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Tangential Tallini sets in finite Grassmannians of lines

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Abstract

A Tallini set in a semilinear space is a set **B** of points, such that each line not contained in **B** intersects **B** in at most two points. In this paper, the following notion of a tangential Tallini set in the Grassmannian $\Gamma_{n,1,q}$, q odd, is investigated: a Tallini set is called tangential when it meets every ruled plane (i.e. the set of lines contained in a plane of $\mathrm{PG}(n,q)$) in either $q+1$ or q^2+q+1 elements. A Tallini set $Q_{\mathbf{B}}$ in $\mathrm{PG}(n,q)$ can be associated with each tangential Tallini set **B** in $\Gamma_{n,1,q}$. Each $\ell \in \mathbf{B}$ is a line of $\mathrm{PG}(n,q)$ intersecting $Q_{\mathbf{B}}$ in either 0, or 1, or $q+1$ points; when $n \neq 4$ and **B** is covered by $(n-2)$-dimensional projective subspaces of $\Gamma_{n,1,q}$ the first case does not occur. If **B** is a tangential Tallini set in $\Gamma_{n,1,q}$ covered by $(n-2)$-dimensional subspaces, any of which is in $\mathrm{PG}(n,q)$ the set of all lines through a point and in a hyperplane, then either $Q_{\mathbf{B}}$ is a quadric, and **B** is the set of all lines contained in, or tangent to, $Q_{\mathbf{B}}$, or **B** is a linear complex.

Keywords: Tallini set; Grassmannian; Quadric; Finite; Projective space

1. Introduction

1.1. Outline

In [10–13], Tallini developed the theory of *k*-sets in Grassmann manifolds as a natural extension of the combinatorial investigations of the finite projective spaces. This was the

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starting point for further work on sets of lines in particular positions with respect to quadrics, such as the secant lines and the self-conjugate lines, i.e. lines which are tangent to, or contained in, a quadric. If Q is a possibly singular quadric and π is a plane in the projective space $\mathrm{PG}(n, q)$, the self-conjugate lines of Q which are contained in π form either a pencil, or a dual conic, or the whole dual plane. This motivated us to carry out a general investigation on the sets of lines satisfying such property, and to call them tangential Tallini sets. They include the linear complexes. It is shown in Theorems 8 and 10 that any tangential Tallini set $\mathbf{B}$ is related to a Tallini set $Q_{\mathbf{B}}$ in $\mathrm{PG}(n, q)$. We characterize by means of a common property the set of self-conjugate lines of a quadric in $\mathrm{PG}(n, q)$, including singular quadrics, and the linear complexes: see Theorem 24. So, several results, given by de Resmini [5,6], Tallini [11] and Venezia [15,16], are unified, improved and generalized. Similar characterizations of different sets of lines related to a quadric are due to de Resmini, Ferri, and Tallini [4,7,10].

1.2. Notation

Let $\mathrm{PG}(n, q) = (\mathcal{P}, \mathcal{R})$, $n \geqslant 3$, be the n-dimensional projective space over the Galois field $\mathbb{F}_q$, q odd, and $\Gamma_{n,1,q} = (\mathcal{R}, \mathcal{F})$ the Grassmann space representing the lines of $\mathrm{PG}(n, q)$. Here, $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{F}$ are the sets of all points, all lines and all pencils in $\mathrm{PG}(n, q)$, respectively, a *pencil* being the set of all lines through a point in a plane. Since $\Gamma_{n,1,q}$ is a semilinear space, the elements of $\mathcal{R}$ and $\mathcal{F}$ are also called *g-points* and *g-lines*, respectively. In a similar way the *g-planes* and *g-subspaces*, that is, projective planes and subspaces contained in $\Gamma_{n,1,q}$, are defined. For background on semilinear spaces, also called partial line spaces, the reader is referred to [14].

If π is a plane of $\mathrm{PG}(n, q)$, denote by T_π the set of lines of π. Such a T_π is called a *ruled plane*, and is a g-plane. The set of all ruled planes will be denoted by $\mathcal{T}$. A *star of lines*, star for short, is the set S_A of all lines in $\mathrm{PG}(n, q)$ through a point A, the *center* of the star. We denote by $\mathcal{S}$ the set of all stars. A *d-dimensional star* is the set of all lines belonging to a common star and contained in a $(d + 1)$-dimensional subspace of $\mathrm{PG}(n, q)$. So, a star is a $(n - 1)$-dimensional star, a pencil is a one-dimensional star.

In this paper, $\mathbf{B}$ will always denote a *tangential Tallini set*, i.e. a set of g-points, such that

(i) for any $\varphi \in \mathcal{F}$, there holds $|\varphi \cap \mathbf{B}| \in \{0, 1, 2, q + 1\}$;
(ii) for any $T \in \mathcal{T}$, there holds $|T \cap \mathbf{B}| \in \{q + 1, q^2 + q + 1\}$.

So, for every $T \in \mathcal{T}$, either $T \subseteq \mathbf{B}$, or $T \cap \mathbf{B}$ is a g-line or a g-$(q + 1)$-arc, where a g–k-arc is a set of k g-points, no three of them collinear. An example of a set of g-points satisfying (i) and (ii) is given by the set of self-conjugate lines of a quadric in $\mathrm{PG}(n, q)$.

An element φ of $\mathcal{F}$ is called an *exterior*, *tangent* or *secant* g-line when $|\varphi \cap \mathbf{B}|$ is equal to 0, 1 or 2, respectively.

The number of points in an i-dimensional projective space is denoted by $\theta_i = (q^{i+1} - 1)/(q - 1)$, $i \in \mathbb{N}$. Let $\mathcal{S}_0$ be the set of all stars S such that either $S \subseteq \mathbf{B}$, or $S \cap \mathbf{B}$ is a g-prime of S; that is, $S \cap \mathbf{B}$ is a hyperplane of the projective space S. Let $Q_{\mathbf{B}}$ be the set of the centers of all stars in $\mathcal{S}_0$, and $V_{\mathbf{B}} \subseteq Q_{\mathbf{B}}$ the set of the centers of all stars which are contained in $\mathbf{B}$. Finally, let $\mathcal{T}^*$ be the set of all ruled planes which are contained in $\mathbf{B}$.

2. General properties of tangential Tallini sets

Proposition 1. *The cardinality of* **B** *is equal to*

$$\frac{\theta_n\theta_{n-1}}{\theta_2} + |\mathcal{T}^*|\,\frac{q^2}{\theta_{n-2}}. \tag{1}$$

Proof. Computing in two ways the pairs (ℓ, T) with $T \in \mathcal{T}$ and $\ell \in \mathbf{B} \cap T$ gives

$$|\mathbf{B}|\theta_{n-2} = |\mathcal{T}^*|\theta_2 + \left(\frac{\theta_n\theta_{n-1}\theta_{n-2}}{\theta_2\theta_1} - |\mathcal{T}^*|\right)\theta_1. \qquad \square$$

By Proposition 1, if $n \equiv 0, 2 \pmod 3$, then θ_{n-2} divides $|\mathcal{T}^*|$.

Proposition 2. *If $S \in \mathcal{S}$ and $\ell \in \mathcal{R} \setminus S$, then there exists precisely one $T \in \mathcal{T}$ such that $\ell \in T$ and $T \cap S \in \mathcal{F}$.*

Lemma 3. *Assume $n = 3$ and let $S \in \mathcal{S}$. Then one of the following holds*:

(a) $|S \cap \mathbf{B}| = 1$ *and, in this case,* $|\mathcal{T}^*| = q + 1$;
(b) $S \cap \mathbf{B}$ *is a g-line, and* $|\mathcal{T}^*| \in \{0, q + 1\}$; *also,* $|\mathcal{T}^*| = q + 1$ *if, and only if, the g-line* $S \cap \mathbf{B}$ *is contained in an element of* $\mathcal{T}^*$;
(c) $S \cap \mathbf{B}$ *is a g-*$(q + 1)$*-arc, and* $\mathcal{T}^* = \emptyset$;
(d) $S \cap \mathbf{B} = \varphi_1 \cup \varphi_2$, *where* φ_1 *and* φ_2 *are two distinct g-lines*; *in this case* $|\mathcal{T}^*| = q + 1$, *and each of both g-lines is a subset of an element of* $\mathcal{T}^*$;
(e) $S \subseteq \mathbf{B}$, *and* $\mathcal{T}^* \neq \emptyset$.

