

Deligne's duality for de Rham realizations of 1-motives

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Dedicated to the Memory of Valentino Cristante

Let M be a 1-motive over a base scheme S and M' its Cartier dual. We show the existence of a canonical duality between the de Rham realizations of M and M' ; this generalizes a result in [5]. Furthermore, we study universal extensions of 1-motives and their relation with \mathfrak{h} -extensions.

1 Introduction

It is known (cf. [9]) that the Lie algebra of the universal extension $A^{\mathfrak{h}}$ of an abelian variety A is canonically isomorphic to the first de Rham cohomology group of the dual abelian variety A' . Deligne defines in [5] the de Rham realization $T_{\mathrm{dR}}(M)$ of a 1-motive $M = [X \rightarrow G]$ over a base scheme S as the Lie algebra of $\mathbf{G}^{\mathfrak{h}}$, where $M^{\mathfrak{h}} = [X \rightarrow \mathbf{G}^{\mathfrak{h}}]$ is a universal extension of M . In this way he gets a (covariant) functor from the category of 1-motives to the category of locally free sheaves over S . In the case S is the spectrum of an algebraically closed field k , Deligne defines a pairing $\Phi : T_{\mathrm{dR}}(M) \otimes T_{\mathrm{dR}}(M') \rightarrow k$ (cf. [5], 10.2.7.3) between the de Rham realizations of a 1-motive and its Cartier dual $M' = [X' \rightarrow G']$; for $k = \mathbb{C}$ the pairing Φ coincides with an analogous (perfect) pairing on Hodge realizations (cf. [5], 10.2.8) and hence it is perfect. In the present paper we construct a pairing Φ between de Rham realizations of dual 1-motives over a general base S and we show that it is perfect; this fact generalizes also a result of Coleman that shows the perfectness of Φ in the case of abelian schemes (over a base flat over \mathbb{Z}) via the comparison with a second (perfect) pairing (cf. [3], 1.1.1). As the existing proofs do not extend to the general case, we show directly the perfectness of Φ proving that this pairing fits in a diagram

$$\begin{array}{ccccc}
 \omega_{G'} & \otimes & \underline{\mathrm{Lie}}(G') & \longrightarrow & \underline{\mathrm{Lie}}(\mathbb{G}_{m,S}) \\
 \downarrow \iota & & \uparrow g' & & \\
 \Phi: T_{\mathrm{dR}}(M) & \otimes & T_{\mathrm{dR}}(M') & \longrightarrow & \underline{\mathrm{Lie}}(\mathbb{G}_{m,S}) \\
 \downarrow g & & \uparrow \iota' & & \\
 \underline{\mathrm{Lie}}(G) & \otimes & \omega_G & \longrightarrow & \underline{\mathrm{Lie}}(\mathbb{G}_{m,S})
 \end{array}$$

where the upper (resp. lower) pairing is the usual duality between the Lie algebra of G' (resp. G) and the sheaf of invariant differentials of G' (resp. G). As the maps ι , g' and ι' , g come out to be transposes of each others, we get the perfectness of Φ with no restriction on the base.

In the last section we describe the relation between a universal extension $v : X \rightarrow \mathbf{G}^{\mathfrak{h}}$ of M and \mathfrak{h} -extensions of M' showing that there is an exact sequence

$$X \xrightarrow{v} \mathbf{G}^{\mathfrak{h}} \longrightarrow \underline{\mathrm{Ext}}^{\mathfrak{h}}(M', \mathbb{G}_{m,S}) \longrightarrow 0.$$

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This result generalizes the fact that the universal extension of an abelian scheme A represents the functor which assigns to an S -scheme S' the \mathfrak{h} -extensions of the dual abelian scheme A' by the multiplicative group over S' .

2 Universal extensions of 1-motives

Let S be a scheme. Recall that an S -1-motive $M = [u: X \rightarrow G]$ is a two term complex (in degree $-1, 0$) of commutative group schemes over S such that X is an S -group scheme that locally for the étale topology on S is isomorphic to a constant group of type \mathbb{Z}^r , G is an S -group scheme extension of an abelian scheme A over S by a torus T , u is an S -homomorphism $X \rightarrow G$. Morphisms of S -1-motives are usual morphisms of complexes. The category of 1-motives can be seen as a full subcategory of the derived category of bounded complexes of fppf sheaves on S (cf. [10]).

