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ALGEBRAIC RICCATI EQUATION AND J-SPECTRAL FACTORIZATION FOR $\mathcal{H}_\infty$ SMOOTHING AND DECONVOLUTION*

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Abstract. This paper deals with a general steady-state estimation problem in the $\mathcal{H}_\infty$ setting. The existence of the stabilizing solution of the related algebraic Riccati equation (ARE) and of the solution of the associated J-spectral factorization problem is investigated. The existence of such solutions is well established if the prescribed attenuation level γ is larger than γ_f (the infimum of the values of γ for which a *causal* estimator with attenuation level γ exists). We consider the case when $\gamma \le \gamma_f$ and show that the stabilizing solution of the ARE still exists (except for a finite number of values of γ) as long as a fixed-lag acausal estimator (smoother) does. The stabilizing solution of the ARE may be employed to derive a state-space realization of a minimum-phase J-spectral factor of the J-spectrum associated with the estimation problem. This J-spectral factor may be used, in turn, to compute the minimum-lag smoothing estimator. Some of the aspects of the J-spectral factorization problem and the properties of its solutions are discussed in correspondence to the (finite number of) values of γ for which the stabilizing solution of the ARE does not exist.

Key words. estimation problems, J-spectral factorization, algebraic Riccati equation, filtering, deconvolution, smoothing

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1. Introduction and problem statement.

Consider the discrete-time linear system

$$x_{k+1} = Ax_k + Bw_k, \tag{1.1a}$$

$$y_k = Cx_k + Dw_k, \tag{1.1b}$$

$$z_k = Lx_k + Mw_k, \tag{1.1c}$$

where $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^p$ are the state and the measurement output vector, respectively, and $w_k \in \mathbb{R}^m$ is the vector of inputs and disturbances. The to-be-estimated signal $z_k \in \mathbb{R}^l$ is an unaccessible linear combination of the state and the input. This is very general and includes, as particular cases, the *state-filtering* problem ($L = I$ and $M = 0$) and the *deconvolution* problem ($L = 0$ and $M = I$). See section 1.3 for a brief discussion and references on these problems.

Let $F(z)$ be the transfer function of a causal filter driven by the observations y_k whose output $\hat{z}_k$ is an estimate of z_k, and let $e_k := z_k - \hat{z}_k$. The infinite-horizon $\mathcal{H}_\infty$ *filtering problem* consists in designing an estimator $F(z)$ guaranteeing a prescribed level of attenuation γ between the ℓ_2-norm of w_k and the ℓ_2-norm of the estimation error e_k. Introducing the transfer functions

$$H_1(z) := C(zI - A)^{-1}B + D, \tag{1.2a}$$

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$$H_2(z) := L(zI - A)^{-1}B + M, \tag{1.2b}$$

this problem is equivalent to that of finding a stable causal transfer function $F(z)$ such that

$$F(z)H_1(z) - H_2(z) \in \mathcal{RH}_\infty, \qquad \|F(z)H_1(z) - H_2(z)\|_\infty < \gamma, \tag{1.3}$$

where $\mathcal{RH}_\infty$ denotes the space of real rational matrix functions whose poles lie all in $\{z \in \mathbb{C} : \ |z| < 1\}$. $\mathcal{RH}_\infty$ is endowed with the infinity norm associating with any $G(z) \in \mathcal{RH}_\infty$, $\|G(z)\|_\infty = \sup_{|z|=1} \|G(z)\|$ with $\|G(z)\|$ being the largest singular value of $G(z)$ for each given z.

The $\mathcal{H}_\infty$ *fixed-lag smoothing problem with preview horizon of length* N (or N-*lag smoothing problem*) may be defined as the problem of estimating (in the $\mathcal{H}_\infty$ framework) the signal z_k given the observations up to time $k + N$, $N \geq 0$. In other words, the only difference with the filtering problem is that the transfer function of the estimator is not required to be causal but has to be of the form $E(z) = z^N F(z)$ with $F(z)$ being stable and causal. This may be easily reformulated as the problem of finding a stable causal transfer function $F(z)$ such that

$$F(z)H_1(z) - H_{2,N}(z) \in \mathcal{RH}_\infty, \qquad \|F(z)H_1(z) - H_{2,N}(z)\|_\infty < \gamma \tag{1.4}$$

with

$$H_{2,N}(z) := z^{-N}H_2(z). \tag{1.5}$$

For a general overview on $\mathcal{H}_\infty$ estimation see to [19, 21] and references therein. For the continuous time version of the $\mathcal{H}_\infty$ smoothing problem see [18].

Assumptions. Let[1]

$$B_1 := B(I - D^\top(DD^\top)^{-1}D), \qquad F := A - BD^\top(DD^\top)^{-1}C. \tag{1.6}$$

We make the following standard assumptions:

$$DD^\top > 0, \tag{1.7a}$$

$$(A, C) \text{ is detectable}, \tag{1.7b}$$

$$(F, B_1) \text{ is reachable}, \tag{1.7c}$$

$$F \text{ is invertible}. \tag{1.7d}$$

Moreover, if $MD^\top \neq 0$, we define the auxiliary signal $\xi_k := z_k - MD^\top(DD^\top)^{-1}y_k = \widetilde{L}x_k + \widetilde{M}w_k$ with $\widetilde{L} := L - MD^\top(DD^\top)^{-1}C$ and $\widetilde{M} := M - MD^\top(DD^\top)^{-1}D$ and observe that, since ξ is obtained from z by subtracting a linear combination of the observed signal y, the estimations of z and ξ are equivalent problems (i.e., the estimation error is the same). In addition, a simple computation shows that $\widetilde{M}D^\top = 0$. Hence, without loss of generality, we assume

$$MD^\top = 0 \tag{1.7e}$$

so that the following identity holds:

$$BM^\top = B_1M^\top. \tag{1.8}$$

Eventually, notice that neither A nor F is assumed to be stable.

[1] We denote by $M^\top$ the transpose of a matrix M.

From the point of view of the motivating $\mathcal{H}_\infty$ estimation problem, we mention that assumptions (1.7a) and (1.7d) do not impair loss of generality. In fact if (1.7a) is not satisfied we may resort to a Silverman transfomation (see [6]) that, employing a spectral interactor matrix, yields an equivalent problem in which $DD^\top > 0$. If (1.7d) is not satisfied, we may perform a preliminary feedback transformation, as described in [5], and obtain an equivalent problem, in which (1.7d) is satisfied.

1.1. *J*-spectral factorization. For a real transfer matrix function $G(z)$, we define $G^\sim(z) := G^\top(z^{-1})$. Let

$$H(z) := \begin{bmatrix} H_1(z) & 0 \\ H_2(z) & I_l \end{bmatrix} = \begin{bmatrix} C \\ L \end{bmatrix} (zI - A)^{-1} [B \mid 0\,] + \begin{bmatrix} D & 0 \\ M & I_l \end{bmatrix} \tag{1.9}$$

and $J_{i,j}(\gamma) := \operatorname{diag}\{I_i, -\gamma^2 I_j\}$. (We shall simply write J and I, instead of $J_{i,j}(\gamma)$ and I_i, when there is no risk of confusion.) The transfer function

$$\Psi(z) := H(z) J_{m,l}(\gamma) H^\sim(z) = \begin{bmatrix} \Psi_{11}(z) & \Psi_{12}(z) \\ \Psi_{12}^\sim(z) & \Psi_{22}(z) \end{bmatrix} \tag{1.10}$$

with

$$\Psi_{11}(z) := H_1(z) H_1^\sim(z), \quad \Psi_{12}(z) := H_1(z) H_2^\sim(z), \quad \Psi_{22}(z) := H_2(z) H_2^\sim(z) - \gamma^2 I \tag{1.11}$$

is called the *J-spectrum* of the system. A *square J-spectral factor* of $\Psi(z)$ is a square matrix function $\Omega(z)$ such that

$$\Omega(z) J_{p,l}(\gamma) \Omega^\sim(z) = \Psi(z). \tag{1.12}$$

A square J-spectral factor is said to be *minimum phase* if all its zeros lay in the open unit disk.[2] See [10] and references therein for further details on J-spectral factorization.

The estimation problem is strictly related to the existence of a J-spectral factor. Precisely, the existence of an estimator with prescribed attenuation level γ is equivalent to the existence of a minimum phase J-spectral factor of $\Psi(z)$ having a realization with the same state matrix A and the same output matrix $[C^\top \mid L^\top]^\top$ of the realization (1.9) of $H(z)$; see [3]. Once obtained such a spectral factor, a simple constructive procedure (described in [3]) furnishes a class of estimators $F(z)$ satisfying (1.3).

1.2. Algebraic Riccati equation. Let

$$H_0 := \begin{bmatrix} C \\ L \end{bmatrix}, \quad J_0 := \begin{bmatrix} D \\ M \end{bmatrix}, \quad J_1 := J_0 J_0^\top - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} = \begin{bmatrix} DD^\top & DM^\top \\ MD^\top & MM^\top - \gamma^2 I \end{bmatrix} \tag{1.13}$$

and consider the algebraic Riccati equation (ARE)

$$\Delta = A\Delta A^\top + BB^\top - (A\Delta H_0^\top + BJ_0^\top)(J_1 + H_0 \Delta H_0^\top)^{-1}(H_0 \Delta A^\top + J_0 B^\top). \tag{1.14}$$

[2] Notice that in some papers and books the definition of minimum phase requires the stability of the spectral factor. Here, since neither A nor F is assumed to be "stable," we are not interested in stable J-spectral factors and we adopt the definition of phase minimality given above.

It is well known (see, e.g., [21]) that the existence of a proper stable filter $F(z)$ satisfying (1.3) is equivalent to the existence of a *symmetric*, *positive definite*, *stabilizing*, *feasible* solution of (1.14), i.e., a solution $\Delta_f = \Delta_f^\top > 0$ such that

$$A_f := A - (A\Delta_f H_0^\top + BJ_0^\top)(J_1 + H_0\Delta_f H_0^\top)^{-1} H_0 \tag{1.15}$$

is stable and $J_1 + H_0\Delta_f H_0^\top$ has the same inertia of $J_{p,l}(\gamma)$. In passing, notice that as long as $(DD^\top + C\Delta_f C^\top) > 0$, the latter condition (feasibility) may be written in the form

$$\gamma^2 I - MM^\top - L\Delta_f L^\top + L\Delta_f C^\top (DD^\top + C\Delta_f C^\top)^{-1} C\Delta_f L^\top > 0. \tag{1.16}$$

If we now define

$$\gamma_f := \inf\{\gamma : \text{ there exists a proper stable filter } F(z) \text{ satisfying (1.3)}\}, \tag{1.17}$$

it is clear that (1.14) admits a feasible symmetric positive semidefinite stabilizing solution if and only if $\gamma > \gamma_f$.