Proof. We apply Proposition 2 in order to compute the number of elements of **B**.

Case 1: $S \cap \mathbf{B}$ *is a g–k-arc* $(0 \leqslant k \leqslant q+1)$. For every $T \in \mathcal{T}$ having non-empty intersection with *S*, we have $|T \cap \mathbf{B}| = q + 1$. This gives the following properties.

1. On the $k(k-1)/2$ ruled planes which meet *S* in a secant g-line there are exactly $(q-1)k(k-1)/2$ g-points of $\mathbf{B} \setminus S$.
2. On the $k(q+2-k)$ ruled planes which meet *S* in a tangent g-line there are $qk(q+2-k)$ g-points of $\mathbf{B} \setminus S$.
3. On the $\theta_2 - k(k-1)/2 - k(q+2-k)$ ruled planes which meet *S* in an exterior g-line, there are $(q+1)(\theta_2 - k(k-1)/2 - k(q+2-k))$ g-points of $\mathbf{B} \setminus S$.

By adding the k g-points of $S \cap \mathbf{B}$, we obtain $|\mathbf{B}| = q^3 + 2q^2 + (2-k)q + 1$. Let $a = |\mathcal{T}^*|/(q+1)$. By Proposition 1, *a* is a natural number such that $q(1-a) = k-1$. Therefore, we have either (a) or (c).

Case 2: $S \cap \mathbf{B}$ *is not a g-arc, and* $S \not\subseteq \mathbf{B}$. In this case $S \cap \mathbf{B}$ contains a g-line φ. If a g-point $\ell \in S \setminus \varphi$ such that $S \cap \mathbf{B} = \varphi \cup \{\ell\}$ exists, then, computing as in Case 1, $|\mathbf{B}| = q^3 + 2q^2 + 1$ when φ is contained in an element of $\mathcal{T}^*$, $|\mathbf{B}| = q^3 + q^2 + 1$ otherwise, contradicting Proposition 1. Therefore, $S \cap \mathbf{B}$ is either a g-line, or the union of two distinct g-lines φ_1 and φ_2. In the latter case, computing once more as in Case 1 gives

$|\mathbf{B}| = q^3 + bq^2 + q + 1$, where $b \in \{0, 1, 2\}$ is the number of elements of $\mathcal{T}^*$ among the ruled planes containing φ_1 or φ_2. Proposition 1 implies $b = 2$ and $|\mathcal{T}^*| = q + 1$.

If $S \cap \mathbf{B}$ is a g-line, then $|\mathbf{B}| = q^3 + (c + 1)q^2 + q + 1$, where $c \in \{0, 1\}$ is the number of elements of $\mathcal{T}^*$ containing $S \cap \mathbf{B}$. This implies (*b*).

Case 3: $S \subseteq \mathbf{B}$. Since there is a g-point ℓ in $\mathbf{B} \setminus S$, the ruled plane through ℓ meeting S is contained in $\mathbf{B}$. □

Notation. 1. The set of all g-points collinear with $\ell \in \mathcal{R}$, including ℓ, is denoted by $\ell^\perp$.

2. Let ε be a two-dimensional star; that is, ε is the set of all lines in a star in a three-dimensional subspace of $\mathrm{PG}(n, q)$, say U_ε. Let $\mathcal{P}_\varepsilon$, $\mathcal{R}_\varepsilon$ and $\Gamma_\varepsilon = (\mathcal{R}_\varepsilon, \mathcal{F}_\varepsilon)$ be the point set, the line set and the Grassmannian of lines of U_ε, respectively. Let $\mathbf{B}_\varepsilon = \mathcal{R}_\varepsilon \cap \mathbf{B}$. Clearly $\mathbf{B}_\varepsilon$ is a tangential Tallini set in Γ_ε.

Proposition 4. *Let $S \in \mathcal{S}$. If S is not contained in* $\mathbf{B}$, *then $S \cap \mathbf{B}$ is a g-quadric in S. When $V_{\mathbf{B}} = \emptyset$, such a g-quadric cannot be the union of two distinct g-primes of S.*

Proof. Let ε be a two-dimensional star contained in S. By Lemma 3, applied to the Grassmannian Γ_ε, either $\varepsilon \subseteq \mathbf{B}$, or $\varepsilon \cap \mathbf{B}$ is a possibly singular g-conic. Any set of points K in a projective space, such that every plane section not contained in K is a conic, is a quadric or the whole space [3,8].

Next, assume that $S \cap \mathbf{B}$ is the union of two distinct g-primes of S, say V_1 and V_2, and $V_{\mathbf{B}} = \emptyset$. Let $\ell_0 \in V_1 \cap V_2$. We now obtain two bounds for $|\ell_0^\perp \cap \mathbf{B}|$.

Let φ be a g-line through ℓ_0. Assume that for an $i \in \{1, 2\}$, φ is contained in V_i and intersects V_{3-i} only in ℓ_0. Then there is a g-plane η in S that contains φ and such that $\eta \cap \mathbf{B}$ is the union of two distinct lines. By Lemma 3, applied to Γ_η, the unique ruled plane containing φ is a subset of $\mathbf{B}$. The ruled planes through ℓ_0 meet pairwise only in ℓ_0, and there exist $2q^{n-3}$ such g-planes meeting S in a line that is contained in exactly one of the primes V_1 and V_2. Each of the remaining ruled planes through ℓ_0 intersects $\mathbf{B}$ in at least $q + 1$ g-points. Therefore

$$\begin{aligned}|\ell_0^\perp \cap \mathbf{B}| &\geqslant 1 + 2q^{n-3}(q^2 + q) + (\theta_{n-2} - 2q^{n-3})q \\ &= \theta_{n-1} + 2q^{n-1}. \end{aligned} \tag{2}$$

Next, every g-point collinear with ℓ_0 belongs to one of the $q + 1$ stars through ℓ_0. By the assumption $V_{\mathbf{B}} = \emptyset$, every star intersects $\mathbf{B}$ in at most $\theta_{n-2} + q^{n-2}$ g-points (the cardinality of two g-primes). Hence

$$\begin{aligned}|\ell_0^\perp \cap \mathbf{B}| &\leqslant 1 + (q + 1)(\theta_{n-2} + q^{n-2} - 1) \\ &= 2\theta_{n-1} + q^{n-2} - q - 1. \end{aligned} \tag{3}$$

From (2) and (3),

$$q^{n-1} - \theta_{n-2} - q^{n-2} + q + 1 \leqslant 0,$$

a contradiction. □

Proposition 5. *Let $\ell^* \in \mathbf{B}$, $S \in \mathcal{S}$ and $S_1, S_2 \in \mathcal{S}_0$ such that $S_1 \neq S_2$ and $S \cap S_1 \cap S_2 = \{\ell^*\}$. Then $S \in \mathcal{S}_0$.*

Proof. First, we prove the statement under the assumption $n = 3$. It is convenient to deal with four cases.