An *extension* of an S -1-motive $M = [u: X \rightarrow G]$ by a group H is an extension E of G by H together with a homomorphism $v: X \rightarrow E$ that lifts u . Two extensions (E_i, v_i) , $i = 1, 2$, are isomorphic if there exists an isomorphism $\varphi: E_1 \rightarrow E_2$ (as extension of G by H) such that $v_2 = \varphi \circ v_1$. As usual, $\text{Ext}^1(M, H)$ denotes the group of isomorphism classes of extensions of M by H . In the following, we will simply speak of 1-motives meaning S -1-motives.

A *universal extension* of M is an extension $M^{\mathfrak{h}} = [X \rightarrow \mathbf{G}^{\mathfrak{h}}]$ of M by a vector group $\mathbb{V}(M)$ over S such that the homomorphism of push-out

$$\epsilon: \text{Hom}_{\mathcal{O}_S}(\mathbb{V}(M), W) \longrightarrow \text{Ext}^1(M, W) \quad (1)$$

is an isomorphism for all vector groups W over S (cf. [5]). Observe that $M^{\mathfrak{h}}$ and $\mathbb{V}(M)$ are determined up to canonical isomorphisms (cf. [9], p. 2). Universal extensions of 1-motives exist (see [5], [2]). As explained in [9], I, 1.7, it is sufficient to show that the following conditions are satisfied:

- a) $\underline{\text{Hom}}(M, \mathbb{G}_{a,S}) = 0$,
- b) $\underline{\text{Ext}}_{\text{Zar}}^1(M, \mathbb{G}_{a,S})$ is a locally free sheaf of \mathcal{O}_S -modules of finite rank,

as sheaves for the Zariski topology over S . If this is the case,

$$\underline{\text{Ext}}_{\text{Zar}}^1(M, \mathcal{W}) = \underline{\text{Ext}}_{\text{Zar}}^1(M, \mathbb{G}_{a,S}) \otimes_{\mathcal{O}_S} \mathcal{W}$$

for any locally free \mathcal{O}_S -module of finite rank \mathcal{W} and one takes as $\mathbb{V}(M)$ the vector group associated to the dual sheaf of $\underline{\text{Ext}}_{\text{Zar}}^1(M, \mathbb{G}_{a,S})$.

For the torus T and the abelian scheme A condition a) is automatically satisfied. Hence the same holds for the semi-abelian scheme G and then for M . As $\text{Ext}^1(T, \mathbb{G}_{a,S}) = 0$ also condition b) holds for tori. For abelian varieties the result is proved in [9], 1.10. Moreover, denote by $A^{\mathfrak{h}}$ a (fixed) universal extension of A ; as $\text{Ext}^1(G, \mathbb{G}_{a,S}) = \text{Ext}^1(A, \mathbb{G}_{a,S})$ one gets $\mathbb{V}(G) = \mathbb{V}(A)$ and a universal extension of G is $G^{\mathfrak{h}} = A^{\mathfrak{h}} \times_A G$ (see [1], 2.2.1).

Lemma 2.1 *Let $M = [u: X \rightarrow G]$ be a 1-motive and W a vector group over S . Then the functor*

$$S' \rightsquigarrow \text{Ext}^1(M_{S'}, W_{S'}) \quad (2)$$

is a sheaf for the flat and Zariski topologies. Here $M_{S'}$ denotes the S' -1-motive obtained via base-change.

