# Robustness of the Hobson-Rogers Model with Respect to the Offset Function

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**Abstract.** In this paper we analyse the robustness of the Hobson–Rogers model with respect to the offset function, which depends on the whole past of the risky asset and is thus not fully observable. We prove that, if the offset function is the realisation of a stationary process, then the error in pricing a derivative asset decreases exponentially with respect to the observation window. We present sufficient conditions on the volatility in order to characterise the invariant density and three examples.

## 1. Introduction

The year 1973 is a milestone in the modeling of financial markets: in fact, in that year the papers of Black and Scholes [2] and Merton [15], where an explicit formula for the price of call and put options was present, saw the light. The formula now known universally as "the Black and Scholes formula" links the price of a call option to quantities which are observed in the market (current price, strike price, time to maturity) and a parameter, the volatility, which gives an idea of how rapidly the asset prices can change. The two papers cited above influenced financial markets so deeply that every investment bank today has to deal with "the Black and Scholes approach": this is also witnessed by the Nobel prize in 1997.

The so-called "Black and Scholes model" is however valid only as a first approximation: in fact, it was soon realised that the assumption of a constant volatility was in contrast with the empirical observations of derivative prices in real markets, which suggest that the volatility is not constant, but rather depends both on time to maturity and on the strike price.

In the last years a growing interest has been raised for models where the asset prices' dynamics do not depend only on their current values, but also on past

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values: these models can be usually seen as generalisations of the so-called level-dependent volatility models, where the volatility is usually a function of time and current price level, and the market is complete. By making the volatility depend also on the past prices of the risky assets, it is also possible to enrich the model by reproducing correlations and dependencies which are observed in practice. Among these models, the one proposed by Hobson–Rogers [12] is the only case (to the authors' knowledge) where the model is equivalent to a 2-dimensional Markov model, thus the problem of pricing and hedging a derivative asset is led to the solution of a linear PDE. In particular, one component of this Markov process represents the price and the other one represents the so-called *offset function of order 1*, which is an integral depending on all the past history of the asset price, and is thus not fully observable.

There are two ways of using the Hobson–Rogers model in practice. One is to consider a finite horizon approximation, where the offset function is defined only on a finite observation interval of the past price. Unfortunately, the authors proved in a previous paper [10] that it is impossible to obtain a Markov system in this way. The other way is to use the pricing PDE with a misspecified initial offset function, thus making a mistake both on the path of the process as on the calculation of the price of the derivative assets. This approach is studied in detail in this paper. One can then search for the initial offset value which minimises this error. We find out that, for all the contingent claims which are Lipschitz continuous functions of the log-price of the asset, this error is proportional to the variance of the offset function at time 0. By assuming that we can observe the past prices of the risky asset on an interval of length $R$, this variance decreases exponentially with respect to $R$, and is proportional to the variance of the offset function at time $-R$. If we also assume that the offset function is a stationary process, we can calculate this variance, which does not depend on $R$: in this way, if one wants an error less than a given $\varepsilon$ in pricing a derivative asset, one only has to observe the past price for a sufficient time $R$.

The paper is organised as follows. In Section 2 we present the Hobson–Rogers model. In Section 3 we make a survey, based on [10], on the reasons why a version of the Hobson–Rogers model with finite observation horizon loses Markovianity. In Section 4 we study the robustness of the Hobson–Rogers model with respect to the misspecification of the offset function, and in Section 5 we provide an estimate of the minimum observation horizon required for having an error less than a given threshold. In Section 6 we provide a way to calculate the variance of the offset function at the beginning of the observation window in terms of the invariant measure of the offset function, and provide sufficient conditions on the volatility in order to have a characterisation of the invariant density. Section 7 presents three examples.

## 2. The Hobson–Rogers model

We define the discounted log-price process $Z(t)$ at time $t$ as

$$Z(t) = \log(S(t)e^{-rt})$$

where $r$ is the (constant) risk-free interest rate, and the *offset function* of order $m$, denoted by $P^{(m)}(t)$, by

$$P^{(m)}(t) = \int_0^\infty \lambda e^{-\lambda u}(Z(t) - Z(t-u))^m du \quad \text{for} \quad m = 0, \ldots, n \tag{2.1}$$

the constant $\lambda$ being a parameter of the model which describes the rate at which past information is discounted. Then, for some value $n$, we assume the following.

*Assumption* 2.1. $Z(t)$ satisfies the SDE

$$dZ(t) = -\frac{1}{2}\sigma^2(P^{(1)}(t), \ldots, P^{(n)}(t))dt + \sigma(P^{(1)}(t), \ldots, P^{(n)}(t))\; dW(t)$$

where $\sigma(\cdot)$ and $\sigma^2(\cdot)$ are globally Liptschitz, $\sigma(\cdot)$ is strictly positive and $(W_t)_{t\in\mathbb{R}}$ is a so-called two-sided Brownian motion [3] under a probability measure $\mathbb{P}$, which is chosen such that $(S(t)e^{-rt})_t$ is a $\mathbb{P}$-martingale.

This probability measure $\mathbb{P}$ is in fact known as *risk-neutral* probability or *martingale measure*), and the existence of such a $\mathbb{P}$ is equivalent to the non-existence of arbitrage opportunities in the market (see [10, 12] and the references therein for details).

This model can be seen as a "good" model because no new Brownian motions (or other source of uncertainty) have been introduced in the specification of the price process. This means that the market is complete and any contingent claim is hedgeable (see [10] for details). On the other hand, it is possible to allow $\sigma(\cdot)$ to be a function of the price level $S(t)$ also. So, this model can be extended to include the class of level-dependent volatility processes as a special case. The reason for the definition of the processes $P^{(m)}(t)$, $m = 0, \ldots, n$, is seen in the following lemma.

**Lemma 2.2.** *$(Z, P^{(1)}, \ldots, P^{(n)})$ is a $(n+1)$-dimensional Markov process, and the offset processes $P^{(m)}(t)$ satisfy the coupled SDEs*

$$\begin{aligned} dP^{(m)}(t) &= mP^{(m-1)}(t)\; dZ(t) + \frac{m(m-1)}{2}P^{(m-2)}(t)\; d\langle Z\rangle(t) - \lambda P^{(m)}(t)\; dt, \\ & \quad m > 1 \\ dP^{(1)}(t) &= dZ(t) - \lambda P^{(1)}(t)\; dt \end{aligned} \tag{2.2}$$

*Proof.* See [12]. □

Being $(Z, P^{(1)}, \ldots, P^{(n)})$ a $(n+1)$-dimensional Markov process, we can easily employ the Kolmogorov equation when pricing a contingent claim with final payoff $h(S(T))$. In fact, (for sake of simplicity consider from now on the case $n = 1$ and denote $P(t) \equiv P^{(1)}(t))$ its price $V(t) = \mathbb{E}[h(S(T))|\mathcal{F}_t]$ is of the form $V(t) = F(t, S(t), P(t))$, where $F$ is the solution of the Kolmogorov equation

$$F_t + rsF_s - \lambda pF_p + \left(\frac{1}{2}s^2 F_{ss} + sF_{ps} + \frac{1}{2}F_{pp} - \frac{1}{2}F_p\right)\sigma^2(p) = rF \tag{2.3}$$

subject to the boundary condition

$$F(p, s, T) = h(s)$$

Besides, the solution of the hedging problem is a closed formula: it is enough to use the Itô formula on $F$ and to make some calculations to obtain that the hedging strategy at time $t$ is given by

$$\Delta(t) = F_s(t, S(t), P(t)) + \frac{F_p(t, S(t), P(t))}{S(t)}$$

In conclusion this model allows to construct a process for the price, but we can see that some difficulties arise. In fact, for the computation of $P(0)$ (or in general $P(t)$), we need to know the path of $S$ on all its past $(-\infty, 0)$ (or $(-\infty, t)$). This requirement is unusual in the modelisation of financial markets, where one usually meets models that start from a certain moment in time (usually 0). In fact, the requirement of an infinite horizon in the past raises mathematical and "practical" (or better economical) complications. From the mathematical side, we would have to define a stochastic calculus with time ranging on all the real line. Once that this is done, we would have to establish that $P$ is well defined: in fact, remember that $P$ is the integral of a process on $(-\infty, 0)$, so one must also prove that this integral is well defined. From the economical side, assets that "existed forever" do not exist in the real market. Thus, one has to establish what can be used instead of the price path of $S$ when the asset still did not exist.

While these problems seem less worrying than stated, mainly due to the exponential weight in (2.1), still theoretical (and practical) solutions to these issues are not present in literature, at least to the authors' knowledge. For this reason, we will explore two different approaches to avoid these problems.