Case 1: *there exist $S', S'' \in \mathcal{S}$ such that $S' \cap S'' = \{\ell^*\}$ and $S' \subseteq \mathbf{B}$, $S'' \subseteq \mathbf{B}$*. Each g-line φ in S through ℓ^* is contained in precisely one ruled plane T. The intersections $T \cap S'$, $T \cap S''$ are two distinct g-lines contained in $\mathbf{B}$. Thus $T \subseteq \mathbf{B}$ and $\varphi \subseteq \mathbf{B}$. As a consequence, $S \subseteq \mathbf{B}$.

Case 2: *there exist $S', \tilde{S} \in \mathcal{S}$ such that $S' \cap \tilde{S} = \{\ell^*\}$, $S' \subseteq \mathbf{B}$, and $\tilde{S} \cap \mathbf{B}$ is a g-line, say ϕ*. Let T_ϕ be the ruled plane containing ϕ. The g-lines ϕ and $T_\phi \cap S'$ are distinct and contained in $T_\phi \cap \mathbf{B}$, so $T_\phi \subseteq \mathbf{B}$. Consider a ruled plane $T \neq T_\phi$ such that $\ell^* \in T$. The g-line $T \cap S'$ is contained in $\mathbf{B}$, whereas $T \cap \tilde{S} \cap \mathbf{B} = \{\ell^*\}$, hence $T \cap \mathbf{B} \subseteq S'$. Therefore, any g-line ψ through ℓ^* either is contained in $\mathbf{B}$, when $\psi \subseteq S'$ or $\psi \subseteq T_\phi$, or is tangent to $\mathbf{B}$. This implies that any star through the g-point ℓ^* and other than S' meets $\mathbf{B}$ in a g-line.

Case 3: *there exists $T \in \mathcal{T}^*$ such that $\ell^* \in T$*. Any $\ell \in (\ell^*)^\perp$ belongs to a ruled plane through ℓ^*. Then $|(\ell^*)^\perp \cap \mathbf{B}| \geqslant 2q^2 + q + 1$. Assume we are neither in Case 1 nor in Case 2. Then $|(\ell^*)^\perp \cap \mathbf{B} \cap S_i| = q + 1$ for $i = 1, 2$, and, for every star S''' through ℓ^* other than S_1 and S_2, $|(\ell^*)^\perp \cap \mathbf{B} \cap S'''| \leqslant 2q + 1$. Therefore $|(\ell^*)^\perp \cap \mathbf{B}| \leqslant 1 + 2q + (q-1)2q = 2q^2 + 1$, a contradiction.

Case 4: *otherwise*. Let $\tilde{T}$ be a ruled plane through ℓ^*. The g-lines $\tilde{T} \cap S_1$ and $\tilde{T} \cap S_2$ are distinct and tangent to, or contained in, $\mathbf{B}$, hence $\tilde{T} \cap \mathbf{B}$ is a g-line. This implies that any g-line through ℓ^* is tangent to, or contained in, $\mathbf{B}$. If two of the $q + 1$ g-lines through ℓ^* are contained in a common star, then, by Lemma 3, Case 3 occurs, which is impossible. Therefore, the intersection of $\mathbf{B}$ with every star containing ℓ^* is a g-line.

Next, assume $n > 3$. Let ε be any g-plane contained in S and such that $\ell^* \in \varepsilon$. The intersection $\varepsilon_i = \mathcal{R}_\varepsilon \cap S_i$ is a g-plane for $i = 1, 2$. Thus $\mathbf{B}_\varepsilon \cap \varepsilon_i = \mathbf{B} \cap \mathcal{R}_\varepsilon \cap S_i$ either is a g-line in ε_i, or is equal to ε_i. Since Proposition 5 has already been proved for $n = 3$, either $\varepsilon \subseteq \mathbf{B}$, or $\varepsilon \cap \mathbf{B}$ is a g-line.

Assume ℓ_1 and ℓ_2 are two distinct g-points in $S \cap \mathbf{B}$. If ε is a g-plane through ℓ_1, ℓ_2 and ℓ^*, then, by the previous argument, the g-line $\ell_1\ell_2$ is contained in $S \cap \mathbf{B}$. Consequently $S \cap \mathbf{B}$ is a g-subspace of S.

Every g-line φ' in S lies on a g-plane of S through ℓ^*, so $\varphi' \cap S \cap \mathbf{B} \neq \emptyset$. Therefore, either $S \subseteq \mathbf{B}$, or $S \cap \mathbf{B}$ is a g-prime of S. □

Definition. For a subspace V of a projective space P and a set $I \subset P \setminus V$, the *cone VI* with *vertex V* is the set of all points on the lines joining a point of V to a point of I.

Proposition 6. *Let Θ be the set of all lines in* PG(n, q) *which are incident with $V_{\mathbf{B}}$. Then*

(i) *$Q_{\mathbf{B}}$ is a cone with vertex $V_{\mathbf{B}}$;*
(ii) *for every $S \in \mathcal{S}$, $S \cap \mathbf{B}$ is a cone with vertex $S \cap \Theta$.*

Proof. (i) Let $A, B \in V_{\mathbf{B}}$, $A \neq B$, and $\{\ell\} = S_A \cap S_B$. Each ruled plane T through ℓ meets $\mathbf{B}$ in at least two distinct g-lines $T \cap S_A$ and $T \cap S_B$; therefore, $T \subseteq \mathbf{B}$. Consequently, the line of $\mathrm{PG}(n, q)$ joining A and B is contained in $\mathbf{B}$ and $V_{\mathbf{B}}$ is a subspace of $\mathrm{PG}(n, q)$. On the other hand, by Proposition 5, every line of $\mathrm{PG}(n, q)$ intersecting $V_{\mathbf{B}}$ is either tangent to, or contained in, $Q_{\mathbf{B}}$.

(ii) Clearly $\Theta \subseteq \mathbf{B}$. Since $V_{\mathbf{B}}$ is a subspace of $\mathrm{PG}(n, q)$, $S \cap \Theta$ is a g-subspace of S. Assume $\ell_0 \in S \cap \Theta$, $\ell_1 \in S \setminus \Theta$. Let φ be the g-line $\ell_0\ell_1$ and C a point of $\mathrm{PG}(n, q)$ incident with both ℓ_0 and $V_{\mathbf{B}}$. There is exactly one ruled plane containing φ, say T. The g-line $T \cap S_C$ is contained in $\mathbf{B}$. Thus, either $T \cap \mathbf{B} = T \cap S_C$, and $\varphi \cap \mathbf{B} = \{\ell_0\}$, or $T \subseteq \mathbf{B}$, and $\varphi \subseteq \mathbf{B}$. So, every g-line in S incident with $S \cap \Theta$ either intersects $\mathbf{B}$ in exactly one point, or is contained in $\mathbf{B}$. □

Proposition 7. *Let V and V' be complementary subspaces of* $\mathrm{PG}(n, q)$ *such that $V \subseteq V_{\mathbf{B}}$, and* $\dim V' \geqslant 3$. *Let $\Gamma' = (\mathcal{R}', \mathcal{F}')$ be the Grassmannian of the lines of V', and $\mathbf{B}' = \mathcal{R}' \cap \mathbf{B}$. Then*

(i) *$\mathbf{B}'$ is a tangential Tallini set of Γ'; if $V = V_{\mathbf{B}}$, then no star of Γ' is contained in $\mathbf{B}'$.*

(ii) *Let $Q_{\mathbf{B}'}$ be the set of points of V' which are centers of stars of Γ' intersecting $\mathbf{B}'$ in g-subspaces of dimension $d \geqslant \dim V' - 2$. Then the following hold.*

  (a) $Q_{\mathbf{B}'} = Q_{\mathbf{B}} \cap V'$;

  (b) *for any line ℓ disjoint to V, the line ℓ belongs to $\mathbf{B}$ if, and only if, the projection in* $\mathrm{PG}(n, q)$ *of ℓ from V onto V' is a g-point of $\mathbf{B}'$.*

Proof. (i) It is clear that $\mathbf{B}'$ is a tangential Tallini set of Γ'. Now assume that $V = V_{\mathbf{B}}$ and there is a star of Γ' contained in $\mathbf{B}'$. Hence a point A of V' exists such that $S_A \cap \mathcal{R}' \subseteq \mathbf{B}'$. In S_A the g-subspaces $S_A \cap \mathcal{R}'$ and $S_A \cap \Theta$ are complementary. Then, by Proposition 6(ii), $S_A \subseteq \mathbf{B}$, a contradiction.

