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# Getting superstring amplitudes by degenerating Riemann surfaces 

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#### Abstract

We explicitly show how the chiral superstring amplitudes can be obtained through factorisation of the higher genus chiral measure induced by suitable degenerations of Riemann surfaces. This powerful tool also allows to derive, at any genera, consistency relations involving the amplitudes and the measure. A key point concerns the choice of the local coordinate at the node on degenerate Riemann surfaces that greatly simplifies the computations. As a first application, starting from recent ansätze for the chiral measure up to genus five, we compute the chiral two-point function for massless Neveu-Schwarz states at genus two, three and four. For genus higher than three, these computations include some new corrections to the conjectural formulae appeared so far in the literature. After GSO projection, the two-point function vanishes at genus two and three, as expected from space-time supersymmetry arguments, but not at genus four. This suggests that the ansatz for the superstring measure should be corrected for genus higher than four.


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## 1. Introduction

In the last years there has been a considerable progress in understanding and deriving explicit formulas for multiloop superstring amplitudes. In a series of papers [1-6], D'Hoker and Phong obtained, from first principles, an explicitly gauge independent expression for the 2-loop chiral superstring measure on the moduli space of Riemann surfaces, given by

[^0]\[

$$
\begin{equation*}
d \mu^{(2)}[\delta]=\Xi^{(2)}[\delta] d \mu_{B o s}^{(2)}, \tag{1.1}
\end{equation*}
$$

\]

where $\delta \in \mathbb{Z}_{2}^{2}$ is an even spin structure, $\Xi^{(2)}[\delta]$ is a modular form of weight 8 for a subgroup of $\operatorname{Sp}(4, \mathbb{Z})$ and $d \mu_{B o s}^{(g)}$ is the genus $g$ bosonic string measure. Based on such a result, they also proved the non-renormalisation of the cosmological constant and of the $n$-point functions, $n \leqslant 3$, up to $g=2$, as expected by space-time supersymmetry arguments [7]. Furthermore, the four-point amplitude has been computed and checked against the constraints coming from S-duality [8].

Direct computations of higher loop corrections to superstring amplitudes have been intensively investigated during the years. In spite of such efforts, direct computations still appear out of reach. Nevertheless, the strong constraints coming from modular invariance and from factorisation under degeneration limits, together with the explicit 2-loop expressions, can lead to reliable conjectures on such corrections. This is the point of view adopted, for example, in [9], where the explicit expression of higher loop contributions to the four-point function has been proposed.

In $[2,3]$, D'Hoker and Phong conjectured that Eq. (1.1) could be extended to genus $g>2$ for a suitable modular form $\Xi^{(g)}[\delta]$ of weight 8 . Such a form is required to fulfill a set of constraints related to holomorphicity, modular invariance and factorisation. In [10] Cacciatori, Dalla Piazza and van Geemen (CDvG) found a solution to such constraints at $g=3$ and in [11] the uniqueness of this solution has been proved.

The CDvG ansatz for the $g=3$ measure has been generalised to any $g$ by Grushevsky [12]. Salvati Manni proved in [13] that such an ansatz provides a solution to the constraints for $g=4,5$. For $g>5$ some issues arise due to the presence of holomorphic roots of modular forms in the definition of the chiral measure, and it is not clear whether such roots are well defined and have the correct modular properties. In [14], Oura, Poor, Salvati Manni and Yuen (OPSMY) proposed an alternative construction for the chiral measure up to $g=5$, using lattice theta series rather than theta constants, as done by Grushevsky. They also proved that the solution to the constraints is unique up to $g=4$. The explicit equivalence of all ansätze up to $g=4$ has been shown in [15]. It is still an open question to understand whether Grushevsky and OPSMY proposals coincide at $g=5$.

There are several consistency conditions that the chiral superstring measure must satisfy. In particular, non-renormalisation theorems from space-time supersymmetry imply that the cosmological constant and the $n$-point functions, for $n \leqslant 3$, must vanish [7]. The vanishing of the $g$-loop correction to the cosmological constant corresponds to the condition

$$
\sum_{\delta \text { even }} \Xi^{(g)}[\delta]=0
$$

where the sum over the spin structures corresponds to the GSO projection [16]. This identity has been proved for the CDvG-Grushevsky (CDvG-G) ansatz for genera 3 [10] and 4 [12]. Remarkably, for $g=4$, the cosmological constant corresponds to a non-zero Siegel modular form of weight 8 (the Igusa-Schottky form), vanishing only on the locus of Jacobians of Riemann surfaces. For $g=5$ the vanishing of the cosmological constant has to be imposed as a further constraint on the chiral measure and it is satisfied by the OPSMY ansatz and by a slight modification of the original Grushevsky's ansatz [17]. It would be interesting to understand whether this further condition implies the uniqueness of the solution also in the case $g=5$.

Consistency conditions related to non-renormalisation of the chiral amplitudes for $n=1,2,3$ Neveu-Schwarz massless states are much more difficult to check. As the two-loop explicit computation shows, these amplitudes are given by a sum of several different contributions that cannot
be easily determined in terms of the chiral measure alone. Very schematically, all such contributions can be collected into two different terms that, following [6], we call the connected ( $\mathcal{B}_{c}[\delta]$ ) and disconnected ( $\mathcal{B}_{d}[\delta]$ ) part (see also [18] for a relevant preliminary investigation of such contributions). The disconnected part can be easily expressed in terms of the chiral measure. In particular, the disconnected part of one-point function vanishes trivially after summing over the spin structures. For $n=2,3, \mathcal{B}_{d}[\delta]$ is given, up to spin-independent factors, by the functions

$$
\begin{align*}
& \hat{A}_{2}[\delta](a, b):=\Xi^{(g)}[\delta] S_{\delta}(a, b)^{2},  \tag{1.2}\\
& \hat{A}_{3}[\delta](a, b, c):=\Xi^{(g)}[\delta] S_{\delta}(a, b) S_{\delta}(b, c) S_{\delta}(c, a), \tag{1.3}
\end{align*}
$$

where $a, b, c$ are the insertion points and $S_{\delta}$ is the Szegö kernel [19]. On the other hand, $\mathcal{B}_{c}[\delta]$ is much more complicated to compute and its precise form is unknown for $g>2$. One possible approach to this problem is to introduce some simplifying assumptions. In this respect, it is useful to analyse the explicit two-loop computation of the two- and three-point functions. In these cases, the connected and disconnected contributions vanish separately after the GSO projection [4]. It is reasonable to conjecture that a similar mechanism occurs at higher genus as well, so that

$$
\sum_{\delta \text { even }} \mathcal{B}_{c}[\delta]=0
$$

and the non-renormalisation theorems would imply that also $\sum_{\delta \text { even }} \mathcal{B}_{d}[\delta]$ vanishes, i.e.

$$
\begin{align*}
& \hat{A}_{2}(a, b):=\sum_{\delta \text { even }} \hat{A}_{2}[\delta](a, b)=0,  \tag{1.4}\\
& \hat{A}_{3}(a, b, c):=\sum_{\delta \text { even }} \hat{A}_{3}[\delta](a, b, c)=0, \tag{1.5}
\end{align*}
$$

for all insertion points $a, b, c$. A strong argument for the identities (1.4) and (1.5) to hold on the hyperelliptic locus for any genus has been given by Morozov in [20], whereas Grushevsky and Salvati Manni proved (1.4) for genus 3 [21]. However, in [22] it has been proved that (1.5) does not hold for any non-hyperelliptic Riemann surface of genus 3 . More precisely, $\hat{A}_{3}(a, b, c)=0$ for all $a, b, c \in C$, where $C$ is a Riemann surface of genus 3 , if and only if $C$ is hyperelliptic. In this paper, we will also prove that (1.4) and (1.5) do not hold at genus four (see Section 2.1). Apparently, these results lead to a contradiction between the chiral measure ansatz at three loop and non-renormalisation theorems. However, as discussed in [22], it is plausible to consider this discrepancy as the evidence that the connected part of the chiral amplitude does not vanish in these cases.

In this paper, we propose a different approach to the computation of the (spin dependent part of the) chiral amplitude for two NS massless states at $g$-loop, for $g=2,3,4$, based on the natural factorisation properties of the chiral measure. More precisely, the two-point function can be obtained by considering the chiral measure at genus $g+1$ in the limit in which one of the handles of the Riemann surface becomes infinitely long. We apply this procedure to the OPSMY ansatz for the chiral measure and show that the two-point function is given by (1.2) plus a correction term. For $g=2,3$ such a term vanishes after summing over the spin structures, so that the complete two-point function vanishes as expected by space-time supersymmetry. This represents a highly non-trivial consistency check for the chiral measure at genus $g+1=3,4$. On the other hand, the two-point function does not vanish at genus 4 , which could be the signal that the OPSMY ansatz must be corrected at $g=5$.

The paper is organised as follows. In Section 2, after reviewing Grushevsky ansatz, we formulate a lemma and proposition based on theta relations, that imply the non-vanishing of the proposed two-point function at genus four. This also easily reproduces the known results in the case of genus lower than four. Another simple consequence is that the proposed three-point amplitude does not vanish at genus four as requested by the non-renormalisation theorem. We conclude this section by considering the OPSMY ansatz for the superstring measure in terms of theta lattices [14].

In Section 3 we consider the degeneration of handles of Riemann surfaces, to provide basic relationships among measure and amplitudes at arbitrary genera. A key point is the choice of a local coordinate at the node of the degenerate Riemann surfaces that greatly simplifies the computations. As an application, we explicitly show that the two-point function corresponds to the leading term in the degeneration parameter. It turns out that the proposed superstring measure actually leads to a vanishing two-point function for genus two and three. In this respect, it should be stressed that while the results in Section 2 are obtained assuming the form (1.2) and (1.3) for the $n$-point functions, here the results are obtained using only the ansatz for the chiral measure, so that this investigation also provides an important check for the ansatz itself at genus three and four. We also directly show that the two-point function at $g=4$, implied by the OPSMY ansatz for the measure, does not vanish as requested by the non-renormalisation theorem. In turn, this also implies that the proposed three-point function does not vanish at the same genus. Section 4 is devoted to our conclusions.

In Appendix A we first fix some notation used in the main text and recall basic facts on Riemann surfaces and Riemann theta functions. Next, we provide a careful analysis of the degeneration of Riemann surfaces which is used in Section 3 to derive the two-point function. We also provide a basic formula for a section of $|2 \Theta|$, with $\Theta$ denoting the theta divisor. In Appendix $B$, after reviewing useful results on unimodular lattices and the associated theta series, we consider the summation on the spin structures. In this context, we obtain some results that, at the best of our knowledge, are new.

## 2. The chiral superstring measure

The chiral superstring measure $d \mu^{(g)}[\delta]$ satisfies some natural consistency conditions coming from modular invariance and factorisation properties [3,10]. Such conditions impose strong constraints on the modular form $\Xi^{(g)}[\delta](\Omega)$ defined in (1.1), which, at least for low genera, are sufficient to uniquely characterise this form. It is easier to first describe the constraints satisfied by $\Xi^{(g)}[0]$ :

1. $\Xi^{(g)}[0](\Omega)$ is a modular form of weight 8 under $\Gamma_{g}(1,2) \subset \Gamma_{g}=\operatorname{Sp}(2 g, \mathbb{Z})$

$$
\begin{equation*}
\Xi^{(g)}[0]\left((A \Omega+B)(C \Omega+D)^{-1}\right)=\operatorname{det}(C \Omega+D)^{8} \Xi^{(g)}[0](\Omega), \tag{2.1}
\end{equation*}
$$

$\left(\begin{array}{ll}A & B \\ C & D\end{array}\right) \in \Gamma_{g}(1,2)$ (see Appendix A for more details on modular forms).
2. In the limit

$$
\Omega_{g} \rightarrow\left(\begin{array}{cc}
\Omega_{k} & 0 \\
0 & \Omega_{g-k}
\end{array}\right)
$$

where $\Omega_{k} \in \mathfrak{H}_{k}, \Omega_{g-k} \in \mathfrak{H}_{g-k}, \Xi^{(g)}[0]\left(\Omega_{g}\right)$ must factorise

$$
\begin{equation*}
\Xi^{(g)}[0]\left(\Omega_{g}\right) \rightarrow \Xi^{(k)}[0]\left(\Omega_{k}\right) \Xi^{(g-k)}[0]\left(\Omega_{g-k}\right) . \tag{2.2}
\end{equation*}
$$

3. For $g=1$, the known result for the chiral measure must be reproduced, so that

$$
\begin{equation*}
\Xi^{(1)}[0](\tau)=\theta[0](\tau)^{4} \prod_{\delta \text { even }} \theta[\delta](\tau)^{4} \tag{2.3}
\end{equation*}
$$

with $\tau \in \mathfrak{H}_{1}$.
Once these properties are satisfied for a certain $\Xi^{(g)}[0]$, then, for any other even spin structure $\delta$ we can define

$$
\begin{equation*}
\Xi^{(g)}[\delta](\Omega):=\operatorname{det}(C \Omega+D)^{-8} \Xi^{(g)}[0]\left((A \Omega+B)(C \Omega+D)^{-1}\right) \tag{2.4}
\end{equation*}
$$

where $\left(\begin{array}{ll}A & B \\ C & D\end{array}\right) \in \Gamma_{g}$ satisfies (see Eqs. (A.2) and (B.2))

$$
\left[\begin{array}{l}
\delta^{\prime}  \tag{2.5}\\
\delta^{\prime \prime}
\end{array}\right]=\left[0 \cdot\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right)\right]=\left[\begin{array}{c}
\left({ }^{t} A C\right)_{0} \\
\left({ }^{t} B D\right)_{0}
\end{array}\right]
$$

(for any matrix $A$, we denote by $A_{0}$ the vector of diagonal entries). With this definition, each $\Xi^{(g)}[\delta]$ can be shown to satisfy all the constraints from modular invariance and factorisation, as an immediate consequence of (2.1), (2.2) and (2.3).

Space-time supersymmetry implies that the cosmological constant must vanish after the GSO projection. In terms of the chiral measure, this condition becomes
4. Vanishing of the cosmological constant

$$
\begin{equation*}
\sum_{\delta \text { even }} \Xi^{(g)}[\delta](\Omega)=0, \quad \Omega \in \mathcal{J}_{g} \subseteq \mathfrak{H}_{g} \tag{2.6}
\end{equation*}
$$

where $\mathcal{J}_{g}$ is the locus of the period matrices of Riemann surfaces of genus $g$.
For $g \leqslant 4$, this last condition is a consequence of (2.1), (2.2) and (2.3), while at genus 5 it must be imposed as an independent constraint. The solution of the above conditions in the case of hyperelliptic Riemann surfaces has been found by Poor and Yuen [23].

### 2.1. Grushevsky ansatz

In [12] an ansatz has been proposed for the chiral superstring measure which satisfies the conditions (2.1), (2.2) and (2.3). At genus 5, a modified version of this ansatz is needed to satisfy also (2.6) [17]. In this subsection, we describe Grushevsky's construction and prove that the functions $\hat{A}_{2}$ and $\hat{A}_{3}$ defined in (1.4) and (1.5) do not vanish at $g=4$.

