tilde{g}_4}{3\tilde{g}_3} \left(t_{11} + \frac{\tilde{g}_4^2}{3\tilde{g}_3^2}(t_6 + t_7 + t_8) \right) \frac{u^3}{M\Lambda} &= 0, \\
 2\delta v_2^T + \frac{\tilde{g}_4}{3\tilde{g}_3} \left(t_{11} + \frac{\tilde{g}_4^2}{3\tilde{g}_3^2}(t_6 + t_7 + t_8) \right) \frac{u^3}{M\Lambda} &= 0, \\
 \frac{g_2\tilde{g}_4}{3\tilde{g}_3}\delta u' + \frac{2g_1\tilde{g}_4}{9\tilde{g}_3}(2\delta v_1 - \delta v_2 - \delta v_3) - \frac{1}{g} \left(\frac{3}{2}s_{10} + \frac{\tilde{g}_4^2}{2\tilde{g}_3^2}s_3 + \frac{\tilde{g}_4}{3\tilde{g}_3}s_6 \right) \frac{Mu}{\Lambda} &= 0, \\
 \frac{g_2\tilde{g}_4}{3\tilde{g}_3}\delta u' + \frac{2g_1\tilde{g}_4}{9\tilde{g}_3}(2\delta v_2 - \delta v_1 - \delta v_3) - \frac{1}{g} \left(\frac{\tilde{g}_4^2}{2\tilde{g}_3^2}s_4 - \frac{\tilde{g}_4}{\tilde{g}_3} \left(\frac{s_6}{6} + \frac{s_8}{4} \right) \right) \frac{Mu}{\Lambda} &= 0, \\
 \frac{g_2\tilde{g}_4}{3\tilde{g}_3}\delta u' + \frac{2g_1\tilde{g}_4}{9\tilde{g}_3}(2\delta v_3 - \delta v_1 - \delta v_2) - \frac{1}{g} \left(\frac{\tilde{g}_4^2}{2\tilde{g}_3^2}s_5 - \frac{\tilde{g}_4}{\tilde{g}_3} \left(\frac{s_6}{6} - \frac{s_8}{4} \right) \right) \frac{Mu}{\Lambda} &= 0, \\
 g_5\delta u' + 2\tilde{g}_3\tilde{g}_4(\delta v_1 + \delta v_2 + \delta v_3) - \frac{\tilde{g}_4}{2g\tilde{g}_3}x_2 \frac{Mu}{\Lambda} &= 0.
 \end{aligned}
 \tag{B.9}$$

These equations are solved by:

$$\begin{aligned}
 \delta v_1^T &= -\frac{3t_3}{2g} \frac{v_T^2}{\Lambda} + \left[-\frac{\tilde{g}_4}{2g\tilde{g}_3t_{11}} - \frac{\tilde{g}_4^3}{6g\tilde{g}_3^3}(t_6 + t_7 + t_8) \right] \frac{u^3}{v_T\Lambda}, \\
 \delta v_2^T &= \delta v_3^T = \left[\frac{\tilde{g}_4}{4g\tilde{g}_3t_{11}} + \frac{\tilde{g}_4^3}{12g\tilde{g}_3^3}(t_6 + t_7 + t_8) \right] \frac{u^3}{v_T\Lambda}, \\
 \delta v_1 &= \left[\frac{g_5s_{10}}{6g_2\tilde{g}_4^2} + \frac{g_5}{18g_2\tilde{g}_3^2}(s_3 + s_4 + s_5) - \frac{g_3\tilde{s}_{10}}{g_1\tilde{g}_4} \right. \\
 &\quad \left. - \frac{\tilde{g}_4}{6g_1\tilde{g}_3}(2s_3 - s_4 - s_5) - \frac{s_6}{3g_1} - \frac{x_2}{18\tilde{g}_3^2} \right] \frac{v_T u}{\Lambda},
 \end{aligned}$$

$$\begin{aligned}
\delta v_2 &= \left[\frac{g_5 s_{10}}{6g_2 \tilde{g}_4^2} + \frac{g_5}{18g_2 \tilde{g}_3^2} (s_3 + s_4 + s_5) + \frac{g_3 \tilde{s}_{10}}{2g_1 \tilde{g}_4} \right. \\
&\quad \left. - \frac{\tilde{g}_4}{6g_1 \tilde{g}_3} (2s_4 - s_3 - s_5) + \frac{s_6}{6g_1} + \frac{s_8}{4g_1} - \frac{x_2}{18\tilde{g}_3^2} \right] \frac{v_T u}{\Lambda}, \\
\delta v_3 &= \left[\frac{g_5 s_{10}}{6g_2 \tilde{g}_4^2} + \frac{g_5}{18g_2 \tilde{g}_3^2} (s_3 + s_4 + s_5) + \frac{g_3 \tilde{s}_{10}}{2g_1 \tilde{g}_4} \right. \\
&\quad \left. - \frac{\tilde{g}_4}{6g_1 \tilde{g}_3} (2s_5 - s_3 - s_4) + \frac{s_6}{6g_1} - \frac{s_8}{4g_1} - \frac{x_2}{18\tilde{g}_3^2} \right] \frac{v_T u}{\Lambda}, \\
\delta u' &= - \left[\frac{g_3 \tilde{s}_{10}}{g_2 \tilde{g}_4} + \frac{\tilde{g}_4}{3g_2 \tilde{g}_3} (s_3 + s_4 + s_5) \right] \frac{v_T u}{\Lambda},
\end{aligned} \tag{B.10}$$

where u remains undetermined. This justifies the corrections (33) to the vacuum alignment adopted in Section 5.

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