Proof. Consider the sequence

$$0 \longrightarrow G \longrightarrow M \longrightarrow [X \rightarrow 0] \longrightarrow 0 \quad (3)$$

and recall that the functor $S' \rightsquigarrow \text{Hom}(X_{S'}, Q_{S'})$ is a sheaf for any S -group scheme Q and that the category $\text{EXT}(G, W)$ is rigid (cf. [9], I, 1.10 proof). \square

From (3) we get an exact sequence

$$0 \longrightarrow X^* = \underline{\text{Hom}}(X, \mathbb{G}_{a,S}) \longrightarrow \underline{\text{Ext}}_{\text{Zar}}^1(M, \mathbb{G}_{a,S}) \longrightarrow \underline{\text{Ext}}_{\text{Zar}}^1(G, \mathbb{G}_{a,S}) \longrightarrow 0 \quad (4)$$

that might not be exact on the right. However, it is exact on a suitable affine étale covering of S where X becomes constant. Since X^* and $\underline{\text{Ext}}_{\text{Zar}}^1(G, \mathbb{G}_{a,S})$ are locally free of finite rank, the same is $\underline{\text{Ext}}_{\text{Zar}}^1(M, \mathbb{G}_{a,S})$ and (4) is exact on the right. In particular, M admits a universal extension M^\natural . Passing to duals on (4) one gets also a sequence of vector groups

$$0 \longrightarrow \mathbb{V}(G) \xrightarrow{i} \mathbb{V}(M) \xrightarrow{\bar{\tau}} X \otimes \mathbb{G}_{a,S} \longrightarrow 0. \quad (5)$$

2.1 A description of $\mathbb{V}(M)$ via invariant differentials.

It is well known that given an abelian scheme A over S the vector group $\mathbb{V}(A)$ corresponds to the locally free sheaf $\underline{\text{Lie}}(A)^* = \omega_{A'}$ of invariant differentials of the dual abelian scheme A' . In the next pages, we will generalize this result to 1-motives showing that if $M' = [u': X' \rightarrow G']$ is the Cartier dual of M , the vector group $\mathbb{V}(M)$ corresponds to the sheaf $\omega_{G'}$ of invariant differentials of the semi-abelian scheme G' . This fact will be of great use in the following sections.

For the definition of the dual motive $M' = [u': X' \rightarrow G']$ of $M = [u: X \rightarrow G]$ we refer to [5]. Denote by $[X \rightarrow A]$ the 1-motive obtained via composition of u with the homomorphism $G \rightarrow A$. We recall that by definition G' represents the sheaf $\underline{\text{Ext}}_{\text{fl}}^1([X \rightarrow A], \mathbb{G}_{m,S})$, the group X' is the group of characters of T and u' is the boundary homomorphism of the long exact sequence of Ext sheaves obtained applying $\underline{\text{Hom}}(-, \mathbb{G}_{m,S})$ to the exact sequence

$$0 \longrightarrow T \longrightarrow M \longrightarrow [X \rightarrow A] \longrightarrow 0. \quad (6)$$

Furthermore, the sequence

$$0 \longrightarrow A \longrightarrow [X \rightarrow A] \longrightarrow [X \rightarrow 0] \longrightarrow 0 \quad (7)$$

provides a short exact sequence

$$0 \rightarrow T' = \underline{\text{Hom}}(X, \mathbb{G}_{m,S}) \rightarrow G' = \underline{\text{Ext}}_{\text{fl}}^1([X \rightarrow A], \mathbb{G}_{m,S}) \rightarrow A' = \underline{\text{Ext}}_{\text{fl}}^1(A, \mathbb{G}_{m,S}) \rightarrow 0 \quad (8)$$

that describes G' as a semi-abelian scheme. In the case $A = 0$ and $M = [u: X \rightarrow T]$ the dual 1-motive $M' = [u': X' \rightarrow T']$ is simply obtained via the usual Cartier duality.

We start relating the Lie algebra of a semi-abelian scheme to vector extensions of its Cartier dual.

Lemma 2.2 *Let B be a semi-abelian scheme over S . Then*

$$\underline{\text{Lie}}(B) = \underline{\text{Lie}}(\underline{\text{Ext}}_{\text{fl}}^1(N, \mathbb{G}_{m,S})) = \underline{\text{Ext}}_{\text{fl}}^1(N, \mathbb{G}_{a,S})$$

where N is the 1-motive Cartier dual of $[0 \rightarrow B]$. In particular, if we think $\underline{\text{Lie}}(B)$ as a sheaf for the Zariski topology then

$$\underline{\text{Lie}}(B) = \underline{\text{Ext}}_{\text{Zar}}^1(N, \mathbb{G}_{a,S}).$$

Proof. The last assertion follows from the first via Lemma 2.1. The first isomorphism is obvious because B is isomorphic to $\underline{\text{Ext}}_{\text{fl}}^1(N, \mathbb{G}_{m,S})$ by Cartier duality. It remains to prove that $\underline{\text{Lie}}(B)$ is isomorphic to $\underline{\text{Ext}}_{\text{fl}}^1(N, \mathbb{G}_{a,S})$.