Let $V$ be a vector subspace of $\mathbb{F}_{2}^{2 g}$, with $\mathbb{F}_{2}:=\{0,1\}$ the field of characteristic 2. Set

$$
P(V):=\prod_{\delta^{\prime} \in V} \theta\left[\delta^{\prime}\right], \quad P_{i, s}^{g}:=\sum_{V, \operatorname{dim} V=i} P(V)^{s}
$$

For each $\delta \in \mathbb{F}_{2}^{2 g}$, consider the affine space $A:=\delta+V$ and define

$$
P(A) \equiv P(V+\delta):=\prod_{\delta^{\prime} \in V} \theta\left[\delta^{\prime}+\delta\right]
$$

and

$$
P_{i, s}^{g}[\delta]:=\sum_{V, \operatorname{dim} V=i} P(V+\delta)^{s}=\sum_{A \ni \delta, \operatorname{dim} A=i} P(A)^{s}
$$

Grushevsky proposal for the modular form $\Xi^{(g)}[\delta]$ appearing in the superstring measure $d \mu[\delta]=$ $\Xi^{(g)}[\delta] d \mu_{B o s}$ is

$$
\Xi_{G}^{(g)}[\delta]:=2^{-g} \sum_{i=0}^{g}(-1)^{i} 2^{\frac{i(i-1)}{2}} P_{i, 2^{4-i}}^{g}[\delta] .
$$

The cosmological constant is

$$
\begin{aligned}
\Xi_{G}^{(g)}:=\sum_{\delta \text { even }} \Xi_{G}^{(g)}[\delta] & =2^{-g} \sum_{i=0}^{g}(-1)^{i} 2^{\frac{i(i+1)}{2}} \sum_{A, \operatorname{dim} A=i} P(A)^{2^{4-i}} \\
& =2^{-g} \sum_{i=0}^{g}(-1)^{i} 2^{\frac{i(i+1)}{2}} S_{i, 2^{4-i}},
\end{aligned}
$$

where

$$
S_{i, s}:=\sum_{A, \operatorname{dim} A=i} P(A)^{s}
$$

Note the factor $2^{\frac{i(i+1)}{2}}$ which differs from $2^{\frac{i(i-1)}{2}}$ in the definition of $\Xi^{(g)}[\delta]$, because in the cosmological constant each affine space $A$ of dimension $i$ is counted $2^{i}$ times, one per each element $\delta \in A$. The cosmological constant up to $g=5$ can be computed using the following relations for the modular forms $S_{i, 2^{4-i}}$ [24]

$$
\begin{aligned}
& \left(2^{2 g}-1\right) S_{0,16}=6 S_{1,8}+24 S_{2,4}, \quad g \geqslant 2, \\
& \left(2^{2 g-2}-1\right) S_{1,8}=18 S_{2,4}+168 S_{3,2}, \quad g \geqslant 3, \\
& \left(2^{2 g-4}-1\right) S_{2,4}=42 S_{3,2}+840 S_{4,1}, \quad g \geqslant 4,
\end{aligned}
$$

together with the following relation which holds on $\mathcal{J}_{g} \subseteq \mathfrak{H}_{g}$ for $g \geqslant 5$ [17]

$$
\left(2^{2 g-6}-1\right) S_{3,2}=90 S_{4,1}+3720 S_{5,1 / 2}, \quad g \geqslant 5 .
$$

It follows that

$$
\Xi_{G}^{(g)}=2^{g-1}\left(2^{g}+1\right) D_{g} J^{(g)}
$$

for some non-vanishing $D_{g} \in \mathbb{C}$, where $2^{g-1}\left(2^{g}+1\right)$ is the number of even spin structures at genus $g$,

$$
\begin{equation*}
J^{(g)}:=\Theta_{E_{8}}^{2}-\Theta_{D_{16}^{+}}=2^{-2 g}\left(\left(1-2^{g}\right) S_{0,16}+2 S_{1,8}\right) \tag{2.7}
\end{equation*}
$$

and $\Theta_{E_{8}}, \Theta_{D_{16}^{+}}$are the theta series corresponding to the even unimodular lattices $\Lambda=E_{8}$ and $\Lambda=D_{16}^{+}$(see Section 2.2). In particular, $\Xi_{G}^{(g)}=0$ for $g=2,3$, because $J^{(g)}$ vanishes identically on $\mathfrak{H}_{g}$ for $g \leqslant 3$, while

$$
\begin{equation*}
D_{4}=-\frac{2^{7} \cdot 3}{7 \cdot 17} \tag{2.8}
\end{equation*}
$$

and

$$
D_{5}=-\frac{2^{11} \cdot 17}{7 \cdot 11 \cdot 31}
$$

For $g=4$ the form $J^{(4)}$ vanishes identically on the locus $\mathcal{J}_{4}$ (in fact, $\mathcal{J}_{4}$ is the divisor of $J^{(4)}$ inside $\mathfrak{H}_{4}$ ), while $J^{(5)} \neq 0$ on $\mathcal{J}_{5}$ [17]. Thus, for the constraint (2.6) to be satisfied, one has to introduce a modified measure at $g=5$

$$
\begin{equation*}
\tilde{\Xi}_{G}^{(5)}[0]:=\Xi_{G}^{(5)}[0]-D_{5} J^{(5)} \tag{2.9}
\end{equation*}
$$

that continues to satisfy the factorisation properties and assures that Eq. (2.6) is satisfied.
Let $C$ be a Riemann surface of genus $g$ and define

$$
\begin{equation*}
\hat{A}_{2}(a, b):=\sum_{\delta \text { even }} \Xi^{(g)}[\delta] S_{\delta}(a, b)^{2}, \tag{2.10}
\end{equation*}
$$

$a, b \in C$, where $S_{\delta}(a, b)$ is the Szegö kernel (see Appendix A.1). It has been proposed that the chiral two-point function for NS states on $C$ corresponds to $\hat{A}_{2}(a, b)$ up to spin independent factors. By space-time supersymmetry, the two-point function is expected to vanish identically on any Riemann surface. It has been proved [21] that with Grushevsky ansatz this condition on (2.10) is satisfied for $g \leqslant 3$. In the following, we will prove that such a condition does not hold for $g=4$. This is an immediate consequence of the following useful lemma.

## Lemma 2.1.

$$
\begin{equation*}
\frac{\partial \Xi_{G}^{(g)}}{\partial \Omega_{i j}}=\frac{2^{4}}{2 \pi i\left(1+\delta_{i j}\right)} \sum_{\delta} \Xi_{G}^{(g)}[\delta] \partial_{i} \partial_{j} \log \theta[\delta] \tag{2.11}
\end{equation*}
$$

Proof. By a direct computation

$$
\begin{aligned}
\frac{\partial S_{k, s}}{\partial \Omega_{i j}} & =\sum_{A, \operatorname{dim} A=k} \frac{\partial}{\partial \Omega_{i j}} \prod_{\delta \in A} \theta[\delta]^{s}=s \sum_{A, \operatorname{dim} A=k} \sum_{\delta \in A} P(A)^{s} \frac{\partial}{\partial \Omega_{i j}} \log \theta[\delta] \\
& =s \sum_{\delta} \sum_{V, \operatorname{dim} V=k} P(\delta+V)^{s} \frac{\partial}{\partial \Omega_{i j}} \log \theta[\delta],
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{\partial \Xi_{G}^{(g)}}{\partial \Omega_{i j}} & =2^{-g} \sum_{k=1}^{g}(-1)^{k} 2^{\frac{k(k+1)}{2}} \frac{\partial S_{k, 2^{4-k}}}{\partial \Omega_{i j}} \\
& =2^{4-g} \sum_{\delta} \sum_{k=1}^{g}(-1)^{k} 2^{\frac{k(k-1)}{2}} \sum_{V, \operatorname{dim} V=k} P(\delta+V)^{2^{4-k}} \frac{\partial}{\partial \Omega_{i j}} \log \theta[\delta] \\
& =2^{4} \sum_{\delta} \Xi_{G}^{(g)}[\delta] \frac{\partial}{\partial \Omega_{i j}} \log \theta[\delta]=2^{4} \sum_{\delta} \frac{\Xi_{G}^{(g)}[\delta]}{2 \pi i\left(1+\delta_{i j}\right)} \partial_{i} \partial_{j} \log \theta[\delta],
\end{aligned}
$$

where, in the last line, we used the heat equation for the theta function

$$
\begin{equation*}
\partial_{i} \partial_{j} \theta[\delta](z, \Omega)=2 \pi i\left(1+\delta_{i j}\right) \frac{\partial}{\partial \Omega_{i j}} \theta[\delta](z, \Omega) \tag{2.12}
\end{equation*}
$$

Proposition 2.2. In the case $\Xi^{(g)}[\delta]$ in (2.10) is identified with $\Xi_{G}^{(g)}[\delta]$, we have

$$
\hat{A}_{2}(a, b)=\omega(a, b) \Xi_{G}^{(g)}+\frac{2 \pi i}{16} \sum_{i \leqslant j}^{g} \frac{\partial \Xi_{G}^{(g)}}{\partial \Omega_{i j}}\left(\omega_{i}(a) \omega_{j}(b)+\omega_{i}(b) \omega_{j}(a)\right) .
$$

Proof. First use the relation (formula (38) in p. 25 of [19], see also Appendix A. 3 for a proof)

$$
\begin{equation*}
S_{\delta}(a, b)^{2}=\omega(a, b)+\sum_{i, j}^{g} \omega_{i}(a) \omega_{j}(b) \partial_{i} \partial_{j} \log \theta[\delta](0), \tag{2.13}
\end{equation*}
$$

to obtain

$$
\begin{align*}
\hat{A}_{2}[\delta](a, b) & \equiv \Xi^{(g)}[\delta] S_{\delta}(a, b)^{2} \\
& =\Xi^{(g)}[\delta] \omega(a, b)+\sum_{i, j}^{g} \Xi^{(g)}[\delta] \omega_{i}(a) \omega_{j}(b) \partial_{i} \partial_{j} \log \theta[\delta](0), \tag{2.14}
\end{align*}
$$

then use the previous lemma.
This result leads immediately to the known results for $g \leqslant 3$ and to a new one for $g=4$.
Corollary 2.3. For $g \leqslant 3$,

$$
\begin{equation*}
\hat{A}_{2}(a, b)=0 \tag{2.15}
\end{equation*}
$$

while for $g=4$

$$
\begin{equation*}
\hat{A}_{2}(a, b) d \mu_{B o s}^{(4)}=c \sum_{i \leqslant j}(-1)^{m_{i j}}\left(\omega_{i}(a) \omega_{j}(b)+\omega_{i}(b) \omega_{j}(a)\right) \bigwedge_{\substack{k \leqslant l,(k, l) \neq(i, j)}}^{4} d \Omega_{k l} \neq 0 \tag{2.16}
\end{equation*}
$$

for some non-zero constant $c \in \mathbb{C}$ and $m_{i j} \in \mathbb{Z}$.
Proof. Eq. (2.15) follows immediately from Proposition 2.2 and the fact that for $g \leqslant 3, \Xi_{G}^{(g)}=0$ identically on $\mathfrak{H}_{g}$. As proved in [13], $\Xi_{G}^{(4)}=D_{4} J^{(4)}$, with $D_{4} \neq 0$ given in (2.8). Furthermore, $\frac{\partial J^{(4)}}{\partial \Omega_{i j}}$ cannot vanish identically on $\mathcal{J}_{4}$ for all $i, j$, because $\mathcal{J}_{4}$ is the divisor of $J^{(4)}$ and is irreducible [25]. Fix some $1 \leqslant i, j \leqslant 4$ and consider the open subset of $\mathcal{J}_{4}$ where $\partial J^{(4)} / \partial \Omega_{i j} \neq 0$. The bosonic string measure on this subset is given (up to a constant) by

$$
d \mu_{B o s}^{(4)}=(-1)^{m_{i j}} \frac{\bigwedge_{k \leqslant l,(k, l) \neq(i, j)}^{4} d \Omega_{k l}}{\partial J^{(4)} / \partial \Omega_{i j}},
$$

where $m_{i j}$ is the position of $d \Omega_{i j}$ with respect to a given ordering of $\left\{d \Omega_{k l}\right\}_{k \leqslant l}$. Then, for each point in $\mathcal{J}_{4}$, we have

$$
\hat{A}_{2}(a, b) d \mu_{\text {Bos }}^{(4)}=\frac{2 \pi i D_{4}}{16} d \mu_{\text {Bos }}^{(4)} \sum_{\substack{i \leqslant j \\ \partial J^{(4)} / \partial \Omega_{i j} \neq 0}} \frac{\partial J^{(4)}}{\partial \Omega_{i j}}\left(\omega_{i}(a) \omega_{j}(b)+\omega_{i}(b) \omega_{j}(a)\right)
$$

$$
\begin{aligned}
& =\frac{2 \pi i D_{4}}{16} \sum_{\substack{i \leqslant j \\
\partial J^{(4)} / \partial \Omega_{i j} \neq 0}}(-1)^{m_{i j}}\left(\omega_{i}(a) \omega_{j}(b)+\omega_{i}(b) \omega_{j}(a)\right) \bigwedge_{\substack{k \leqslant l,(k, l) \neq(i, j)}}^{4} d \Omega_{k l} \\
& =\frac{2 \pi i D_{4}}{16} \sum_{i \leqslant j}(-1)^{m_{i j}}\left(\omega_{i}(a) \omega_{j}(b)+\omega_{i}(b) \omega_{j}(a)\right) \bigwedge_{k \leqslant l,(k, l) \neq(i, j)}^{4} d \Omega_{k l},
\end{aligned}
$$

where we used the fact that $d \Omega_{11} \wedge \cdots \wedge d \hat{\Omega}_{i j} \wedge \cdots \wedge d \Omega_{44}=0$ when $\partial J^{(4)} / \partial \Omega_{i j}=0$.
This corollary also implies that the proposed three-point function

$$
\begin{equation*}
\hat{A}_{3}(a, b, c):=\sum_{\delta \text { even }} \Xi^{(g)}[\delta] S_{\delta}(a, b) S_{\delta}(b, c) S_{\delta}(c, a) \tag{2.17}
\end{equation*}
$$

does not vanish for $g=4$, as expected from space-time supersymmetry. To see this, note that, in the limit $c \rightarrow a$, the coefficient of the term $(c-a)^{-2}$ coincides with $\hat{A}_{2}(a, b)$. As discussed in [22], the fact that (2.10) and (2.17) do not vanish for $g=4$ does not really rule out the proposals $\Xi^{(4)}[\delta]$ for the chiral measure, because it is reasonable that the two- and three-point functions receive other contributions different from $\hat{A}_{2}$ and $\hat{A}_{3}$. For the same reasons, however, the fact that $\hat{A}_{2}$ vanishes at $g=3$ cannot be considered as a real argument in favor of this ansatz. In the following sections, we will consider a more reliable computation for the two-point function based on the factorisation of vacuum amplitudes.