The first one consists in specifying a model with finite horizon and to make the volatility depend on integrals of the price path. Unfortunately up to now all the models of this kind present in literature [1, 8] do not give a Markovian structure as the Hobson-Rogers model does, unless one uses from the beginning a level-dependent volatility model: in the next Section 3 we present a survey, based on [10], of these results.

The second one is the following. The problem of pricing a contingent claim with the Hobson-Rogers model is equivalent to solve the PDE (2.3), once the initial conditions $S(0) = s$, $P(0) = p$ are specified. While the price $S(0)$ is observed in the market, in order to calculate the true value $P(0)$ one would have to observe the asset in all its past. Since this is impossible, one has to use the model with a misspecification $\tilde{P}(0)$. Our aim will be then to search for the initial condition $\tilde{P}(0)$ which minimizes the error of pricing the contingent claim $h(S(T))$. This will be done from Section 4 on.

## 3. A finite delay model

Now we analyse a modification of the Hobson-Rogers model where we consider a finite time horizon and we make the risky asset's dynamics depend on integrals of the price path. Inspired by a model in [8], the model that we study is

$$dS(t) = S(t)\sigma(Y(t), Z(t))\ dW(t)$$

where the processes $Y$ and $Z$ are defined as

$$Y(t) = \int_0^\tau e^{-\lambda v} f(S(t-v))\ dv = \int_{t-\tau}^t e^{\lambda(u-t)} f(S(u))\ du, \qquad Z(t) = S(t-\tau)$$

where $f$ is a strictly monotone function and $\tau$ is a given finite delay. Notice that for $f(x) = \log x$ and $\tau = +\infty$ one has that $\lambda Y(t) = \log S(t) - P^{(1)}(t)$, $P^{(1)}$ being the first offset function of the Hobson-Rogers model. Our scope is now to find a self-financing portfolio $V$ which replicates the option with payoff $h(S(T))$ (or more generally $h(S(T), Y(T))$). Unlike in the Hobson-Rogers model, here the process $(S, Y)$ is not Markov, and this is more due to the finite horizon nature of $Y$ rather then to the specification of the volatility, more general than the Hobson-Rogers' one.

One can immediately think to use the state variables $(S(t), Y(t), Z(t))$, but this entails the use of anticipative stochastic calculus. In fact, by making use of the Itô formula on a deterministic function of $(S(t), Y(t), S(t-\tau))$, we end up with stochastic differentials of the kind $G(t, S(t), Y(t), S(t-\tau))dS(t-\tau)$, where $G(t, S(t), Y(t), S(t-\tau))$ is not adapted to the filtration of the differential $dS(t-\tau)$ but "anticipates" (see [14] and the references therein). Conversely, we would have to prove that the portfolio dynamics could be written in the form $dV(t) = \Delta(t)\ dS(t)$, with $\Delta$ adapted to the filtration of $S$. In doing this, we will surely lose the Markovianity of the original Hobson-Rogers model.

One can be tempted to explore the following shortcut: though $(S, Y)$ is not in general a Markov process, we make the strong assumption that for every final payoff of the form $h(S(T), Y(T))$ there exists a deterministic function $F$ such that

$$V(t) = \mathbb{E}[h(S(T), Y(T))|\mathcal{F}_t] = F(t, S(t), Y(t)) \tag{3.1}$$

If this assumption is true, then the self-financing portfolio depends in a deterministic way only on the current values of $S$ and $Y$. Unfortunately, the next result states that the assumption (3.1) is equivalent to $\sigma$ not depending on $y, z$, that is to $S$ to be Markov; moreover, in this case, (3.1) is only true for $h$ not depending on $Y$ and the function $F$ depends on $t, s$ only.

**Theorem 3.1.** *If assumption* (3.1) *is true, then* $\sigma_z = \sigma_y = 0$.

The interested reader can find the proof in [10].

*Remark* 3.2. In this failed try, we were inspired by the positive results in [8]. We however have to say that in that paper the authors analyse a controlled system (which gives more degrees of freedom in reaching Markovianity), and also in that situation the authors succeed in reducing the system to the current values of $S$

and $Y$ only when the dynamics of $S$ is linear and with some restriction on the coefficients.

## 4. Robustness of the Hobson-Rogers model

As already announced, now we focus ourselves in establishing what happens if our Markov process $(P, Z)$ starts from a misspecified initial condition $(\tilde{P}(0), Z(0))$ instead of the true initial condition $(P(0), Z(0))$.

From now on, denote with $\Sigma := (P, Z)$ the process with the correct (but not known) initial conditions and by $\tilde{\Sigma} = (\tilde{P}, \tilde{Z})$ the process starting from the misspecified initial conditions $(\tilde{P}(0), Z(0))$. Then the evolution of (both $\Sigma$ and) $\tilde{\Sigma}$ is given by

$$\begin{cases} d\tilde{P}(t) &=& -\left(\frac{1}{2}\sigma^2(\tilde{P}(t), \tilde{Z}(t)) + \lambda \tilde{P}(t)\right) \, dt + \sigma(\tilde{P}(t), \tilde{Z}(t)) \, dW(t), \\ \tilde{P}(0) &\neq& P(0) \\ d\tilde{Z}(t) &=& -\frac{1}{2}\sigma^2(\tilde{P}(t), \tilde{Z}(t)) \, dt + \sigma(\tilde{P}(t), \tilde{Z}(t)) \, dW(t), \\ \tilde{Z}(0) &=& Z(0) \end{cases}$$

the dynamics of $\Sigma$ being driven by the same differential equation with the "right" initial conditions.

Now we present two estimates on the dependence of the process $\Sigma$ (or $\tilde{\Sigma}$) on the initial condition: the first one is an $L^2$-estimate on $\sup_{0 \le u \le T} |\Sigma(u) - \tilde{\Sigma}(u)|$, and the second one is an $L^2$-estimate on $|\Sigma(T) - \tilde{\Sigma}(T)|$. Assume that the functions $\sigma(p, z)$ and $\sigma^2(p, z)$ are globally Lipschitz in $(p, z)$ with respect to the Euclidean norm, in the sense that for $f = \sigma, \sigma^2$ there exists $K \ge 0$ (called *Lipschitz constant* of $f$) such that

$$|f(p, z) - f(\tilde{p}, \tilde{z})| \le K|(p, z) - (\tilde{p}, \tilde{z})| = K\sqrt{(p - \tilde{p})^2 + (z - \tilde{z})^2} \quad \forall (p, z), (\tilde{p}, \tilde{z})$$

**Theorem 4.1.** *If $\sigma$, $\sigma^2$ are globally Lipschitz with Lipschitz constants respectively $L$, $M$, then for $t \in [0, T]$ we have*

$$\mathbb{E}\left[\sup_{0 \le u \le t} |\Sigma(u) - \tilde{\Sigma}(u)|^2\right] \le 3\mathbb{E}[|P(0) - \tilde{P}(0)|^2] e^{c(L,M,T)t}$$

*where $c(L, M, T) = 2M^2T + 6\lambda^2 T + 20L^2$, and*

$$\mathbb{E}[|\Sigma(t) - \tilde{\Sigma}(t)|^2] \le 3\mathbb{E}[|P(0) - \tilde{P}(0)|^2] e^{C(L,M,T)t}$$

*where $C(L, M, T) = 2M^2T + 6\lambda^2 T + 5L^2$.*

Results of this kind are classical in the theory of SDEs: we present the proof in order to show that the constants $C(L, M, T)$ and $c(L, M, T)$ are the best possible for our equations.