If this is the case, there exists a square J-spectral factor of $\Psi(z)$ having all zeros in the open unit disk and satisfying the stability conditions described in [3, Theorem 3.1], ensuring the existence of a proper stable filter $F(z)$.

On the other hand, it is also well known that there exists a value γ_0 such that for $\gamma < \gamma_0$, the ARE (1.14) does not admit any symmetric solution satisfying the feasibility condition and hence $\Psi(z)$ does not admit any minimal square J-spectral factor, namely, a J-spectral factor having the last possible McMillan degree. The constant γ_0 has the following interpretation from the J-spectral factorization point of view. A necessary condition for the existence of a square spectral factor is that $\Psi(e^{j\omega})$ has p positive eigenvalues and l negative eigenvalues for all $\omega \in (-\pi, \pi]$. Since under the present assumptions $\Psi_{11}(e^{j\omega}) > 0$ for all $\omega \in (-\pi, \pi]$, this condition is equivalent to the negative definiteness of the Schur complement

$$S(z) := \Psi_{22}(z) - \Psi_{12}^{\sim}(z)[\Psi_{11}(z)]^{-1}\Psi_{12}(z) \tag{1.18}$$

on the unit circle, i.e., to

$$S(e^{j\omega}) < 0, \qquad \omega \in (-\pi, \pi]. \tag{1.19}$$

By employing (1.11), we may write $S(z)$ as $S(z) = W(z) - \gamma^2 I$ with

$$\begin{aligned} W(z) &:= H_2(z)H_2^{\sim}(z) - H_2(z)H_1^{\sim}(z)[H_1(z)H_1^{\sim}(z)]^{-1}H_1(z)H_2^{\sim}(z) \\ &= H_2(z)[I - H_1^{\sim}(z)[H_1(z)H_1^{\sim}(z)]^{-1}H_1(z)]H_2^{\sim}(z). \end{aligned} \tag{1.20}$$

It is now easy to see that (1.19) is satified for γ sufficiently large: γ_0 is the infimum of the values of γ for which (1.19) is satisfied, i.e., γ_0^2 is the $\mathcal{L}_\infty$ norm of $W(z)$:

$$\gamma_0^2 = \|W(z)\|_\infty. \tag{1.21}$$

For a general overview on the connections between ARE and spectral and J-spectral factorization see [11, 12, 17, 1, 15] and references therein.

Remark 1.1. From the estimation problem point of view, the constant γ_0 is the infimum of the values of γ for which there exists a fixed-lag smoother (for some preview horizon N) achieving the attenuation level γ. To see this, first recall that under the detectability assumption of the pair (A, C), without any loss of generality, one can assume from the very beginning the stability of $H_1(z)$ and $H_2(z)$ and transform the given problem in a unilateral model matching problem in a new stable free filter parameter [21]. Now, let $V_1(z)$ be a square spectral factor of $H_1(z)$. A simple computation shows that

$$\|F(z)H_1(z) - H_2(z)\|_\infty^2 = \|F(z)V_1(z) - H_2(z)H_1^\sim(z)V_1^\sim(z)^{-1}\|_\infty^2 + \|W(z)\|_\infty. \tag{1.22}$$

Hence, the attenuation level cannot be lower than γ_0 and this infimum value is achieved by the acausal estimator

$$F_\infty(z) = H_2(z)H_1^\sim(z)\left(H_1(z)H_1^\sim(z)\right)^{-1} = \sum_{i=-\infty}^{\infty} F_i z^i. \tag{1.23}$$

Now the question arises of how to construct a stable fixed-lag smoother when an attenuation level $\gamma > \gamma_0$ is prescribed. This problem can be solved directly from expression (1.22) by taking a finite length expansion of the anticausal part of $F_\infty(z)$ given by (1.23). Indeed, let

$$F_N(z) = F_\infty(z) - \sum_{i=N+1}^{\infty} F_i z^i.$$

If $F_N(z)$ is the (N steps acausal, yet stable) estimator so constructed, then

$$\begin{aligned}\|F_N(z)H_1(z) - H_2(z)\|_\infty^2 &\le \gamma_0^2 + \left\|\sum_{i=N+1}^{\infty} F_i z^i V_1(z)\right\|_\infty^2 \\ &\le \gamma_0^2 + \left(\sum_{i=N+1}^{\infty} \|F_i\|\right)^2 \|H_1(z)\|_\infty^2\end{aligned}$$

can be rendered arbitrarily close to γ_0^2 by selecting a sufficiently large N. The estimator can be realized from the relevant factor following the results in [3] or directly from the solution of a suitable Riccati equation as described in [20].

1.3. Contribution of the paper. In this paper, we investigate the existence of a feasible stabilizing solution of (1.14) and the existence and the construction of a minimum phase square J-spectral factor $\Omega(z)$ for values of γ in the interval $(\gamma_0, \gamma_f]$. Clearly, for such values of γ the filter cannot exist, but there exists a fixed-lag smoother associated with a certain preview horizon length N achieving the desired attenuation level. As shown in [5], the computation of the stabilizing solution Δ of (1.14) and of the corresponding J-spectral factor $\Omega(z)$ are crucial steps to obtain efficiently the minimum-lag smoothing filter achieving the desired attenuation level. Notably, the solution Δ can be directly used to initialize an iterative algorithm to work out a minimum-lag central smoother; see [13, 2].

The contribution of the present paper is to prove that for $\gamma \in (\gamma_0, \gamma_f]$, equation (1.14) still admits a stabilizing feasible solution except for a finite number (at most $2[n + \text{rank}(M)]$) of values of γ. This result generalizes a result in [4] where it was assumed that $DD^\top = I$, $M = 0$, and $DB^\top = 0$. While removing the assumption $DD^\top = I$ impairs only slightly longer formulas, the presence of the matrix M gives rise to a much more difficult problem. Also, the presence of the matrix M implies that having assumed (1.7e) we cannot assume $DB^\top = 0$ (only one of the two conditions (1.7e) and $DB^\top = 0$ can be assumed without loss of generality) and this adds some other technical difficulties to the problem. On the other hand the estimation problem becomes much more interesting and of practical importance if we consider the presence of the matrix M. Indeed, in this case, not only can we attack deconvolution problems for the importance of which we refer to [9] and references therein, but we can, more generally, address the problem of estimating an arbitrary linear combination of input and state vectors; see the discussion in [7] for the practical importance of the latter problem.

1.4. Paper organization. In section 2 we analyze some auxiliary spectral factorization problems and the corresponding AREs. In section 3 we show that for generic values of γ, the antistabilizing solution of an ARE considered in section 2 is nonsingular. Such preliminary results are employed in section 4, where we prove our main result on the existence of a stabilizing feasible solution of (1.14). In section 5 we explicitly derive a minimum phase J-spectral factor of $\Psi(z)$. In section 6 we analyze some of the issues arising in correspondence of the values of γ for which we cannot guarantee the existence of the stabilizing solution of (1.14) and we discuss some peculiar aspects of the J-spectral factorization.

2. Preliminary spectral factorizations and auxiliary AREs. In this section we compute a state-space realization of a spectral factor $T(z)$ of $W(z)$ defined in (1.20) and a spectral factorization of $-S(z) = \gamma^2 I - W(z)$. These factorizations are related to the stabilizing solutions of a pair of coupled AREs. These solutions will be used to compute the solution (and hence to prove constructively its existence) of a third ARE that is strictly related to (1.14). This procedure will be carried over through various steps.

2.1. Computation of a square spectral factor of $H_1(z)H_1^\sim(z)$. Consider the following standard filtering ARE:

$$P = APA^\top + BB^\top - (APC^\top + BD^\top)(DD^\top + CPC^\top)^{-1}(CPA^\top + DB^\top). \tag{2.1}$$

By assumptions (1.7b) and (1.7c), it admits a positive definite stabilizing solution $P_s = P_s^\top > 0$. Such a solution corresponds to spectral factorization of $H_1(z)H_1^\sim(z)$. In fact, by defining

$$D_1 := (DD^\top + CP_sC^\top)^{1/2}, \qquad G_1 := (AP_sC^\top + BD^\top)D_1^{-1}, \tag{2.2}$$

it is easy to check (see [12]) that

$$H_1(z)H_1^\sim(z) = V_1(z)V_1^\sim(z) \tag{2.3}$$

with

$$V_1(z) := C(zI - A)^{-1}G_1 + D_1. \tag{2.4}$$

2.2. Computation of $V_1(z)^{-1}H_1(z)$. First notice that

$$V_1(z)^{-1} = D_1^{-1} - D_1^{-1}C(zI - \Gamma)^{-1}G_1D_1^{-1} \tag{2.5}$$

with

$$\Gamma := A - G_1D_1^{-1}C \tag{2.6}$$

being the closed-loop matrix corresponding to the stabilizing solution P_s of the ARE (2.1) (so that all the eigenvalues of Γ lay in the open unit disk).