### 2.2. The OPSMY ansatz

In this section, we define $\Xi^{(g)}[\delta]$ in terms of theta series of 16 -dimensional unimodular lattices, following Oura, Poor, Salvati Manni and Yuen (OPSMY) [14]. A $d$-dimensional lattice $\Lambda \subset \mathbb{R}^{d}$ is called unimodular (or self-dual) if it is isomorphic to its dual $\Lambda \cong \Lambda^{*}$, where

$$
\Lambda^{*}:=\left\{\lambda \in \mathbb{R}^{d} \mid \lambda \cdot \mu \in \mathbb{Z} \text { for all } \mu \in \Lambda\right\}
$$

A unimodular lattice is called even if the norm $\lambda \cdot \lambda$ of all its vectors is an even integer, and odd otherwise. There are eight 16 -dimensional unimodular lattices, listed in Table 1, where $E_{8}^{2}$ and $D_{16}^{+}$are even and the others odd [26]. The genus $g$ theta series of a lattice $\Lambda$ is a holomorphic function on $\mathfrak{H}_{g}$ defined as

$$
\begin{equation*}
\Theta_{\Lambda}^{(g)}(\Omega):=\sum_{\lambda_{1}, \ldots, \lambda_{g} \in \Lambda} e^{\pi i \sum_{i, j}^{g} \lambda_{i} \cdot \lambda_{j} \Omega_{i j}} \tag{2.18}
\end{equation*}
$$

Following [14], let us define $\xi^{j}:=\left(\xi_{0}^{j}, \ldots, \xi_{5}^{j}\right) \in \mathbb{C}^{6}, j=0, \ldots, 5$, by ${ }^{1}$

$$
\xi^{0}:=(1,1,1,1,1), \quad \xi^{j}:=\left(0, \frac{1}{8^{j}}, \frac{1}{4^{j}}, \frac{1}{2^{j}}, 1,2^{j}\right), \quad j=1, \ldots, 5
$$

and the dual basis $\left\{\left(c_{0}^{i}, \ldots, c_{5}^{i}\right)\right\}_{i=0, \ldots, 5} \subset \mathbb{C}^{6}$, so that

[^1]Table 1
The 16-dimensional unimodular lattices. The vectors of norm 2 form the root system of the Lie algebra $\mathfrak{g}_{k}$. Each lattice can be decomposed as $\Lambda_{k}=\tilde{\Lambda}_{k} \oplus \mathbb{Z}^{n_{k}}$, where $\tilde{\Lambda}_{k}$ has minimal norm 2, and the associated Lie algebras decompose accordingly $\mathfrak{g}_{k}=\tilde{\mathfrak{g}}_{k} \oplus D_{n_{k}}$ (with the identification $D_{2} \equiv A_{1} \oplus A_{1}$ ). $l_{k}$ is twice the dual Coxeter number of $\tilde{\mathfrak{g}}_{k}$ and $N_{k}$ is the number of roots in $\mathfrak{g}_{k}$. The value $l_{5}=92$ is chosen for later convenience. See Appendix B for the definition of $\Lambda_{k}^{(1)}, \Lambda_{k}^{(2)}$.

| $k$ | $\Lambda_{k}$ | Parity | $n_{k}$ | $\mathfrak{g}_{k}=\tilde{\mathfrak{g}}_{k} \oplus D_{n_{k}}$ | $l_{k}$ | $N_{k}$ | $\Lambda_{k}^{(1)}$ | $\Lambda_{k}^{(2)}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | $\left(D_{8} \oplus D_{8}\right)^{+}$ | odd | 0 | $\left(D_{8} \oplus D_{8}\right) \oplus 0$ | 28 | 224 | $D_{16}^{+}$ | $E_{8}^{2}$ |
| 1 | $\mathbb{Z} \oplus A_{15}^{+}$ | odd | 1 | $A_{15} \oplus 0$ | 32 | 240 | $D_{16}^{+}$ | $D_{16}^{+}$ |
| 2 | $\mathbb{Z}^{2} \oplus\left(E_{7} \oplus E_{7}\right)^{+}$ | odd | 2 | $2 E_{7} \oplus 2 A_{1}$ | 36 | 256 | $E_{8}^{2}$ | $E_{8}^{2}$ |
| 3 | $\mathbb{Z}^{4} \oplus D_{12}^{+}$ | odd | 4 | $D_{12} \oplus D_{4}$ | 44 | 288 | $D_{16}^{+}$ | $D_{16}^{+}$ |
| 4 | $\mathbb{Z}^{8} \oplus E_{8}$ | odd | 8 | $E_{8} \oplus D_{8}$ | 60 | 352 | $E_{8}^{2}$ | $E_{8}^{2}$ |
| 5 | $\mathbb{Z}^{16}$ | odd | 16 | $0 \oplus D_{16}$ | 92 | 480 | $D_{16}^{+}$ | $D_{16}^{+}$ |
| 6 | $E_{8} \oplus E_{8}$ | even | 0 | $\left(E_{8} \oplus E_{8}\right) \oplus 0$ | 60 | 480 | $E_{8}^{2}$ | $E_{8}^{2}$ |
| 7 | $D_{16}^{+}$ | even | 0 | $D_{16} \oplus 0$ | 60 | 480 | $D_{16}^{+}$ | $D_{16}^{+}$ |

$$
\sum_{k=0}^{5} c_{k}^{i} \xi_{k}^{j}=\delta_{i j}
$$

For $g<5$, the theta series of the 16-dimensional unimodular lattices

$$
\Theta_{k}^{(g)}:=\Theta_{\Lambda_{k}}^{(g)}
$$

$k=0, \ldots, 7$, are not linearly independent and the linear relations can be easily expressed using the coefficients $c_{k}^{i}$. In particular [14],

$$
\begin{align*}
& \sum_{k=0}^{5} c_{k}^{i} \Theta_{k}^{(g)}=0, \quad \text { for } g \leqslant 3, g<i \leqslant 5  \tag{2.19}\\
& \sum_{i=0}^{5} c_{k}^{5} \Theta_{k}^{(4)}=C J^{(4)} \tag{2.20}
\end{align*}
$$

where $J^{(g)}$ is defined in (2.7) and

$$
\begin{equation*}
C=-\frac{2^{5} \cdot 3}{7} \tag{2.21}
\end{equation*}
$$

There are also well-known relations between the theta series of even lattices

$$
\begin{equation*}
J^{(g)}=0, \quad g \leqslant 3, \quad J_{\mid J_{4}}^{(4)}=0 . \tag{2.22}
\end{equation*}
$$

Set

$$
\Xi_{O P S M Y}^{(g)}[0](\Omega):=\sum_{k=0}^{5} c_{k}^{g} \Theta_{k}^{(g)}(\Omega)
$$

By (2.4), $\Xi_{\text {OPSMY }}^{(g)}[\delta]$ for every even $\delta \in \mathbb{F}_{2}^{2 g}$ can be easily expressed in terms of the corresponding lattice theta series

$$
\begin{equation*}
\Theta_{k}^{(g)}[\delta](\Omega)=\operatorname{det}(C \Omega+D)^{-8} \Theta_{k}^{(g)}\left((A \Omega+B)(C \Omega+D)^{-1}\right) \tag{2.23}
\end{equation*}
$$

where $\left(\begin{array}{ll}A & B \\ C & D\end{array}\right)$ satisfies (2.5). Therefore

$$
\Xi_{O P S M Y}^{(g)}[\delta](\Omega)=\sum_{k=0}^{5} c_{k}^{g} \Theta_{k}^{(g)}[\delta](\Omega)
$$

A useful expression for $\Theta_{\Lambda}^{(g)}[\delta](\Omega)$ is (see Appendix B for a derivation)

$$
\begin{align*}
\Theta_{\Lambda}^{(g)}[\delta](\Omega)= & \sum_{\lambda_{1}, \ldots, \lambda_{g} \in \Lambda} \exp \left(\pi i \sum_{i, j}^{g}\left(\lambda_{i}+\frac{\delta_{i}^{\prime} u}{2}\right) \cdot\left(\lambda_{j}+\frac{\delta_{j}^{\prime} u}{2}\right) \Omega_{i j}\right. \\
& \left.+2 \pi i \sum_{i}^{g}\left(\lambda_{i}+\frac{\delta_{i}^{\prime} u}{2}\right) \cdot\left(\delta_{i}^{\prime \prime} \frac{u}{2}\right)\right), \tag{2.24}
\end{align*}
$$

where $u \in \Lambda$ is a parity vector for $\Lambda$, i.e. $u \cdot \lambda \equiv \lambda \cdot \lambda \bmod 2$ for all $\lambda \in \Lambda$. We take (2.24) to be the definition of $\Theta_{\Lambda}^{(g)}[\delta]$ for a general integral lattice $\Lambda$ and for every theta characteristic $\delta \in \mathbb{F}_{2}^{2 g}$. Note that, even though this definition makes sense more generally, Eq. (2.23) holds for some $\left(\begin{array}{ll}A & B \\ C & D\end{array}\right) \in \operatorname{Sp}(2 g, \mathbb{Z})$ only if $\Lambda$ is a 16 -dimensional unimodular lattice and $\delta$ is even.

Summing $\Xi_{O P S M Y}^{(g)}[\delta]$ over spin structures yields (see Appendix B for the derivation)

$$
\sum_{\delta \text { even }} \Xi_{O P S M Y}^{(g)}[\delta](\Omega)=2^{g-1}\left(2^{g}+1\right) B_{g} J^{(g)},
$$

where

$$
\begin{equation*}
B_{4}=\frac{2^{2} \cdot 3^{3} \cdot 5 \cdot 11}{7 \cdot 17} \tag{2.25}
\end{equation*}
$$

and

$$
B_{5}=-\frac{2^{5} \cdot 17}{7 \cdot 11}
$$

Thus, in analogy with (2.9), we define a modified measure for $g=5$

$$
\tilde{\Xi}_{O P S M Y}^{(5)}[0](\Omega):=\sum_{k=0}^{7} c_{k}^{5} \Theta_{k}^{(5)}(\Omega)=\Xi_{O P S M Y}^{(5)}[0](\Omega)-B_{5} J^{(5)},
$$

where, for all $g$ (this modification is irrelevant for $g \leqslant 4$ ), we set

$$
\begin{equation*}
c_{6}^{g}=-c_{7}^{g}=-B_{5}, \tag{2.26}
\end{equation*}
$$

so that Eq. (2.6) is satisfied.
It is known that two forms $\Xi^{(g)}$, satisfying (2.1), (2.2) and (2.3), must be the same for $g=2,3$ while for $g=4$ differ by a multiple of the Igusa-Schottky form $J^{(4)}$, which vanishes on the Jacobian locus, so that, by (2.8) and (2.25),

$$
\begin{equation*}
\Xi_{O P S M Y}^{(g)}[\delta]=\Xi_{G}^{(g)}[\delta], \quad g \leqslant 3, \quad \Xi_{O P S M Y}^{(4)}[\delta]=\Xi_{G}^{(4)}[\delta]+\left(B_{4}-D_{4}\right) J^{(4)} \tag{2.27}
\end{equation*}
$$

It is an open question whether $\tilde{\Xi}_{O P S M Y}^{(5)}[\delta]$ and $\tilde{\Xi}_{G}^{(5)}[\delta]$ coincide on the Jacobian locus (see [15] for a discussion on this point). From now on, we will drop the subscripts in $\Xi_{G}^{(g)}$ and $\Xi_{O P S M Y}^{(g)}$ when the result is independent of the particular definition.

## 3. Two-point function from factorisation

Consider a family of Riemann surfaces $\tilde{C}_{t}, 0<|t|<1$ of genus $g+1$ such that, in the limit $t \rightarrow 0$, one of the handles becomes infinitely long or, in the conformally equivalent picture, the cycle $\tilde{\alpha}_{g+1}$ around this handle is pinched to form a node. This family of surfaces can be defined using the standard plumbing fixture procedure, see Section A.2.

It is an old idea, both in conformal field theory and string theory, that, in the limit $t \rightarrow 0$, the amplitudes defined on $\tilde{C}_{t}$ must satisfy suitable factorisation properties [27]. Let us give a rough description of the physical picture behind this idea in the simple case of the zero-point amplitude $Z$ in some conformal field theory. A genus $g+1$ Riemann surface $\tilde{C}_{t}$ can be obtained by "gluing" a long thin cylinder to a Riemann surface $C$ of genus $g$ with two holes. In the limit $t \rightarrow 0$, the cylinder becomes infinitely thin and the holes collapse to two punctures $a, b \in C$. Then, the boundary conditions of the fields around these punctures can be described in terms of vertex operators at $a$ and $b$. In a state-operator formalism, the propagation of states along the cylinder is given by an operator $t^{L_{0}} \bar{t}^{L_{0}}$, where $L_{0}$ and $\bar{L}_{0}$ generate the world-sheet dilatations and rotations. Thus, the zero-point function $Z$ can be expanded as

$$
\begin{equation*}
Z \rightarrow \sum_{\phi}\langle\phi| t^{L_{0}} \bar{t}^{\bar{L}_{0}}|\phi\rangle\left\langle V(\phi, a) V\left(\phi^{*}, b\right)\right\rangle_{C}, \tag{3.1}
\end{equation*}
$$

where the sum runs over a complete set of states of the theory and $\phi^{*}$ denote the conjugated of $\phi$.
In the following, we will apply this procedure to obtain the (spin dependent part of the) chiral two-point function for two NS massless states on a surface of genus $g$ from factorisation of the chiral measure at genus $g+1$. Specialising Eq. (3.1) to the case where $Z$ is the superstring chiral zero-point function before GSO projection, one obtains

$$
d \mu^{(g+1)}[\tilde{\delta}] \rightarrow \sum_{\phi} t^{h_{\phi}}\left\langle V(\phi, a) V\left(\phi^{*}, b\right)\right\rangle_{C}
$$

where the sum is over a complete set of chiral $L_{0}$-eigenstates with $L_{0} \phi=h_{\phi} \phi$. The spin structure $\tilde{\delta} \in \mathbb{F}_{2}^{2 g+2}$ is given by

$$
\tilde{\delta}=\left[\begin{array}{ccc}
\tilde{\delta}_{1}^{\prime} & \ldots & \tilde{\delta}_{g+1}^{\prime} \\
\tilde{\delta}_{1}^{\prime \prime} & \ldots & \tilde{\delta}_{g+1}^{\prime \prime}
\end{array}\right]=\left[\begin{array}{cccc}
\delta_{1}^{\prime} & \ldots & \delta_{g}^{\prime} & \epsilon^{\prime} \\
\delta_{1}^{\prime \prime} & \ldots & \delta_{g}^{\prime \prime} & \epsilon^{\prime \prime}
\end{array}\right] \in \mathbb{F}_{2}^{2 g+2}
$$

where each $\tilde{\delta}_{i}^{\prime}$ (respectively, $\tilde{\delta}_{i}^{\prime \prime}$ ), $i=1, \ldots, g+1$, determines the periodicity of the world-sheet fermionic fields around the cycle $\tilde{\alpha}_{i}$ (resp., $\tilde{\beta}_{i}$ ) of $\tilde{C}_{t}$. In particular, the cycle $\tilde{\alpha}_{g+1}$ encircles the infinitely long cylinder in the limit $t \rightarrow 0$, so that, when $\epsilon^{\prime}=0$ (respectively, $\epsilon^{\prime}=1$ ), the sum in (3.1) runs only over the Neveu-Schwarz (resp., Ramond) sector. Thus, the two-point function at genus $g$ for NS states and for an arbitrary even spin structure $\delta \in \mathbb{F}_{2}^{2 g}$ can be obtained from the degeneration limit of

$$
d \mu^{(g+1)}\left[\begin{array}{cc}
\delta^{\prime} & 0 \\
\delta^{\prime \prime} & 0
\end{array}\right] \quad \text { or } \quad d \mu^{(g+1)}\left[\begin{array}{cc}
\delta^{\prime} & 0 \\
\delta^{\prime \prime} & 1
\end{array}\right]
$$

To project out the NS tachyon, we consider a linear combination such that its leading term corresponds to massless states. Because the GSO projection is implemented by summing the chiral
measure over all spin structures without phases, the correct linear combination to consider is

$$
\frac{1}{2}\left(d \mu^{(g+1)}\left[\begin{array}{cc}
\delta^{\prime} & 0 \\
\delta^{\prime \prime} & 0
\end{array}\right]+d \mu^{(g+1)}\left[\begin{array}{cc}
\delta^{\prime} & 0 \\
\delta^{\prime \prime} & 1
\end{array}\right]\right)=d \mu_{B o s}^{(g+1)} X_{N S}\left[\begin{array}{c}
\delta^{\prime} \\
\delta^{\prime \prime}
\end{array}\right]
$$

where

$$
X_{N S}\left[\begin{array}{c}
\delta^{\prime} \\
\delta^{\prime \prime}
\end{array}\right]:=\frac{1}{2}\left(\tilde{\Xi}^{(g+1)}\left[\begin{array}{cc}
\delta^{\prime} & 0 \\
\delta^{\prime \prime} & 0
\end{array}\right]+\tilde{\Xi}^{(g+1)}\left[\begin{array}{cc}
\delta^{\prime} & 0 \\
\delta^{\prime \prime} & 1
\end{array}\right]\right)
$$

is the spin dependent part. Indeed, the tachyon contribution would correspond to the $t^{1 / 2}$ power in the expansion of $X_{N S}$, but it can be verified that all half-integer powers of $t$ are canceled by summing over the two spin structures. It follows that, as $t \rightarrow 0$,

$$
\begin{equation*}
X_{N S}[\delta]=t \frac{A_{2}[\delta](a, b)}{d z(a) d z(b)}+O\left(t^{2}\right) \tag{3.2}
\end{equation*}
$$

up to an irrelevant spin-independent factor, where $A_{2}[\delta]$ is the chiral two-point function for NS massless states. Note that $A_{2}[\delta](a, b)$ is a meromorphic 1 -differential in $a, b$ and $A_{2}[\delta](a, b) /(d z(a) d z(b))$ corresponds to its evaluation in the local coordinates around $a$ and $b$ used in the plumbing fixture construction, see Section A.2.