*Proof.* We have that

$$\sup_{0 \le u \le t} |\Sigma(u) - \tilde{\Sigma}(u)|^2 \le \sup_{0 \le u \le t} |Z(u) - \tilde{Z}(u)|^2 + \sup_{0 \le u \le t} |P(u) - \tilde{P}(u)|^2$$

which yields

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le u\le t}|\Sigma(u)-\tilde{\Sigma}(u)|^2\right]\\
&\le \mathbb{E}\left[\sup_{0\le u\le t}|Z(u)-\tilde{Z}(u)|^2\right]+\mathbb{E}\left[\sup_{0\le u\le t}|P(u)-\tilde{P}(u)|^2\right]=(1)+(2)
\end{aligned}$$

For the first term in the right hand side, applying Doob's inequality and the Lipschitz property of $\sigma$ and $\sigma^2$, we have

$$\begin{aligned}
(1) &= \mathbb{E}\left[\sup_{0\le u\le t}\left|Z(0)-\tilde{Z}(0))+\int_0^u \frac{1}{2}(\sigma^2-\tilde{\sigma}^2)\ ds+\int_0^u(\sigma-\tilde{\sigma})\ dW(s)\right|^2\right]\\
&\le 2\mathbb{E}\left[\sup_{0\le u\le t}\left|\int_0^t \frac{1}{2}(\sigma^2-\tilde{\sigma}^2)ds\right|^2\right]+2\mathbb{E}\left[\sup_{0\le u\le t}\left|\int_0^u(\sigma-\tilde{\sigma})dW(s)\right|^2\right]\\
&\le \frac{1}{2}T\int_0^t \mathbb{E}|\sigma^2-\tilde{\sigma}^2|^2ds+8\int_0^t\mathbb{E}|\sigma-\tilde{\sigma}|^2ds\\
&\le \left(\frac{1}{2}M^2T+8L^2\right)\int_0^t\mathbb{E}\left[|Z-\tilde{Z}|^2+|P-\tilde{P}|^2\right]ds
\end{aligned}$$

where $\sigma$, $\tilde{\sigma}$ is a shorthand notation for $\sigma(P(s),Z(s))$, $\tilde{\sigma}(s)=\sigma(\tilde{P}(s),\tilde{Z}(s))$. For the second term we have

$$\begin{aligned}
(2) &\le 3\mathbb{E}|P(0)-\tilde{P}(0)|^2+3T\int_0^t\mathbb{E}\left[\left|\frac{1}{2}(\sigma^2-\tilde{\sigma}^2)+\lambda(P-\tilde{P})\right|^2\right]ds\\
&\quad +3\mathbb{E}\left[\sup_{0\le u\le t}\left|\int_0^u(\sigma-\tilde{\sigma})dW(s)\right|^2\right]\\
&\le 3\mathbb{E}|P(0)-\tilde{P}(0)|^2+3T\int_0^t\mathbb{E}\left[\frac{1}{2}|\sigma^2-\tilde{\sigma}^2|^2+2\lambda^2|P-\tilde{P}|^2\right]ds\\
&\quad +12\int_0^t\mathbb{E}|\sigma-\tilde{\sigma}|^2ds\\
&\le 3\mathbb{E}|P(0)-\tilde{P}(0)|^2+\left(\frac{3}{2}M^2T+12L^2\right)\int_0^t\mathbb{E}|Z-\tilde{Z}|^2ds\\
&\quad +(3M^2T+6\lambda^2T+12L^2)\int_0^t\mathbb{E}|P-\tilde{P}|^2ds\\
&\le 3\mathbb{E}|P(0)-\tilde{P}(0)|^2\\
&\quad +\left(\frac{3}{2}M^2T+6\lambda^2T+12L^2\right)\int_0^t\mathbb{E}\left[|P-\tilde{P}|^2+|Z-\tilde{Z}|^2\right]ds
\end{aligned}$$

then

$$\mathbb{E}\left[\sup_{0\le u\le t}|\Sigma(u)-\tilde{\Sigma}(u)|^2\right]$$
$$\le\ 3\mathbb{E}|P(0)-\tilde{P}(0)|^2+(2M^2+6\lambda^2T+20L^2)\int_0^t \mathbb{E}|\Sigma(s)-\tilde{\Sigma}(s)|^2 ds$$

and the theorem follows from the Gronwall lemma applied to

$$v(t)=\mathbb{E}[\sup_{0\le u\le t}|\Sigma(u)-\tilde{\Sigma}(u)|^2].$$

For the estimate on $\mathbb{E}[|\Sigma(t)-\tilde{\Sigma}(t)|^2]$, the proof proceeds in a similar way, without applying Doob's inequality to the term $\mathbb{E}[\sup_{0\le u\le t}|\int_0^u(\sigma-\tilde{\sigma})dW(s)|^2]$. □

**Corollary 4.2.** *If $h: C^0[0,T]\to\mathbb{R}$ is the payoff of a path-dependent claim such that the function $z(\cdot)\to h(e^{z(\cdot)})$ is globally Lipschitz, then*

$$\left|\mathbb{E}[h(S_T)]-\mathbb{E}[h(\tilde{S}_T)]\right|^2\le 3J^2\mathbb{E}|P(0)-\tilde{P}(0)|^2e^{c(L,M,T)T} \tag{4.1}$$

*where $J$ is the Lipschitz constant of $z(\cdot)\to h(e^{z(\cdot)})$. If $h$ is a simple European claim, then an analogous estimate holds, with $C(L,M,T)$ instead of $c(L,M,T)$ and $J$ the Lipschitz constant of $z\to h(e^z)$.*

*Proof.* We have that

$$\begin{aligned}\mathbb{E}|h(S(T))-h(\tilde{S}(T))|^2 &\le J^2\mathbb{E}\|Z(\cdot)-\tilde{Z}(\cdot)\|^2_{C^0}\\ &\le J^2\mathbb{E}\left[\sup_{0\le t\le T}(|Z(t)-\tilde{Z}(t)|^2+|P(t)-\tilde{P}(t)|^2)\right]\end{aligned}$$

and from Theorem 4.1 we obtain Equation (4.1). □

We can see that the difference between the processes $\Sigma$ and $\tilde{\Sigma}$ depends on the difference between the initial conditions $P(0)$ and $\tilde{P}(0)$. Unfortunately, we cannot obtain any improvement on the coefficients $c(L,M,T)$ or $C(L,M,T)$ in the case $\sigma=\sigma(P)$.

*Remark* 4.3. Notice that in Corollary 4.2 the function $z\to h(e^z)$ is required to be globally Lipschitz, so a little caution must be used. For example, if the function $h:\mathbb{R}\to\mathbb{R}$ is globally Lipschitz and piecewise $C^1$, then

$$\frac{\partial h(e^z)}{dz}=e^zh'(e^z)$$

is bounded (thus $z\to h(e^z)$ is globally Lipschitz) if and only if $h'$ decreases faster than $e^z$.

Consider now some examples.

*Example* (European put). The payoff is $h(s) = (K - s)^+$. We have

$$\frac{\partial h(e^z)}{\partial z} = -e^z \mathbf{I}_{z<\log K}$$

then the Lipschitz constant in this case is less or equal than

$$I = \sup_z \left| \frac{\partial h(s)}{\partial s} \right| = e^{\log K} = K$$

*Example* (European call). The payoff is now given by $h(s) = (s - K)^+$. We can write this expression as $h(s) = s - K - (K - s)^+$, so the error is the same as when pricing the put.

*Example* (Asian put). The payoff is now given by $h(s(\cdot)) = (K - \int_0^T s(t)dt)^+$. For two generic paths $z, \bar{z} \in C^0([0,T])$, if both $\int_0^T e^{z(u)}\, du, \int_0^T e^{\bar{z}(u)}\, du$ are less than $K$, then

$$\begin{aligned}
\left| h(e^{z(\cdot)}) - h(e^{\bar{z}(\cdot)}) \right| &= \left| \left( K - \int_0^T e^{z(u)}\, du \right)^+ - \left( K - \int_0^T e^{\bar{z}(u)}\, du \right)^+ \right| \\
&\leq \left| \int_0^T e^{z(u)}\, du - \int_0^T e^{\bar{z}(u)}\, du \right| \\
&\leq \left| \int_0^T |z(u) - \bar{z}(u)| e^{\sup(z(u),\bar{z}(u))}\, du \right| \\
&\leq \|z - \bar{z}\|_{C^0} \left| \int_0^T e^{\sup(z(u),\bar{z}(u))}\, du \right| \leq 2K \|z - \bar{z}\|_{C^0} \quad (4.2)
\end{aligned}$$

where in the last line we applied the inequality

$$\int_0^T e^{\sup(z(u),\bar{z}(u))}\, du = \int_{\{u: z(u) > \bar{z}(u)\}} e^{z(u)}\, du + \int_{\{u: \bar{z}(u) > z(u)\}} e^{\bar{z}(u)}\, du \leq 2K$$

If (say) $\int_0^T e^{\bar{z}(u)}\, du > K$ and $\int_0^T e^{z(u)}\, du \leq K$, then we can choose $\tilde{z} \in C^0$ such that $\int_0^T e^{\tilde{z}(u)}\, du = K$ and $\|\tilde{z} - z\|_{C^0} \leq \|\bar{z} - z\|_{C^0}$ (for example, $\tilde{z} := tz + (1-t)\bar{z}$ for a suitable $t \in (0,1)$). Then

$$\begin{aligned}
\left| h(e^{z(\cdot)}) - h(e^{\bar{z}(\cdot)}) \right| &= K - \int_0^T e^{z(u)}\, du = \left| \int_0^T e^{\tilde{z}(u)}\, du - \int_0^T e^{z(u)}\, du \right| \\
&\leq 2K \|\tilde{z} - z\|_{C^0} \leq 2K \|\bar{z} - z\|_{C^0}
\end{aligned}$$

by Equation (4.2). If both $\int_0^T e^{z(u)}\, du, \int_0^T e^{\bar{z}(u)}\, du$ are greater than $K$, there is nothing to prove. Then the Lipschitz constant in this case is equal to $2K$.