Then we have

$$\begin{aligned} V_2(z) &:= V_1(z)^{-1}H_1(z) \\ &= D_1^{-1}C(zI-A)^{-1}B - D_1^{-1}C(zI-\Gamma)^{-1}G_1D_1^{-1}D + D_1^{-1}D \\ &\quad -D_1^{-1}C(zI-\Gamma)^{-1}\underbrace{G_1D_1^{-1}C}_{(zI-\Gamma)-(zI-A)}(zI-A)^{-1}B \\ &= D_1^{-1}C(zI-\Gamma)^{-1}(B-G_1D_1^{-1}D) + D_1^{-1}D \\ &= C_2(zI-\Gamma)^{-1}G_2 + D_2 \end{aligned} \tag{2.7}$$

with

$$C_2 := D_1^{-1}C, \qquad G_2 := B - G_1D_1^{-1}D, \qquad D_2 := D_1^{-1}D. \tag{2.8}$$

2.3. Computation of $V_2(z)H_2^{\sim}(z)$. We have

$$\begin{aligned} V_2(z)H_2^{\sim}(z) &= [C_2(zI-\Gamma)^{-1}G_2 + D_2][B^\top(z^{-1}I-A^\top)^{-1}L^\top + M^\top] \\ &= C_2(zI-\Gamma)^{-1}G_2M^\top + D_2B^\top(z^{-1}I-A^\top)^{-1}L^\top + D_2M^\top \\ &\quad + C_2(zI-\Gamma)^{-1}G_2B^\top(z^{-1}I-A^\top)^{-1}L^\top. \end{aligned} \tag{2.9}$$

Taking the ARE (2.1) into account, it is easy to check that

$$G_2B^\top = P_s - \Gamma P_s A^\top, \tag{2.10}$$

which may be easily rewritten in the form

$$G_2B^\top = (zI-\Gamma)P_s(z^{-1}I-A^\top) + (zI-\Gamma)P_sA^\top + \Gamma P_s(z^{-1}I-A^\top). \tag{2.11}$$

Plugging this expression in (2.9) we get

$$V_2(z)H_2^{\sim}(z) = C_2(zI-\Gamma)^{-1}G_3 + G_1^\top(z^{-1}I-A^\top)^{-1}L^\top + D_3 \tag{2.12}$$

with $G_3 := \Gamma P_s L^\top + G_2M^\top$ and $D_3 := C_2P_sL^\top + D_2M^\top$ or, taking into account that $DM^\top = 0$,

$$G_3 = \Gamma P_s L^\top + BM^\top, \qquad D_3 = C_2P_sL^\top \tag{2.13}$$

2.4. Computation of $H_2(z)H_1^{\sim}(z)[H_1(z)H_1^{\sim}(z)]^{-1}H_1(z)H_2^{\sim}(z)$. It is clear that $H_2(z)H_1^{\sim}(z)[H_1(z)H_1^{\sim}(z)]^{-1}H_1(z)H_2^{\sim}(z) = H_2(z)V_2^{\sim}(z)V_2(z)H_2^{\sim}(z)$. We have

$$\begin{aligned} H_2(z)V_2^{\sim}(z)V_2(z)H_2^{\sim}(z) &= D_3^\top D_3 + G_3^\top(z^{-1}I-\Gamma^\top)^{-1}C_2^\top C_2(zI-\Gamma)^{-1}G_3 \\ &\quad + D_3^\top C_2(zI-\Gamma)^{-1}G_3 + D_3^\top G_1^\top(z^{-1}I-A^\top)^{-1}L^\top \\ &\quad + [D_3^\top C_2(zI-\Gamma)^{-1}G_3 + D_3^\top G_1^\top(z^{-1}I-A^\top)^{-1}L^\top]^{\sim} \\ &\quad + L(zI-A)^{-1}G_1C_2(zI-\Gamma)^{-1}G_3 \\ &\quad + [L(zI-A)^{-1}G_1C_2(zI-\Gamma)^{-1}G_3]^{\sim} \\ &\quad + L(zI-A)^{-1}G_1G_1^\top(z^{-1}I-A^\top)^{-1}L^\top. \end{aligned} \tag{2.14}$$

The last three terms of such expression may be expanded as follows: first taking into account that

$$G_1C_2 = G_1D_1^{-1}C = A - \Gamma = (zI - \Gamma) - (zI - A) \tag{2.15}$$

we have

$$L(zI - A)^{-1}G_1C_2(zI - \Gamma)^{-1}G_3 = L(zI - A)^{-1}G_3 - L(zI - \Gamma)^{-1}G_3. \tag{2.16}$$

Second, taking into account the ARE (2.1), we have

$$\begin{aligned} G_1G_1^\top = AP_sA^\top - P_s + BB^\top = BB^\top - (zI - A)P_s(z^{-1}I - A^\top) - (zI - A)P_sA^\top \\ -AP_s(z^{-1}I - A^\top) \end{aligned} \tag{2.17}$$

so that

$$\begin{aligned} L(zI-A)^{-1}G_1G_1^\top(z^{-1}I-A^\top)^{-1}L^\top \\ = L(zI - A)^{-1}BB^\top(z^{-1}I - A^\top)^{-1}L^\top - LP_sL^\top \\ - L(zI - A)^{-1}AP_sL^\top - LP_sA^\top(z^{-1}I - A^\top)^{-1}L^\top. \end{aligned} \tag{2.18}$$

2.5. Computation of $W(z)$. From the definition (1.20) of $W(z)$ it follows that

$$W(z) = H_2(z)H_2^\sim(z) - H_2(z)V_2^\sim(z)V_2(z)H_2^\sim(z). \tag{2.19}$$

Moreover, by computing $H_2(z)H_2^\sim(z)$, we get the following expression:

$$\begin{aligned} H_2(z)H_2^\sim(z) = L(zI - A)^{-1}BB^\top(z^{-1}I - A^\top)^{-1}L^\top + MM^\top \\ + L(zI - A)^{-1}BM^\top + [L(zI - A)^{-1}BM^\top]^\sim. \end{aligned} \tag{2.20}$$

By subtracting (2.14) from (2.20), taking (2.16), (2.18), and (2.19) into account, we get

$$\begin{aligned} W(z) = (L - D_3^\top C_2)(zI - \Gamma)^{-1}G_3 + [(L - D_3^\top C_2)(zI - \Gamma)^{-1}G_3]^\sim \\ - G_3^\top(z^{-1}I - \Gamma^\top)^{-1}C_2^\top C_2(zI - \Gamma)^{-1}G_3 + LP_sL^\top + MM^\top - D_3^\top D_3 \\ + L(zI - A)^{-1}(BM^\top - G_1D_3 + AP_sL^\top - G_3) \\ + [L(zI - A)^{-1}(BM^\top - G_1D_3 + AP_sL^\top - G_3)]^\sim. \end{aligned} \tag{2.21}$$

We now prove that $BM^\top - G_1D_3 + AP_sL^\top - G_3 = 0$, so that the last two terms of the latter expression vanish. We have

$$\begin{aligned} BM^\top-G_1D_3+AP_sL^\top-G_3 \\ = BM^\top - \underbrace{G_1C_2}_{A-\Gamma}P_sL^\top + AP_sL^\top - \Gamma P_sL^\top - BM^\top = 0. \end{aligned} \tag{2.22}$$

2.6. Spectral factorization of $W(z)$. To factorize $W(z)$ we first need to rewrite the term

$$G_3^\top(z^{-1}I - \Gamma^\top)^{-1}C_2^\top C_2(zI - \Gamma)^{-1}G_3.$$

To this aim we derive a new expression for $C_2^\top C_2$. Standard computations show that the solution P_s of the ARE (2.1) satisfies the following identity [15, p. 271]:

$$P_s = FP_sF^\top - FP_sC^\top(CP_sC^\top + DD^\top)^{-1}CP_sF^\top + B_1B_1^\top, \tag{2.23}$$

where F and B_1 are defined in (1.6). We also have

$$
\begin{aligned}
\Gamma &= A - G_1 D_1^{-1} C = F + [BD^\top (DD^\top)^{-1} D_1^2 - AP_s C^\top - BD^\top] D_1^{-1} C_2 \\
&= F - FP_s C^\top D_1^{-1} C_2 = F - FP_s C_2^\top C_2 \\
&= F(I - P_s C_2^\top C_2) \qquad (2.24)
\end{aligned}
$$

so that (2.23) may be rewritten in the form

$$P_s - B_1 B_1^\top = \Gamma P_s F^\top = FP_s \Gamma^\top. \qquad (2.25)$$

From $DD^\top > 0$ and $P_s > 0$ it easily follows that $(I - P_s C_2^\top C_2) = P_s(P_s^{-1} - C^\top (CP_s C^\top + DD^\top)^{-1} C)$ is nonsingular (in fact, $(P_s^{-1} - C^\top (CP_s C^\top + DD^\top)^{-1} C) > 0$). Thus Γ, and hence $FP_s\Gamma^\top$, are nonsingular so that the left-hand side of (2.25) is nonsingular as well, and we have

$$(P_s - B_1 B_1^\top)^{-1} = \Gamma^{-\top} P_s^{-1} F^{-1}, \qquad (2.26)$$

which may be rewritten as

$$P_s^{-1} = \Gamma^\top (P_s - B_1 B_1^\top)^{-1} F, \qquad (2.27)$$

and, by multiplying on the right side by $(I - P_s C_2^\top C_2)$,

$$P_s^{-1}(I - P_s C_2^\top C_2) = \Gamma^\top (P_s - B_1 B_1^\top)^{-1} \Gamma. \qquad (2.28)$$

Therefore

$$
\begin{aligned}
C_2^\top C_2 &= P_s^{-1} - \Gamma^\top (P_s - B_1 B_1^\top)^{-1} \Gamma \qquad (2.29\text{a}) \\
&= P_s^{-1} - \Gamma^\top P_s^{-1} \Gamma - \Gamma^\top P_s^{-1} B_1 V^{-1} B_1^\top P_s^{-1} \Gamma \qquad (2.29\text{b}) \\
&= (z^{-1} I - \Gamma^\top) P_s^{-1} (zI - \Gamma) + \Gamma^\top P_s^{-1} (zI - \Gamma) + (z^{-1} I - \Gamma^\top) P_s^{-1} \Gamma \\
&\quad - \Gamma^\top P_s^{-1} B_1 V^{-1} B_1^\top P_s^{-1} \Gamma, \qquad (2.29\text{c})
\end{aligned}
$$

where

$$V := I - B_1^\top P_s^{-1} B_1. \qquad (2.30)$$

Notice that

$$P_s^{-1} - C_2^\top C_2 = P_s^{-1} - C^\top (DD^\top + CP_s C^\top)^{-1} C = P_s^{-1} [P_s^{-1} + C^\top (DD^\top)^{-1} C]^{-1} P_s^{-1} > 0 \qquad (2.31)$$

so that from (2.29a) it follows that $P_s - B_1 B_1^\top > 0$, which, together with the positive definiteness of P_s, implies that V is positive definite as well.