### 3.1. Computation of the two-point function

Here we compute $A_{2}[\delta](a, b)$ with the mentioned coordinate choice (see Eq. (3.2)), using the OPSMY ansatz for the chiral measure at genus $g+1$, for $g=2,3,4$. As we will see, such a choice of the local coordinate at the node of degenerate Riemann surfaces leads to a considerable simplification of the calculations.

For a general $\tilde{\Omega} \in \mathfrak{H}_{g+1}$ define

$$
\tilde{\Omega}=\left(\begin{array}{cc}
\Omega & { }^{t} z \\
z & \frac{1}{2 \pi i} \log q
\end{array}\right)
$$

where $\Omega \in \mathfrak{H}_{g}, z \in \mathbb{C}^{g}$ and $q \in \mathbb{C}, 0<|q|<1$. Consider a Riemann surface of genus $g+1$ and take the degeneration limit in which the cycle $\tilde{\alpha}_{g+1}$ is pinched. One obtains a singular surface of genus $g$ with two points $a, b$ identified to form a node. Let $t$ be the degeneration parameter and $\tilde{\Omega}(t) \in \mathfrak{H}_{g+1}$ the corresponding period matrix. As $t \rightarrow 0$, for a suitable choice of local coordinates (see Appendix A.2), we have

$$
\begin{align*}
& q(t):=t+O\left(t^{2}\right), \quad z_{i}(t)=\int_{a}^{b} \omega_{i}+O\left(t^{2}\right), \quad i=1, \ldots, g  \tag{3.3}\\
& \Omega_{i j}(t)=\Omega_{i j}+2 \pi i t E(a, b)^{2}\left(\omega_{i}(a) \omega_{j}(b)+\omega_{i}(b) \omega_{j}(a)\right)+O\left(t^{2}\right), \\
& i, j=1, \ldots, g \tag{3.4}
\end{align*}
$$

where $\Omega \in \mathcal{J}_{g}$. With respect to this choice of local coordinates, (3.2) becomes

$$
\begin{equation*}
X_{N S}[\delta]=t E(a, b)^{2} A_{2}[\delta](a, b)+O\left(t^{2}\right) \tag{3.5}
\end{equation*}
$$

For a generic $\tilde{\Omega} \in \mathfrak{H}_{g+1}$, let us take the expansion of $X_{N S}[\delta] \equiv X_{N S}[\delta](q, z, \Omega)$ as $q \rightarrow 0$

$$
X_{N S}[\delta](q, z, \Omega)=G^{(g)}[\delta](\Omega)+q F^{(g)}[\delta](\Omega, z)+O\left(q^{2}\right)
$$

The effect of summing over the two spin structures $\left[\begin{array}{cc}\delta^{\prime} & 0 \\ \delta^{\prime \prime} & 0\end{array}\right]$ and $\left[\begin{array}{cc}\delta^{\prime} & 0 \\ \delta^{\prime \prime} & 1\end{array}\right]$ is to project out the halfinteger powers in $q$. Modular properties of $\tilde{\Xi}^{(g+1)}\left[\begin{array}{cc}\delta^{\prime \prime} & 0 \\ \delta^{\prime \prime} & 0\end{array}\right](\tilde{\Omega})$ and $\tilde{\Xi}^{(g+1)}\left[\begin{array}{cc}\delta^{\prime} & 0 \\ \delta^{\prime \prime} & 1\end{array}\right](\tilde{\Omega})$ imply that $G^{(g)}[\delta](\Omega)$ is independent of $z$ and $F^{(g)}[\delta](\Omega, z)$ is a section of $|2 \Theta|$ (see Appendix A.3). It follows that $G^{(g)}[\delta](\Omega)$ can be computed for $z=0$. The factorisation properties of the theta series

$$
\Theta_{k}^{(g+1)}\left[\begin{array}{ll}
\delta^{\prime} & 0 \\
\delta^{\prime \prime} & *
\end{array}\right](\tilde{\Omega}) \xrightarrow{z \rightarrow 0} \Theta_{k}^{(1)}\left[\begin{array}{c}
0 \\
*
\end{array}\right]\left((2 \pi i)^{-1} \log q\right) \Theta_{k}^{(g)}\left[\begin{array}{c}
\delta^{\prime} \\
\delta^{\prime \prime}
\end{array}\right](\Omega)
$$

yield

$$
X_{N S}[\delta](q, z=0, \Omega)=\sum_{k=0}^{7} c_{k}^{g+1}\left(1+N_{k} q+O\left(q^{2}\right)\right) \Theta_{k}^{(g)}[\delta](\Omega),
$$

where $N_{k}$ is the number of vectors of norm 2 in the lattice $\Lambda_{k}$ (see Table 1). By (2.19), (2.20), (2.22) and (2.26), it follows that

$$
G^{(g)}[\delta](\Omega)=\sum_{k=0}^{7} c_{k}^{g+1} \Theta_{k}^{(g)}[\delta](\Omega)=\left(C-B_{5}\right) J^{(g)}(\Omega)
$$

Similarly, one can find the expansion of $\left(c_{0}^{g+1} N_{0}, \ldots, c_{5}^{g+1} N_{5}\right) \in \mathbb{C}^{6}$ with respect to the basis $c^{0}, \ldots, c^{5}$ by computing the scalar products with the dual basis $\xi^{0}, \ldots, \xi^{5}$

$$
\sum_{k=0}^{5} c_{k}^{g+1} N_{k} \xi_{k}^{i}=0, \quad i<g, \quad \sum_{k=0}^{5} c_{k}^{g+1} N_{k} \xi_{k}^{g}=128, \quad \sum_{k=0}^{5} c_{k}^{5} N_{k} \xi_{k}^{5}=720
$$

to obtain (note that $N_{6}=N_{7}=480$ )

$$
\begin{equation*}
F^{(g)}[\delta](\Omega, 0)=128 \Xi_{\text {OPSMY }}^{(g)}[\delta](\Omega)+\left(720 C-480 B_{5}\right) J^{(g)} . \tag{3.6}
\end{equation*}
$$

It follows that, as $t \rightarrow 0$,

$$
\begin{aligned}
X_{N S}[\delta]= & t\left(\sum_{i, j}^{g} 2 \pi i E(a, b)^{2} \omega_{i}(a) \omega_{j}(b)\left(1+\delta_{i j}\right)\left(C-B_{5}\right) \frac{\partial J^{(g)}}{\partial \Omega_{i j}}+F^{(g)}[\delta](\Omega, b-a)\right) \\
& +O\left(t^{2}\right) .
\end{aligned}
$$

Since $F^{(g)}$ is a section of $|2 \Theta|$, we can use the standard result (see Appendix A. 3 for a proof)

$$
\begin{aligned}
& F^{(g)}[\delta](\Omega, b-a) \\
& \quad=E(a, b)^{2}\left(F^{(g)}[\delta](\Omega, 0) \omega(a, b)+\frac{1}{2} \sum_{i, j}^{g} \partial_{i} \partial_{j} F^{(g)}[\delta](\Omega, 0) \omega_{i}(a) \omega_{j}(b)\right) .
\end{aligned}
$$

When $\Omega \in \mathcal{J}_{g}$, Eq. (3.6) is equivalent to $F^{(g)}[\delta](\Omega, 0)=128 \Xi^{(g)}[\delta](\Omega)$ that, by (2.27), holds in both cases $\Xi^{(g)}[\delta](\Omega) \equiv \Xi_{\text {OPSMY }}^{(g)}[\delta](\Omega)$ and $\Xi^{(g)}[\delta](\Omega) \equiv \Xi_{G}^{(g)}[\delta](\Omega)$. It follows that

$$
\begin{aligned}
A_{2}[\delta](a, b)= & 128 \Xi^{(g)}[\delta](\Omega) \omega(a, b) \\
& +\sum_{i, j}^{g} \omega_{i}(a) \omega_{j}(b)\left(2 \pi i\left(C-B_{5}\right)\left(1+\delta_{i j}\right) \frac{\partial J^{(g)}}{\partial \Omega_{i j}}+\frac{1}{2} \partial_{i} \partial_{j} F^{(g)}[\delta](\Omega, 0)\right) .
\end{aligned}
$$

Notice that, by (2.22), the term $\frac{\partial J^{(g)}}{\partial \Omega_{i j}}$ vanishes for $g \leqslant 3$ but not for $g=4$.

It remains to compute $\partial_{i} \partial_{j} F^{(g)}[\delta](\Omega, 0)$. For a general lattice $\Lambda$, set

$$
F_{\Lambda}^{(g)}[\delta](\Omega, z):=\sum_{\substack{\lambda_{1}, \ldots, \lambda_{g} \\ \lambda_{k} \in \Lambda+u \frac{\delta_{k}^{\prime}}{2}}} \sum_{\tilde{\tilde{\lambda}} \cdot \tilde{\lambda}=2} e^{\pi i \lambda_{k} \cdot \lambda_{l} \Omega_{k l}+2 \pi i \sum_{k} \lambda_{k} \cdot\left(\tilde{\lambda} z_{k}+u \frac{\delta_{k}^{\prime \prime}}{2}\right)}
$$

and, in particular, $F_{k}[\delta](\Omega, z) \equiv F_{\Lambda_{k}}[\delta](\Omega, z), k=0, \ldots, 7$, with $\Lambda_{k}$ listed in Table 1. Note that $F_{\Lambda}[\delta](\Omega, 0)=N_{\Lambda} \Theta_{\Lambda}[\delta](\Omega)$, where $N_{\Lambda}$ is the number of vectors of norm 2 in $\Lambda$. Since, by (2.24),

$$
\begin{aligned}
& \frac{1}{2}\left(\Theta_{k}^{(g+1)}\left[\begin{array}{ll}
\delta^{\prime} & 0 \\
\delta^{\prime \prime} & 0
\end{array}\right](\tilde{\Omega})+\Theta_{k}^{(g+1)}\left[\begin{array}{cc}
\delta^{\prime} & 0 \\
\delta^{\prime \prime} & 1
\end{array}\right](\tilde{\Omega})\right) \\
& \quad=\Theta_{k}^{(g)}[\delta](\Omega)+q F_{k}^{(g)}[\delta](\Omega, z)+O\left(q^{2}\right)
\end{aligned}
$$

we have

$$
F^{(g)}[\delta](\Omega, z)=\sum_{k=0}^{7} c_{k}^{g+1} F_{k}^{(g)}[\delta](\Omega, z),
$$

and

$$
\partial_{i} \partial_{j} F_{\Lambda}[\delta](\Omega, 0)=\sum_{\substack{\lambda_{1}, \ldots, \lambda_{g} \\ \lambda_{k} \in \Lambda+u \frac{\delta_{k}^{\prime}}{2}}}(2 \pi i)^{2} \sum_{\tilde{\lambda} \cdot \tilde{\lambda}=2}\left(\lambda_{i} \cdot \tilde{\lambda}\right)\left(\tilde{\lambda} \cdot \lambda_{j}\right) e^{\pi i \sum_{k, l} \lambda_{k} \cdot \lambda_{l} \Omega_{k l}+\pi i \sum_{k} \lambda_{k} \cdot u \delta_{k}^{\prime \prime}}
$$

In general, the lattice $\Lambda_{k}$ is a direct sum $\Lambda_{k}=\tilde{\Lambda}_{k} \oplus \mathbb{Z}^{n_{k}}$, where $\tilde{\Lambda}_{k}$ has no vectors of norm 1 [26]. It follows that the set of vectors of norm 2 in $\Lambda$ splits into a disjoint union

$$
\{\lambda \in \Lambda \mid \lambda \cdot \lambda=2\}=\{\lambda \in \tilde{\Lambda} \mid \lambda \cdot \lambda=2\} \sqcup\left\{\lambda \in \mathbb{Z}^{n} \mid \lambda \cdot \lambda=2\right\} .
$$

Hence,

$$
F_{\Lambda}[\delta](\Omega, z)=\Theta_{\tilde{\Lambda}}[\delta](\Omega) F_{\mathbb{Z}^{n}}[\delta](\Omega, z)+\Theta_{\mathbb{Z}^{n}}[\delta](\Omega) F_{\tilde{\Lambda}}[\delta](\Omega, z)
$$

The vectors of norm 2 in $\tilde{\Lambda}_{k}$ are the roots of a semi-simple Lie algebra $\tilde{\mathfrak{g}}_{k}$ (see Table 1). Let $\Delta$ be the set of roots of a simple Lie algebra of rank $r$, a standard result is

$$
\sum_{\alpha \in \Delta} \alpha^{t} \alpha=l_{\Delta} \mathbb{I}_{r}
$$

where $l_{\Delta}$ is a constant depending on the Lie algebra. This can be proved by noting that the matrix on the left-hand side is invariant under the action of the Weyl group, so that it must be proportional to the identity. The constant $l_{\Delta}$ can be easily computed by taking the trace of both sides

$$
\sum_{\alpha \in \Delta} \alpha \cdot \alpha=r l_{\Delta}
$$

In the case of simply-laced algebras one obtains

$$
l_{\Delta}=\frac{2 N}{r}
$$

where $N$ is the number of roots (more generally, $l_{\Delta}$ is twice the dual Coxeter number of the algebra). The Lie algebra $\tilde{g}_{k}$ associated to $\tilde{\Lambda}_{k}$ is either simple or the sum of two copies of the same simple algebra, so that

$$
\sum_{\tilde{\lambda} \cdot \tilde{\lambda}=2}\left(\lambda_{i} \cdot \tilde{\lambda}\right)\left(\tilde{\lambda} \cdot \lambda_{j}\right)=l_{k} \lambda_{i} \cdot \lambda_{j}
$$

with $l_{k}$ given in Table 1. From this identity, one sees that $\tilde{F}_{k}:=F_{\tilde{\Lambda}_{k}}$ satisfies an analog of the heat-kernel equation (2.12)

$$
\partial_{i} \partial_{j} \tilde{F}_{k}[\delta](\Omega, 0)=2 \pi i\left(1+\delta_{i j}\right) l_{k} \frac{\partial}{\partial \Omega_{i j}} \tilde{\Theta}_{k}[\delta](\Omega)
$$

Furthermore, since $\theta[\delta](\Omega, z)$ is even in $z$, one has

$$
F_{\mathbb{Z}^{n}}[\delta](\Omega, z)=\sum_{\substack{\tilde{\lambda} \in \mathbb{Z}^{n} \\ \tilde{\lambda} \cdot \tilde{\lambda}=2}} \prod_{i=1}^{n} \theta[\delta]\left(\Omega, \tilde{\lambda}_{i} z\right)=2 n(n-1) \theta[\delta](\Omega, 0)^{n-2} \theta[\delta](\Omega, z)^{2},
$$

and

$$
\begin{aligned}
\partial_{i} \partial_{j} F_{\mathbb{Z}^{n}}[\delta](\Omega, 0) & =4 n(n-1) \theta[\delta](\Omega, 0)^{n-1} \partial_{i} \partial_{j} \theta[\delta](\Omega, 0) \\
& =2 \pi i\left(1+\delta_{i j}\right)(4 n-4) \frac{\partial}{\partial \Omega_{i j}} \Theta_{\mathbb{Z}^{n}}[\delta](\Omega)
\end{aligned}
$$