*Example* (Lookback put). The payoff is now given by

$$h(s(\cdot)) = \left( K - \max_{0 \leq t \leq T} s(t) \right)^+ .$$

As above, if both $\max e^{z(\cdot)}$, $\max e^{\bar{z}(\cdot)}$ are less than $K$, then we calculate

$$\begin{aligned} \left| h(e^{z(\cdot)}) - h(e^{\bar{z}(\cdot)}) \right| &\leq \left| \max_{0\leq u\leq T} e^{z(u)} - \max_{0\leq u\leq T} e^{\bar{z}(u)} \right| \leq \|e^{z(\cdot)} - e^{\bar{z}(\cdot)}\|_{C^0} \\ &\leq \|z-\bar{z}\|_{C^0} \|e^{\max(z,\bar{z})}\|_{C^0} \leq K\|z-\bar{z}\|_{C^0} \end{aligned}$$

If at least one of the quantities $\max e^{z(\cdot)}$, $\max e^{\bar{z}(\cdot)}$ is greater than $K$, an argument similar to the one of the previous example applies. Thus, in this case the Lipschitz constant is equal to $K$.

## 5. Using past information

We have seen in Section 4 that the error in pricing derivative assets depends on the difference between the true offset function $P(0)$ and the misspecified value $\tilde{P}(0)$, which we can choose. Of course, our aim will be to choose it in order to minimise the final error. In doing this, we are entitled to use not only the current value of $S(0)$, but also past values.

More in detail, we assume (as it is reasonable) that we know all the past values of the price $S(t)$ (thus, of $Z(t)$) for $t \in [-R, 0]$, where $R > 0$ is a given real number which represents the width of an observation window in the past. As before, the process $P(t)$ remains unobserved also in the past. However, it turns out that we can make the uncertainty on $P$ decay exponentially with respect to the width $R$ of the observation window. Again, we represent this uncertainty by defining the process $\tilde{P}$, starting from the misspecified condition $\tilde{P}(-R)$ and following the dynamics

$$\begin{aligned} d\tilde{P}(t) &= -\lambda\tilde{P}(t)\ dt + dZ(t), \quad t \in (-R, 0] & (5.1)\\ \tilde{P}(-R) &\neq P(-R) & (5.2) \end{aligned}$$

while the process $P$ always follows the dynamics given by Equation (2.2). Notice that this time, as we can observe $Z$ in the interval $[-R, 0]$, we have no uncertainty on this process.

The following lemma shows that, as both the dynamics of $\tilde{P}$ and $P$ depend on the known values of $Z$, the difference between $P(0)$ and $\tilde{P}(0)$ decays exponentially with respect to the width $R$, as announced.

**Lemma 5.1.** *For every choice of $\tilde{P}(-R)$, we have*

$$|P(0) - \tilde{P}(0)| = e^{-\lambda R}|P(-R) - \tilde{P}(-R)| \tag{5.3}$$

*Proof.* By calculating the Itô differential of the process $(e^{\lambda t}P(t))_t$, we have

$$\begin{aligned} d(e^{\lambda t}P(t)) &= e^{\lambda t}\ dP(t) + \lambda e^{\lambda t}P(t)\ dt \\ &= e^{\lambda t}(dZ(t) - \lambda P(t)\ dt) + \lambda e^{\lambda t}P(t)\ dt = e^{\lambda t}\ dZ(t) \end{aligned}$$

and, analogously,

$$de^{\lambda t}\tilde{P}(t) = e^{\lambda t}\ dZ(t)$$

This means that, calculating the two processes in the two points $t = -R, 0$, we have

$$\begin{aligned} P(0) &= e^{-\lambda R} P(-R) + \int_{-R}^{0} e^{\lambda t}\, dZ(t), \\ \tilde{P}(0) &= e^{-\lambda R} \tilde{P}(-R) + \int_{-R}^{0} e^{\lambda t}\, dZ(t). \end{aligned}$$

The lemma follows by calculating the difference. □

*Remark* 5.2. Notice that Equation (5.1) entails

$$\begin{aligned} \tilde{P}(0) &= e^{-\lambda R} \tilde{P}(-R) + Z(0) - e^{-\lambda R} Z(-R) - \int_{-R}^{0} \lambda e^{\lambda t} Z(t)\, dt \\ &= \int_{0}^{R} \lambda e^{-\lambda u} (Z(0) - Z(-u))\, du + e^{-\lambda R} (Z(0) - Z(-R) + \tilde{P}(-R)) \end{aligned}$$

This can be seen by the properties of stochastic integrals of deterministic functions, or directly from Equation (2.1) (which obviously extends to $\tilde{P}$).

Now we are in the position of solving the following problem: for a given $\varepsilon > 0$ we want to find a minimum observation time $R$ such that the error when pricing a contingent claim $h$ is less than $\varepsilon$.

**Corollary 5.3.** *If $h$ is a general path-dependent claim as in Corollary* 4.2 *and*

$$R > \frac{\log(\frac{3J^2 E|P(-R) - \tilde{P}(-R)|^2}{\varepsilon^2}) + c(L, M, T)T}{2\lambda} \tag{5.4}$$

*then*

$$|E[h(S_T)] - E[h(\tilde{S}_T)]| < \varepsilon \tag{5.5}$$

*Moreover, if $h(S(T))$ is the payoff of a simple European claim, then to obtain the same estimate it is sufficient that*

$$R > \frac{\log(\frac{3I^2 E|P(-R) - \tilde{P}(-R)|^2}{\varepsilon^2}) + C(L, M, T)T}{2\lambda}$$

*Proof.* From (5.4) we have

$$2\lambda R > \log\Big(\frac{3J^2 \mathbb{E}|P(-R) - \tilde{P}(-R)|^2)}{\varepsilon^2}\Big) + c(L, M, T)T$$

that yields

$$[c(L, M, T)T - 2\lambda R] + \log(3J^2 \mathbb{E}|P(-R) - \tilde{P}(-R)|^2) < \log \varepsilon^2$$

By taking the exponential of both the members we obtain

$$3J^2 \mathbb{E}|P(-R) - \tilde{P}(-R)|^2 e^{c(L,M,T)T - 2\lambda R} < \varepsilon^2$$

From (4.1) and (5.3) we have

$$|\mathbb{E}[h(Z_T)] - \mathbb{E}[h(\tilde{Z}_T)]|^2 \leq 3J^2 \mathbb{E}|P(-R) - \tilde{P}(-R)|^2 e^{C(L,M,T)T - 2\lambda R} \tag{5.6}$$

this implies that (5.5) is verified. For the case of a European claim, the proof is the same with $c(L, M, T)$ instead of $C(L, M, T)$. □

## 6. Stationarity

So far, we have seen that the problem of estimating the pricing error when we misspecify the offset function $\tilde{P}$ is led to the knowledge of $\mathbb{E}[|P(-R) - \tilde{P}(-R)|^2]$, which is in general not allowed as we do not know the initial distribution of $P(-R)$, even if we can decide the value $\tilde{P}(-R)$.

The situation can be simplified by much if we make the crucial assumption that the 2-dimensional process $(P, Z)$ is stationary, or that the process $P$ itself is stationary. In this case, if we want the error to be (for example) less than a given $\varepsilon > 0$, it is sufficient to fix $\tilde{P}(-R)$ be equal to the mean of the invariant measure of $P$ (this minimises the quantity $\mathbb{E}[|P(-R) - \tilde{P}(-R)|^2]$, which is thus equal to the variance of $P(-R)$) and to observe the risky asset in the past for a sufficiently long time $R$. In fact, if the process $P$ is stationary and admits a unique invariant measure, under suitable assumptions the marginal distribution of $P(t)$ converges, for $t \to +\infty$, to the invariant measure, regardless of the initial condition of $P$. This means that, if we assume that the process $P$ started in the past at a time $T << -R$ from an arbitrary initial condition, the distribution of $P(-R)$ can be approximated very well by the invariant measure. Thus, the situation of finding $\mathbb{E}[|P(-R) - \tilde{P}(-R)|^2]$ boils down to finding the variance of the invariant measure for $P$, provided we let $\tilde{P}(-R)$ be equal to the mean of the invariant measure.

