

Precise Toppling Balance, Quenched Disorder, and Universality for Sandpiles

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A single sandpile model with quenched random toppling matrices captures the crucial features of different models of self-organized criticality. With symmetric matrices avalanche statistics falls in the multiscaling Bak-Tang-Wiesenfeld universality class. In the asymmetric case the simple scaling of the Manna model is observed. The presence or absence of a precise toppling balance between the amount of sand released by a toppling site and the total quantity the same site receives when all its neighbors topple once determines the appropriate universality class.

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Self-organized criticality (SOC) occurs in the nonlinear transport of some physical entity, like energy, sand, or stress, through a system of linear size L under a constant, slow external drive [1–4]. The transport has breakdown features, with intermittent bursts of activity called avalanches. At long times the probability distribution of avalanche sizes becomes critical since they do not reveal characteristic scales between L and the minimal length specified in the model. In spite of many efforts, the physical mechanisms underlying the different forms of scaling realized in such models and the related universality issues remain poorly understood. The probability distribution of the number of topplings in an avalanche of the Bak-Tang-Wiesenfeld (BTW) [5] sandpile, historically the prototype of SOC, has been shown recently to violate the finite size scaling (FSS) ansatz, assumed for many years, and to obey a peculiar form of multiscaling [6,7]. On the other hand, in the Manna stochastic sandpile [8] the corresponding distribution is known to obey FSS [7,9,10]. Understanding the key mechanisms at the basis of these radically different forms of scaling remains a major challenge which should be faced also in the perspective of new concrete applications [11].

In this Letter we study sandpile models [2] on lattices with quenched disorder. This means that different bonds of the lattice allow flow of different, but constant numbers of sand grains through them. We further distinguish between an undirected case, in which the flow is identical in the two directions for each bond, and a directed one, in which this flow can be asymmetric. We show that, once averaged over the possible realizations of the quenched disorder, the avalanche size distribution of the model with symmetric flow in each bond obeys the same multiscaling as the BTW model, while asymmetry in the flow leads to FSS with the exponents of the Manna sandpile. Thus, for a disordered sandpile the two main SOC universality classes can both be realized simply upon enforcing or releasing a local symmetry requirement.

The deterministic sandpile model on a square lattice of size L is described using an integer “toppling matrix”

(TM) Δ of size $L^2 \times L^2$ [2]. At lattice site k there is a column of h_k sand grains. The system is externally driven by adding a single grain at a time at a randomly selected site i : $h_i \rightarrow h_i + 1$. Each site has a threshold height H_i of stability. If $h_i > H_i$, the sand column at i topples and grains are distributed to other sites. Consequently, all heights are updated as $h_j \rightarrow h_j - \Delta_{ij}$ where $\Delta_{ii} > 0$ and $\Delta_{ij} \leq 0$ for $i \neq j$. The threshold heights are chosen as $H_i = \Delta_{ii}$. Grain conservation is assured by putting $\Delta_{ii} = -\sum_{j \neq i} \Delta_{ij}$. Grains must flow out of the system through the boundary sites to maintain a stationary state. The BTW model is a special case of this formulation where $\Delta_{ii} = 4$, $\Delta_{ij} = -1$ for $|i - j| = 1$, and $\Delta_{ij} = 0$ for $|i - j| > 1$ [5]. In contrast, in the stochastic sandpile model [8], annealed randomness enters in the grain distribution process upon toppling. Indeed, in this case each grain of the toppling site is transferred to a randomly selected neighboring site. For both BTW and stochastic sandpiles the grain number is conserved, boundaries are open, and, when a site topples, grains are distributed to its neighborhood. In spite of these common features and similarities [12], the difference in the behavior of the two models concerns even the type of scaling.

In the Manna sandpile, the randomness in the choice of the neighbors getting a grain from the toppling site can be regarded as an “annealed” disorder. Indeed, the random choice of the TM elements is constantly updated during the course of a given avalanche. In contrast, a “quenched” disorder is realized in the models discussed here, for which a random realization of Δ is maintained for a full samples of avalanches, and an average over the distributions of many samples created is performed eventually.