Using these results, one gets

$$
\begin{aligned}
& \partial_{i} \partial_{j} F_{k}^{(g)}[\delta](\Omega, 0) \\
& \quad=2 \pi i\left(1+\delta_{i j}\right) l_{k} \frac{\partial \Theta_{k}^{(g)}[\delta](\Omega)}{\partial \Omega_{i j}}+\left(4 n_{k}-4-l_{k}\right) n_{k} \Theta_{k}^{(g)}[\delta](\Omega) \partial_{i} \partial_{j} \log \theta[\delta](\Omega, 0)
\end{aligned}
$$

so that

$$
\begin{aligned}
\partial_{i} \partial_{j} F^{(g)}[\delta](\Omega, 0)= & \sum_{k=0}^{7} c_{k}^{g+1} \partial_{i} \partial_{j} F_{k}^{(g)}[\delta](\Omega, 0) \\
= & 2 \pi i\left(1+\delta_{i j}\right) \frac{\partial}{\partial \Omega_{i j}}\left(\sum_{k=0}^{7} s_{k}^{g} \Theta_{k}^{(g)}[\delta](\Omega)\right) \\
& -\left(\sum_{k=0}^{5} t_{k}^{g} \Theta_{k}^{(g)}[\delta](\Omega)\right) \partial_{i} \partial_{j} \log \theta[\delta](\Omega, 0)
\end{aligned}
$$

where

$$
s_{k}^{g}:=c_{k}^{g+1} l_{k}, \quad t_{k}^{g}:=c_{k}^{g+1} n_{k}\left(l_{k}-4 n_{k}+4\right)
$$

By an explicit computation, one can verify that

$$
\sum_{k=0}^{5} \xi_{k}^{i} s_{k}^{g}=0, \quad i<g, \quad \sum_{k=0}^{5} \xi_{k}^{g} s_{k}^{g}=32, \quad \sum_{k=0}^{5} \xi_{k}^{5} s_{k}^{4}=152
$$

for $g=2,3,4$. By (2.19) and (2.20), and noting that $l_{6}=l_{7}=60$, one obtains

$$
\sum_{k=0}^{7} s_{k}^{g} \Theta_{k}^{(g)}[\delta](\Omega)=32 \Xi_{O P S M Y}^{(g)}[\delta](\Omega)+\left(152 C-60 B_{5}\right) J^{(g)}
$$

Analogously,

$$
\sum_{k=0}^{5} \xi_{k}^{i} t_{k}^{g}=0, \quad i<g, \quad \sum_{k=0}^{5} \xi_{k}^{g} t_{k}^{g}=256
$$

so that

$$
\sum_{k=0}^{5} t_{k}^{g} \Theta_{k}^{(g)}[\delta](\Omega)=256 \Xi^{(g)}[\delta](\Omega), \quad \Omega \in \mathcal{J}_{g}
$$

The final expression for the chiral two-point function is

$$
\begin{align*}
& A_{2}[\delta](a, b) \\
&= 128 \Xi^{(g)}[\delta](\Omega) \omega(a, b)+\sum_{i, j}^{g} \omega_{i}(a) \omega_{j}(b)\left[-\frac{256}{2} \Xi^{(g)}[\delta](\Omega) \partial_{i} \partial_{j} \log \theta[\delta](\Omega, 0)\right. \\
&\left.+2 \pi i\left(1+\delta_{i j}\right) \frac{\partial}{\partial \Omega_{i j}}\left(\frac{32}{2} \Xi_{O P S M Y}^{(g)}[\delta](\Omega)+\frac{1}{2}\left(152 C-60 B_{5}\right) J^{(g)}+\left(C-B_{5}\right) J^{(g)}\right)\right] \\
&= 128 \hat{A}_{2}[\delta](a, b)+\sum_{i, j}^{g} \omega_{i}(a) \omega_{j}(b)\left[-256 \Xi^{(g)}[\delta](\Omega) \partial_{i} \partial_{j} \log \theta[\delta](\Omega, 0)\right.  \tag{3.7a}\\
&\left.+2 \pi i\left(1+\delta_{i j}\right) \frac{\partial}{\partial \Omega_{i j}}\left(16 \Xi_{O P S M Y}^{(g)}[\delta](\Omega)+\left(77 C-31 B_{5}\right) J^{(g)}\right)\right], \tag{3.7b}
\end{align*}
$$

where Eq. (2.14) has been used. We stress that, in the last line of (3.7a) and (3.7b), $\Xi_{O P S M Y}^{(g)}[\delta]$ has been used instead of $\Xi_{G}^{(g)}[\delta]$. This is important for $g=4$, because the difference is proportional to $J^{(4)}$, whose derivatives $\partial J^{(4)} / \partial \Omega_{i j}$ in the directions transverse to the Jacobian locus $\mathcal{J}_{4}$ are not zero. Such an issue does not arise for $\Xi^{(g)}[\delta]$ on the first line of (3.7a) and (3.7b), since $\Xi_{\text {OPSMY }}^{(4)}[\delta]-\Xi_{G}^{(4)}[\delta]=0$ on $\mathcal{J}_{4}$.

### 3.2. Vanishing of the two-point function

For $g=2,3, J^{(g)}=0$ identically on $\mathfrak{H}_{g}$, so that Eq. (3.7b) simplifies to

$$
\begin{aligned}
A_{2}[\delta](a, b)= & 128 \hat{A}_{2}[\delta](a, b)+16 \cdot 2 \pi i\left(1+\delta_{i j}\right) \sum_{i, j}^{g} \omega_{i}(a) \omega_{j}(b) \\
& \times\left(\frac{\partial \Xi^{(g)}[\delta](\Omega)}{\partial \Omega_{i j}}-16 \Xi^{(g)}[\delta](\Omega) \frac{\partial}{\partial \Omega_{i j}} \log \theta[\delta](\Omega, 0)\right) .
\end{aligned}
$$

This reproduces (up to an irrelevant factor) the ansatz (2.10) for the two-point function, plus a correction. After summing over the spin structures, by (2.11) the correction vanishes, so that,
by (2.15),

$$
\begin{equation*}
\sum_{\delta \text { even }} A_{2}[\delta](a, b)=0, \quad g=2,3 \tag{3.8}
\end{equation*}
$$

as expected from space-time supersymmetry. Note, however, that the meaning of this result is quite different from the analogous results obtained so far in the literature. In fact, here we have made no further assumptions on the form of the two-point function, beyond the ansatz for the chiral measure and the natural factorisation properties of string amplitudes. Furthermore, the fact that the two-point function vanishes at genus $g$ is really a check for the chiral measure at genus $g+1$ rather than $g$. For such reasons, (3.8) is a strong argument supporting the ansatz for the chiral measure at genus three and four.

We now directly show that the two-point function at $g=4$, implied by the OPSMY ansatz for the measure, does not vanish at $g=4$. To this end, it is convenient to choose $\Xi_{G}^{(4)}[\delta]$ on the second line of (3.7b), so that, after summing over the even spin structures, we can use (2.11) to simplify this expression (note that the coefficient of the polar part is proportional to the cosmological constant and vanishes)

$$
A_{2}(a, b)=2^{3}\left(2^{4}+1\right)\left(-8 D_{4}+16 B_{4}+77 C-31 B_{5}\right) \sum_{i, j}^{g} \omega_{i}(a) \omega_{j}(b) 2 \pi i\left(1+\delta_{i j}\right) \frac{\partial J^{(4)}}{\partial \Omega_{i j}}
$$

and being

$$
-8 D_{4}+16 B_{4}+77 C-31 B_{5}=-\frac{2^{14}}{7 \cdot 11 \cdot 17} \neq 0
$$

we conclude that the two-point function obtained by factorisation from the OPSMY ansatz at $g=5$ does not vanish.

## 4. Conclusions

The renewed recent interest in trying to solve long standing questions in superstring theory, mainly due to basic papers by D'Hoker and Phong, led to a parallel deeper analysis of the structure of moduli space of Riemann surfaces involving Riemann theta functions, Siegel modular forms and theta series associated to unimodular lattices.

In the present paper we have seen that a careful use of such mathematical results, combined with the old idea of factorisation of string and conformal field theory amplitudes under degeneration limits of Riemann surfaces, provide powerful tools to analyse the structure of superstring amplitudes that would be inaccessible to a direct calculation. A key point that simplifies considerably the computations, concerns the choice of the local coordinate at the node on degenerate Riemann surfaces. Our techniques lead to several advantages with respect to other approaches to the problem, which were based on strong assumptions about the form of these amplitudes. On one hand, one can obtain information on the connected part of the $n$-point function at a certain genus, once the chiral superstring measure is known at higher loop. This could lead to a major advance in the so far prohibitive task of computing higher loop $n$-point functions in the RNS formulation of superstrings. On the other hand, one can use the non-renormalisation theorems for one-, two- and three-point functions to check consistency of the chiral measure at higher genus, without introducing any further assumptions.

We applied this procedure to obtain a general expression the (spin dependent part of the) chiral two-point function for two NS massless states on a surface of genus $g$, for every $g$, from factorisation of the chiral measure at genus $g+1$. Then, we specialised our result to the recent ansätze for the chiral superstring measure and explicitly compute the two-point function up to genus 4. We proved that, after GSO projection, the two-point function vanishes at $g=2,3$ as expected from space-time supersymmetry and, in particular, that the connected and the disconnected part of the amplitude vanish separately.

We also showed that the same result does not hold for the genus four two-point function obtained from the OPSMY ansatz for the chiral measure at genus five. In this case, the connected and disconnected part, after summing over the spin structures, give the same non-vanishing contribution up to a factor, but these contributions do not cancel each other. This probably means that OPSMY ansatz has to be modified. For such a reason, it would be very interesting to understand whether Grushevsky expression for the chiral measure is equivalent to OPSMY at genus five. If they are different, we can conjecture that a certain linear combination of the two ansätze exists, leading to a vanishing two-point function at genus four. If this is the case, then the vanishing of the two-point function at genus $g$ should be imposed as an additional constraint for the chiral measure at genus $g+1$. On the contrary, if Grushevsky and OPSMY expressions are equivalent, it would be interesting to understand whether they are the unique solutions to the constraints.

Another direction for further investigation concerns the computation of the three-point functions at genus $g$ by multiple factorisation of the chiral measure at genus $g+2$. In this respect, it is interesting to observe that the disconnected part of the three-point function vanish at genus $g=2$ but not at genus $g=3$ [22]. Because these amplitudes can be obtained from multiple factorisation of the chiral measure at genera $g+2=4,5$, it is tempting to conjecture that this is related to the vanishing of the disconnected part of the two-point function at genera $g+1=3,4$, respectively. Finally, our techniques could be checked by computing the four-point function at genus two and comparing it with the results of [4]. All such computations involve, however, multiple degenerations limits and are technically more complicated.

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## Appendix A. Theta functions and Riemann surfaces

Here we first provide some background on theta functions and Riemann surfaces (see [19, 28,29 ] for proofs and details). Next, we consider the degeneration of Riemann surfaces which is used in section three to derive the two-point function. We also derive a basic formula for a section of $|2 \Theta|$, with $\Theta$ denoting the theta divisor.

## A.1. Definitions and basic results

Let $\mathfrak{H}_{g}$ denote the Siegel upper half-space, i.e. the space of $g \times g$ complex symmetric matrices with positive definite imaginary part

$$
\mathfrak{H}_{g}:=\left\{\left.\Omega \in M_{g \times g}(\mathbb{C})\right|^{t} \Omega=\Omega, \operatorname{Im} \Omega>0\right\} .
$$

Let $\operatorname{Sp}(2 g, \mathbb{Z})$ be the symplectic modular group, i.e. the group of $2 g \times 2 g$ complex matrices $M:=\left(\begin{array}{cc}A & B \\ C & D\end{array}\right)$, where $A, B, C, D$ are $g \times g$ blocks satisfying

$$
{ }^{t} A C={ }^{t} C A, \quad{ }^{t} B D={ }^{t} D B, \quad{ }^{t} D A-{ }^{t} B C=\mathbb{I}_{g} .
$$

Let us define the action of $\operatorname{Sp}(2 g, \mathbb{Z})$ on $\mathbb{C}^{g} \times \mathfrak{H}_{g}$ by

$$
\begin{equation*}
(M \cdot z, M \cdot \Omega):=\left({ }^{t}(C \Omega+D)^{-1} z,(A \Omega+B)(C \Omega+D)^{-1}\right), \tag{A.1}
\end{equation*}
$$

where $M:=\left(\begin{array}{ll}A & B \\ C & D\end{array}\right) \in \operatorname{Sp}(2 g, \mathbb{Z})$ and $(z, \Omega) \in \mathbb{C}^{g} \times \mathfrak{H}_{g}$.
For each $\delta^{\prime}, \delta^{\prime \prime} \in \mathbb{F}_{2}^{g}$, the theta function $\theta[\delta]:=\theta\left[\begin{array}{c}\delta^{\prime} \\ \delta^{\prime \prime}\end{array}\right]: \mathbb{C}^{g} \times \mathfrak{H}_{g} \rightarrow \mathbb{C}$ with characteristics $[\delta]:=\left[\begin{array}{c}\delta^{\prime} \\ \delta^{\prime \prime}\end{array}\right]$ is defined by

$$
\theta[\delta](z, \Omega):=\sum_{k \in \mathbb{Z}^{g}} \exp \pi i\left[{ }^{t}\left(k+\frac{\delta^{\prime}}{2}\right) \Omega\left(k+\frac{\delta^{\prime}}{2}\right)+2^{t}\left(k+\frac{\delta^{\prime}}{2}\right)\left(z+\frac{\delta^{\prime \prime}}{2}\right)\right]
$$

where $(z, \Omega) \in \mathbb{C}^{g} \times \mathfrak{H}_{g}$. For each fixed $\Omega, \theta[\delta](z, \Omega)$ is an even or odd function on $\mathbb{C}^{g}$ depending whether $(-1)^{\delta^{\prime} \cdot \delta^{\prime \prime}}$ is +1 or -1 , respectively. Correspondingly, there are $2^{g-1}\left(2^{g}+1\right)$ even and $2^{g-1}\left(2^{g}-1\right)$ odd theta characteristics. Under translations $z \mapsto z+\lambda, z \in \mathbb{C}^{g}, \lambda \in$ $\mathbb{Z}^{g}+\Omega \mathbb{Z}^{g} \subset \mathbb{C}^{g}$, theta functions get multiplied by a nowhere vanishing factor

$$
\theta\left[\begin{array}{c}
\delta^{\prime} \\
\delta^{\prime \prime}
\end{array}\right](z+n+\Omega m, \Omega)=e^{-\pi i^{t} m \Omega m-2 \pi i^{t} m z+\pi i\left(\delta^{t} \delta^{\prime} n-\delta^{t} \delta^{\prime \prime} m\right)} \theta\left[\begin{array}{c}
\delta^{\prime} \\
\delta^{\prime \prime}
\end{array}\right](z, \Omega),
$$

$m, n \in \mathbb{Z}^{g}$. It follows that, for any fixed $\Omega$, the theta functions can be seen as sections of line bundles on the complex torus $A_{\Omega}:=\mathbb{C}^{g} /\left(\mathbb{Z}^{g}+\Omega \mathbb{Z}^{g}\right)$, with a well defined divisor on $A_{\Omega}$. We denote by $\Theta$ the divisor of $\theta(z)=\theta[0](z, \Omega)=\theta\left[\begin{array}{l}0 \\ 0\end{array}\right](z, \Omega)$.