(iia) Let $C \in Q_{\mathbf{B}'}$. Either $S_C \cap \mathbf{B}'$ is a g-prime of $S_C \cap \mathcal{R}'$, or $S_C \cap \mathcal{R}' \subseteq \mathbf{B}'$. This implies, by Proposition 6(ii), that $S_C \in \mathcal{S}_0$.

Conversely, if $D \in Q_{\mathbf{B}} \cap V'$, then $S_D \cap \mathbf{B}$ is a g-prime of S_D or $S_D \subseteq \mathbf{B}$, thus $S_D \cap \mathbf{B}'$ is a g-prime of $S_D \cap \mathcal{R}'$ or $S_D \cap \mathcal{R}' \subseteq \mathbf{B}'$.

(iib) Let ℓ' be the projection of ℓ on V'. If ℓ and ℓ' have a common point, then there is a plane π in $\mathrm{PG}(n, q)$ containing ℓ, ℓ' and a point E of V. The pencil of lines φ with center E on π is a subset of $\mathbf{B}$. Then either $T_\pi \subseteq \mathbf{B}$, and $\ell, \ell' \in \mathbf{B}$, or $T_\pi \cap \mathbf{B} = \varphi$ and $\ell, \ell' \notin \mathbf{B}$.

Now assume that ℓ and ℓ' are skew. Take a line m incident with both ℓ and ℓ', but not with V. Let π_1 be the plane containing the lines ℓ and m, and π_2 the plane containing ℓ' and m. Each π_i meets V in a point, $i = 1, 2$. The same argument as above proves that $\ell \in \mathbf{B}$ if, and only if, $m \in \mathbf{B}$, as well as $m \in \mathbf{B}$ if, and only if, $\ell' \in \mathbf{B}$. □

Theorem 8. (i) *Every $\ell \in \mathbf{B}$ is a line of* $\mathrm{PG}(n, q)$ *that intersects $Q_{\mathbf{B}}$ in either* 0, *or* 1, *or* $q + 1$ *points.*

(ii) *If* $\dim V_{\mathbf{B}} \neq n - 3$, $n \neq 4$ *and $\mathbf{B}$ is covered by g-subspaces of dimension $n - 2$, then each g-point of $\mathbf{B}$ is a line of* $\mathrm{PG}(n, q)$ *contained in, or tangent to, $Q_{\mathbf{B}}$.*

Proof. (i) is a straightforward consequence of Proposition 5. As to (ii), it is enough to prove that any g-point ℓ in $\mathbf{B} \setminus \Theta$ is on a star S^* such that $S^* \in \mathcal{S}_0$.

Assume $\dim V_{\mathbf{B}} > n-3$ and let π be a plane through ℓ. Since π meets $V_{\mathbf{B}}$, it contains $q+1$ lines of $\mathbf{B}$ other than ℓ. Thus $T_\pi \subseteq \mathbf{B}$. Hence, any star containing ℓ is in $\mathcal{S}_0$.

Next, assume $\dim V_{\mathbf{B}} < n-3$. Let U be a g-subspace of dimension $n-2$ such that $\ell \in U \subseteq \mathbf{B}$. Since $\dim U \neq 2$, there is a star S_A such that $\ell \in U \subseteq S_A \cap \mathbf{B}$. Let V' be a subspace of $\mathrm{PG}(n,q)$ containing ℓ and complementary to $V_{\mathbf{B}}$. We have $\dim V' \geqslant 3$. By Propositions 7 and 4, keeping the notation in Proposition 7 (with $V = V_{\mathbf{B}}$), $S_A \cap \mathcal{R}' \cap \mathbf{B}$ is a g-quadric in $S_A \cap \mathcal{R}'$ containing the g-prime $U \cap \mathcal{R}'$. By Proposition 4, $A \in Q_{\mathbf{B}'}$, then Proposition 7 gives $A \in Q_{\mathbf{B}}$. □

It will turn out that $\dim V_{\mathbf{B}} = n-3$ is no real exception (cf. Proposition 18).

Proposition 9. *Let $\ell^* \in \mathcal{R}$, $S \in \mathcal{S}$ and $S_1, S_2, S_3 \in \mathcal{S}_0$ such that $S_1 \neq S_2 \neq S_3 \neq S_1$ and $\ell^* \in S \cap S_1 \cap S_2 \cap S_3$. Then $S \in \mathcal{S}_0$.*

Proof. By Proposition 5 we may assume $\ell^* \notin \mathbf{B}$. Let T be a ruled plane through ℓ^*. By assumption, the g-lines $T \cap S_i$ are three distinct tangent g-lines to $\mathbf{B}$. Thus $T \cap \mathbf{B}$ is a g-line. Therefore, any g-line containing ℓ^* is a tangent g-line to the g-quadric $S \cap \mathbf{B}$. The statement follows. □

Theorem 10. *The set $Q_{\mathbf{B}}$ is a Tallini set in* $\mathrm{PG}(n,q)$.

Proof. If A, B, C are three distinct points in $Q_{\mathbf{B}}$ on a line $\ell^* \in \mathcal{R}$, then $S_A, S_B, S_C \in \mathcal{S}_0$. The statement follows from Proposition 9. □

Theorem 11. *If $\mathbf{B} \neq \mathcal{R}$ and there are no secant g-lines, then $\mathbf{B}$ is a linear complex. Conversely, every linear complex is a tangential Tallini set admitting no secant g-lines.*

Proof. Any g-line φ is contained in precisely one ruled plane T. By assumption, either $T \subseteq \mathbf{B}$, or $T \cap \mathbf{B}$ is a g-line. Thus, any $\varphi \in \mathcal{F}$ is either contained in $\mathbf{B}$, or intersects $\mathbf{B}$ in exactly one g-point. This property implies that $\mathbf{B}$ is a linear complex [11,2]. □

3. The case $n = 3$

Proposition 12. *Let $n = 3$ and $T \in \mathcal{T}^*$. Then there exists a star S^* such that $S^* \subseteq \mathbf{B}$ and $S^* \cap T \neq \emptyset$.*

Proof. By Proposition 1, a $T' \in \mathcal{T}^*$ such that $T' \neq T$ exists. Let $\{\ell^*\} = T \cap T'$. Let T'' be a further ruled plane through ℓ^*, and $\ell \in T'' \cap \mathbf{B} \setminus \{\ell^*\}$. Let S^* be the star containing both ℓ and ℓ^*; it meets T, T' in two g-lines contained in $\mathbf{B}$. Then $|S^* \cap \mathbf{B}| > 2q + 1$. □

Proposition 13. *Assume $n = 3$ and $V_{\mathbf{B}} = \emptyset$. Then $|U \cap \mathbf{B}| = q + 1$ for any $U \in \mathcal{S} \cup \mathcal{T}$. Also, $|\mathbf{B}| = \theta_3$.*

Proof. The statement follows from Propositions 1, 12 and Lemma 3. □

We now consider a *ruled* tangential Tallini set $\mathbf{B}$; this means that each g-point in $\mathbf{B}$ lies on a g-line contained in $\mathbf{B}$.