Plugging in the expression in the right-hand side of (2.29c) in place of $C_2^\top C_2$ in the third term of (2.21), we get

$$
\begin{aligned}
W(z) = {} & (L - D_3^\top C_2 - G_3^\top P_s^{-1} \Gamma)(zI - \Gamma)^{-1} G_3 \\
& + [(L - D_3^\top C_2 - G_3^\top P_s^{-1} \Gamma)(zI - \Gamma)^{-1} G_3]^\sim \\
& + G_3^\top (z^{-1} I - \Gamma^\top)^{-1} \Gamma^\top P_s^{-1} B_1 V^{-1} B_1^\top P_s^{-1} \Gamma (zI - \Gamma)^{-1} G_3 \\
& + LP_s L^\top + MM^\top - D_3^\top D_3 - G_3^\top P_s^{-1} G_3. \qquad (2.32)
\end{aligned}
$$

Now we are ready to prove that the following spectral factorization of $W(z)$ holds:

$$W(z) = T(z) T^\sim(z), \qquad (2.33)$$

where

$$T(z) := [-G_3^\top (z^{-1}I - \Gamma^\top)^{-1}\Gamma^\top P_s^{-1}B_1 - G_3^\top P_s^{-1}B_1 + M]V^{-1/2}. \tag{2.34}$$

To this aim it is sufficient to show that

$$\begin{aligned}(M - G_3^\top P_s^{-1}B_1)V^{-1}(M - G_3^\top P_s^{-1}B_1)^\top \\ - (LP_sL^\top + MM^\top - D_3^\top D_3 - G_3^\top P_s^{-1}G_3) = 0\end{aligned} \tag{2.35}$$

and

$$\Gamma^\top P_s^{-1}B_1V^{-1}(M - G_3^\top P_s^{-1}B_1)^\top + (L - D_3^\top C_2 - G_3^\top P_s^{-1}\Gamma)^\top = 0. \tag{2.36}$$

As for (2.35) notice that

$$M - G_3^\top P_s^{-1}B_1 = M - LP_s\Gamma^\top P_s^{-1}B_1 - MB_1^\top P_s^{-1}B_1 = MV - LP_s\Gamma^\top P_s^{-1}B_1 \tag{2.37}$$

so that, taking (1.8) into account, we may expand the left-hand side of (2.35) as

$$\begin{aligned}&MVM^\top - MB_1^\top P_s^{-1}\Gamma P_sL^\top - (MB_1^\top P_s^{-1}\Gamma P_sL^\top)^\top \\ &+ LP_s\Gamma^\top P_s^{-1}B_1V^{-1}B_1^\top P_s^{-1}\Gamma P_sL^\top - LP_sL^\top - MM^\top \\ &+ LP_sC_2^\top C_2P_sL^\top + LP_s\Gamma^\top P_s^{-1}\Gamma P_sL^\top + MB_1^\top P_s^{-1}B_1M^\top \\ &+ MB_1^\top P_s^{-1}\Gamma P_sL^\top + (MB_1^\top P_s^{-1}\Gamma P_sL^\top)^\top \\ &= LP_s(\Gamma^\top P_s^{-1}B_1V^{-1}B_1^\top P_s^{-1}\Gamma - P_s^{-1} + C_2^\top C_2 + \Gamma^\top P_s^{-1}\Gamma)P_sL^\top = 0,\end{aligned} \tag{2.38}$$

where the latter equality follows from the expression (2.29b) for $C_2^\top C_2$.

Similarly, we may expand the left-hand side of (2.36) as

$$\begin{aligned}&L^\top - C_2^\top C_2P_sL^\top - \Gamma^\top P_s^{-1}\Gamma P_sL^\top - \Gamma^\top P_s^{-1}B_1V^{-1}B_1^\top P_s^{-1}\Gamma P_sL^\top \\ &= (P_s^{-1} - C_2^\top C_2 - \Gamma^\top P_s^{-1}\Gamma - \Gamma^\top P_s^{-1}B_1V^{-1}B_1^\top P_s^{-1}\Gamma)P_sL^\top = 0.\end{aligned} \tag{2.39}$$

We now rewrite $T(z)$ in a more convenient form. Notice that

$$(z^{-1}I - \Gamma^\top)^{-1} = -\Gamma^{-\top} - \Gamma^{-\top}(zI - \Gamma^{-\top})^{-1}\Gamma^{-\top}$$

so that

$$\begin{aligned}T(z) &= [G_3^\top\Gamma^{-\top}(zI - \Gamma^{-\top})^{-1}P_s^{-1}B_1 + M]V^{-1/2} \\ &= [G_3^\top\Gamma^{-\top}P_s^{-1}(zI - P_s\Gamma^{-\top}P_s^{-1})^{-1}B_1 + M]V^{-1/2} \\ &= L_1(zI - F_a)^{-1}B_1V^{-1/2} + MV^{-1/2}\end{aligned} \tag{2.40}$$

with

$$L_1 := G_3^\top\Gamma^{-\top}P_s^{-1} = L + MB_1^\top\Gamma^{-\top}P_s^{-1}, \qquad F_a := P_s\Gamma^{-\top}P_s^{-1}. \tag{2.41}$$

Notice that since Γ is stable, F_a is antistable. As a direct consequence of (2.26) we have the following relation that will be useful in what follows:

$$F_a = (I - B_1B_1^\top P_s^{-1})^{-1}F. \tag{2.42}$$

2.7. Spectral factorization of $-S(z)$. We have

$$-S(z) = \gamma^2 I - W(z) = \gamma^2 I - T(z)T^{\sim}(z).$$

As we have done for the factorization in section 2.1 consider the following ARE for the spectral factorization of $-S(z)$:

$$X = F_a X F_a^\top - B_1 V^{-1} B_1^\top - (F_a X L_1^\top - B_1 V^{-1} M^\top)(R + L_1 X L_1^\top)^{-1}(L_1 X F_a^\top - M V^{-1} B_1^\top) \tag{2.43}$$

with

$$R := \gamma^2 I - M V^{-1} M^\top. \tag{2.44}$$

In view of (1.21), we have $\|T(z)\|_\infty < \gamma$ for any $\gamma > \gamma_0$ so that as a consequence of a generalized version of the discrete-time bounded real lemma (see [8, Theorem 2.1]), (2.43) admits a stabilizing solution, namely, a solution $X_s = X_s^\top$ such that

$$\Gamma_a := F_a - (F_a X_s L_1^\top - B_1 V^{-1} M^\top)(R + L_1 X_s L_1^\top)^{-1} L_1 \tag{2.45}$$

is a stability matrix; moreover $(R + L_1 X_s L_1^\top) > 0$ and the function

$$T_1(z) := [I + L_1(zI - F_a)^{-1}(F_a X_s L_1^\top - B_1 V^{-1} M^\top)(R + L_1 X_s L_1^\top)^{-1}](R + L_1 X_s L_1^\top)^{1/2} \tag{2.46}$$

is a square spectral factor of $-S(z) = \gamma^2 I_l - W(z)$, namely, it is a square matrix function such that $-S(z) = T_1(z)T_1^{\sim}(z)$. Moreover, the numerator matrix of $T_1(z)$ is given by (2.45) and hence $[T_1(z)]^{-1}$ is stable.

We now show that X_s is positive definite. In fact, since $(R + L_1 X_s L_1^\top) > 0$, we have

$$X_s \le F_a X_s F_a^\top - B_1 V^{-1} B_1^\top. \tag{2.47}$$

But (F, B_1) is, by assumption, reachable, (F_a, B_1) is such (because F_a is obtained from F by state feedback as it is apparent from (2.42)), and then also $(F_a, B_1 V^{-1/2})$ is reachable. Therefore, since F_a is antistable, a standard Lyapunov argument, shows that $X_s > 0$.

2.8. The ARE for $P_s^{-1} - X_s^{-1}$. Define

$$R_1 := I - \frac{M^\top M}{\gamma^2} \tag{2.48}$$

and

$$F_y := A - B J_0' J_1^{-1} H_0, \tag{2.49}$$

where J_0, J_1, H_0 are defined in (1.13). Notice that F_y is well defined if and only if J_1 is nonsingular or, equivalently, if and only if R_1 is nonsingular. Let

$$\mathcal{G}_1 := \{\gamma > \gamma_0 : \text{ at least one of the matrices } R,\ R_1, \text{ and } F_y, \text{ is singular}\}. \tag{2.50}$$

Remark 2.1. Notice that since F_y may be written in the form

$$F_y = F - B_1 M^\top (M M^\top - \gamma^2 I)^{-1} L, \tag{2.51}$$

where F is nonsingular, $\mathcal{G}_1$ contains a finite number of points. More precisely, it contains, at most, $n + 2\,\text{rank}(M)$ points.

We consider now the following ARE:

$$
\begin{aligned}
Y = {} & F^\top Y F + \left(F^\top Y B_1 + \frac{L^\top M}{\gamma^2}\right)(R_1 - B_1^\top Y B_1)^{-1}\left(B_1^\top Y F + \frac{M^\top L}{\gamma^2}\right) \\
& - C^\top (DD^\top)^{-1} C + \frac{L^\top L}{\gamma^2}
\end{aligned} \tag{2.52}
$$

and prove that it admits a solution that can be explicitely computed. To this aim we need a technical lemma.