Our Δ_{ij} is nonzero only for $i = j$ or $|i - j| = 1$. However, unlike in the BTW model, Δ_{ij} takes random values. These are chosen among the negative integers $-1, -2, \dots, -m$ when $|i - j| = 1$. Thus, when one of the sites joined by a nearest neighbor bond topples, this bond can allow the flow of more than one grain, unlike in the BTW case. When the TM is symmetric, we call the model undirected and assign only one random integer

$\Delta_{ij} = \Delta_{ji}$ independently to each nearest neighbor bond [Fig. 1(a)]. In the directed case the TM is asymmetric and both Δ_{ij} and Δ_{ji} are assigned independently drawn random integers [Fig. 1(b)]. In this way a toppling at one end of a bond may send, through the same bond, a different number of grains from that at the other end. At any site i we put $H_i = \Delta_{ii} = -\sum_{j \neq i} \Delta_{ij}$. The total number of grains received by the site i when all neighboring sites topple once is $H'_i = -\sum_{j \neq i} \Delta_{ji}$. We call H_i and H'_i the out degree and the in degree of site i , respectively.

Irrespective of whether Δ is symmetric or not, the models enjoy the Abelian property of the BTW sandpile [13]. This allows us to establish a number of general exact results concerning the stationary state, recurrent configurations, etc., which hold in both the directed and the undirected cases [2]. An important notion is that an avalanche can be decomposed into waves of toppling [14,15]. To this purpose one considers the sequence of topplings following the addition of the seed grain at site 0. The first wave is the set of all topplings that follows the first toppling at 0 while 0 itself is prevented from a possible second toppling. If 0 is still unstable after the first wave, the second wave starts by allowing a second toppling at 0, and so on.

A peculiar property of waves with a symmetric TM is that the set of lattice sites which topple has no holes. For example, a single untoppled site cannot be fully surrounded by toppled sites. Indeed, in the undirected model the equality $H_i = H'_i$ is strictly maintained at all sites except at the boundary, which implies that a site must

topple irrespective of its height, if all its neighbors topple once. Thus, waves in the undirected model, like in the BTW model, have no holes. In addition, one can show that all sites involved by a wave topple only once, like the seed site. All this is not true anymore with asymmetric TM. In this case the waves can have holes and can include sites that topple more than once. Indeed, in the directed model, $H_i \neq H'_i$ in general and a site with $H_i > H'_i$ does not topple even if all its neighbors topple once. This creates a single site hole in the avalanche. On the other hand, if a site i has an out degree sufficiently smaller than its in degree, it may topple for the second time even if none of its neighbors have toppled for the second time. Thus, in general, different sites belonging to a given wave topple a different number of times.

The compactness and uniformity of waves in the undirected model leads us to expect for them and for the resulting avalanches a structure similar to that of the BTW model [Fig. 2(a)]. The nonuniformity introduced by disorder should not be relevant at large scales when one counts toppling events. The situation is quite different for the directed model: the structures of waves and of avalanches in this case [Fig. 2(b)] look, in fact, similar to those of the Manna model [16] where the numbers of grains transmitted upon in the two directions of a given bond are different in general, and this imbalance is maintained dynamically due to the stochastic distribution of sand grains.

In the BTW model, the probability distribution $\text{Prob}(s, L)$ of the total number of topplings, s , in an avalanche has been found recently to obey a multiscaling ansatz [7]. On the other hand, it is pretty well established [9] by now that in the Manna stochastic sandpile this distribution obeys simple FSS as