Proposition 14. *Assume that $\mathbf{B}$ is ruled, $n = 3$ and $V_{\mathbf{B}} = \emptyset$. Then $|Q_{\mathbf{B}}| \geqslant q^2 + 1$. The equality holds if, and only if, each g-point in $\mathbf{B}$ lies on precisely one g-line contained in $\mathbf{B}$.*

Proof. Compute in two ways the number N of pairs (A, ℓ) such that $A \in Q_{\mathbf{B}}$ and $\ell \in S_A \cap \mathbf{B}$. Clearly, $N = |Q_{\mathbf{B}}|\theta_1$. By Theorem 8(ii), we have $N \geqslant |\mathbf{B}|$, thus the statement follows from Proposition 13. □

Proposition 15. *Assume $n = 3$ and $V_{\mathbf{B}} = \emptyset$. Let $\ell \in \mathbf{B}$ be a line in* PG$(3, q)$ *and contained in $Q_{\mathbf{B}}$. For any point X on ℓ, let $\alpha(X)$ be the plane of the pencil $S_X \cap \mathbf{B}$. Then α is a one-to-one map defined on the set of points of ℓ.*

Proof. If $X \neq Y$ and $\alpha(X) = \alpha(Y)$, then the plane $\alpha(X)$ contains at least $2q + 1$ lines in $\mathbf{B}$, a contradiction. □

Proposition 16. *Assume $n = 3$, $V_{\mathbf{B}} = \emptyset$ and $Q_{\mathbf{B}} \neq \mathcal{P}$. Then no plane in* PG$(3, q)$ *is contained in $Q_{\mathbf{B}}$.*

Proof. Assume on the contrary that there is a plane π contained in $Q_{\mathbf{B}}$ and a point A in PG$(3, q)$ not belonging to $Q_{\mathbf{B}}$. Thus A is not on π.

For any point X on π, the lines through X which belong to $\mathbf{B}$ form a pencil. Further, since $|T_\pi \cap \mathbf{B}| = q + 1$, so $T_\pi \cap \mathbf{B}$ is a pencil with a center point on π, say C.

The lines of $\mathbf{B}$ passing through A intersect π in the points of a $(q + 1)$-arc Ω. Let ℓ be a line through C intersecting Ω in two distinct points D and E. Let ε be the plane through A, D and E. Among the lines of $T_\varepsilon \cap \mathbf{B}$ there are: (i) ℓ and AD, hence all lines of the pencil on ε with center D; (ii) ℓ and AE, hence all lines of the pencil with center E. Therefore, $|T_\varepsilon \cap \mathbf{B}| \geqslant 2q + 1$, a contradiction. □

Theorem 17. *Assume that* $\mathbf{B} \neq \mathcal{R}$, *that* $\mathbf{B}$ *is ruled and that* $n = 3$. *Then one of the following holds*: (i) $Q_{\mathbf{B}}$ *is a quadric of* $\mathrm{PG}(3, q)$ *and* $\mathbf{B}$ *is the set of all self-conjugate lines of* $Q_{\mathbf{B}}$; (ii) $Q_{\mathbf{B}} = \mathcal{P}$ *and* $\mathbf{B}$ *is a linear complex.*

Proof. By Theorem 11, we may deal with just the case in which there exist secant g-lines.

Case 1.1: $\dim V_{\mathbf{B}} = -1$, $|Q_{\mathbf{B}}| \leqslant q^2 + 1$. By Theorem 10 and Proposition 14, $Q_{\mathbf{B}}$ is a Tallini set in $\mathrm{PG}(3, q)$ of size $q^2 + 1$ and not containing lines; hence [1] an elliptic quadric. The g-points in $\mathbf{B}$ are tangent lines to $Q_{\mathbf{B}}$ (cf. Theorem 8). A cardinality argument implies the converse.

Case 1.2: $\dim V_{\mathbf{B}} = -1$, $|Q_{\mathbf{B}}| > q^2 + 1$. Since there are secant g-lines, $|Q_{\mathbf{B}}| < \theta_3$. By Propositions 14, 13 and 5, there is a g-point $\ell^* \in \mathbf{B}$ such that each star S containing ℓ^* intersects $\mathbf{B}$ in a g-line. As a line of $\mathrm{PG}(3, q)$, ℓ^* is contained in $Q_{\mathbf{B}}$. Let A be a point of $Q_{\mathbf{B}}$ not on ℓ^*. The g-line $S_A \cap \mathbf{B}$ is a pencil of lines which are non-secant to $Q_{\mathbf{B}}$. One line in the pencil, say ℓ, intersects ℓ^* in a point A'. Thus ℓ is contained in $Q_{\mathbf{B}}$. Since $\ell \cup \ell^* \neq Q_{\mathbf{B}}$, there are a point B in $Q_{\mathbf{B}}$ but not on $\ell \cup \ell^*$, and a line m that is contained in $Q_{\mathbf{B}}$ and is incident with both B and ℓ^*, by the above argument. If B is on the plane $A\ell^*$, then by Theorem 10 the whole plane is contained in $Q_{\mathbf{B}}$, contradicting Proposition 16. Thus m is not on the plane $A\ell^*$. On the other hand, since $\ell^*, \ell \in \mathbf{B}$, so $S_{A'} \cap \mathbf{B}$ is the pencil on the plane $A\ell^*$ with center A'. Therefore A' and m are not incident, and ℓ and m are skew lines.

By a similar argument, any point X on ℓ belongs to a line of $\mathbf{B}$, which is contained in $Q_{\mathbf{B}}$ and meets m. More precisely, such a line is contained in $\alpha(X)$ (cf. Proposition 15). Since α is a one-to-one map, we obtain $q + 1$ lines contained in $Q_{\mathbf{B}}$ which are pairwise skew. So, $|Q_{\mathbf{B}}| \geqslant \theta_1^2$.

In [9] it is proved that a Tallini set K in $\mathrm{PG}(n, q)$ (q odd, $n \geqslant 3$), distinct from $\mathrm{PG}(n, q)$ and such that $|K| \geqslant \theta_{n-1}$ is either the union of a prime and a t-dimensional subspace ($-1 \leqslant t \leqslant n - 1$), or a non-singular quadric in a space of even dimension, or a cone projecting such a quadric, or a non-singular hyperbolic quadric in a space of odd dimension, or a cone projecting a non-singular hyperbolic quadric. Since $Q_{\mathbf{B}}$ contains $q + 1$ pairwise skew lines, $Q_{\mathbf{B}}$ is a non-singular hyperbolic quadric. The self-conjugate lines of such quadric are exactly θ_3; so they are precisely the elements of $\mathbf{B}$.

Case 2: $\dim V_{\mathbf{B}} = 0$. Since in this case Theorem 8 does not apply, we have to prove that every g-point in $\mathbf{B}$ is a line of $\mathrm{PG}(3, q)$ that meets $Q_{\mathbf{B}}$. This is clear for the g-points in Θ. If $\tilde{\ell} \in \mathbf{B} \setminus \Theta$ and $\tilde{\ell}$ is external to $Q_{\mathbf{B}}$, then by Proposition 6(ii) each star intersects $\mathbf{B}$ in the union of two lines. This implies $|\tilde{\ell}^{\perp} \cap \mathbf{B}| = 2q^2 + 2q + 1$. On the other hand, writing α for the number of ruled planes through $\tilde{\ell}$ which are contained in $\mathbf{B}$, one obtains $|\tilde{\ell}^{\perp} \cap \mathbf{B}| = (\alpha + 1)q^2 + q + 1$, contradicting the previous equality. Thus $\tilde{\ell}$ is a line in $\mathrm{PG}(3, q)$ that is either tangent to or contained in $Q_{\mathbf{B}}$.