Given a scheme S' , denote by S'_ϵ the fibre product $S' \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Z}[\epsilon]/(\epsilon^2))$. Recall that by definition of Lie algebras we have an exact sequence

$$0 \longrightarrow \text{Lie}(B_{S'}) \longrightarrow B(S'_\epsilon) = \text{Ext}^1(N_{S'_\epsilon}, \mathbb{G}_{m,S'_\epsilon}) \xrightarrow{f_B} B(S') = \text{Ext}^1(N_{S'}, \mathbb{G}_{m,S'}) \quad (9)$$

where f_B is the composition with the closed immersion $S' \rightarrow S'_\epsilon$ induced by

$$\mathbb{Z}[\epsilon]/(\epsilon^2) \rightarrow \mathbb{Z}, \quad \epsilon \mapsto 0,$$

(or the base-change on exact sequences). Let now $\mathfrak{R}_{S'_\epsilon/S'}(\mathbb{G}_{m,S'_\epsilon})$ be the Weil restriction of $\mathbb{G}_{m,S'_\epsilon}$ with respect to the base-change morphism $S'_\epsilon \rightarrow S'$. For any S' -scheme Z it holds $\mathfrak{R}_{S'_\epsilon/S'}(\mathbb{G}_{m,S'_\epsilon})(Z) = \mathbb{G}_{m,S'_\epsilon}(Z_\epsilon) = \mathbb{G}_{m,S'}(Z_\epsilon)$. Moreover, there is an exact sequence

$$0 \longrightarrow \mathbb{G}_{a,S'} \longrightarrow \mathfrak{R}_{S'_\epsilon/S'}(\mathbb{G}_{m,S'_\epsilon}) \xrightarrow{f} \mathbb{G}_{m,S'} \longrightarrow 0 \quad (10)$$

where the homomorphism $f: \mathbb{G}_{m,S'}(Z_\epsilon) \rightarrow \mathbb{G}_{m,S'}(Z)$ is obtained via composition with the closed immersion $Z \rightarrow Z_\epsilon$. From (10) we deduce an exact sequence

$$0 \longrightarrow \text{Ext}^1(N_{S'}, \mathbb{G}_{a,S'}) \longrightarrow \text{Ext}^1(N_{S'}, \mathfrak{R}_{S'_\epsilon/S'}(\mathbb{G}_{m,S'_\epsilon})) \xrightarrow{f_N} \text{Ext}^1(N_{S'}, \mathbb{G}_{m,S'}) \quad (11)$$

where f_N is the push-out with respect to f .

In order to prove that $\underline{\text{Lie}}(B) = \underline{\text{Ext}}^1_{\mathfrak{R}}(N, \mathbb{G}_{a,S})$, it is sufficient to check that

$$\underline{\text{Lie}}(B_{S'}) = \text{Ext}^1(N_{S'}, \mathbb{G}_{a,S'})$$

for any S -scheme S' . Comparing (9) and (11) we are reduced to see that

$$\text{Ext}^1(N_{S'}, \mathfrak{R}_{S'_\epsilon/S'}(\mathbb{G}_{m,S'_\epsilon})) = \text{Ext}^1(N_{S'_\epsilon}, \mathbb{G}_{m,S'_\epsilon})$$

and that f_N coincides with f_B . This is not hard using properties of Weil restriction. \square

Recall that $\mathbb{V}(M)$ is the vector group associated to the dual sheaf of $\underline{\text{Ext}}^1_{\text{Zar}}(M, \mathbb{G}_{a,S})$. We will prove now that it corresponds to the sheaf of invariant differentials of G' .