LEMMA 2.1. *Let $\gamma > \gamma_0$ and $\gamma \notin \mathcal{G}_1$ and define*

$$F_1 := F_a + B_1 V^{-1} M^\top R^{-1} L_1. \tag{2.53}$$

Then the following relation holds:

$$X_s^{-1} = F_1^\top X_s^{-1}(I - B_1 R_1^{-1} B_1^\top (P_s^{-1} - X_s^{-1}))^{-1} F_y - L_1^\top R^{-1} L_1. \tag{2.54}$$

Proof. First we establish the following useful identity:

$$F_y = (I - B_1 R_1^{-1} B_1^\top P_s^{-1}) F_1. \tag{2.55}$$

To show identity (2.55), we write F_1 in the form

$$
\begin{aligned}
F_1 &= (I + B_1 V^{-1} M^\top R^{-1} M B_1^\top P_s^{-1}) F_a + B_1 V^{-1} M^\top R^{-1} L \\
&= (I + B_1 V^{-1} M^\top R^{-1} M B_1^\top P_s^{-1})(I - B_1 B_1^\top P_s^{-1})^{-1} F + B_1 V^{-1} M^\top R^{-1} L,
\end{aligned} \tag{2.56}
$$

which gives

$$
\begin{aligned}
& (I - B_1 R_1^{-1} B_1^\top P_s^{-1}) F_1 \\
&= (I - B_1 R_1^{-1} B_1^\top P_s^{-1})(I + B_1 V^{-1} M^\top R^{-1} M B_1^\top P_s^{-1})(I - B_1 B_1^\top P_s^{-1})^{-1} F \\
&\quad + (I - B_1 R_1^{-1} B_1^\top P_s^{-1}) B_1 V^{-1} M^\top R^{-1} L,
\end{aligned} \tag{2.57}
$$

so that, taking (2.51) into account, it is sufficient to prove that

$$(I - B_1 R_1^{-1} B_1^\top P_s^{-1})(I + B_1 V^{-1} M^\top R^{-1} M B_1^\top P_s^{-1})(I - B_1 B_1^\top P_s^{-1})^{-1} = I \tag{2.58}$$

and

$$(I - B_1 R_1^{-1} B_1^\top P_s^{-1}) B_1 V^{-1} M^\top R^{-1} = -B_1 M^\top (M M^\top - \gamma^2 I)^{-1}. \tag{2.59}$$

As for (2.58), we have

$$
\begin{aligned}
& (I - B_1 R_1^{-1} B_1^\top P_s^{-1})(I + B_1 V^{-1} M^\top R^{-1} M B_1^\top P_s^{-1})(I - B_1 B_1^\top P_s^{-1})^{-1} \\
&= (I - B_1 R_1^{-1} B_1^\top P_s^{-1})(I - B_1 B_1^\top P_s^{-1} + B_1 (I + V^{-1} M^\top R^{-1} M) B_1^\top P_s^{-1})(I - B_1 B_1^\top P_s^{-1})^{-1} \\
&= (I - B_1 R_1^{-1} B_1^\top P_s^{-1})\left[I - B_1 B_1^\top P_s^{-1} + B_1 \left(V - \frac{M^\top M}{\gamma^2}\right)^{-1} V B_1^\top P_s^{-1}\right](I - B_1 B_1^\top P_s^{-1})^{-1} \\
&= (I - B_1 R_1^{-1} B_1^\top P_s^{-1})\left[I + B_1 \left(V - \frac{M^\top M}{\gamma^2}\right)^{-1} V B_1^\top P_s^{-1}(I - B_1 B_1^\top P_s^{-1})^{-1}\right] \\
&= (I - B_1 R_1^{-1} B_1^\top P_s^{-1})\left[I + B_1 \left(V - \frac{M^\top M}{\gamma^2}\right)^{-1} B_1^\top P_s^{-1}\right] \\
&= (I - B_1 R_1^{-1} B_1^\top P_s^{-1})(I - B_1 R_1^{-1} B_1^\top P_s^{-1})^{-1} = I.
\end{aligned} \tag{2.60}
$$

As for (2.59), we have

$$\begin{aligned}(I - B_1R_1^{-1}B_1^\top P_s^{-1})B_1V^{-1}M^\top R^{-1} &= B_1(I - R_1^{-1}B_1^\top P_s^{-1}B_1)V^{-1}M^\top R^{-1}\\ &= B_1R_1^{-1}(\underbrace{R_1 - B_1^\top P_s^{-1}B_1}_{V-\frac{M^\top M}{\gamma^2}})V^{-1}M^\top R^{-1}\\ &= B_1R_1^{-1}M^\top\left(I - \frac{MV^{-1}M^\top}{\gamma^2}\right)R^{-1}\\ &= -B_1M^\top(MM^\top - \gamma^2 I)^{-1}. \qquad (2.61)\end{aligned}$$

Since X_s is a solution of the ARE (2.43), as a direct consequence of standard equivalence of Riccati equations [15, p. 271], we have the following identity:

$$X_s = F_1X_sF_1^\top - F_1X_sL_1^\top(L_1X_sL_1^\top + R)^{-1}L_1X_sF_1^\top - B_1\left(V - \frac{M^\top M}{\gamma^2}\right)^{-1}B_1^\top \tag{2.62}$$

with F_1 being defined in (2.53). The closed-loop matrix may thus be written in the form

$$\Gamma_a = F_1 - F_1X_sL_1^\top(L_1X_sL_1^\top + R)^{-1}L_1 = F_1(I + X_sL_1^\top R^{-1}L_1)^{-1}. \tag{2.63}$$

Notice that $V - \frac{M^\top M}{\gamma^2}$ is nonsingular because V and R are such. In the same way, $X_s^{-1}+L_1^\top R^{-1}L_1$, and hence $(I+X_sL_1^\top R^{-1}L_1)$, are nonsingular because X_s, $L_1X_sL_1^\top+R$, and R are such. From (2.62) we easily get

$$X_s + B_1\left(V - \frac{M^\top M}{\gamma^2}\right)^{-1}B_1^\top = F_1(X_s - X_sL_1^\top(L_1X_sL_1^\top + R)^{-1}L_1X_s)F_1^\top. \tag{2.64}$$

As an immediate consequence of (2.55) we have that F_1 is nonsingular, so that the right-hand side of (2.64) (which may be written in the form $F_1(I+X_sL_1^\top R^{-1}L_1)^{-1}F_1^\top$) is nonsingular. Hence $X_s + B_1(V - \frac{M^\top M}{\gamma^2})^{-1}B_1^\top$ is nonsingular as well and then $\frac{M^\top M}{\gamma^2} - V - B_1^\top X_s^{-1}B_1 = -\left(R_1 - B_1^\top(P_s^{-1} - X_s^{-1})B_1\right)$ is such. (Actually, a more detailed analysis allows to conclude that $\left(R_1 - B_1^\top(P_s^{-1} - X_s^{-1})B_1\right) > 0$.) Then we have

$$\left[X_s + B_1\left(V - \frac{M^\top M}{\gamma^2}\right)^{-1}B_1^\top\right]^{-1} = [F_1(X_s - X_sL_1^\top(L_1X_sL_1^\top + R)^{-1}L_1X_s)F_1^\top]^{-1}, \tag{2.65}$$

which yields

$$X_s^{-1} + X_s^{-1}B_1\left(\frac{M^\top M}{\gamma^2} - V - B_1^\top X_s^{-1}B_1\right)^{-1}B_1^\top X_s^{-1} = F_1^{-\top}(X_s^{-1} + L_1^\top R^{-1}L_1)F_1^{-1} \tag{2.66}$$

and hence

$$X_s^{-1} = F_1^\top X_s^{-1}F_1 - F_1^\top X_s^{-1}B_1\left(R_1 - B_1^\top(P_s^{-1} - X_s^{-1})B_1\right)^{-1}B_1^\top X_s^{-1}F_1 - L_1^\top R^{-1}L_1 \tag{2.67a}$$

$$= F_1^\top X_s^{-1}(I - B_1R_1^{-1}B_1^\top(P_s^{-1} - X_s^{-1}))^{-1}(I - B_1R_1^{-1}B_1^\top P_s^{-1})F_1 - L_1^\top R^{-1}L_1. \tag{2.67b}$$

Finally, by plugging (2.55) in (2.67b) we obtain (2.54). ∎

PROPOSITION 2.1. *Let $\gamma > \gamma_0$ and $\gamma \notin \mathcal{G}_1$. Then the ARE (2.52) admits the (necessarily unique) antistabilizing solution. In fact, such solution is given by*[3]

$$Y_a := P_s^{-1} - X_s^{-1}. \tag{2.68}$$

Proof. We have to show that when $Y = Y_a$, (2.52) is an identity and that all the eigenvalues of

$$\Gamma_y := F - B_1(B_1^\top Y_a B_1 - R_1)^{-1}\left(B_1^\top Y_a F + \frac{M^\top L}{\gamma^2}\right) \tag{2.69}$$

lay in $\{z \in \mathbb{C} : \ |z| > 1\}$.

We first prove that Y_a is a solution of the ARE

$$Y = F_y^\top Y F_y + F_y^\top Y B_1(R_1 - B_1^\top Y B_1)^{-1} B_1^\top Y F_y - H_0^\top J_1^{-1} H_0. \tag{2.70}$$

From (2.23) we get

$$(P_s - B_1 B_1^\top)^{-1} = F^{-\top}(P_s - P_s C^\top(C P_s C^\top + DD^\top)^{-1} C P_s)^{-1} F^{-1}, \tag{2.71}$$

which, using the same procedure that lead to (2.67a), yields

$$P_s^{-1} = F^\top P_s^{-1} F - F^\top P_s^{-1} B_1(B_1^\top P_s^{-1} B_1 - I)^{-1} B_1^\top P_s^{-1} F - C^\top(DD^\top)^{-1} C, \tag{2.72}$$

which, taking (2.42) into account, may be also written in the form

$$P_s^{-1} = F_a^\top P_s^{-1} F - C^\top(DD^\top)^{-1} C \tag{2.73}$$

and, finally, in the form

$$P_s^{-1} = F_1^\top P_s^{-1} F_y + F_a^\top P_s^{-1}(F - F_y) - (F_1 - F_a)^\top P_s^{-1} F_y - C^\top(DD^\top)^{-1} C. \tag{2.74}$$

By subtracting (2.54) from (2.74) we now get

$$Y_a = F_1^\top[P_s^{-1} - X_s^{-1}(I - B_1 R_1^{-1} B_1^\top Y_a)^{-1}] F_y + R_2 \tag{2.75}$$

with

$$R_2 := F_a^\top P_s^{-1}(F - F_y) - (F_1 - F_a)^\top P_s^{-1} F_y - C^\top(DD^\top)^{-1} C + L_1^\top R^{-1} L_1. \tag{2.76}$$

We may rewrite (2.75) as follows:

$$\begin{aligned} Y_a &= F_1^\top[(Y_a - P_s^{-1} B_1 R_1^{-1} B_1^\top Y_a)(I - B_1 R_1^{-1} B_1^\top Y_a)^{-1}] F_y + R_2 \\ &= F_1^\top(I - P_s^{-1} B_1 R_1^{-1} B_1^\top) Y_a (I - B_1 R_1^{-1} B_1^\top Y_a)^{-1} F_y + R_2 \\ &= F_y^\top Y_a (I - B_1 R_1^{-1} B_1^\top Y_a)^{-1} F_y + R_2 \\ &= F_y^\top Y_a F_y + F_y^\top Y_a B_1(R_1 - B_1^\top Y_a B_1)^{-1} B_1^\top Y_a F_y + R_2, \end{aligned} \tag{2.77}$$

where we have taken (2.55) into account.