By Proposition 1 and Lemma 3, $|\mathbf{B}| = \theta_3 + q^2$. By assumption there is a secant g-line, which is contained in a ruled plane T_0. Thus $T_0 \cap \mathbf{B}$ is a dual $(q + 1)$-arc on a plane π. Let $\ell_0 \in T_0 \cap \mathbf{B}$. Any g-line which contains ℓ_0 and is secant to $T_0 \cap \mathbf{B}$ is contained in a star not in $\mathcal{S}_0$. In this way we obtain that q points on ℓ_0 do not belong to $Q_{\mathbf{B}}$. So, in $\mathrm{PG}(3, q)$ ℓ_0 is tangent to $Q_{\mathbf{B}}$. The lines of $\mathrm{PG}(3, q)$ belonging to $T_0 \cap \mathbf{B}$ are tangent to $Q_{\mathbf{B}}$, and $\mathcal{C} = \pi \cap Q_{\mathbf{B}}$ is a non-singular conic. By Proposition 6(i), $Q_{\mathbf{B}}$ is a cone projecting $\mathcal{C}$. The number of self-conjugate lines of such a cone is $\theta_3 + q^2$, i.e. they are precisely the elements of $\mathbf{B}$.

Case 3: $\dim V_{\mathbf{B}} = 1$. In this case $|\mathbf{B}| = \theta_3 + q^2$. The lines incident with $V_{\mathbf{B}}$ are exactly $\theta_3 + q^2$. Thus $\mathbf{B}$ is a special linear complex (contradicting the existence of secant g-lines).

Case 4: $\dim V_{\mathbf{B}} > 1$. This implies $\mathbf{B} = \mathcal{R}$, a contradiction. □

For more information on the case in which $\mathbf{B}$ is not ruled the reader is referred to [15,16]. It is an open problem, however, whether such a possibility occurs.

Proposition 18. *The assumption* $\dim V_{\mathbf{B}} \neq n-3$ *in Theorem* 8(ii) *is superfluous.*

Proof. Assume that $\mathbf{B}$ is covered by g-subspaces of dimension $n-2$ and $\dim V_{\mathbf{B}} = n-3$. Also assume that there are secant g-lines. Let V and V' be complementary subspaces of $PG(n,q)$ such that $V \subseteq V_{\mathbf{B}}$ and $\dim V = n-4$, $\dim V' = 3$. Let $Q_{\mathbf{B}'}$ be defined as in Proposition 7. By Theorem 17, $Q_{\mathbf{B}'}$ is a quadric and by Propositions 6 and 7, $Q_{\mathbf{B}}$ is a singular quadric. By Proposition 7(iib), $\mathbf{B}$ is the set of all self-conjugate lines of $Q_{\mathbf{B}}$. □

4. The general case

Proposition 19. *Let* $S \in \mathcal{S}$. (i) *If* $S \cap \mathbf{B}$ *is a g-subspace of dimension* $n-3$, *then for each* $\ell \in S \cap \mathbf{B}$ *there is a* $S^* \in \mathcal{S}$ *such that* $\ell \in S^* \subseteq \mathbf{B}$. (ii) *If* $V_{\mathbf{B}} = \emptyset$, *then* $\theta_{n-2} - q^{n-3} \leqslant |S \cap \mathbf{B}| \leqslant \theta_{n-2} + q^{n-3}$.

Proof. (i) We prove the statement by induction on n. First, let $n = 3$. If $S \cap \mathbf{B}$ is a g-point, then, by Lemma 3, $\mathcal{T}^* \neq \emptyset$. By Proposition 12 there exists $S^* \in \mathcal{S}$ such that $S^* \subseteq \mathbf{B}$. Obviously, $S^* \cap S = S \cap \mathbf{B}$.

Next, assume $n > 3$. Let $S \cap \mathbf{B}$ be a g-subspace of dimension $n-3$. Let $S_1, S_2, \dots, S_q$ be the stars through ℓ other than S. Such stars are $q+1$ g-subspaces of dimension $n-1$.

Denote by $\mathcal{U}$ the set of all g-primes of S through ℓ which do not contain $S \cap \mathbf{B}$. There are θ_{n-2} g-primes of S through ℓ. Exactly $q+1$ of these g-primes contain $S \cap \mathbf{B}$, so $|\mathcal{U}| = q^2\theta_{n-4}$.

For any $\Sigma \in \mathcal{U}$, the lines of $PG(n,q)$ belonging to Σ are contained in a prime of $PG(n,q)$, say $\beta(\Sigma)$. We obtain a one-to-one map β defined on $\mathcal{U}$.

Let $\mathcal{W}_i$ be the set of all g-primes of S_i, $i = 1, 2, \dots, q$. Let $\alpha_i(\Sigma)$ be the set of all g-points in S_i which are lines contained in $\beta(\Sigma)$. We obtain q maps $\alpha_i : \mathcal{U} \to \mathcal{W}_i$ $(i = 1, 2, \dots, q)$. Each α_i is one-to-one, and if $i \neq j$, $\Sigma, \Sigma' \in \mathcal{U}$, then $\alpha_i(\Sigma) \neq \alpha_j(\Sigma')$. For any $\Sigma \in \mathcal{U}$, let $\Gamma_\Sigma = (\mathcal{R}_\Sigma, \mathcal{F}_\Sigma)$ be the Grassmannian of the lines of $\beta(\Sigma)$. In Γ_Σ, Σ is a star, and $\Sigma \cap \mathbf{B}$ is a g-subspace of dimension $n-4$. By induction assumption, in Γ_Σ there is a star Σ^* such that $\ell \in \Sigma^* \subseteq \mathcal{R}_\Sigma \cap \mathbf{B}$. Such Σ^* is of type $\alpha_i(\Sigma)$ for some i. Therefore there are at least $q^2\theta_{n-4}$ distinct g-subspaces of this kind which are contained in $\mathbf{B}$. This implies that for some i, $S_i \cap \mathbf{B}$ contains at least $q\theta_{n-4} > 2$ distinct g-primes and then $S_i \subseteq \mathbf{B}$ (cf. Proposition 4).

(ii) The g-quadric $S \cap \mathbf{B}$ is different from the union of two distinct primes, whether rational over $\mathbb{F}_q$ (cf. Proposition 4) or not (by the above arguments). Then the statement is a consequence of the following general property of the quadrics: if $\mathcal{Q}$ is a quadric in $\mathrm{PG}(d,q)$ ($d \geqslant 2$) and $\mathcal{Q}$ is different from the union of two distinct primes (rational over $\mathbb{F}_q$ or in a quadratic extension), then $\theta_{d-1} - q^{d-2} \leqslant |\mathcal{Q}| \leqslant \theta_{d-1} + q^{d-2}$. This may be seen by induction on d. If $\mathcal{Q}$ is a non-singular quadric in $\mathrm{PG}(d,q)$, $d > 2$, then $|\mathcal{Q}| = \theta_{d-1}$ for q even and $|\mathcal{Q}| = \theta_{d-1} \pm q^{(d-1)/2}$ for d odd, and the assertion holds. If $\mathcal{Q}$ is singular, then it is a cone $\{P\}\mathcal{Q}'$, where P is a point and $\mathcal{Q}'$ is a quadric in a prime for which, by induction assumption, $\theta_{d-2} - q^{d-3} \leqslant |\mathcal{Q}'| \leqslant \theta_{d-2} + q^{d-3}$. In this case the statement follows from $|\mathcal{Q}| = 1 + q|\mathcal{Q}'|$. □

Proposition 20. *If $V_{\mathbf{B}} = \emptyset$ and $\ell \in \mathbf{B}$, then*

$$1 + (q+1)(\theta_{n-2} - q^{n-3} - 1) \leqslant |\ell^{\perp} \cap \mathbf{B}| \leqslant 1 + (q+1)(\theta_{n-2} + q^{n-3} - 1). \tag{4}$$

Proof. Each g-point in $\ell^{\perp} \setminus \{\ell\}$ belongs to precisely one star S such that $\ell \in S$, and there are $q+1$ stars through S. Then the statement follows from Proposition 19(ii). □

Proposition 21. *If $n \geqslant 4$, $V_{\mathbf{B}} = \emptyset$ and $S \in \mathcal{S}$, then the singular g-subspace W* (*i.e. the set of singular g-points*) *of the g-quadric $S \cap \mathbf{B}$ is of dimension different from $n-4$.*

Proof. Assume on the contrary that $S \cap \mathbf{B}$ is a cone projecting from W a non-singular g-conic of a 2-dimensional star ε. The vertex W is non-empty, so let ℓ be a g-point of W.