While the general case when the volatility $\sigma$ depends on both $P$ and $Z$ seems more difficult to analyse, much can be said in the case when $\sigma$ depends only on $P$. In this case the process $P$ is a Markov process with the following evolution

$$dP(t) = m(P(t))\ dt + \sigma(P(t))\ dW(t) \tag{6.1}$$

where $m(x) = -\frac{1}{2}\sigma^2(x) - \lambda x$. We now give sufficient conditions for the existence and uniqueness for the invariant distribution. For this purpose we use the following theorem from [11], that gives condition for the existence of the invariant measure.

**Theorem 6.1.** *Assume that there exists a function $V \in C^2(\mathbb{R})$ such that*

$$V(x) \geq 0, \qquad \sup_{|x|>R} LV(x) := -A_R \to -\infty \quad \textit{as} \quad R \to \infty$$

*where $LV(x) := m(x)V'(x) + \frac{1}{2}\sigma^2(x)V''(x)$ and $R$ is arbitrary. Then there exists a solution of Equation* (6.1) *which is a stationary Markov process.*

Take $V(x) = x^2$, then

$$LV(x) = \left(-\frac{1}{2}\sigma^2(x) - \lambda x\right) x + \frac{1}{2}\sigma^2(x) = \frac{1}{2}(1-x)\sigma^2(x) - \lambda x^2$$

Now if we assume

$$\sigma^2(x) \leq a|x| + b \tag{6.2}$$

it follows that

$$\begin{aligned} LV(x) &\geq -\frac{1}{2}(x-1)(a|x|+b) - \lambda x^2 = \\ &= -\frac{1}{2}ax|x| - \frac{1}{2}bx + \frac{1}{2}(a|x|+b) - \lambda x^2 \end{aligned} \tag{6.3}$$

If $x > 0$ then $LV(x) \to -\infty$ when $R \to \infty$. If $x < 0$ then

$$LV(x) \geq \left(\frac{1}{2}a - \lambda\right)x^2 - \frac{1}{2}bx + \frac{1}{2}(a|x|+b) \to -\infty$$

if $a < 2\lambda$. We can thus conclude with the following result.

**Theorem 6.2.** *If Assumption* (6.2) *holds with* $a < 2\lambda$*, there exists an invariant measure for the process* (6.1).

In order to obtain also uniqueness results, we will need additional assumptions. If the process $P$ has an invariant probability with density $\mu(x)$, from the backward Kolmogorov equation we have

$$\begin{aligned} 0 &= -\frac{d[m(x)\mu(x)]}{dx} + \frac{1}{2}\frac{d^2[\sigma^2(x)\mu(x)]}{dx^2} \\ 0 &= \frac{d}{dx}\left[-m(x)\mu(x) + \frac{1}{2}\frac{d\sigma^2(x)\mu(x)}{dx}\right] \end{aligned} \tag{6.4}$$

this implies that

$$\frac{1}{2}\frac{d\sigma^2(x)\mu(x)}{dx} = m(x)\mu(x) + c$$

Assume that $c = 0$ and $y(x) = \sigma^2(x)\mu(x)$: then we have

$$\begin{aligned} \int \frac{dy}{y} &= \int \frac{2m(x)}{\sigma^2(x)}dx \\ \ln y &= \int_{x_0}^{x} \frac{2m(u)}{\sigma^2(u)}du + \ln C \\ y(x) &= Ce^{\int_{x_0}^{x} \frac{2m(u)}{\sigma^2(u)}du} \end{aligned} \tag{6.5}$$

where $C$ is an arbitrary constant and $x_0$ is an arbitrary point. If the following relation

$$\mu(x) = C\frac{e^{G(x)}}{\sigma^2(x)} \tag{6.6}$$

where $G(x) = \int_{x_0}^{x} \frac{2m(u)}{\sigma^2(u)}du$, gives a density, this is the invariant density for our process $P$.

Now we study the conditions for existence and uniqueness of the invariant measure for the process $P$ when $\sigma$ satisfies the following assumption:

*Assumption* 6.3. There exist $a \in [0, 2\lambda)$, $b, \varepsilon > 0$ such that

$$\varepsilon \leq \sigma^2(x) \leq a|x| + b$$

**Theorem 6.4.** *If $\sigma$ satisfies Assumption* (6.3)*, then there exists an unique invariant measure for $P$, with density given by* (6.6)*. Moreover, if $P^{-T,\eta}$ follows the dynamics* (6.1) *with initial condition $P^{-T,\eta}(-T) = \eta$ with $-T < -R$, then for every initial distribution $\eta$ and $E \in \mathbb{R}$, we have*

$$\lim_{T\to\infty} \mathbb{E}[(P^{-T,\eta}(-R) - E)^2] = \int_{\mathbb{R}} (x - E)^2 \mu(x)\, dx$$

*Proof.* By results contained in [11], it is sufficient to prove that

$$\int_{-\infty}^{\infty} \frac{e^{G(x)}}{\sigma^2(x)}\, dx < \infty$$

and that

$$\int_{-\infty}^{0} \frac{e^{-G(x)}}{\sigma^2(x)}\, dx = \int_{0}^{\infty} \frac{e^{-G(x)}}{\sigma^2(x)}\, dx = +\infty$$

where

$$G(x) = \int_0^x \left(-1 - \frac{2\lambda u}{\sigma^2(u)}\right) du = -x - \int_0^x \frac{2\lambda u}{\sigma^2(u)}\, du$$

If $x \geq 0$,

$$\begin{aligned} G(x) &\leq -x - \frac{2\lambda}{a}\int_0^x \frac{au + b - b}{au + b} du + C = -x - \frac{2\lambda}{a}x + \frac{2\lambda}{a^2}\int_0^x \frac{-b}{au+b} du + C \\ &= -x - \frac{2\lambda}{a}x + \frac{2\lambda b}{a^2}\ln(ax+b) + C_1 =: n_1(x) \end{aligned}$$

If $x < 0$,

$$G(x) \leq -x - 2\lambda \int_0^x \frac{u}{\varepsilon}\, du = -x - \frac{\lambda}{\varepsilon}x^2 =: n_2(x)$$

where as usual $C$, $C_1$, are some constants. Then $e^{G(x)} \leq e^{n_1(x)}$ if $x \geq 0$ and $e^{G(x)} \leq e^{n_2(x)}$ if $x < 0$. So, we can write

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{e^{G(x)}}{\sigma^2(x)} dx \leq \int_{-\infty}^{\infty} \frac{e^{G(x)}}{\varepsilon} dx \\ &\leq K_1 \int_{-\infty}^{0} \frac{e^{-x-\frac{\lambda}{\varepsilon}x^2}}{\varepsilon}\, dx + K_2 \int_0^{+\infty} \frac{e^{-x(1+\frac{2\lambda}{a})}(ax+b)^{\frac{2\lambda b}{a^2}}}{\varepsilon} dx < +\infty \end{aligned}$$

where $K_1$ and $K_2$ are constants. Besides,

$$\begin{aligned} \int_{-\infty}^{0} e^{-G(x)}\, dx &\geq \int_{-\infty}^{0} e^{-n_2(x)}\, dx = K_1 \int_{-\infty}^{0} e^{x+\frac{\lambda}{\varepsilon}x^2}\, dx = +\infty, \\ \int_0^{\infty} e^{-G(x)}\, dx &\geq \int_0^{+\infty} e^{-n_1(x)}\, dx = K_2 \int_0^{+\infty} e^{x(1+\frac{2\lambda}{a})}(ax+b)^{-\frac{2\lambda b}{a^2}}\, dx = +\infty \end{aligned}$$

□

## 7. Some examples

Now we analyse some particular specifications for $\sigma$. The first two are present in the original Hobson-Rogers paper and in other related papers (see [12]), while the third is suggested by the fact that affine processes are very often used in Mathematical Finance, and they have a well-established theory.