[3] We shall denote such a solution by $Y_a(\gamma)$ when we want to stress its dependence upon γ.

To prove that Y_a is a solution of the ARE (2.70), it remains only to show that

$$R_2 = -H_0^\top J_1^{-1} H_0. \tag{2.78}$$

Indeed, taking into account (2.51), (2.53), and the first of (2.41), we have

$$\begin{aligned}
&R_2 + H_0^\top J_1^{-1} H_0 \\
&= F_a^\top P_s^{-1}(F - F_y) - (F_1 - F_a)^\top P_s^{-1} F_y + L_1^\top R^{-1} L_1 + L^\top (MM^\top - \gamma^2 I)^{-1} L \\
&= F_a^\top P_s^{-1} B_1 M^\top (MM^\top - \gamma^2 I)^{-1} L - L_1^\top R^{-1} M V^{-1} B_1^\top P_s^{-1} F \\
&\quad + L_1^\top R^{-1} M \underbrace{V^{-1} B_1^\top P_s^{-1} B_1}_{V^{-1} - I} M^\top (MM^\top - \gamma^2 I)^{-1} L \\
&\quad + L_1^\top R^{-1} L_1 + L^\top (MM^\top - \gamma^2 I)^{-1} L \\
&= F_a^\top P_s^{-1} B_1 M^\top (MM^\top - \gamma^2 I)^{-1} L - L_1^\top R^{-1} M V^{-1} B_1^\top P_s^{-1} F \\
&\quad + L_1^\top R^{-1} \underbrace{M V^{-1} M^\top}_{-R + \gamma^2 I} (MM^\top - \gamma^2 I)^{-1} L \\
&\quad - L_1^\top R^{-1} \underbrace{MM^\top}_{(MM^\top - \gamma^2 I) + \gamma^2 I} (MM^\top - \gamma^2 I)^{-1} L \\
&\quad + L_1^\top R^{-1} L_1 + L^\top (MM^\top - \gamma^2 I)^{-1} L \\
&= F_a^\top P_s^{-1} B_1 M^\top (MM^\top - \gamma^2 I)^{-1} L - L_1^\top R^{-1} M V^{-1} B_1^\top P_s^{-1} F \\
&\quad + \underbrace{(L^\top - L_1^\top)}_{-F_a^\top P_s^{-1} B_1 M^\top} (MM^\top - \gamma^2 I)^{-1} L + L_1^\top R^{-1} \underbrace{(L_1 - L)}_{M B_1^\top P_s^{-1} F_a} \\
&= L_1^\top R^{-1} M (B_1^\top P_s^{-1} F_a - V^{-1} B_1^\top P_s^{-1} F) \\
&= L_1^\top R^{-1} M (B_1^\top P_s^{-1} F_a - B_1^\top P_s^{-1} \underbrace{(I - B_1 B_1^\top P_s^{-1})^{-1} F}_{F_a}) = 0.
\end{aligned} \tag{2.79}$$

We now prove that Y_a is a solution of (2.52) as well. In fact, we have

$$\begin{aligned}
F_y &= F - B_1 M^\top (MM^\top - \gamma^2 I)^{-1} L = F + B_1 M^\top \left(\frac{I}{\gamma^2} + \frac{M R_1^{-1} M^\top}{\gamma^4} \right) L \\
&= F + B_1 R_1^{-1} \frac{M^\top L}{\gamma^2}
\end{aligned} \tag{2.80}$$

and

$$\begin{aligned}
H_0^\top J_1^{-1} H_0 &= C^\top (DD^\top)^{-1} C + L^\top (MM^\top - \gamma^2 I)^{-1} L \\
&= C^\top (DD^\top)^{-1} C - \frac{L^\top L}{\gamma^2} - \frac{L^\top M}{\gamma^2} R_1^{-1} \frac{M^\top L}{\gamma^2}
\end{aligned} \tag{2.81}$$

so that, by using standard manipulations [15, p. 271], (2.70) may be rewritten in the form (2.52).

It remains to prove that the solution Y_a is indeed antistabilizing, i.e., that all the eigenvalues of Γ_y lay in $\{z \in \mathbb{C} : \ |z| > 1\}$. Notice that Γ_y may be written in the form

$$\Gamma_y = F_y + B_1 (R_1 - B_1^\top Y_a B_1)^{-1} B_1^\top Y_a F_y = (I - B_1 R_1^{-1} B_1^\top Y_a)^{-1} F_y. \tag{2.82}$$

Moreover, we rewrite (2.54) in the form

$$X_s^{-1} + L_1^\top R^{-1} L_1 = F_1^\top X_s^{-1} \Gamma_y, \tag{2.83}$$

which, together with (2.63) yields

$$I = X_s(I + L_1^\top R^{-1} L_1 X_s)^{-1} F_1^\top X_s^{-1} \Gamma_y = X_s \Gamma_a^\top X_s^{-1} \Gamma_y, \tag{2.84}$$

so that Γ_y is clearly antistable. ∎

3. A monotonicity result. In this section we prove that the values of γ for which $Y_a(\gamma)$ is singular are finitely many. This will be usefull in order to prove existence of the stabilizing solution of (1.14).

PROPOSITION 3.1. *Let $Y_a(\gamma)$ be as in* (2.68). *The set*

$$\mathcal{G}_2 := \{\gamma > \gamma_0 : \ Y_a(\gamma) \text{ is singular}\} \tag{3.1}$$

contains, at most, n points.

Proof. The solution X_s of (2.43) is a continuous function of γ and its first derivative with respect to γ^2 exists and is continuous; see, e.g., [15, Theorem 14.2.2]. Since P_s does not depend on γ, the derivative $\frac{dY_a}{d\gamma^2}$ exists and is continuos as well. Consider an open set $\mathcal{I} = (a, b)$ such that $a \geq \gamma_0$, $b > a$ (possibly $b = \infty$) and $\mathcal{I} \cap \mathcal{G}_1 = \emptyset$. (Notice that the set $\{\gamma \in \mathbb{R} : \ \gamma > \gamma_0\}$ may be written as the union of a finite number of sets of this form and of a finite number of isolated points.) For $\gamma \in \mathcal{I}$, we may take derivatives in both sides of (2.52) and, by defining

$$Y_\gamma := \frac{dY_a}{d\gamma^2}, \tag{3.2}$$

we get

$$\begin{aligned} Y_\gamma = {} & F^\top Y_\gamma F - \left(F^\top Y_\gamma B_1 - \frac{L^\top M}{\gamma^4}\right)(R_1 - B_1^\top Y_a B_1)^{-1}\left(B_1^\top Y_a F + \frac{M^\top L}{\gamma^2}\right) \\ & - \left(F^\top Y_a B_1 + \frac{L^\top M}{\gamma^2}\right)(R_1 - B_1^\top Y_a B_1)^{-1}\left(B_1^\top Y_\gamma F - \frac{M^\top L}{\gamma^4}\right) \\ & - \left(F^\top Y_a B_1 + \frac{L^\top M}{\gamma^2}\right)(R_1 - B_1^\top Y_a B_1)^{-1}\frac{M^\top M}{\gamma^4} \\ & \quad (R_1 - B_1^\top Y_a B_1)^{-1}\left(B_1^\top Y_a F + \frac{M^\top L}{\gamma^2}\right) \\ & + \left(F^\top Y_a B_1 + \frac{L^\top M}{\gamma^2}\right)(R_1 - B_1^\top Y_a B_1)^{-1} B_1^\top Y_\gamma B_1 \\ & \quad (R_1 - B_1^\top Y_a B_1)^{-1}\left(B_1^\top Y_a F + \frac{M^\top L}{\gamma^2}\right) \\ & - \frac{L^\top L}{\gamma^4} \end{aligned} \tag{3.3}$$

and, taking (2.69) into account,

$$Y_\gamma = \Gamma_y^\top Y_\gamma \Gamma_y - \frac{L_2^\top L_2}{\gamma^4} \tag{3.4}$$

with

$$L_2 := L - M(R_1 - B_1^\top Y_a B_1)^{-1}\left(B_1^\top Y_a F + \frac{M^\top L}{\gamma^2}\right). \tag{3.5}$$

Taking (2.82) into account, we may rewrite (2.70) (with $Y+Y_a$) in the form

$$Y_a = \Gamma_y^\top Y_a \Gamma_y - \Gamma_y^\top Y_a B_1 R_1^{-1} B_1^\top Y_a \Gamma_y - H_0^\top J_1^{-1} H_0. \tag{3.6}$$

Define

$$Z := Y_a + \gamma^2 Y_\gamma. \tag{3.7}$$

By adding (3.4) multiplied by γ^2 to (3.6), we get

$$Z = \Gamma_y^\top Z \Gamma_y - \Gamma_y^\top Y_a B_1 R_1^{-1} B_1^\top Y_a \Gamma_y - H_0^\top J_1^{-1} H_0 - \frac{L_2^\top L_2}{\gamma^2}. \tag{3.8}$$

By taking into account the definitions (2.69) and (3.5) of Γ_y and L_2, respectively, and identity (2.81), it is not difficult to see that

$$-\Gamma_y^\top Y_a B_1 R_1^{-1} B_1^\top Y_a \Gamma_y - H_0^\top J_1^{-1} H_0 - \frac{L_2^\top L_2}{\gamma^2} = -C_1^\top C_1 - C^\top (DD^\top)^{-1} C \tag{3.9}$$

with

$$C_1 := (R_1 - B_1^\top Y_a B_1)^{-1} \left(B_1^\top Y_a F + \frac{M^\top L}{\gamma^2} \right). \tag{3.10}$$

Therefore, Z satisfies

$$Z = \Gamma_y^\top Z \Gamma_y - C_1^\top C_1 - C^\top (DD^\top)^{-1} C. \tag{3.11}$$