Now, if φ is a g-line such that $\ell \in \varphi \subseteq \mathbf{B}$ and $\varphi \not\subseteq W$, then the ruled plane T_{φ} containing φ is contained in $\mathbf{B}$. This may be seen in this way: $W\varphi$, the span of $W \cup \varphi$, is a g-subspace that meets ε in a g-point $\ell_1 \in \mathbf{B}$. If U is a g-subspace of dimension $n-3$ containing W and intersecting ε in a g-point $\ell_2 \in \mathbf{B}$, $\ell_2 \neq \ell_1$, and φ' is a g-line such that $\ell \in \varphi' \subseteq U$ and $\varphi' \not\subseteq W$, then $\varphi' \subseteq \mathbf{B}$. The intersection $(W\varphi\varphi') \cap \varepsilon$ is a g-line and is secant to $\mathbf{B}$. As a consequence, the g-plane $\eta = \varphi\varphi'$ satisfies $\eta \cap \mathbf{B} = \varphi \cup \varphi'$. By Lemma 3, with respect to Γ_{η} (cf. Section 2), $T_{\varphi} \subseteq \mathbf{B}$.

In each of the $q+1$ g-subspaces joining W with a g-point in $\varepsilon \cap \mathbf{B}$ there are q^{n-4} g-lines through ℓ which are contained in $\mathbf{B}$ but not in W. All these lines are contained in elements of $\mathcal{T}^*$. Each of the remaining $q^{n-2} + \theta_{n-5}$ g-lines through ℓ in S is contained in a ruled plane that shares at least $q+1$ g-points with $\mathbf{B}$. This allows to give a bound on the size of $\ell^{\perp} \cap \mathbf{B}$. Since any two distinct ruled planes through ℓ meet only in ℓ, we have

$$\begin{aligned} |\ell^{\perp} \cap \mathbf{B}| &\geqslant 1 + q^{n-4}(q+1)(q^2+q) + (q^{n-2} + \theta_{n-5})q \\ &= \theta_{n-1} + q^{n-2}(q+1). \end{aligned} \tag{5}$$

The right-hand inequality in (4) and (5) together give $q^{n-1} + q + 1 \leqslant \theta_{n-2} + q^{n-3}$, a contradiction. □

We now state, for future reference, a simple general property of the quadrics.

Proposition 22. *Let $\mathcal{Q}$ and H be a quadric and a prime in* $\mathrm{PG}(d,q)$*, respectively. If $\mathcal{Q}\cap H$ is a $(d-2)$-dimensional subspace of* $\mathrm{PG}(d,q)$*, then the singular space of $\mathcal{Q}$ has dimension at least $d-3$.*

Proof. Let A be the matrix associated with $\mathcal{Q}$. The linear mapping $L:\mathbb{F}_q^{d+1}\to\mathbb{F}_q^{d+1}$ related to A maps a $(d-1)$-dimensional subspace of $\mathbb{F}_q^{d+1}$ (the one associated with $\mathcal{Q}\cap H$) onto a subspace of dimension at most one. Then the kernel of L has dimension at least $d-2$. □

Theorem 23. *If* **B** *is covered by $(n-2)$-dimensional stars and there are secant g-lines, then $Q_{\mathbf{B}}$ is a quadric and* **B** *is the set of all self-conjugate lines of $Q_{\mathbf{B}}$.*

Proof. For $n=3$ the result is contained in Theorem 17. Then assume that the theorem holds for $n-1\geqslant 3$.

Case 1: $V_{\mathbf{B}}=\emptyset$. Let U be any prime in $\mathrm{PG}(n,q)$. Let $\Gamma_U=(\mathcal{R}_U,\mathcal{F}_U)$ be the Grassmannian of lines of U. When X is a point in U, $S_{X,U}=S_X\cap\mathcal{R}_U$ will denote the set of all lines through X and contained in U. Also define $\mathbf{B}_U=\mathcal{R}_U\cap\mathbf{B}$. Let Q_U be the set of all points X in U such that $S_{X,U}\cap\mathbf{B}$ either is a g-prime of $S_{X,U}$ or is equal to $S_{X,U}$.

We claim that $Q_U=Q_{\mathbf{B}}\cap U$. For, if $X\in Q_{\mathbf{B}}\cap U$, then $S_{X,U}\cap\mathbf{B}=\mathcal{R}_U\cap S_X\cap\mathbf{B}$ is a g-subspace of dimension at least $n-3$. So, the inclusion $Q_{\mathbf{B}}\cap U\subseteq Q_U$ is clear. Next, let $Y\in Q_U$. The set $S_{Y,U}$ is a g-prime of S_Y, and either $S_{Y,U}\subseteq\mathbf{B}$, or $S_{Y,U}\cap\mathbf{B}$ is a g-prime of $S_{Y,U}$. In the former case the g-quadric $S_Y\cap\mathbf{B}$ contains the g-prime $S_{Y,U}$, so it is a g-prime (cf. Proposition 4), and $Y\in Q_{\mathbf{B}}$. In the latter case, by Proposition 22 the singular g-space of the g-quadric $S_Y\cap\mathbf{B}$ has dimension $\delta\geqslant n-4$. The equality is ruled out by Proposition 21. Since the g-quadric can be neither the union of two distinct g-primes (Proposition 4), nor a g-subspace of dimension $n-3$ (Proposition 19(i)), so $\delta\neq n-3$. Therefore $\delta=n-2$, $S_Y\cap\mathbf{B}$ is a g-prime of S_Y, and $Y\in Q_{\mathbf{B}}$.

Next, we prove that $Q_U\neq U$. There is a secant g-line to **B**, say φ. The plane π of $\mathrm{PG}(n,q)$ containing the pencil φ shares with U at least one line ℓ^*. Furthermore $T_\pi\cap\mathbf{B}$ is a g-$(q+1)$-arc. If $Q_U=U$, then all stars through ℓ^* intersect **B** in g-subspaces of dimension $n-2$. On the other hand, such stars intersect T_π in exactly the $q+1$ g-lines on T_π through ℓ^*. Among such g-lines there are secant g-lines, a contradiction. So, $Q_U\neq U$. In particular Γ_U contains g-lines that are secant to $\mathbf{B}_U$.

Let $\tilde{\ell}\in\mathbf{B}_U$. The g-point $\tilde{\ell}$ belongs to a $(n-2)$-dimensional star contained in **B**. So, $\tilde{\ell}$ belongs to a $(n-3)$-dimensional star contained in $\mathbf{B}_U$.

We proved so far that $\mathbf{B}_U$ is a tangential Tallini set in the Grassmannian of lines of U covered by $(n-3)$-dimensional stars, that there exist secant g-lines to $\mathbf{B}_U$, and $Q_U=Q_{\mathbf{B}}\cap U$. By the induction assumption, $Q_{\mathbf{B}}\cap U$ is a quadric in U and $\mathbf{B}_U$ is the set of all self-conjugate lines of $Q_{\mathbf{B}}\cap U$. Since U is arbitrary, $Q_{\mathbf{B}}$ itself is a quadric. If $\overline{\ell}\in\mathbf{B}$, take any prime $\overline{U}$ of $\mathrm{PG}(n,q)$ containing $\overline{\ell}$; since $\overline{\ell}$ is a self-conjugate line of $Q_{\mathbf{B}}\cap\overline{U}$, $\overline{\ell}$ is a self-conjugate line of $Q_{\mathbf{B}}$, too. Conversely, each self-conjugate line of $Q_{\mathbf{B}}$ belongs to some $\mathbf{B}_U$, by the induction assumption.