### 7.1. The case $\sigma(P) = \min\{\sqrt{a + bP^2}, N\}$

This example comes from the original Hobson-Rogers paper [12]:

$$\sigma(P) = \min\{\sqrt{a + bP^2}, N\} \tag{7.1}$$

where $a > 0$, $b > 0$ and $N > 0$ are some constants. As $\sigma$ satisfies Assumption 6.3 for each possible value of $a, b, N > 0$, we can calculate the function $G(x)$:

$$G(x) = -(x - x_0) - \int_{x_0}^{x} \frac{2\lambda u}{\sigma^2(u)} du$$

When $x < \sqrt{\frac{N^2-a}{b}}$ the function $G(x)$ becomes

$$G(x) = -(x - x_0) - \int_{x_0}^{x} \frac{2\lambda u}{N^2}\, du = (x_0 - x) - \frac{2\lambda}{N^2}\Big(\frac{x^2}{2} - \frac{x_0^2}{2}\Big) = -x - \frac{\lambda}{N^2}x^2 + L_1$$

where $L_1$ is a constant. In this case the function $\mu(x)$ is equal to

$$\mu(x) = C\frac{e^{G(x)}}{N^2} = K\frac{e^{-\frac{\lambda}{N^2}x^2 - x}}{N^2} = K_1\frac{e^{-\frac{\lambda}{N^2}(x + \frac{N^2}{2\lambda})^2}}{N^2}$$

where $K$ and $K_1$ are constants. When $x \in \left[-\sqrt{\frac{N^2-a}{b}}, +\sqrt{\frac{N^2-a}{b}}\right]$, the function $G(x)$ is

$$G(x) = -(x - x_0) - \int_{-\sqrt{\frac{N^2-a}{b}}}^{x} \frac{2\lambda u}{a + bu^2} du = -x - \frac{\lambda}{b}\ln(a + bx^2) + L_2$$

where $L_2$ is a constant. The function $\mu(x)$ is equal to

$$\mu(x) = K_2\frac{e^{-x}(a + bx^2)^{-\frac{\lambda}{b}}}{a + bx^2} = K_2 e^{-x}(a + bx^2)^{-\frac{\lambda}{b}-1}$$

where $K_2$ is a constant. Now we see the case when $x > \sqrt{\frac{N^2-a}{b}}$. In this case the function $G(x)$ is

$$G(x) = -(x - x_0) - \int_{\sqrt{\frac{N^2-a}{b}}}^{x} \frac{2\lambda u}{N^2} du = -x - \frac{\lambda}{N^2}x^2 + L_3$$

for some constant $L_3$. Then

$$\mu(x) = K_3\frac{e^{-\frac{\lambda}{N^2}(x + \frac{N^2}{2\lambda})^2}}{N^2}$$

where $K_3$ is a constant. The function $\mu(x)$ must be continuous at the points $x_1 = -\sqrt{\frac{N^2-a}{b}}$ and $x_2 = \sqrt{\frac{N^2-a}{b}}$, so that at this point we have

$$\lim_{x\to x_1^-} \mu(x) = \lim_{x\to x_1^+} \mu(x), \qquad \text{and } \lim_{x\to x_2^-} \mu(x) = \lim_{x\to x_2^+} \mu(x)$$

that implies

$$K_2 = K_1 e^{-\frac{\lambda(N^2-a)}{bN^2} - \frac{N^2}{4\lambda}} N^{\frac{2\lambda}{b}}, \quad K_3 = K_1$$

In conclusion, the invariant density is

$$\mu(x) = \begin{cases} K_1 e^{-\frac{\lambda(N^2-a)}{bN^2} - \frac{N^2}{4\lambda}} N^{\frac{2\lambda}{b}} e^{-x}(a+bx^2)^{-\frac{\lambda}{b}-1} & |x| \le \sqrt{\frac{N^2-a}{b}} \\ K_1 \frac{e^{-\frac{\lambda}{N^2}(x+\frac{N^2}{2\lambda})^2}}{N^2} & |x| \ge \sqrt{\frac{N^2-a}{b}} \end{cases}$$

For the mean and the covariance of the process $P$ under the invariant measure, there is not an explicit form. For this reason, a numerical calculation is required.

*Example.* As in [9], we take

$$a = 0.04, \quad b = 0.2, \quad \lambda = 1, \quad N = 1$$

so we have

$$L = \sup_{x\in\mathbb{R}} \left| \frac{\partial \sigma}{\partial x} \right| = \frac{\sqrt{b(N^2-a)}}{N} = 0.438178$$

and

$$M = \sup_{x\in\mathbb{R}} \left| \frac{\partial \sigma^2}{\partial x} \right| = 2\sqrt{b(N^2-a)} = 0.876356$$

then we have

$$\mathbb{E}[P] = -0.022293, \qquad \mathrm{Var}[P] = 0.022437$$

We want to find $R$ such that (5.5) is verified for $\varepsilon = 10^{-2}$. If $J = 1$ (as is often the case), by taking different maturities, we find these results both for a general path-dependent claim as for a European one:

| | path-dependent claim | | European claim | |
|---|---|---|---|---|
| $T$ | $c(L, M, T)$ | $R$ | $c(L, M, T)$ | $R$ |
| 0.25 | 5.724000 | 3.971457 | 2.844000 | 3.611457 |
| 0.5 | 7.608000 | 5.157957 | 4.728000 | 4.437957 |
| 1.0 | 11.376000 | 8.943957 | 8.496000 | 7.503957 |
| 2.0 | 18.912000 | 22.167957 | 16.032000 | 19.287957 |
| 3.0 | 26.448000 | 42.927957 | 23.568000 | 38.607957 |
| 4.0 | 33.984000 | 71.223957 | 31.104000 | 65.463957 |
| 5.0 | 41.520000 | 107.055957 | 38.640000 | 99.855957 |

In this case, if we want to make an error of less than $\varepsilon = 10^{-2}$ in pricing (for example) a 6-months contingent claim, we have to observe the underlying asset for at least 5.15 years in the case of a path-dependent contingent claim and at least 4.43 years in the case of a European contingent claim.

Of course the situation can change, depending on the parameters. Take for example (always from [9])

$$a = 0.49, \quad b = 2.45, \quad \lambda = 1, \quad N = 2.236068$$

Now we have

$$L = \sup_{x\in\mathbb{R}} \left| \frac{\partial \sigma}{\partial x} \right| = 1.486573, \qquad M = \sup_{x\in\mathbb{R}} \left| \frac{\partial \sigma^2}{\partial x} \right| = 6.648158$$

and

$$\mathbb{E}[P] = 1.281530, \qquad \mathrm{Var}[P] = 2.674600$$

If again we want to find $R$ such that (5.5) is verified for $\varepsilon = 10^{-2}$ and $J = 1$, this time we find these results both for a general path-dependent claim as for a European one:

| | path-dependent claim | | European claim | |
|---|---|---|---|---|
| $T$ | $c(L,M,T)$ | $R$ | $c(L,M,T)$ | $R$ |
| 0.25 | 67.797000 | 14.121001 | 34.648500 | 9.977439 |
| 0.5 | 91.396000 | 28.495376 | 58.247500 | 20.208251 |
| 1.0 | 138.594000 | 74.943376 | 105.445500 | 58.369126 |
| 2.0 | 232.990000 | 238.636376 | 199.841500 | 205.487876 |
| 3.0 | 327.386000 | 496.725376 | 294.237500 | 447.002626 |
| 4.0 | 421.782000 | 849.210376 | 388.633500 | 782.913376 |
| 5.0 | 516.178000 | 1296.091376 | 483.029500 | 1213.220126 |

In this case, if we want to make an error of less than $\varepsilon = 10^{-2}$ in pricing (for example) a 6-months contingent claim, we have to observe the underlying asset for at least 28.49 years in the case of a path-dependent contingent claim and at least 20.20 years in the case of a European contingent claim.

**7.2. The case $\sigma^2(P) = \frac{a+bP^2}{c+d'P^2}$**

Consider $\sigma$ of the form

$$\sigma^2(P) = \frac{a + bP^2}{c + d'P^2}$$

where $a$, $b$, $c$, $d'$ are some positive numbers. As $\sigma$ satisfies Assumption 6.3 for each possible value of $a, b, c, d' > 0$, as in the previous section we calculate the function $G$:

$$\begin{aligned} G(x) &= -(x - x_0) - 2\lambda \int_{x_0}^{x} \frac{(c + d'u^2)u}{a + bu^2} du = -x - \lambda \int_{x_0^2}^{x^2} \frac{c + d'u^2}{a + bu^2} du^2 + c_0 \\ &= -x - \frac{\lambda c}{b} \int_{bx_0^2+a}^{bx^2+a} \frac{1}{y} dy - \frac{\lambda d'}{b^2} \int_{bx_0^2+a}^{bx^2+a} \frac{a + bu^2 - a}{a + bu^2} d(bu^2 + a) + c_1 \\ &= -x - \frac{\lambda c}{b} \ln(bx^2) - \frac{\lambda d'}{b^2}(bx^2 + a) + \frac{\lambda d' a}{b^2} \ln(bx^2 + a) + c_2 \\ &= -x - \frac{\lambda(bc - ad')}{b^2} \ln(bx^2 + a) - \frac{\lambda d'}{b^2}(bx^2 + a) + c_2 \end{aligned} \tag{7.2}$$

The function $\mu$ is

$$\begin{aligned}\mu(x) &= C\frac{e^{G(x)}}{\sigma^2(x)} = C\frac{e^{-x}(bx^2+a)^{-\frac{\lambda}{b^2}(bc-ad')}e^{-\frac{\lambda d'}{b^2}(bx^2+a)+c_1}}{\frac{a+bx^2}{c+d'x^2}} \\ &= K\frac{e^{-\frac{\lambda d'}{b}(x+\frac{b}{2\lambda d'})^2}(bx^2+a)^{-\frac{\lambda}{b^2}(bc-ad')-1}}{c+d'x^2} \qquad (7.3)\end{aligned}$$

and it is the density of the unique invariant measure of the process $P$. Also in this case, we cannot calculate explicitly the mean and the variance of the process $P$, so a numerical integration is again required.