Notice that since Γ_y is antistable, (3.4) and (3.11) imply that

$$Y_\gamma \geq 0, \tag{3.12a}$$
$$Z \geq 0. \tag{3.12b}$$

Moreover, by multiplying (3.4) and (3.11) on the left side by $\Gamma_y^{-\top}$ and on the right side by Γ_y^{-1}, it is easy to see that both $\ker Y_\gamma$ and $\ker Z$ are invariant for Γ_y^{-1} and hence for Γ_y. Then

$$\ker Y_\gamma \subseteq \ker L_2 \tag{3.13}$$

and

$$\ker Z \subseteq \ker \begin{bmatrix} C_1 \\ C \end{bmatrix}. \tag{3.14}$$

Therefore,

$$\mathcal{K} := \ker \begin{bmatrix} Y_a \\ Y_\gamma \end{bmatrix} = \ker \begin{bmatrix} Z \\ Y_\gamma \end{bmatrix} \tag{3.15}$$

is Γ_y-invariant and satisfies

$$\mathcal{K} \subseteq \ker \begin{bmatrix} C \\ C_1 \\ L_2 \end{bmatrix}. \tag{3.16}$$

We now prove that $\mathcal{K} = \{0\}$. To this end it is sufficient to show that

$$\ker \begin{bmatrix} \Gamma_y - \lambda I \\ C \\ C_1 \\ L_2 \end{bmatrix} = \{0\} \quad \forall \lambda \in \mathbb{C}. \tag{3.17}$$

Assume by contradiction that there exist $v \neq 0$ and $\lambda \in \mathbb{C}$ such that

$$\begin{bmatrix} \Gamma_y - \lambda I \\ C \\ C_1 \\ L_2 \end{bmatrix} v = 0. \tag{3.18}$$

Then, since Γ_y is antistable, $|\lambda| > 1$. Moreover, the definitions (1.6) of F and (2.69) of Γ_y yield

$$\Gamma_y = A - BD^\top (DD^\top)^{-1} C - B_1 C_1 \tag{3.19}$$

so that $\Gamma_y v = Av$ and we get

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} v = 0, \quad |\lambda| > 1, \tag{3.20}$$

which is in contradiction with detectability of the pair (A, C). Therefore $\mathcal{K} = 0$ and then, when one eigenvalue of Y_a (that is a continuous function of Y_a and hence of γ) is zero, its derivative with respect to γ is positive. Thus, if $Y_a(\bar{\gamma})$ is singular and (counting with multiplicity) has, say, k_+ positive, k_0 zero, and k_- negative eigenvalues, then there exists a positive value δ such that $Y_a(\gamma)$ has k_+ positive and $k_0 + k_-$ negative eigenvalues for $\gamma \in (\bar{\gamma} - \delta, \bar{\gamma})$ and $k_+ + k_0$ positive and k_- negative eigenvalues for $\gamma \in (\bar{\gamma}, \bar{\gamma} + \delta)$. Clearly, this may happen for, at most, n different values of γ.

So far we have assumed that $\gamma \in \mathcal{I}$. As already observed, the set $\{\gamma \in \mathbb{R} : \gamma > \gamma_0\}$ may be written as the union of a finite number of sets of the same type of $\mathcal{I}$ and of a finite number of values of γ. In correspondence of such values, Y_a remains a continuous function of γ so that we can extend the conclusion to the whole set $\{\gamma \in \mathbb{R} : \gamma > \gamma_0\}$. □

4. Existence of a stabilizing solution of the ARE (1.14). Next we show that when Y_a is nonsingular, then Y_a^{-1} is the (necessarily unique) stabilizing solution of (1.14), so that, in view of Propositions 2.1 and 3.1, we get that, except for a finite number of values of $\gamma > \gamma_0$, the ARE (1.14) admits the stabilizing solution. We shall prove in the next section that such solution is also feasible.

Consider the set $\mathcal{G}_1 \cup \mathcal{G}_2$ and observe that it contains, at most, $2[n + \operatorname{rank}(M)]$ points. Then, the set of *regular* values of γ defined as

$$\mathcal{G}_r := \{\gamma > \gamma_0 : \gamma \notin \mathcal{G}_1 \cup \mathcal{G}_2\} \tag{4.1}$$

is generic in $\{\gamma \in \mathbb{R} : \gamma > \gamma_0\}$.

The following theorem is our main result.

THEOREM 4.1. *Let $\gamma \in \mathcal{G}_r$ and let $Y_a(\gamma)$ be the corresponding antistabilizing solution of* (2.52). *Then* (1.14) *admits a unique symmetric stabilizing solution. Such solution is given by*

$$\Delta_s = Y_a^{-1}. \tag{4.2}$$

Proof. As shown in Proposition 2.1, Y_a is a solution of (2.70) so that we have

$$\begin{aligned} Y_a + H_0^\top J_1^{-1} H_0 &= F_y^\top Y_a F_y + F_y^\top Y_a B_1 (R_1 - B_1^\top Y_a B_1)^{-1} B_1^\top Y_a F_y \\ &= F_y^\top (Y_a + Y_a B_1 (R_1 - B_1^\top Y_a B_1)^{-1} B_1^\top Y_a) F_y. \end{aligned} \tag{4.3}$$

Since $\gamma \in \mathcal{G}_r$, Y_a, R_1 and $(R_1 - B_1^\top Y_a B_1)$ are nonsingular so that $Y_a + Y_a B_1 (R_1 - B_1^\top Y_a B_1)^{-1} B_1^\top Y_a$ is such. Moreover, F_y is nonsingular so that the left-hand side of (4.3) is nonsingular and the same procedure that led to (2.67a) gives

$$Y_a^{-1} = F_y Y_a^{-1} F_y^\top - F_y Y_a^{-1} H_0^\top (J_1 + H_0 Y_a^{-1} H_0^\top)^{-1} H_0 Y_a^{-1} F_y^\top + B_1 R_1^{-1} B_1^\top, \tag{4.4}$$

which, taking (4.2) into account, yields, after some standard manipulations [15, p. 271]

$$\Delta_s = A \Delta_s A^\top + B B^\top - (A \Delta_s H_0^\top + B J_0^\top)(J_1 + H_0 \Delta_s H_0^\top)^{-1} (H_0 \Delta_s A^\top + J_0 B^\top). \tag{4.5}$$

It remains to show that

$$\Gamma_s := A - (A \Delta_s H_0^\top + B J_0^\top)(J_1 + H_0 \Delta_s H_0^\top)^{-1} H_0 = F_y - F_y \Delta_s H_0^\top (J_1 + H_0 \Delta_s H_0^\top)^{-1} H_0 \tag{4.6}$$

is stable. To this aim we rewrite (4.4) in the form

$$Y_a^{-1} - B_1 R_1^{-1} B_1^\top = \Gamma_s Y_a^{-1} F_y^\top, \tag{4.7}$$

which, taking (2.82) into account, yields

$$I = \Gamma_s Y_a^{-1} F_y^\top (Y_a^{-1} - B_1 R_1^{-1} B_1^\top)^{-1} = \Gamma_s Y_a^{-1} \Gamma_y^\top Y_a, \tag{4.8}$$

so that, since Γ_y is antistable, Γ_s is stable. □

Remark 4.1. It is worth noticing that the computation of the stabilizing solution of the Riccati equation (1.14) can be numerically obtained by resorting to standard routines available in most control packages without the need of computing the solution the auxiliary Riccati equations (2.1) and (2.43). Indeed, those Riccati equations were only instrumental to the purpose of proving the existence of the stabilizing solution of (1.14).

5. Construction of the minimum phase J-spectral factor and $\mathcal{H}_\infty$ estimator design. The following important theorem gives a constructive procedure to obtain, from the solution Δ_s, a minimum-phase J-spectral factor of $\Psi(z)$ having a realization with the same state matrix A and the same output matrix H_0 of the realization (1.9) of $H(z)$.

THEOREM 5.1. *Let*

$$\gamma \in \mathcal{G}_r \tag{5.1}$$

and let Δ_s be the corresponding stabilizing solution of equation (1.14). *Then,*

1. *The solution Δ_s is feasible, i.e., $J_1 + H_0 \Delta_s H_0^\top$ has the same inertia of $J_{p,l}(\gamma)$ and hence there exists a nonsingular matrix Λ such that*

$$\Lambda J_{p,l}(\gamma) \Lambda^\top = J_1 + H_0 \Delta_s H_0^\top. \tag{5.2}$$

2. *The transfer matrix*

$$(5.3)\ \Omega_s(z) := H_0(zI-A)^{-1}(A\Delta_s H_0^\top + BJ_0^\top)(J_1 + H_0\Delta_s H_0^\top)^{-1}\Lambda + \Lambda$$

is a square J-spectral factor of the J-spectral density $\Psi(z)$.

3. *All the zeros of $\Omega_s(z)$ lay in the open unit disk, i.e., $\Omega_s(z)^{-1}$ is stable.*

Proof. The proof of this theorem may be obtained following the same lines of Theorem 4.1 in [4]. □

We recall from [3] that there is a simple procedure that, from $\Omega_s(z)$, furnishes a realization of an $\mathcal{H}_\infty$ smoothing filter with attenuation level γ.

6. The critical values of γ: A peculiar feature of the J-spectral factorization. In this section, we analyze the ARE (1.14) and the associated J-spectral factorization problem in the case when $\gamma \in \mathcal{G}_s := \mathcal{G}_1 \cup \mathcal{G}_2$. To this aim we write the set $\mathcal{G}_s$ as the union of disjoint sets $\mathcal{G}_s = \mathcal{G}_1' \cup \mathcal{G}_2$ with $\mathcal{G}_1' := \mathcal{G}_1 \cap \overline{\mathcal{G}_2}$ (usually $\mathcal{G}_1' = \mathcal{G}_1$) and we consider the cases $\gamma \in \mathcal{G}_1'$ and $\gamma \in \mathcal{G}_2$ separately.