The action of $\operatorname{Sp}(2 g, \mathbb{Z})$ on the space of theta characteristics $\mathbb{F}_{2}^{2 g}$ is defined by

$$
[\delta \cdot M]=\left[\binom{\delta^{\prime}}{\delta^{\prime \prime}} \cdot M\right]:=\left(\begin{array}{cc}
{ }^{t} A & { }^{t} C  \tag{A.2}\\
{ }^{t} B & { }^{t} D
\end{array}\right)\left[\begin{array}{c}
\delta^{\prime} \\
\delta^{\prime \prime}
\end{array}\right]+\left[\begin{array}{c}
\left({ }^{t} A C\right)_{0} \\
\left({ }^{t} B D\right)_{0}
\end{array}\right] \bmod 2,
$$

where, for any matrix $A$, we denote by $A_{0}$ the vector of diagonal entries. Theta characteristics are invariant under the action of the subgroup $\Gamma(2) \subset \operatorname{Sp}(2 g, \mathbb{Z})$, where

$$
\Gamma(n):=\left\{M \in \operatorname{Sp}(2 g, \mathbb{Z}) \mid M=\mathbb{I}_{2 g} \bmod n\right\}
$$

is the subgroup of elements of $\operatorname{Sp}(2 g, \mathbb{Z})$ congruent to the $2 g \times 2 g$ identity matrix mod $n$. The theta characteristic $[0]:=\left[\begin{array}{l}0 \\ 0\end{array}\right]$ is fixed by the subgroup

$$
\Gamma(1,2):=\left\{\left.\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right) \in \operatorname{Sp}(2 g, \mathbb{Z}) \right\rvert\,\left({ }^{t} A C\right)_{0} \equiv\left({ }^{t} B D\right)_{0} \equiv 0 \bmod 2\right\}
$$

Symplectic transformations preserve the parity of the characteristics and, for any two $\delta, \epsilon \in \mathbb{Z}_{2}^{2 g}$ of the same parity, there exists $M \in \operatorname{Sp}\left(2 g, \mathbb{Z}_{2}\right)$ such that $\epsilon=M \cdot \delta$.

A (Siegel) modular form $f$ of weight $k \in \mathbb{Z}$ for a subgroup $\Gamma \in \operatorname{Sp}(2 g, \mathbb{Z})$ is a holomorphic function on $\mathfrak{H}_{g}$ such, for all $M \in \Gamma$, that

$$
f(M \cdot \Omega)=\operatorname{det}(C \Omega+D)^{k} f(\Omega)
$$

A condition of regularity, automatically satisfied for $g>1$, is also required for $g=1$.
Let $C$ be a Riemann surface of genus $g>1$. The choice of a marking for $C$ provides a set of generators $\left\{\alpha_{1}, \ldots, \alpha_{g}, \beta_{1}, \ldots, \beta_{g}\right\}$ for the first homology group $H_{1}(C, \mathbb{Z})$ on $C$, with symplectic
intersection matrix, that is

$$
\begin{equation*}
\alpha_{i} \cdot \alpha_{j}=0=\beta_{i} \cdot \beta_{j}, \quad \alpha_{i} \cdot \beta_{j}=\delta_{i j} \tag{A.3}
\end{equation*}
$$

for all $i, j=1, \ldots, g$. The choice of such generators canonically determines a basis $\left\{\omega_{1}, \ldots, \omega_{g}\right\}$ for the space $H^{0}\left(K_{C}\right)$ of holomorphic 1-differentials on $C$, with normalised $\alpha$-periods $\oint_{\alpha_{i}} \omega_{j}=\delta_{i j}$, for all $i, j=1, \ldots, g$. The $\beta$-periods define the Riemann period matrix $\Omega_{i j}:=$ $\oint_{\beta_{i}} \omega_{j}$, which is symmetric and with positive-definite imaginary part, so that $\Omega \in \mathfrak{H}_{g}$. By Torelli's theorem, the complex structure of $C$ is completely determined by its Riemann period matrix.

The conditions (A.3) determine the basis of $H_{1}(C, \mathbb{Z})$ up to a symplectic transformation

$$
\binom{\alpha}{\beta} \mapsto\binom{\tilde{\alpha}}{\tilde{\beta}}:=\left(\begin{array}{cc}
D & C \\
B & A
\end{array}\right)\binom{\alpha}{\beta}, \quad M=\left(\begin{array}{cc}
A & B \\
C & D
\end{array}\right) \in \operatorname{Sp}(2 g, \mathbb{Z})
$$

under which $\left(\omega_{1}, \ldots, \omega_{g}\right) \mapsto\left(\tilde{\omega}_{1}, \ldots, \tilde{\omega}_{g}\right):=\left(\omega_{1}, \ldots, \omega_{g}\right)(C \Omega+D)^{-1}$, whereas $\Omega \mapsto \tilde{\Omega}:=$ $M \cdot \Omega$ transforms as in (A.1).

The complex torus $A_{C}:=\mathbb{C}^{g} /\left(\mathbb{Z}^{g}+\Omega \mathbb{Z}^{g}\right)$ associated to the Riemann period matrix of $C$ is called the Jacobian torus of $C$. For a fixed base-point $p_{0} \in C$, let $I: C \rightarrow A_{C}$ denote the AbelJacobi map, defined by

$$
p \mapsto I(p):=\left(\int_{p_{0}}^{p} \omega_{1}, \ldots, \int_{p_{0}}^{p} \omega_{g}\right) \in A_{C}
$$

Note that different choices of the path of integration from $p_{0}$ to $p$ correspond, by the formula above, to points in $\mathbb{C}^{g}$ differing by elements in the lattice $\mathbb{Z}^{g}+\Omega \mathbb{Z}^{g}$, so that $I$ is well-defined only on $\mathbb{C}^{g} /\left(\mathbb{Z}^{g}+\Omega \mathbb{Z}^{g}\right)$. The Abel-Jacobi map extends to a map from the Abelian group of divisors on $C$ to $A_{C}$ by

$$
I\left(\sum_{i} p_{i}-\sum_{i} q_{i}\right):=\sum_{i} I\left(p_{i}\right)-\sum_{i} I\left(q_{i}\right)
$$

Such a map is independent of the base-point $p_{0}$ when restricted to zero degree divisors. When no confusion is possible, we will identify such zero degree divisors with their image in $A_{C}$ through $I$. In particular, we will omit both $I$ when considering the theta functions on the Jacobian evaluated at (the image of) some zero degree divisor on $C$ and the argument $\Omega$ for theta functions associated to a marked Riemann surface.

Fix a non-singular odd theta characteristic $v \in \mathbb{F}_{2}^{2 g}$ and consider $\sum_{i=1}^{g} \partial_{i} \theta[\nu](0) \omega_{i}$, which is a holomorphic 1-differential with $g-1$ double zeroes and is the square $h_{\nu}^{2}$ of a holomorphic $1 / 2$-differential with odd spin structure $v$. This differential defines the prime form

$$
\begin{equation*}
E(a, b):=\frac{\theta[v](b-a)}{h_{v}(a) h_{v}(b)} \tag{A.4}
\end{equation*}
$$

$a, b \in C$, which is a section of a line bundle on $C \times C$, antisymmetric in its arguments, vanishing only on the diagonal $a=b$ and independent of the choice of $v$.

For each non-singular even characteristic $\delta \in \mathbb{F}_{2}^{2 g}$, the Szegö kernel is the meromorphic 1/2differential

$$
S_{\delta}(a, b):=\frac{\theta[\delta](a-b)}{\theta[\delta](0) E(a, b)}
$$

with a single pole at $a=b$ and holomorphic elsewhere.

Finally, we denote by

$$
\begin{equation*}
\omega_{a-b}(x):=\frac{d}{d x} \log \frac{E(x, a)}{E(x, b)}, \tag{A.5}
\end{equation*}
$$

$a, b, x \in C$, the Abelian 1-differential of the third kind with single poles on $a$ and $b$ with residue +1 and -1 , respectively, holomorphic elsewhere and with vanishing $\alpha$-periods, and with

$$
\begin{equation*}
\omega(a, b):=\frac{d^{2}}{d a d b} \log E(a, b) \tag{A.6}
\end{equation*}
$$

the Abelian 1-differential of the second kind with a double pole of residue 1 at $a=b$, holomorphic elsewhere and with vanishing $\alpha$-periods.

## A.2. Degeneration formulae

Here we derive the degeneration formulae for the Riemann period matrix. A key point concerns the local coordinate at the node of degenerate Riemann surfaces whose choice leads to a considerable simplification of the calculations to derive the two-point function from the chiral measure.

Consider two distinct points $p_{1}, p_{2} \in C$ and let $z_{1}, z_{2}$ be local coordinates

$$
z_{i}:\{z \in \mathbb{C}| | z \mid<1\} \rightarrow U_{i} \subset C, \quad z_{i}\left(p_{i}\right)=0, \quad i=1,2
$$

centered at $p_{1}$ and $p_{2}$, respectively. Then, a family

$$
\left\{\tilde{C}_{t}|t \in \mathbb{C}, 0<|t|<1\}\right.
$$

of Riemann surfaces of genus $g+1$ is defined, where

$$
\tilde{C}_{t}=C \backslash\left(U_{1, t} \cup U_{2, t}\right)
$$

with $U_{i, t}:=\left\{p \in U_{i}| | z_{i}(p)|\leqslant|t|\}, i=1,2\right.$, and two points $p \in U_{1} \backslash U_{1, t}$ and $q \in U_{2} \backslash U_{2, t}$ are identified if

$$
z_{1}(p) z_{2}(q)=t
$$

Let $\alpha_{1}, \ldots, \alpha_{g}, \beta_{1}, \ldots, \beta_{g}$ be a symplectic basis for the homology of $C$, with representatives in $C \backslash\left(U_{1} \cup U_{2}\right)$ and $\omega_{1}, \ldots, \omega_{g}$ the basis of canonically normalised Abelian differentials. We can choose a basis $\tilde{\alpha}_{1}(t), \ldots, \tilde{\alpha}_{g+1}(t), \tilde{\beta}_{1}(t), \ldots, \tilde{\beta}_{g+1}(t)$ of $H_{1}\left(\tilde{C}_{t}, \mathbb{Z}\right)$ such that

$$
\tilde{\alpha}_{i}(t)=\alpha_{i}, \quad \tilde{\beta}_{i}(t)=\beta_{i}, \quad i=1, \ldots, g .
$$

As representatives of $\tilde{\alpha}_{g+1}(t)$ and $\tilde{\beta}_{g+1}(t)$ we can consider, respectively, the circle $\left|z_{1}\right|=\sqrt{|t|}$ and a suitable path on $C$ from $z_{1}^{-1}(x)$ to $z_{2}^{-1}(t / x)$ for some $x \in \mathbb{C},|t|<|x|<1$. Then, the Riemann period matrix $\tilde{\Omega}(t)$ of $\tilde{C}_{t}$ with respect to this basis is [19,30]

$$
\tilde{\Omega}(t)=\left(\begin{array}{cc}
\Omega_{i j}+2 \pi i t \sigma_{i j} & \int_{p_{1}}^{p_{2}} \omega_{i}+t \sigma_{i} \\
\int_{p_{1}}^{p_{2}} \omega_{j}+t \sigma_{j} & \frac{1}{2 \pi i} \log t+c_{0}+c_{1} t
\end{array}\right)+O\left(t^{2}\right),
$$

where

$$
\begin{array}{ll}
\sigma_{i j}=-\frac{\omega_{i}\left(p_{1}\right) \omega_{j}\left(p_{2}\right)+\omega_{i}\left(p_{2}\right) \omega_{j}\left(p_{1}\right)}{d z_{1}\left(p_{1}\right) d z_{2}\left(p_{2}\right)}, & \sigma_{i}=-\left(\gamma_{1} \frac{\omega_{i}}{d z_{2}}\left(p_{2}\right)+\gamma_{2} \frac{\omega_{i}}{d z_{1}}\left(p_{1}\right)\right), \\
c_{0}=\frac{1}{2 \pi i} \lim _{x \rightarrow 0}\left(\int_{z_{1}^{-1}(x)}^{z_{2}^{-1}(x)} \omega_{\left.p_{2}-p_{1}-2 \log x\right),}\right. & c_{1}=\frac{i}{\pi} \gamma_{1} \gamma_{2}, \tag{A.8}
\end{array}
$$

where $\omega_{p_{2}-p_{1}}$ is the 1-differential of the third kind on $C$ defined in (A.5), and

$$
\begin{equation*}
\gamma_{i}=\lim _{x \rightarrow p_{i}}\left(\omega_{p_{1}-p_{2}}(x)-(-1)^{i} \frac{d z_{i}}{z_{i}}(x)\right), \quad i=1,2 . \tag{A.9}
\end{equation*}
$$

All these parameters can be exactly computed for a suitable choice of the coordinates $z_{1}$ and $z_{2}$. Choose $2 g$ curves on $C$ which are representatives of the basis of homology and consider the canonical dissection of $C$ along these curves. We can identify $C$ with a fundamental domain $\hat{C}$ in the upper half-plane $\mathbb{H}$ with respect to the Fuchsian uniformisation on $C$. Let us choose such a dissection so that $p_{1}, p_{2}$ and the paths $\tilde{\alpha}_{g+1}$ and $\tilde{\beta}_{g+1}$ lie in the interior of $\hat{C}$. Fix an arbitrary point $c \in \hat{C}$, distinct from $p_{1}, p_{2}$, and set

$$
z_{1}(p):=\frac{E\left(p, p_{1}\right) E\left(c, p_{2}\right)}{E\left(p, p_{2}\right) E\left(c, p_{1}\right)}=e^{\int_{c}^{p} \omega_{p_{1}-p_{2}}}, \quad z_{2}(q):=\frac{E\left(q, p_{2}\right) E\left(c, p_{1}\right)}{E\left(q, p_{1}\right) E\left(c, p_{2}\right)}=e^{\int_{c}^{q} \omega_{p_{2}-p_{1}}}
$$

These coordinates, that represent the higher genus generalisations of cross-ratios on the sphere, satisfy the following properties

$$
\begin{array}{ll}
d z_{1}(p)=\omega_{p_{1}-p_{2}}(p) z_{1}(p), & d z_{1}\left(p_{1}\right)=\frac{E\left(c, p_{2}\right)}{E\left(p_{2}, p_{1}\right) E\left(c, p_{1}\right)}, \\
d z_{2}(q)=\omega_{p_{2}-p_{1}}(q) z_{2}(q), & d z_{2}\left(p_{2}\right)=\frac{E\left(c, p_{1}\right)}{E\left(p_{1}, p_{2}\right) E\left(c, p_{2}\right)}
\end{array}
$$

where $p, q$ are distinct from $p_{1}, p_{2}$. Replacing these expressions in (A.9), it follows immediately that $\gamma_{i}=0$ in such coordinates. Furthermore, if in (A.8) we choose a path from $z_{1}^{-1}(x)$ to $z_{2}^{-1}(x)$ in $\hat{C}$ passing through $c$, we obtain

$$
\int_{z_{1}^{-1}(x)}^{z_{2}^{-1}(x)} \omega_{p_{2}-p_{1}}=\int_{c}^{z_{1}^{-1}(x)} \omega_{p_{1}-p_{2}}+\int_{c}^{z_{2}^{-1}(x)} \omega_{p_{2}-p_{1}}=\int_{1}^{x} \frac{d z_{1}}{z_{1}}+\int_{1}^{x} \frac{d z_{2}}{z_{2}}=2 \log x
$$

where we used $z_{1}(c)=1=z_{2}(c)$. It follows that $c_{0}=0$ and we finally obtain

$$
\tilde{\Omega}(t)=\left(\begin{array}{cc}
\Omega_{i j}+2 \pi i t \sigma_{i j} & \int_{p_{1}}^{p_{2}} \omega_{i} \\
\int_{p_{1}}^{p_{2}} \omega_{j} & \frac{1}{2 \pi i} \log t
\end{array}\right)+O\left(t^{2}\right),
$$

where

$$
\sigma_{i j}=-\frac{\omega_{i}\left(p_{1}\right) \omega_{j}\left(p_{2}\right)+\omega_{i}\left(p_{2}\right) \omega_{j}\left(p_{1}\right)}{d z_{1}\left(p_{1}\right) d z_{2}\left(p_{2}\right)}=E\left(p_{1}, p_{2}\right)^{2}\left(\omega_{i}\left(p_{1}\right) \omega_{j}\left(p_{2}\right)+\omega_{i}\left(p_{2}\right) \omega_{j}\left(p_{1}\right)\right)
$$