*Example.* We take

$$a = 0.452, \quad b = 3.012, \quad c = 1.0, \quad d' = 0.261, \quad \lambda = 1.02$$

We calculate the Lipschitz constants $L$ and $M$ for the functions $\sigma$ and $\sigma^2$. We have

$$L = \sup_{x\in\mathbb{R}}\left|\frac{\partial\sigma}{\partial x}\right| = 1.22302 \quad \text{and} \quad M = \sup_{x\in\mathbb{R}}\left|\frac{\partial\sigma^2}{\partial x}\right| = 3.67938$$

In fact, denote

$$k(x) := \frac{\partial\sigma(x)}{\partial x} = \frac{(bc-ad')x}{(c+d'x^2)^{\frac{3}{2}}\sqrt{a+bx^2}}$$

which reaches its maximum for $x = \pm\sqrt{\frac{-d'a+\sqrt{d'^2a^2+4abcd'}}{4d'b}}$. Then

$$L = \left|k\Big(\pm\sqrt{\frac{-d'a+\sqrt{d'^2a^2+4abcd'}}{4d'b}}\Big)\right| = 1.22302$$

Similarly, let us denote

$$g(x) := \frac{\partial\sigma^2}{\partial x} = \frac{2(bc-ad')x}{(c+d'x^2)^2}$$

which reaches its maximum for $x = \pm\sqrt{\frac{c}{3d'}}$. Then

$$M = \left|g\Big(\pm\sqrt{\frac{c}{3d'}}\Big)\right| = \frac{2|bc-ad'|\sqrt{\frac{c}{3d'}}}{(c+d'\frac{c}{3d'})^2} = 3.67938$$

We obtain

$$\mathbb{E}[P] = -0.324053, \qquad \text{Var}[P] = 0.612203$$

and we have these results respectively for a path-dependent and for a European contingent claim:

| | path-dependent claim | | European claim | |
|---|---|---|---|---|
| $T$ | $c(L, M, T)$ | $R$ | $c(L, M, T)$ | $R$ |
| 0.25 | 38.245077 | 9.499770 | 15.808408 | 6.750178 |
| 0.5 | 46.574596 | 16.228215 | 24.137927 | 10.729032 |
| 1.0 | 63.233633 | 35.809752 | 40.796964 | 24.811385 |
| 2.0 | 96.551707 | 99.471410 | 74.115038 | 77.474675 |
| 3.0 | 129.869782 | 195.797846 | 107.433113 | 162.802745 |
| 4.0 | 163.187856 | 324.789061 | 140.751187 | 280.795593 |
| 5.0 | 196.505930 | 486.445055 | 174.069261 | 431.453220 |

**7.3. The case $\sigma^2(P) = a + bP$**

Suppose that the process $P$ is a so-called affine process [5], i.e. $\sigma$ is given by

$$\sigma^2(P) = a + bP \tag{7.4}$$

where $a$ and $b$ are two arbitrary constants. So, Equation (6.1) becomes:

$$dP(t) = \left(-\left(\frac{b}{2} + \lambda\right) P(t) - \frac{a}{2}\right)\, dt + \sqrt{a + bP(t)}dB(t) \tag{7.5}$$

Clearly, there is a solution to (7.5) when the process $a + bP(t)$ is non-negative for all $t$. So, the domain $D$ implied by the non-negativity is

$$D = \{x \in R : a + bx > 0\}$$

We will therefore need to assume, in effect, that the process $a + bP(t)$ has a sufficiently strong positive drift on the boundary point $x = -\frac{a}{b}$. Under the following assumption, we have a unique (strong) solution for the stochastic equation (7.5).

*Assumption* 7.1. We assume that $2\lambda a > b^2$.

In fact, for $x$ such that $a + bx = 0, b[-(\frac{1}{2}b + \lambda)x - \frac{1}{2}a] > \frac{b^2}{2}$, i.e. equivalently $(1 + \frac{2\lambda}{b})x + (1 + \frac{a}{b}) < 0$, this implies $2\lambda a > b^2$. See [5].

**Theorem 7.2.** *Under Assumption* (7.1)*, there is a unique (strong) solution $P$ to the stochastic differential equation* (7.5) *in the domain $D$. Moreover, for this solution $P$, we have $a + bP(t) > 0$ for all $t$ almost surely.*

Since $\sigma$ is not Lipschitz, we cannot apply Theorem 4.1, but we have to formulate an analogous result here.

**Theorem 7.3.** *If the coefficient $\sigma$ satisfies* (7.4)*, then for $t \in [0, T]$ we have*

$$\mathbb{E}\left[\sup_{0\le u\le t} |\Sigma(u) - \tilde{\Sigma}(u)|^2\right] \le \left[3\mathbb{E}|P(0) - \tilde{P}(0)|^2 + \frac{10b^2}{\theta}t\right] e^{c(\theta,T)t} \tag{7.6}$$

*where $\theta$ is an arbitrary parameter and $c(\theta, T) = \left[3(\frac{b}{2}+\lambda)^2 + \frac{b^2}{2}\right] T + 10b^2\theta$, and*

$$\mathbb{E}|\Sigma(t) - \tilde{\Sigma}(t)|^2 \leq \left[3\mathbb{E}|P(0) - \tilde{P}(0)|^2 + \frac{5b^2}{2\theta'} t\right] e^{C(\theta', T)t} \tag{7.7}$$

*where $\theta'$ is an arbitrary parameter and $C(\theta', T) = \frac{b^2}{2}(T + 5\theta') + 3(\frac{b}{2}+\lambda)^2 T$.*

*Proof.* We have that

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq u\leq t} |\Sigma(u) - \tilde{\Sigma}(u)|^2\right] \\ \leq\ & \mathbb{E}\left[\sup_{0\leq u\leq t} |Z(u) - \tilde{Z}(u)|^2\right] + \mathbb{E}\left[\sup_{0\leq u\leq t} |P(u) - \tilde{P}(u)|^2\right] = (1) + (2) \end{aligned}$$

For the first term we have

$$\begin{aligned} (1) &= \mathbb{E}\left[\sup_{0\leq u\leq t}\left| -\frac{1}{2}\int_0^u b(P(s) - \tilde{P}(s))ds \right.\right. \\ &\quad \left.\left. + \int_0^u \left(\sqrt{a + bP(s)(s)} - \sqrt{a + b\tilde{P}(s)}\right) dW(s)\right|^2\right] \\ &\leq 2\mathbb{E}\left[\sup_{0\leq u\leq t}\left|\frac{1}{2}\int_0^u b(P(s) - \tilde{P}(s))\, ds\right|^2\right] \\ &\quad + 2\mathbb{E}\left[\sup_{0\leq u\leq t}\left|\int_0^u \left(\sqrt{a + bP(s)} - \sqrt{a + b\tilde{P}(s)}\right) dW(s)\right|^2\right] \\ &\leq \frac{b^2}{2} T \int_0^t \mathbb{E}|P(s) - \tilde{P}(s)|^2\, ds + 8b^2 \int_0^t \mathbb{E}|P(s) - \tilde{P}(s)|\, ds \\ &\leq \left(\frac{b^2}{2} T + 4b^2\theta\right) \int_0^t \mathbb{E}|P(s) - \tilde{P}(s)|^2\, ds + \frac{4b^2}{\theta} t. \end{aligned}$$

where in the third line we apply the inequality

$$\left|\sqrt{a + bP} - \sqrt{a + b\tilde{P}}\right| \leq b\sqrt{|P - \tilde{P}|}$$