$$\text{Prob}(s, L) \sim s^{-\tau} f\left(\frac{s}{L^D}\right), \quad (1)$$

where the scaling function $f(x) \sim \text{const}$ in the limit of $x \rightarrow 0$ and $f(x)$ approaches zero very fast for $x \gg 1$. The exponent τ and the dimension D fully characterize the scaling of Prob in this case. One immediate way to check the validity of Eq. (1) is to attempt a data collapse by plotting $s^\tau \text{Prob}$ vs s/L^D with trial values of the exponents. We collected extensive data for both our models ($m = 4$) for $L = 128, 256, 512, 1024, 2048,$ and 4096 , namely, 50×10^6 avalanches in 500 independent configurations for $L = 128$ down to $\approx 1.1 \times 10^6$ avalanches for nine configurations for $L = 4096$. The first $\sim 4L^2$ avalanches were skipped to reach the steady state. For the directed case collapse works very nicely giving $\tau \approx 1.28$ and $D \approx 2.75$, close to the most reliable estimates of the Manna sandpile exponents [9]. For the undirected model the collapse does not work for a single set of τ and D and for all values of s and L . This is found also in the BTW sandpile.

A more reliable and quantitative check of the validity, or violation, of FSS, is based on the evaluation of the

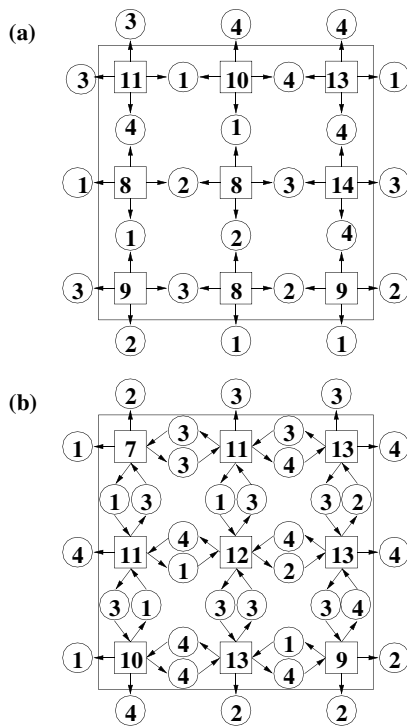


FIG. 1. Random grain flow distribution along the bonds of a 3×3 square lattice: (a) undirected case, (b) directed case.

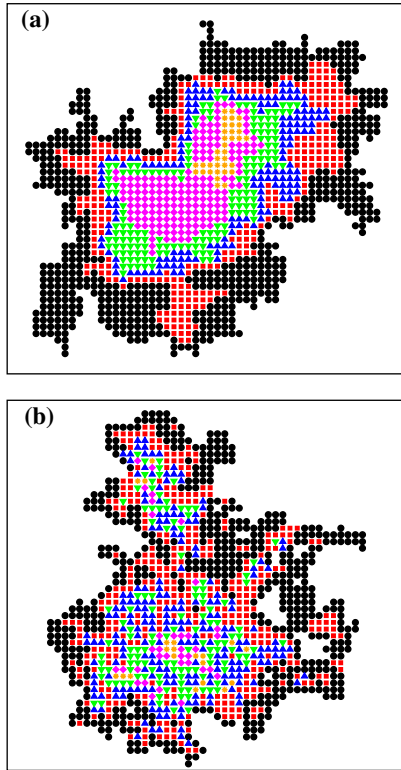


FIG. 2 (color online). Multiply toppled sites within avalanches are shown by different colored symbols: 1 (circles), 2 (squares), 3 (up triangles), 4 (down triangles), 5 (diamonds), and 6 (stars) for the (a) undirected model, and (b) directed model.