Case 2: $V_{\mathbf{B}} \neq \emptyset$. Let $A \in V_{\mathbf{B}}$. Taking the notation of Proposition 7, with $V = \{A\}$, we investigate the tangential Tallini set $\mathbf{B}'$. Since $\mathbf{B}$ is covered by $(n-2)$-dimensional stars, $\mathbf{B}'$ is covered by $(n-3)$-dimensional stars.

We claim that $\mathbf{B}'$ has secant g-lines in $\mathcal{R}'$. By assumption there is a g-line ϕ such that $|\phi \cap \mathbf{B}| = 2$. Such g-line is a pencil lying on a plane ρ. The point A is not on ρ, since otherwise $T_\rho \cap \mathbf{B}$ would be a g-line or the whole T_ρ. The projection of ϕ from A onto V' is a pencil ϕ', and by Proposition 7(iib) such ϕ' is secant to $\mathbf{B}'$.

By the induction assumption $Q_{\mathbf{B}'}$ is a quadric in V' and $\mathbf{B}'$ is the set of all self-conjugate lines of $Q_{\mathbf{B}'}$. By Proposition 6, since $A \in V_{\mathbf{B}}$, $Q_{\mathbf{B}}$ is a cone with vertex A projecting $\mathcal{Q} = Q_{\mathbf{B}} \cap V'$. Proposition 7(iia) states that $\mathcal{Q} = Q_{\mathbf{B}'}$, then $Q_{\mathbf{B}}$ is a quadratic cone.

Now, we claim $\mathbf{B}$ is the set of all self-conjugate lines of $Q_{\mathbf{B}}$. Each line of $\mathrm{PG}(n,q)$ through A belongs to $\mathbf{B}$ and is self-conjugate with respect to $Q_{\mathbf{B}}$. Let $\ell_1 \in \mathcal{R} \setminus S_A$, and let ℓ_1' be the projection of ℓ_1 from A onto V'. Since $Q_{\mathbf{B}}$ is a cone, the projection of $\ell_1 \cap Q_{\mathbf{B}}$ is $\ell_1' \cap Q_{\mathbf{B}'}$. Hence ℓ_1 is a self-conjugate line of $Q_{\mathbf{B}}$ if and only if $\ell_1' \in \mathbf{B}'$. By Proposition 7(iib) this is equivalent to $\ell_1 \in \mathbf{B}$. $\square$

Now we are able to summarize Theorems 11 and 23.

Theorem 24. *If* $\mathbf{B} \neq \mathcal{R}$, *and* $\mathbf{B}$ *is covered by* $(n-2)$*-dimensional stars, then either* (i) $Q_{\mathbf{B}}$ *is a quadric and* $\mathbf{B}$ *is the set of all self-conjugate lines of* $Q_{\mathbf{B}}$, or (ii) $Q_{\mathbf{B}} = \mathcal{P}$ *and* $\mathbf{B}$ *is a linear complex.*

It is still an open problem, whether the assumption on the $(n-2)$-dimensional stars can be removed. For $n = 3$ a counterexample could be a set K of lines such that K intersects every ruled plane in a dual conic and every star in the lines of a quadratic cone.

If a set K with the above properties exists, then it is possible to give an interesting counterexample also for $n = 4$. Assume that K and $\mathrm{PG}(3,q)$ are embedded in $\mathrm{PG}(4,q)$, and let A be a point in $\mathrm{PG}(4,q)$ off $\mathrm{PG}(3,q)$. Next, let K' be the union of S_A and the set of all lines projecting from A a line belonging to K. For a plane π of $\mathrm{PG}(4,q)$ two cases can occur. (i) If A belongs to π, then the projection of π on $\mathrm{PG}(3,q)$ is a line, say ℓ; in case $\ell \in K$, we have $T_\pi \subseteq K'$, otherwise $T_\pi \cap K'$ is a pencil with center A. (ii) If A does not lie on π, then the projection of π on $\mathrm{PG}(3,q)$ is a plane π', so $T_\pi \cap K'$ is a dual conic. This implies that K' is a tangential Tallini set. By (ii), in every star $S_B \neq S_A$ there are secant g-lines. Therefore $Q_{K'} = \{A\}$. It should be noted that if such a set K' exists, then it is covered by g-planes, so that in Theorem 24 the words "$(n-2)$-dimensional stars" cannot be replaced by "$(n-2)$-dimensional g-subspaces".

References

[1] A. Barlotti, Un'estensione del teorema di Segre-Kustaanheimo, Boll. Un. Mat. Ital. 10 (1955) 498–506.

[2] A. Bichara, J. Misfeld, G. Tallini, C. Zanella, On the order structure in the line geometry of a projective space, J. Geom. 41 (1991) 16–32.

[3] F. Buekenhout, Ensembles quadratiques des espaces projectifs, Math. Z. 110 (1969) 306–318.

[4] M.J. de Resmini, A characterization of the secants of an ovaloid in $\mathrm{PG}(3,q)$, q even, $q > 2$, Ars Combin. 16B (1983) 33–49.

[5] M.J. de Resmini, A characterization of linear complexes in PG$(3, q)$, Ars Combin. 18 (1984) 99–102.
[6] M.J. de Resmini, A characterization of the set of lines either tangent to or lying on a nonsingular quadric in PG$(4, q)$, q odd, Lecture Notes in Pure Appl. Math. 103 (1985) 271–288.
[7] O. Ferri, G. Tallini, A characterization of the family of secant lines of an elliptic quadric in PG$(3, q)$, q odd, Rend. Sem. Mat. Brescia 7 (1984) 297–305.
[8] J.W.P. Hirschfeld, J.A. Thas, General Galois Geometries, Oxford Science Publications, Oxford, 1991.
[9] G. Tallini, Sulle k-calotte di uno spazio lineare finito, Ann. Mat. 42 (4) (1956) 119–164.
[10] G. Tallini, Graphic characterization of algebraic varieties in a Galois space, Atti Convegni Lincei 17: Teorie Combin. II Acc. Naz. Lincei (1976) 153–166.
[11] G. Tallini, I k-insiemi di rette di uno spazio di Galois studiati rispetto ai fasci di rette, Quaderni del Seminario Geometrie Combinatorie, vol. 28, Roma, Giugno, 1980.
[12] G. Tallini, The geometry on Grassmann manifolds representing subspaces in a Galois space, Combinatorial and geometric structures and their applications (Trento, 1980), Ann. Discrete Math. 14 (1982) 9–38.
[13] G. Tallini, On line k-sets of type $(0, n)$ with respect to pencils of lines in PG(d, q), Ann. Discrete Math. 14 (1982) 283–292.
[14] G. Tallini, Partial line spaces and algebraic varieties, Symp. Math. 28 (1986) 203–217.
[15] A. Venezia, Sui k-insiemi di punti della grassmanniana delle rette di PG$(3, q)$ di tipo $(0, 1, 2, q+1)_1$ rispetto alle rette e ad un sol carattere rispetto ai piani, Rend. Mat. Appl. VII Ser. 3 (1983) 9–16.
[16] A. Venezia, On a characterization of the set of lines which either belong to or are tangent to a non-singular quadric in PG$(3, q)$, q odd, Rend. Semin. Mat. Brescia 7 (1984) 617–623.