and in the last line the inequality

$$|P - \tilde{P}| \leq \frac{\theta}{2}|P - \tilde{P}|^2 + \frac{1}{2\theta}$$

which holds for any real number $\theta > 0$. Then,

$$\begin{aligned}
(2) &= \mathbb{E}\left[\sup_{0\le u\le t} |P(0) - \tilde{P}(0) - \left(\frac{b}{2}+\lambda\right)\int_0^u (P-\tilde{P})ds \right.\\
&\quad \left. + \int_0^u \left(\sqrt{a+bP} - \sqrt{a+b\tilde{P}}\right)dW(s)|^2\right]\\
&\le 3\mathbb{E}|P(0)-\tilde{P}(0)|^2 + 3\left(\frac{b}{2}+\lambda\right)^2 T\int_0^t \mathbb{E}|P-\tilde{P}|^2 ds\\
&\quad +12\int_0^t \mathbb{E}|\sqrt{a+bP}-\sqrt{a+b\tilde{P}}|^2 ds\\
&\le 3\mathbb{E}|P(0)-\tilde{P}(0)|^2 + 3\left(\frac{b}{2}+\lambda\right)^2 T\int_0^t \mathbb{E}|P-\tilde{P}|^2\ ds\\
&\quad +6b^2\theta\int_0^t \mathbb{E}|P-\tilde{P}|^2 ds + \frac{6b^2}{\theta}t\\
&= 3\mathbb{E}|P(0)-\tilde{P}(0)|^2 + \left[3\left(\frac{b}{2}+\lambda\right)^2 T + 6b^2\theta\right]\int_0^t \mathbb{E}|P-\tilde{P}|^2 ds + \frac{6b^2}{\theta}t.
\end{aligned}$$

Then

$$\mathbb{E}\left[\sup_{0\le u\le t}|\Sigma(u)-\tilde{\Sigma}(u)|^2\right] \le 3\mathbb{E}|P(0)-\tilde{P}(0)|^2 + \frac{10b^2}{\theta}t + c(\theta,T)\int_0^t \mathbb{E}|\Sigma(s)-\tilde{\Sigma}(s)|^2 ds$$

Similarly as in Theorem 4.1, the result follows from Gronwall lemma applied to $v(t) = \mathbb{E}\left[\sup_{0\le u\le t}|\Sigma(u)-\tilde{\Sigma}(u)|^2\right]$. □

The parameters $\theta$ and $\theta'$ which minimize the right hand side of Equation (7.6) and Equation (7.7) are

$$\theta = \frac{-5b^2t + \sqrt{25b^4t^2 + 3\mathbb{E}|P(0)-\tilde{P}(0)|^2}}{3\mathbb{E}|P(0)-\tilde{P}(0)|^2}, \qquad \theta' = \frac{1}{4}\theta.$$

Now we calculate the function $G(x)$. The inequality $a+bx \ge 0$ is equivalent to $x \ge -\frac{a}{b}$ if $b>0$ and to $x \le -\frac{a}{b}$ if $b<0$. Consider the case $b>0$.

$$\begin{aligned}
G(x) &= -\left(x+\frac{a}{b}\right) - \int_{-\frac{a}{b}}^x \frac{2\lambda u}{a+bu}du\\
&= -\left(1+\frac{2\lambda}{b}\right)x + \frac{2\lambda a}{b^2}\ln(a+bx) - \frac{4\lambda a}{b^2} - \frac{a}{b}
\end{aligned}$$

So, the function $\mu(x)$ is

$$\mu(x) = C\frac{e^{G(x)}}{\sigma^2(x)} = K\frac{e^{-(1+\frac{2\lambda}{b})x}(a+bx)^{\frac{2\lambda a}{b^2}}}{a+bx}$$

where $K = Ce^{-\frac{4\lambda a}{b^2} - \frac{a}{b}}$ is constant. For $\mu(x)$ to be a density, the quantity

$$\int_{-\frac{a}{b}}^{\infty} e^{-(1+\frac{2\lambda}{b})x}(a+bx)^{\frac{2\lambda a}{b^2}-1}dx$$

must be finite. This is true if $(1 + \frac{2\lambda}{b}) > 0$ which is always true, and $\frac{2\lambda a}{b^2} - 1 > -1$ i.e. $a > 0$.

Now we analyze the case $b < 0$. In this case $x \leq -\frac{a}{b}$ then,

$$\begin{aligned} G(x) &= -(x - x_0) - \int_{x_0}^{x} \frac{2\lambda u}{a + bu}\, du \\ &= \Big(\frac{2\lambda}{b} - 1\Big)x - \frac{2\lambda a}{b^2} \ln(bx + a) + C_1 \end{aligned}$$

where in the first line we change the variable of integration to $y = a + bu$ and $C_1$ is some constant. Similarly, as in the case $b > 0$, the function $\mu(x)$ is a density if

$$\int_{-\infty}^{-\frac{a}{b}} e^{-(1-\frac{2\lambda}{b})x}(a+bx)^{-\frac{2\lambda a}{b^2}-1}dx$$

is finite. This is true when $(1 - \frac{2\lambda}{b}) < 0$ (is equivalently when $b > 2\lambda > 0$), and $-\frac{2\lambda a}{b^2} - 1 > -1$. But this is absurd because we supposed that $b < 0$. In conclusion,

$$\mu(x) = K' e^{-(1+\frac{2\lambda}{b})x}(a+bx)^{\frac{2\lambda a}{b^2}-1}$$

is an invariant density for our process $P$ in $(-\frac{a}{b}, +\infty)$ if and only if $a > 0$ and $b > 0$. In this case we can calculate the marginal mean and variance for the process $P$ under the invariant measure. For the mean we have that for all $t \in \mathbb{R}$,

$$\begin{aligned} E[P(t)] &= \int_{-\frac{a}{b}}^{\infty} xKe^{-(1+\frac{2\lambda}{b})x}(a+bx)^{\frac{2\lambda a}{b^2}-1}dx \\ &= \frac{1}{b}\int_{-\frac{a}{b}}^{\infty} K(bx+a-a)e^{-(1+\frac{2\lambda}{b})x}(a+bx)^{\frac{2\lambda a}{b^2}-1}dx \\ &= \frac{1}{b}\int_{-\frac{a}{b}}^{\infty} Ke^{-(1+\frac{2\lambda}{b})x}(a+bx)^{\frac{2\lambda a}{b^2}}dx - \frac{a}{b} \\ &= \frac{2\lambda a}{b(b+2\lambda)}\int_{-\frac{a}{b}}^{\infty} Ke^{-(1+\frac{2\lambda}{b})x}(a+bx)^{\frac{2\lambda a}{b^2}-1}dx - \frac{a}{b} \\ &= \frac{2\lambda a}{b(b+2\lambda)} - \frac{a}{b} = -\frac{a}{b+2\lambda} \end{aligned}$$

Now we calculate $E[P^2(t)]$:

$$\begin{aligned}
E[P^2(t)] &= \int_{-\frac{a}{b}}^{\infty} Kx^2 e^{-(1+\frac{2\lambda}{b^2})x}(a+bx)^{\frac{2\lambda a}{b}-1}dx \\
&= \frac{1}{b^2}\int_{-\frac{a}{b}}^{\infty} K(bx+a-a)^2 e^{-(1+\frac{2\lambda}{b^2})x}(a+bx)^{\frac{2\lambda a}{b}-1}dx \\
&= \frac{1}{b^2}\int_{-\frac{a}{b}}^{\infty} Ke^{-(1+\frac{2\lambda}{b^2})x}(a+bx)^{\frac{2\lambda a}{b}+1}dx \\
&\quad -\frac{2a}{b^2}\int_{-\frac{a}{b}}^{\infty} Ke^{-(1+\frac{2\lambda}{b^2})x}(a+bx)^{\frac{2\lambda a}{b}}dx + \frac{a^2}{b^2} \\
&= \frac{2\lambda a+b^2}{b^2(b+2\lambda)(1+\frac{2\lambda}{b})}\frac{2\lambda a}{b^2}b - \frac{4\lambda a^2}{b^2(b+2\lambda)} + \frac{a^2}{b^2} \\
&= \frac{a^2+2\lambda a}{(b+2\lambda)^2} = \frac{a(a+2\lambda)}{(b+2\lambda)^2}
\end{aligned}$$

So that, the variance of the invariant measure of the process $P$ is equal to

$$\mathrm{Var}[P(t)] = E[P^2(t)] - E[P(t)]^2 = \frac{2\lambda a}{(b+2\lambda)^2}$$

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