various moments of Prob [6,7,9]. The q th moment is defined as $\langle s^q \rangle = \int s^q \text{Prob}(s, L) ds$. Assuming that FSS holds, it is easy to show that $\langle s^q \rangle \sim L^{\sigma(q)}$ with the moment exponent given by $\sigma(q) = D(q - \tau + 1)$ for $q > \tau - 1$ and $\sigma(q) = 0$ for $0 < q < \tau - 1$. In the case of multiscaling σ should have a nonlinear dependence on q . A comparison of $\sigma(q)$ for two given models is also a key to establish if they belong to the same universality class or not. The value of $\sigma(q)$ was determined from the slope of the plot of $\log\langle s^q(L) \rangle$ vs $\log L$ for $L = 1024, 2048,$ and 4096 with an error ≈ 0.01 and for 251 values of q between 0 and 5 [Fig. 3(a)]. The derivative of σ is determined by the finite difference method. A slow but monotonic increase of $d\sigma(q)/dq$ with q clearly indicates the multiscaling in BTW as well as in undirected models [Fig. 3(b)], whereas a saturation of $d\sigma(q)/dq$ indicates the validity of FSS for Manna and in directed models [Fig. 3(c)]. To measure deviations quantitatively we define a quantity $X_{a,b} = 2| [d\sigma(q)/dq]_a - [d\sigma(q)/dq]_b | / \{ [d\sigma(q)/dq]_a + [d\sigma(q)/dq]_b \}$. It is observed that after some initial fluctuations $X_{\text{BTW,undir}}$ has a maximum value 0.33% within $q = 2$ and 5 whereas $X_{\text{BTW,direc}}$ gradually increases to 5.5% at $q = 5$. This analysis implies that the undirected model is almost negligibly different from the BTW model in the $d\sigma(q)/dq$ vs q plot, whereas the deviation of the directed model from the BTW model is much larger and gradually

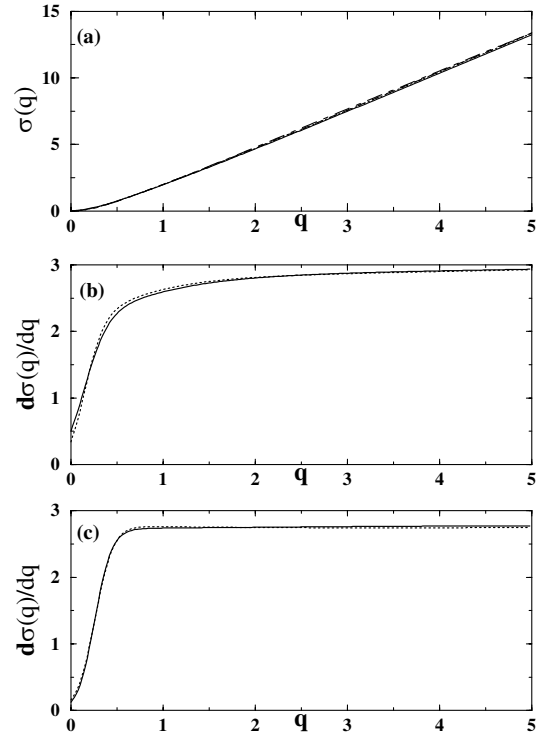


FIG. 3. (a) Plot of $\sigma(q)$ vs q for the BTW (solid line), undirected (dotted line), Manna (dashed line), directed (dot-dashed line) models. Comparison of $d\sigma(q)/dq$ vs q plots between (b) BTW (solid lines) and undirected (dotted lines) models and for the (c) Manna (solid lines) and directed (dotted lines) models.

increases with q . Similarly, $X_{\text{Manna,direc}}$ is limited within 0.91%, whereas $X_{\text{Manna,undir}}$ gradually increases to 6.5% at $q = 5$ which also implies that the directed model behaves very similarly to the Manna model, and is very much different from the BTW model. The above described results concerning X for the various couples of models are altogether strongly supporting the conclusion that while the directed model belongs most likely to the Manna universality class, the undirected one has the multiscaling features known to be peculiar to the standard BTW sandpile.

By decomposing a large sequence of successive avalanches into waves in the undirected and directed cases, we obtained global wave size distributions which obey FSS with the exponents expected for the BTW model [15] and the Manna model [16], respectively. For the BTW sandpile globally sampled waves have a size distribution with a form as in Eq. (1), with $\tau_w = 1$ and $D_w = 2$. In the case of the Manna stochastic sandpile, waves cannot be defined as in the deterministic Abelian sandpiles, but a wavelike decomposition was proposed in Ref. [16]. The global size distribution of the corresponding waves obeys FSS with the same exponents obtained for the avalanche distribution [16]. As already noted above, waves can be consistently defined [2] in the same way for each quenched disorder realization of our directed and undirected models. The