## A.3. A formula for the sections of $|2 \Theta|$

Fix an element $\Omega \in \mathfrak{H}_{g}$ an consider the complex torus $A_{\Omega}:=\mathbb{C}^{g} /\left(\mathbb{Z}^{g}+\Omega \mathbb{Z}^{g}\right)$. A section of $|k \Theta|$ on $A_{\Omega}, k \in \mathbb{N}$, corresponds to a holomorphic function $F$ on $\mathbb{C}^{g}$ obeying the quasiperiodicity conditions

$$
F(z+\Omega m+n)=e^{-k\left(\pi i^{t} m \Omega m+2 \pi i^{t} m z\right)} F(z), \quad m, n \in \mathbb{Z}^{g} .
$$

In the following we will be interested in the space $H^{0}\left(A_{\Omega},|2 \Theta|\right)$ of sections of $|2 \Theta|$, which is spanned by the squares $\theta[\delta]^{2}(z)$ of theta functions with characteristics. In particular, all such sections are even functions of $z$. Let $\Omega$ be the period matrix of a Riemann surface $C$ and $A_{C}$ its Jacobian, and consider the restriction of a section $F \in H^{0}\left(A_{C},|2 \Theta|\right)$ to the locus

$$
C-C:=\left\{(b-a):=\left(\int_{a}^{b} \omega_{1}, \ldots, \int_{a}^{b} \omega_{g}\right) \mid a, b \in C\right\} \subseteq A_{C}
$$

It is easy to see that

$$
\frac{F(b-a)}{E(a, b)^{2}}=\frac{F(b-a)}{\theta[\nu]^{2}(b-a)} h_{\nu}^{2}(a) h_{\nu}^{2}(b)
$$

is a single-valued meromorphic 1-differential with respect to $(a, b) \in C \times C$, with a double pole on the diagonal $a=b$ and holomorphic elsewhere. The space of such differentials is generated by $\left\{\omega_{i}(a) \omega_{j}(b)\right\}_{i, j=1, \ldots, g}$ and by the normalised differential of the second kind $\omega(a, b)$ defined in (A.6), so that

$$
\begin{equation*}
F(b-a)=E(a, b)^{2}\left(c_{0} \omega(a, b)+\sum_{i, j}^{g} c_{i j} \omega_{i}(a) \omega_{j}(b)\right) \tag{A.10}
\end{equation*}
$$

To compute the coefficients $c_{0}, c_{i j}$, let us compare the expansion of both sides of (A.10) in the limit of $b \rightarrow a$. Since $F$ is even, we have

$$
F(b-a)=F(0)+\frac{1}{2}(b-a)^{2} \sum_{i, j}^{g} \partial_{i} \partial_{j} F(0) \omega_{i}(a) \omega_{j}(a)+O(b-a)^{4} .
$$

On the other hand [19],

$$
\begin{aligned}
& d a^{1 / 2} d b^{1 / 2} E(a, b)=(b-a)-\frac{1}{12} S(a)(b-a)^{3}+O(b-a)^{5}, \\
& \omega(a, b)=d a d b\left((b-a)^{-2}+\frac{1}{6} S(a)+O(b-a)^{2}\right),
\end{aligned}
$$

with $S(a)$ a holomorphic projective connection, so that the right-hand side of (A.10) becomes

$$
c_{0}+(b-a)^{2} \sum_{i, j}^{g} c_{i j} \omega_{i}(a) \omega_{j}(a)+O(b-a)^{4} .
$$

It follows that

$$
F(b-a)=E(a, b)^{2}\left(F(0) \omega(a, b)+\frac{1}{2} \sum_{i, j}^{g} \partial_{i} \partial_{j} F(0) \omega_{i}(a) \omega_{j}(b)\right)
$$

Note that, since $\theta[\delta]^{2}(z) \in H^{0}\left(A_{C},|2 \Theta|\right)$, this relation implies Eq. (2.13).

## Appendix B. Theta series and lattices

We collect here some useful results about unimodular lattices and their theta series.

## B.1. Proof of formula (2.24)

To each even theta characteristic $\delta:=\left[\begin{array}{c}\delta^{\prime} \\ \delta^{\prime \prime}\end{array}\right] \in \mathbb{F}_{2}^{2 g}$ associate an element $M_{\delta} \in \operatorname{Sp}(2 g, \mathbb{Z})$ such that

$$
\left[0 \cdot M_{\delta}\right]:=\left[\begin{array}{c}
\left({ }^{t} A C\right)_{0}  \tag{B.1}\\
\left({ }^{t} B D\right)_{0}
\end{array}\right]=[\delta] .
$$

In particular, we can choose

$$
M_{\delta}=\left(\begin{array}{cc}
\operatorname{diag}\left(\delta^{\prime}\right) & -\mathbb{I}_{g}  \tag{B.2}\\
\mathbb{I}_{g} & 0
\end{array}\right)\left(\begin{array}{cc}
\mathbb{I}_{g} & S \\
0 & \mathbb{I}_{g}
\end{array}\right)=\left(\begin{array}{cc}
\operatorname{diag}\left(\delta^{\prime}\right) & \operatorname{diag}\left(\delta^{\prime}\right) S-\mathbb{I}_{g} \\
\mathbb{I}_{g} & S
\end{array}\right),
$$

where, for any vector $v=\left(v_{1}, \ldots, v_{g}\right), \operatorname{diag}(v)$ denotes the diagonal matrix with $\operatorname{diag}(v)_{i i}=v_{i}$. Here, $S$ is an integer $g \times g$ matrix satisfying

$$
\delta^{\prime \prime}=S \delta^{\prime}+S_{0}, \quad S_{0} \cdot \delta^{\prime}=0
$$

For example, if

$$
\left[\begin{array}{l}
\delta^{\prime} \\
\delta^{\prime \prime}
\end{array}\right]=\left[\begin{array}{llllllllll}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0
\end{array}\right]
$$

then

$$
S=\left(\begin{array}{ccccccccc}
0 & 1 & & & & & & & \\
1 & 0 & & & & & & & \\
\\
& & 0 & 1 & & & & & \\
\\
& & 1 & 0 & & & & & \\
\\
& & & & 0 & & & & \\
\\
& & & & & 0 & & & \\
& & & & & & 0 & & \\
& & & & & & & 1 & \\
& & & & & & & & \\
& & & & & & & & \\
& \\
&
\end{array}\right)
$$

and this construction can be easily generalised to every even $\delta$. Note that

$$
M_{\delta} \cdot \Omega=\left(\operatorname{diag}\left(\delta^{\prime}\right)(\Omega+S)-\mathbb{I}\right)(\Omega+S)^{-1}=\operatorname{diag}\left(\delta^{\prime}\right)+\hat{\Omega},
$$

where

$$
\hat{\Omega}=-(\Omega+S)^{-1}
$$

Let $\Lambda$ be a $d$-dimensional unimodular lattice, with $d \equiv 0 \bmod 8$. Choose a basis $\lambda^{(1)}, \ldots$, $\lambda^{(d)} \in \mathbb{R}^{d}$ of generators of $\Lambda$ and let $E$ be the $d \times d$ matrix whose $i$-th column is the vector $\lambda^{(i)}, i=1, \ldots, d$, and $Q$ the Gram matrix

$$
E:=\left(\lambda^{(1)}, \ldots, \lambda^{(d)}\right), \quad Q_{i j}:=\lambda^{(i)} \cdot \lambda^{(j)}
$$

Then, by construction, $Q={ }^{t} E E$ is an integral unimodular matrix. Let $\Theta_{\Lambda}$ be the theta series (2.18) of $\Lambda$ and, as in (2.23), set

$$
\Theta_{\Lambda}[\delta](\Omega)=(\operatorname{det} \hat{\Omega})^{d / 2} \Theta_{\Lambda}\left(\operatorname{diag}\left(\delta^{\prime}\right)+\hat{\Omega}\right)
$$

Let us define $r^{\prime \prime} \in(\mathbb{Z} / 2 \mathbb{Z})^{d}$ and $u \in \Lambda$ by

$$
r^{\prime \prime} \equiv Q_{0} \bmod 2, \quad u:=E Q^{-1} r^{\prime \prime}={ }^{t} E^{-1} r^{\prime \prime}
$$

and notice that for any $\lambda=E n \in \Lambda, n \in \mathbb{Z}^{d}$, we have

$$
\lambda \cdot \lambda={ }^{t} n Q n=\sum_{i}^{d} n_{i}^{2} Q_{i i}+2 \sum_{i<j}^{d} n_{i} Q_{i j} n_{j} \equiv \sum_{i}^{d} n_{i} Q_{i i} \bmod 2,
$$

so that

$$
\begin{equation*}
\lambda \cdot \lambda \equiv n \cdot r^{\prime \prime} \equiv \lambda \cdot u \bmod 2, \quad \text { for all } \lambda \in \Lambda . \tag{B.3}
\end{equation*}
$$

A vector $u \in \Lambda$ satisfying this property is called a parity (or characteristic) vector for $\Lambda$. Thus

$$
\begin{aligned}
& \Theta_{\Lambda}\left(\operatorname{diag}\left(\delta^{\prime}\right)+\hat{\Omega}\right) \\
& \quad=\sum_{\lambda_{1}, \ldots, \lambda_{g} \in \Lambda} e^{\pi i \sum_{i, j} \lambda_{i} \cdot \lambda_{j} \hat{\Omega}_{i j}+\pi i \sum_{i} \delta_{i}^{\prime} \lambda_{i} \cdot \lambda_{i}}=\sum_{\lambda_{1}, \ldots, \lambda_{g} \in \Lambda} e^{\pi i \sum_{i, j} \lambda_{i} \cdot \lambda_{j} \hat{\Omega}_{i j}+\pi i \sum_{i} \delta_{i}^{\prime} \lambda_{i} \cdot u},
\end{aligned}
$$

that can be rewritten as

$$
\Theta_{\Lambda}\left(\operatorname{diag}\left(\delta^{\prime}\right)+\hat{\Omega}\right)=\sum_{n_{1}, \ldots, n_{g} \in \mathbb{Z}^{d}} e^{\left.\pi i \sum_{i, j}{ }^{t}{ }^{t} n_{i} Q n_{j}\right) \hat{\Omega}_{i j}+2 \pi i \sum_{i} \frac{\delta_{i}^{\prime} n_{i}}{2} \cdot r^{\prime \prime}}
$$

so that making a Poisson resummation with respect to $\left(n_{1}, \ldots, n_{g}\right) \in \mathbb{Z}^{g \times d}$ becomes

$$
\Theta_{\Lambda}\left(\operatorname{diag}\left(\delta^{\prime}\right)+\hat{\Omega}\right)=(\operatorname{det} \hat{\Omega})^{-d / 2} \sum_{m_{1}, \ldots, m_{g} \in \mathbb{Z}^{d}} e^{\pi i \sum_{i, j}^{t}\left(m_{i}+\frac{\delta_{i}^{\prime}}{2} r^{\prime \prime}\right) Q^{-1}\left(m_{j}+\frac{\delta_{j}^{\prime}}{2} r^{\prime \prime}\right)\left(\Omega_{i j}+S_{i j}\right)}
$$

Set

$$
n_{i}=Q^{-1} m_{i}, \quad r^{\prime}=Q^{-1} r^{\prime \prime}
$$

and use $Q^{-1} \in \mathrm{GL}(d, \mathbb{Z})$ and $u=E r^{\prime}$ to obtain

$$
\begin{aligned}
\Theta_{\Lambda}[\delta](\Omega) & =\sum_{n_{1}, \ldots, n_{g} \in \mathbb{Z}^{d}} e^{\pi i \sum_{i, j} t\left(n_{i}+\frac{\delta_{i}^{\prime}}{2} r^{\prime}\right) Q\left(n_{j}+\frac{\delta_{j}^{\prime}}{2} r^{\prime}\right)\left(\Omega_{i j}+S_{i j}\right)} \\
& =\sum_{\lambda_{1}, \ldots, \lambda_{g} \in \Lambda} e^{\pi i \sum_{i, j}\left(\lambda_{i}+\frac{\delta_{i}^{\prime}}{2} u\right) \cdot\left(\lambda_{j}+\frac{\delta_{j}^{\prime}}{2} u\right)\left(\Omega_{i j}+S_{i j}\right)}
\end{aligned}
$$

Observe that

$$
\sum_{i, j} \lambda_{i} \cdot \lambda_{j} S_{i j} \equiv \sum_{i} \lambda_{i} \cdot \lambda_{i} S_{i i} \equiv \sum_{i} \lambda_{i} \cdot u S_{i i} \bmod 2,
$$

because $S$ is integral and symmetric. Furthermore, any parity vector $u$ satisfies [31]

$$
u \cdot u \equiv d \bmod 8 \equiv 0 \bmod 8
$$

(for $d=16$ this can also be checked by an explicit case by case calculation), so that the following congruences mod 2 hold

$$
\begin{aligned}
\sum_{i, j}\left(\lambda_{i}+\frac{\delta_{i}^{\prime}}{2} u\right) \cdot\left(\lambda_{j}+\frac{\delta_{j}^{\prime}}{2} u\right) S_{i j} & \equiv \sum_{i} \lambda_{i} \cdot u\left(\left(S \delta^{\prime}\right)_{i}+\left(S_{0}\right)_{i}\right)+\sum_{i, j} \frac{u \cdot u}{4} \delta_{i}^{\prime} S_{i j} \delta_{j}^{\prime} \\
& \equiv \sum_{i} \lambda_{i} \cdot u \delta_{i}^{\prime \prime} \equiv \sum_{i}\left(\lambda_{i}+\frac{\delta_{i}^{\prime}}{2} u\right) \cdot u \delta_{i}^{\prime \prime} \bmod 2
\end{aligned}
$$

and (2.24) follows. Note that the set of parity vectors, i.e. the vectors in $\Lambda$ satisfying (B.3), is given by $u+2 \Lambda$, and (2.24) does not change if we replace $u$ by an arbitrary $\tilde{u} \in u+2 \Lambda$. Also note that $\tilde{u} \cdot \tilde{u} \equiv u \cdot u \bmod 8$ (this property holds for unimodular lattices of any dimension [31]). The definition (2.24) makes sense also for $\delta$ an odd theta characteristic, but in this case there is no $M \in \operatorname{Sp}(2 g, \mathbb{Z})$ satisfying (B.1).