For $\gamma \in \mathcal{G}_1'$, $Y_a(\gamma)$ is still nonsingular and we can define $\Delta_s := Y_a(\gamma)^{-1}$. In all the manifold examples that we have worked out, it turns out that, even for $\gamma \in \mathcal{G}_1'$, Δ_s defined in this way continues to be the stabilizing solution of (1.14). This leads us to conjecture that Theorem 4.1 holds (possibly with the Moore–Penrose pseudoinverse in place of the inverse in the ARE (1.14)) for all $\gamma \in \mathcal{G}_r \cup \mathcal{G}_1'$. In view of continutity, to prove this generalization it would be sufficient to show that for $\gamma \in \mathcal{G}_1'$, $(J_1 + H_0\Delta_s H_0^\top)$ is invertible or (using the pseudoinverse in place of the inverse) that $(A\Delta_s H_0^\top + BJ_0^\top)(J_1 + H_0\Delta_s H_0^\top)^\sharp(H_0\Delta_s A^\top + J_0 B^\top)$ is a continuous function of γ for each $\gamma \in \mathcal{G}_1'$.

Much more interesting is the behavior associated with $\gamma \in \mathcal{G}_2$. In this case, as shown in the following example, the ARE does not admit a stabilizing solution nor does the J-spectral density admit a minimum phase J-spectral factor having a realization with state matrix equal to A. This is a peculiar (and in the authors' opinion, rather counterintuitive) feature of the J-spectral factorization. In fact, in the standard (positive) spectral factorization this phenomenon cannot occur.

Let us consider the following very simple example: $A = 2$, $C = L = 1$, $B = [1 \mid 0]$, $D = [0 \mid 1]$, and $M = [0 \mid 0]$. In this case $W(z)$ defined in (1.20) is given by

$$W(z) = \frac{1}{2(3 - z - z^{-1})} \tag{6.1}$$

so that

$$\gamma_0^2 = \sup_\vartheta \left| \frac{1}{2(3 - 2\cos(\vartheta))} \right| = \frac{1}{2}. \tag{6.2}$$

Moreover, the stabilizing solution of the ARE (2.1) is easily computed to be $P_s = 2 + \sqrt{5}$ and the ARE (2.43) assumes the form

$$X^2 + \left(\frac{3 + \sqrt{5} - \gamma^2(10 + 6\sqrt{5})}{4} \right) X + \gamma^2 \left(\frac{3 + \sqrt{5}}{4} \right) = 0, \tag{6.3}$$

whose stabilizing solution $X_s(\gamma)$ equals P_s for $\gamma = 1$. Moreover, $X_s(\gamma) > P_s$ for $\gamma > 1$ and $X_s(\gamma) < P_s$ for $\gamma < 1$. Thus, in this case, $\mathcal{G}_2 = \{1\}$. The ARE (1.14) assumes the form

$$3\Delta + 1 - \frac{4\Delta^2}{\Delta(1 - \gamma^2) - \gamma^2}(1 - \gamma^2) = 0. \tag{6.4}$$

For $\gamma < 1$, (6.4) admits a negative stabilizing solution Δ_s that tends to $-\infty$ as $\gamma \to 1_-$. For $\gamma > 1$, (6.4) admits a positive stabilizing solution Δ_s that tends to $+\infty$ as $\gamma \to 1_+$.

For $\gamma = 1$, the order of such equation collapses and the stabilizing solution does not exist any more: the only solution is $\Delta_a = -1/3$ and the corresponding closed-loop matrix is $A - 0 = A = 2$. Notice that $\Delta_a = -1/3$ is feasible. In fact

$$J_1 + H_0 \Delta_a H_0^\top = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & -4 \end{bmatrix} \tag{6.5}$$

has a positive and a negative eigenvalue so that there exists a matrix Λ such that[4]

$$J_1 + H_0 \Delta_a H_0^\top = \Lambda J_{1,1}(1) \Lambda^\top. \tag{6.6}$$

The same computation used in the proof of Theorem 5.1 shows that

$$\begin{aligned} \Omega_a(z) &:= H_0(zI - A)^{-1}(A\Delta_a H_0^\top + BJ_0^\top)(J_1 + H_0\Delta_a H_0^\top)^{-1}\Lambda + \Lambda \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z-2)^{-1} \left[-\frac{2}{3} \mid \frac{2}{3} \right] \Lambda + \Lambda \end{aligned} \tag{6.7}$$

is a J-spectral factor of the J-spectrum $\Psi(z)$ that, in this case, is given by

$$\Psi(z) = \begin{bmatrix} \frac{1}{(z-2)(z^{-1}-2)} + 1 & \frac{1}{(z-2)(z^{-1}-2)} \\ \frac{1}{(z-2)(z^{-1}-2)} & \frac{1}{(z-2)(z^{-1}-2)} - 1 \end{bmatrix}. \tag{6.8}$$

The numerator matrix of $\Omega_a(z)$ is $2 - \left[-\frac{2}{3} \mid \frac{2}{3}\right] \Lambda\Lambda^{-1} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = 2$: the unique zero of $\Omega_a(z)$ lies outside the closed unit disk. (Such an $\Omega_a(z)$ is said to be a *maximum phase* J-spectral factor.) The set of all J-spectral factors of $\Psi(z)$ having a realization with state matrix $A = 2$ may be obtained as follows. We set

$$\Omega(z) = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} (z-2)^{-1}[g_1 \mid g_2] + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \tag{6.9}$$

and impose

$$\Omega(z) J_{1,1}(1) \Omega^\sim(z) = \Psi(z). \tag{6.10}$$

The corresponding solutions, up to a noninteresting change of basis in the state space, may be parametrizied as follows:

$$\Omega(z) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z-2)^{-1}[g \mid \pm g] + \begin{bmatrix} \frac{-2-3g^2}{6g} & \pm\frac{2-3g^2}{6g} \\ \frac{-1+3g^2}{3g} & \pm\frac{1+3g^2}{3g} \end{bmatrix}, \qquad g \in \mathbb{R} \setminus \{0\}, \tag{6.11}$$

which is readily seen to be equal to the right-hand side of (6.7) as Λ varies among the solutions of (6.6). In conclusion, we have produced a maximum-phase J-spectral

[4] The set of solutions of (6.6) may be parametrized as

$$\Lambda = \begin{bmatrix} \frac{-2-3g^2}{6g} & \pm\frac{2-3g^2}{6g} \\ \frac{-1+3g^2}{3g} & \pm\frac{1+3g^2}{3g} \end{bmatrix}, g \in \mathbb{R} \setminus \{0\}$$

(where the $\pm$ signs are either both $+$ or both $-$).

factor and have shown that the corresponding minimum-phase one (i.e., a minimum-phase J-spectral factor having a realization with the same state-space matrix) does not exist. This is a very peculiar behavior that has no counterpart in the regular spectral factorization.

Notice that to obtain (6.11) we have imposed the McMillan degree[5] of $\Omega(z)$ to be equal to 1 or, equivalently, we have restricted our search to *minimal* J-spectral factors, namely, J-spectral factors having the last possible McMillan degree. (Such McMillan degree is clearly equal to one-half of the McMillan degree of the J-spectrum $\Psi(z)$.) See [16] for a discussion on the minimality of spectral factors. It is interesting to observe that extending the search to nonminimal J-spectral factors (i.e., J-spectral factors having larger McMillan degree), a minimum-phase J-spectral factor having a unique pole in $z = 2$ does exist. Indeed, it is not difficult to check that

$$\Omega_m(z) = (zI_2 - 2I_2)^{-1} \begin{bmatrix} \frac{-11}{2\sqrt{2}} & \frac{-3}{2\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} \frac{-5\sqrt{2}}{3} & \frac{-\sqrt{2}}{3} \\ \frac{-1}{3\sqrt{2}} & \frac{7}{3\sqrt{2}} \end{bmatrix} \tag{6.12}$$

is a minimal realization of a J-spectral factor of $\Psi(z)$. As $H(z)$, $\Omega_m(z)$ has a unique pole in $z = 2$ and the corresponding numerator matrix

$$2I_2 - \begin{bmatrix} \frac{-11}{2\sqrt{2}} & \frac{-3}{2\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{-5\sqrt{2}}{3} & \frac{-\sqrt{2}}{3} \\ \frac{-1}{3\sqrt{2}} & \frac{7}{3\sqrt{2}} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} \end{bmatrix} \tag{6.13}$$

has a unique eigenvalue (of multiplicity 2) in $z = 1/2$. Again, this fact has no counterpart in classical spectral factorization.

Eventually, notice that these discrepancies between J-spectral factorization and classical spectral factorization are associated to very particular J-spectra. In fact they may occur only for a finite number of values of γ.

6.1. Example with $\gamma \in \mathcal{G}_r$. In the following we consider the previous example in the case when $\gamma_0 < \gamma < \gamma_f$ and $\gamma \in \mathcal{G}_r$. We design a smoothing filter starting from the stabilizing solution Δ_s of (1.14). Let $\gamma^2 = 3/4$. The stabilizing solution Δ_s is then given by $-4 - \sqrt{13}$ and the corresponding J-spectral factor is given by

$$\Omega_s(z) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z-2)^{-1} \begin{bmatrix} 2.942 & | & 6.374 \end{bmatrix} + \begin{bmatrix} -0.3124 & 2.99 \\ 0.2785 & 3.353 \end{bmatrix}. \tag{6.14}$$

From $\Omega_s(z)$, by following the procedure described in [3], we easily get the following transfer function of a 1-step lag smoother,

$$S_1(z) = z[0.2063(z - 0.02824)^{-1}2.958 + 0.1906], \tag{6.15}$$

and it is easy to check that

$$\|S_1(z)H_1(z) - H_2(z)\| = 0.8383 < \gamma \simeq 0.8660. \tag{6.16}$$

7. Conclusions. In this paper a general J-spectral factorization problem was considered and its relation with the existence of the stabilizing solution of the associated Riccati equation was investigated. The stabilizing solution depends on a positive

[5] We recall that the *McMillan degree* of a rational proper matrix function $P(z)$ is the state-space dimension of a minimal realization of $P(z)$; see [14] for more details.

parameter which represents the prescribed attenuation level for the underlying estimation problem. We have shown that the stabilizing solution of the ARE still exists (except for a finite number of values of γ) as long as a fixed-lag acausal estimator (smoother) does. A few aspects of the J-spectral factorization problem and the properties of its solutions are discussed in correspondence to the (finite number of) values of γ for which the stabilizing solution of the ARE does not exist.

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