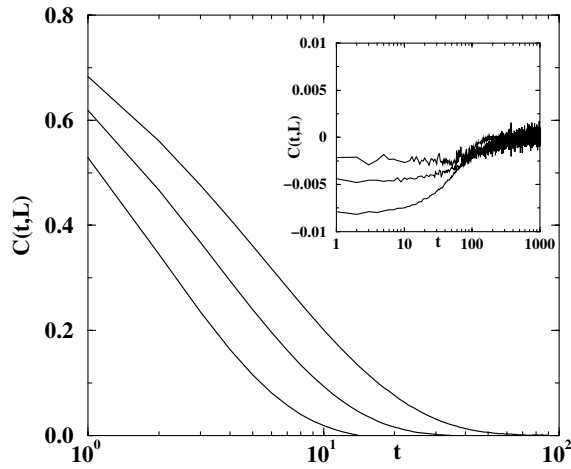


FIG. 4. Autocorrelation function of the wave time series of the undirected and the directed (inset) sandpile for $L = 128, 256,$ and 512 .

global wave scalings obtained here further support the expectation that they fall in the Manna and BTW universality classes, respectively.

Further insight into the different behaviors of the directed and undirected models can be obtained by analyzing the wave time series $\{s_1, s_2, s_3, \dots\}$ of the sizes of successive waves as in Ref. [16]. In Fig. 4 we plot the autocorrelation function:

$$C(t, L) = \frac{\langle s_{k+t} s_k \rangle_L - \langle s_k \rangle_L^2}{\langle s_k^2 \rangle_L - \langle s_k \rangle_L^2}, \quad (2)$$

where the expectation values refer to samples with different L and include quenched disorder averaging. The plots in Fig. 4 are fully consistent with similar ones for the BTW and Manna sandpiles [16]. While in the directed case the autocorrelation function is essentially zero as soon as $t > 0$, in the undirected model it grows steadily with L and approximately scales as $C(t, L) \sim t^{-\tau_c} \mathcal{G}(t/L^{D_c})$ with $\tau_c \approx 0.35$ and $D_c \approx 1$. These exponents should be compared to 0.40 and 1.02, respectively, as determined for the BTW model [16]. This long range autocorrelation must be a consequence of the coherent and uniform spatial structure of each wave in the undirected case. In the directed model, correlations are destroyed by the much more irregular pattern of topplings, with inhomogeneities and holes, in each wave. The correlation patterns show marked self-averaging, being reproducible on the basis of very few disorder realizations.

The local out or in degree balance $H_i = H'_i$ at all sites in the undirected model is essential for the BTW multiscaling behavior to prevail. Numerically, with quenched disorder realized as described above, we find that the density of unbalanced sites with $H_i \neq H'_i$ in the directed model is around 0.88 and those of the sites with $H'_i > H_i$ and $H'_i < H_i$ are equal to 0.44. Now we ask if there is any critical density of unbalanced sites which demarcates the behav-

iors of the undirected and directed models. To study this we first generated an asymmetric TM in which the fraction of the bonds with unequal Δ values is found to be ≈ 0.75 . We tuned this fraction, and thus the density of unbalanced sites, by randomly selecting these bonds and making their Δ values equal by assigning a random integer number between -1 and $-m$. We find that even the presence of as low as 5% bonds with unequal Δ values is sufficient to destroy the multiscaling and to ensure FSS as in the directed model. Thus, as soon as the precise toppling balance is broken, FSS holds, and the universality class turns into that of the Manna sandpile. The transition to Manna behavior does not require a nonzero threshold density of unbalanced sites. Thus, for the undirected model the symmetry of the precise toppling balance is a crucial requisite for the multiscaling to hold. This requisite is, of course, satisfied also by the ordinary BTW sandpile.

To conclude, we studied sandpile models with quenched disorder where the elements of the TM are randomly assigned. With asymmetric TM the precise toppling balance between in and out degrees at each site is not maintained. This imbalance suppresses the wave correlations leading to the BTW-like multiscaling behavior of the avalanche size distribution and results in a FSS regime in the universality class of the Manna stochastic sandpile. Thus, a symmetry mechanism underlies the puzzling difference between BTW and Manna scalings.

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