## B.2. Sums over spin structures

Let $\Lambda$ be an odd unimodular lattice and $\Lambda_{e} \subset \Lambda$ the sublattice of vectors of even norm, so that $\Lambda_{e} \subset \Lambda \subset \Lambda_{e}^{*}$. If $u \in \Lambda$ is a parity vector, i.e. satisfies (B.3), then $u / 2 \in \Lambda_{e}^{*}$ and it maps to a nontrivial element of $\Lambda_{e}^{*} / \Lambda \cong \mathbb{Z}_{2}$. Let $\lambda_{o} \in \Lambda$ be an arbitrary vector of odd norm and $\Lambda_{o}=\lambda_{o}+\Lambda_{e}$ the set of vectors of odd norm, so that $\Lambda=\Lambda_{e} \cup \Lambda_{o}$. We have the decomposition

$$
\Lambda_{e}^{*}=\Lambda \cup\left(\frac{u}{2}+\Lambda\right)=\Lambda_{e} \cup \Lambda_{o} \cup\left(\frac{u}{2}+\Lambda_{e}\right) \cup\left(\frac{u}{2}+\Lambda_{o}\right) .
$$

Set

$$
\begin{equation*}
\Lambda^{(1)}:=\Lambda_{e} \cup\left(\frac{u}{2}+\Lambda_{e}\right), \quad \Lambda^{(2)}:=\Lambda_{e} \cup\left(\frac{u}{2}+\Lambda_{o}\right) . \tag{B.4}
\end{equation*}
$$

Proposition B.1. If $\Lambda$ is a d-dimensional unimodular lattice, with $d \equiv 0 \bmod 8$, then $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are d-dimensional even unimodular lattices.

Proof. For $d$-dimensional unimodular lattices, the norm of a parity vector satisfies $u \cdot u \equiv$ $d \bmod 8$ [31]. It follows that, for $d \equiv 0 \bmod 8$,

$$
\frac{u}{2} \cdot \frac{u}{2}=\frac{u \cdot u}{4} \in 2 \mathbb{Z}, \quad\left(\frac{u}{2}+\lambda_{o}\right) \cdot\left(\frac{u}{2}+\lambda_{o}\right)=\frac{u \cdot u}{4}+u \cdot \lambda_{o}+\lambda_{o} \cdot \lambda_{o} \in 2 \mathbb{Z}
$$

In particular, $u$ and $u+2 \lambda_{o}$ are elements of $\Lambda_{e}$, so that $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are closed under the sum. Furthermore, they are integral $\left(\Lambda^{(i)} \subseteq \Lambda^{(i)^{*}}\right)$ and even. To prove that they are self-dual, first observe that $\Lambda^{(i)^{*}} \subset \Lambda_{e}^{*}$ because $\Lambda_{e} \subset \Lambda^{(i)}$. Since

$$
\lambda_{o} \cdot \frac{u}{2}, \lambda_{o} \cdot\left(\frac{u}{2}+\lambda_{o}\right), \frac{u}{2} \cdot\left(\frac{u}{2}+\lambda_{o}\right) \in \frac{1}{2}+\mathbb{Z},
$$

we conclude that $\Lambda^{(i)^{*}} \cap\left(\Lambda_{e}^{*} \backslash \Lambda^{(i)}\right)$ is empty.
In particular, when $d=16, \Lambda^{(1)}$ and $\Lambda^{(2)}$ must be isomorphic to either $D_{16}^{+}$or $E_{8}^{2}$. The even lattices corresponding to each $\Lambda_{k}$ can be found by considering its set of vectors of norm 2 , which
is the root system of the Lie algebras $\mathfrak{g}_{k}$ (see Table 1). This root system must be contained in $\Lambda_{k}^{(i)}$ and in most cases, namely for $k>0$, there is only one even unimodular lattice satisfying this constraint, so that $\Lambda_{k}^{(1)} \cong \Lambda_{k}^{(2)}$. The only exception is $\Lambda_{0}$, because $\mathfrak{g}_{0} \cong D_{8} \oplus D_{8}$ can be embedded both in $E_{8} \oplus E_{8}$ or in $D_{16}$. In this case, a more detailed analysis of the root systems shows that $\Lambda^{(1)}=\Lambda_{D_{16}^{+}}$and $\Lambda^{(2)}=\Lambda_{E_{8}^{2}}$.

Proposition B.2. Let $\Lambda$ be a d-dimensional unimodular lattice, with $d \equiv 0 \bmod 8, \Lambda^{(1)}$ and $\Lambda^{(2)}$ defined as in (B.4) and $\Theta_{\Lambda}^{(g)}[\delta]$ the theta series (2.24) for every (even or odd) theta characteristic $\delta$. Then

$$
\sum_{\delta \text { even }} \Theta_{\Lambda}^{(g)}[\delta]=2^{g-1}\left(\Theta_{\Lambda^{(1)}}^{(g)}+\Theta_{\Lambda^{(2)}}^{(g)}\right),
$$

and

$$
\sum_{\delta \text { odd }} \Theta_{\Lambda}^{(g)}[\delta]=2^{g-1}\left(\Theta_{\Lambda^{(1)}}^{(g)}-\Theta_{\Lambda^{(2)}}^{(g)}\right)
$$

Proof. For any $0 \leqslant k \leqslant g, \lambda_{k+1}, \ldots, \lambda_{g} \in \mathbb{R}^{d}, \Omega \in \mathfrak{H}_{g}$ and $2 k$-dimensional theta characteristic $[\delta]=\left[\begin{array}{c}\delta^{\prime} \\ \delta^{\prime \prime}\end{array}\right], \delta^{\prime}, \delta^{\prime \prime} \in \mathbb{F}_{2}^{k}$, let us define

$$
R_{k}^{(g)}[\delta]\left(\lambda_{k+1}, \ldots, \lambda_{g}, \Omega\right):=\sum_{\substack{\lambda_{1}, \ldots, \lambda_{k} \\ \lambda_{i} \in \Lambda+\delta_{i}^{\prime} \frac{u}{2}}}(-1)^{\sum_{i=1}^{k} \delta_{i}^{\prime \prime} \lambda_{i} \cdot u} e^{\pi i \sum_{i, j}^{g} \lambda_{i} \cdot \lambda_{j} \Omega_{i j}}, \quad k>0,
$$

and $R_{0}^{(g)}\left(\lambda_{1}, \ldots, \lambda_{g}, \Omega\right):=e^{\pi i \sum_{i, j}^{g} \lambda_{i} \cdot \lambda_{j} \Omega_{i j}}$. We will prove that, for all $1 \leqslant k \leqslant g$

$$
\begin{align*}
& \sum_{\delta \in \mathbb{F}_{2}^{2 k} \text { even }} R_{k}^{(g)}[\delta]=2^{k-1}\left(\sum_{\lambda_{1}, \ldots, \lambda_{k} \in \Lambda^{(1)}} R_{0}^{(g)}+\sum_{\lambda_{1}, \ldots, \lambda_{k} \in \Lambda^{(2)}} R_{0}^{(g)}\right), \\
& \sum_{\delta \in \mathbb{F}_{2}^{2 k} \text { odd }} R_{k}^{(g)}[\delta]=2^{k-1}\left(\sum_{\lambda_{1}, \ldots, \lambda_{k} \in \Lambda^{(1)}} R_{0}^{(g)}-\sum_{\lambda_{1}, \ldots, \lambda_{k} \in \Lambda^{(2)}} R_{0}^{(g)}\right) . \tag{B.5}
\end{align*}
$$

The proposition corresponds to the particular case $k=g$. For all $1 \leqslant k \leqslant g$ and $[\hat{\delta}]=$ $\left[\begin{array}{c}\hat{\delta}^{\prime} \\ \hat{\delta}^{\prime \prime}\end{array}\right] \in \mathbb{F}_{2}^{2(k-1)}$, we have

$$
\begin{aligned}
& R_{k}^{(g)}\left[\begin{array}{cc}
\hat{\delta}^{\prime} & 0 \\
\hat{\delta}^{\prime \prime} & 0
\end{array}\right]=\sum_{\lambda_{k} \in \Lambda_{e}} R_{k-1}^{(g)}[\hat{\delta}]+\sum_{\lambda_{k} \in \Lambda_{o}} R_{k-1}^{(g)}[\hat{\delta}], \\
& R_{k}^{(g)}\left[\begin{array}{cc}
\hat{\delta}^{\prime} & 0 \\
\hat{\delta}^{\prime \prime} & 1
\end{array}\right]=\sum_{\lambda_{k} \in \Lambda_{e}} R_{k-1}^{(g)}[\hat{\delta}]-\sum_{\lambda_{k} \in \Lambda_{o}} R_{k-1}^{(g)}[\hat{\delta}], \\
& R_{k}^{(g)}\left[\begin{array}{cc}
\hat{\delta}^{\prime} & 1 \\
\hat{\delta}^{\prime \prime} & 0
\end{array}\right]=\sum_{\lambda_{k} \in \Lambda_{e}+\frac{u}{2}} R_{k-1}^{(g)}[\hat{\delta}]+\sum_{\lambda_{k} \in \Lambda_{o}+\frac{u}{2}} R_{k-1}^{(g)}[\hat{\delta}], \\
& R_{k}^{(g)}\left[\begin{array}{cc}
\hat{\delta}^{\prime} & 1 \\
\hat{\delta}^{\prime \prime} & 1
\end{array}\right]=\sum_{\lambda_{k} \in \Lambda_{e}+\frac{u}{2}} R_{k-1}^{(g)}[\hat{\delta}]-\sum_{\lambda_{k} \in \Lambda_{o}+\frac{u}{2}} R_{k-1}^{(g)}[\hat{\delta}],
\end{aligned}
$$

so that

$$
R_{k}^{(g)}\left[\begin{array}{cc}
\hat{\delta}^{\prime} & 0 \\
\hat{\delta}^{\prime \prime} & 0
\end{array}\right]+R_{k}^{(g)}\left[\begin{array}{cc}
\hat{\delta}^{\prime} & 0 \\
\hat{\delta}^{\prime \prime} & 1
\end{array}\right]+R_{k}^{(g)}\left[\begin{array}{cc}
\hat{\delta}^{\prime} & 1 \\
\hat{\delta}^{\prime \prime} & 0
\end{array}\right]=\sum_{\lambda_{k} \in \Lambda^{(1)}} R_{k-1}^{(g)}[\hat{\delta}]+\sum_{\lambda_{k} \in \Lambda^{(2)}} R_{k-1}^{(g)}[\hat{\delta}],
$$

and

$$
R_{k}^{(g)}\left[\begin{array}{cc}
\hat{\delta}^{\prime} & 1 \\
\hat{\delta}^{\prime \prime} & 1
\end{array}\right]=\sum_{\lambda_{k} \in \Lambda^{(1)}} R_{k-1}^{(g)}[\hat{\delta}]-\sum_{\lambda_{k} \in \Lambda^{(2)}} R_{k-1}^{(g)}[\hat{\delta}] .
$$

From these formulas, Eq. (B.5) for $k=1$ follows immediately. Now, suppose that Eq. (B.5) holds for $k-1$. Then

$$
\begin{aligned}
& \sum_{\delta \in \mathbb{F}_{2}^{2 k} \text { even }} R_{k}^{(g)}[\delta]=\sum_{\hat{\delta} \in \mathbb{F}_{2}^{2(k-1)} \text { even }}\left(R_{k}^{(g)}\left[\begin{array}{cc}
\hat{\delta}^{\prime} & 0 \\
\hat{\delta}^{\prime \prime} & 0
\end{array}\right]+R_{k}^{(g)}\left[\begin{array}{cc}
\hat{\delta}^{\prime} & 0 \\
\hat{\delta}^{\prime \prime} & 1
\end{array}\right]+R_{k}^{(g)}\left[\begin{array}{cc}
\hat{\delta}^{\prime} & 1 \\
\hat{\delta}^{\prime \prime} & 0
\end{array}\right]\right) \\
& +\sum_{\hat{\delta} \in \mathbb{F}_{2}^{2(k-1)} \text { odd }} R_{k}^{(g)}\left[\begin{array}{ll}
\hat{\delta}^{\prime} & 1 \\
\hat{\delta}^{\prime \prime} & 1
\end{array}\right] \\
& =2^{k-2}\left(\sum_{\substack{\lambda_{k} \in \Lambda^{(1)} \\
\lambda_{1}, \ldots, \lambda_{k-1} \in \Lambda^{(1)}}} R_{0}^{(g)}+\sum_{\substack{\lambda_{k} \in \Lambda^{(1)} \\
\lambda_{1}, \ldots, \lambda_{k-1} \in \Lambda^{(2)}}} R_{0}^{(g)}+\sum_{\substack{\lambda_{k} \in \Lambda^{(2)} \\
\lambda_{1}, \ldots, \lambda_{k-1} \in \Lambda^{(1)}}} R_{0}^{(g)}\right. \\
& +\sum_{\substack{\lambda_{k} \in \Lambda^{(2)} \\
\lambda_{1}, \ldots, \lambda_{k-1} \in \Lambda^{(2)}}} R_{0}^{(g)}+\sum_{\substack{\lambda_{k} \in \Lambda^{(1)} \\
\lambda_{1}, \ldots, \lambda_{k-1} \in \Lambda^{(1)}}} R_{0}^{(g)}-\sum_{\substack{\lambda_{k} \in \Lambda^{(1)} \\
\lambda_{1}, \ldots, \lambda_{k-1} \in \Lambda^{(2)}}} R_{0}^{(g)} \\
& \left.-\sum_{\substack{\lambda_{k} \in \Lambda^{(2)} \\
\lambda_{1}, \ldots, \lambda_{k-1} \in \Lambda^{(1)}}} R_{0}^{(g)}+\sum_{\substack{\lambda_{k} \in \Lambda^{(2)} \\
\lambda_{1}, \ldots, \lambda_{k-1} \in \Lambda^{(2)}}} R_{0}^{(g)}\right) \\
& =2^{k-1}\left(\sum_{\lambda_{1}, \ldots, \lambda_{k} \in \Lambda^{(1)}} R_{0}^{(g)}+\sum_{\lambda_{1}, \ldots, \lambda_{k} \in \Lambda^{(2)}} R_{0}^{(g)}\right) .
\end{aligned}
$$

An analogous computation gives the case with odd spin structures.
Corollary B.3. For the lattices $\Lambda_{k}, k=1, \ldots, 5$ in Table $1, \Lambda^{(1)} \cong \Lambda^{(2)}$.
Proof. Notice that for these lattices

$$
\Theta_{k}[\delta]=\theta[\delta]^{n_{k}} \Theta_{\tilde{\Lambda}_{k}}[\delta],
$$

with $n_{k}>0$. It follows that $\Theta_{k}[\delta]=0$ if $\delta$ is odd. By Proposition B.2, this implies

$$
\Theta_{\Lambda^{(1)}}^{(g)}=\Theta_{\Lambda^{(2)}}^{(g)}
$$

for all $g$ and, since $E_{8}^{2}$ and $D_{16}^{+}$have different theta series at $g=4$, one gets $\Lambda^{(1)} \cong \Lambda^{(2)}$.
As an application, we can use this result to compute the constant $C$ in (2.21). By summing both sides of (2.20) over all even spin structures, we obtain

$$
2^{3}\left(c_{0}^{5}+2 c_{2}^{5}+2 c_{4}^{5}\right) \Theta_{E_{8}^{2}}+2^{3}\left(c_{0}^{5}+2 c_{1}^{5}+2 c_{3}^{5}+2 c_{5}^{5}\right) \Theta_{D_{16}^{+}}=2^{3}\left(2^{4}+1\right) C\left(\Theta_{E_{8}^{2}}-\Theta_{D_{16}^{+}}\right)
$$

so that

$$
C=\frac{c_{0}^{5}+2 c_{2}^{5}+2 c_{4}^{5}}{17}=-\frac{c_{0}^{5}+2 c_{1}^{5}+2 c_{3}^{5}+2 c_{5}^{5}}{17}=-\frac{2^{5} \cdot 3}{7}
$$

An analogous calculation gives the constants $B_{4}$ and $B_{5}$.

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[^1]:    1 We use a different normalisation with respect to [14], so that $\Xi_{O P S M Y}^{(g)}[\delta]$ and $\Xi_{G}^{(g)}[\delta]$ have the same normalisation. In particular, $c_{k}^{i}$ is $2^{4 i}$ times the corresponding coefficient in [14].