Proposition 2.3 *Let M be a 1-motive. It holds*

$$\underline{\text{Ext}}^1(M, \mathbb{G}_{a,S}) = \underline{\text{Ext}}^1([X \rightarrow A], \mathbb{G}_{a,S}) = \underline{\text{Lie}}(G') \quad (12)$$

for the flat and Zariski topologies. Hence $\mathbb{V}(M)$ is (the vector group associated to) the sheaf of invariant differentials $\omega_{G'}$. Moreover, the sequence of vector groups in (5) is the sequence

$$0 \longrightarrow \omega_{A'} \longrightarrow \omega_{G'} \longrightarrow \omega_{T'} \longrightarrow 0 \quad (13)$$

of invariant differentials of (8).

Proof. Denote by M_A the 1-motive $[X \rightarrow A]$. The first isomorphism $\underline{\text{Ext}}^1(M, \mathbb{G}_{a,S}) = \underline{\text{Ext}}^1(M_A, \mathbb{G}_{a,S})$ comes from the exact sequence in (6) using the vanishing $\underline{\text{Hom}}(T, \mathbb{G}_{a,S}) = 0 = \underline{\text{Ext}}^1(T, \mathbb{G}_{a,S})$. The second isomorphism was proved in the previous lemma for $B = G'$ and $N = M_A$.

For the second assertion, observe that using the isomorphisms

$$\underline{\text{Ext}}^1_{\text{Zar}}(M, \mathbb{G}_{a,S}) = \underline{\text{Ext}}^1_{\text{Zar}}(M_A, \mathbb{G}_{a,S}), \quad \underline{\text{Ext}}^1_{\text{Zar}}(G, \mathbb{G}_{a,S}) = \underline{\text{Ext}}^1_{\text{Zar}}(A, \mathbb{G}_{a,S}),$$

the sequence (4) coincides with the sequence

$$0 \longrightarrow X^* = \underline{\text{Hom}}(X, \mathbb{G}_{a,S}) \longrightarrow \underline{\text{Ext}}^1_{\text{Zar}}(M_A, \mathbb{G}_{a,S}) \longrightarrow \underline{\text{Ext}}^1_{\text{Zar}}(A, \mathbb{G}_{a,S}) \longrightarrow 0 \quad (14)$$

obtained from (7). Now, the proof of Lemma 2.2 says that (14) is the sequence of Lie algebras

$$0 \longrightarrow \underline{\text{Lie}}(T') \longrightarrow \underline{\text{Lie}}(G') \longrightarrow \underline{\text{Lie}}(A') \longrightarrow 0$$

of the sequence (8). Passing to duals we get the desired result. \square

2.2 A description of \mathbf{G}^{\natural} as push-out

By the universal property of the universal extension of G and Proposition 2.3, the group scheme \mathbf{G}^{\natural} is (isomorphic to) the push-out

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{V}(G) = \omega_{A'} & \longrightarrow & \mathbf{G}^{\natural} & \xrightarrow{\rho} & G \longrightarrow 0 \\
 & & \downarrow i & & \downarrow & & \parallel \\
 0 & \longrightarrow & \mathbb{V}(M) = \omega_{G'} & \xrightarrow{\iota} & \mathbf{G}^{\natural} & \longrightarrow & G \longrightarrow 0 \\
 & & \downarrow \bar{\tau} & & \downarrow \tau & & \\
 & & X \otimes \mathbb{G}_{a,S} = \omega_{T'} & \xlongequal{\quad} & \omega_{T'} & &
 \end{array} \tag{15}$$

where the vertical sequence on the left is (5) or (13); this fact was firstly observed in [2] (without the contribution of invariant differentials). There is then a useful criterion to test when a homomorphism $X \rightarrow \mathbf{G}^{\natural}$ provides a universal extension of M :

Lemma 2.4 *Let $M = [u: X \rightarrow G]$ be a 1-motive as above and let \mathbf{G}^{\natural} be the group scheme defined in diagram (15). A homomorphism $v: X \rightarrow \mathbf{G}^{\natural}$ such that $\rho \circ v = u$ is a universal extension of M if and only if $\tau \circ v: X \rightarrow X \otimes \mathbb{G}_{a,S}$ is a universal extension of the 1-motive $[X \rightarrow 0]$.*

Proof. Let W be a vector group over S and consider the following diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_S}(X \otimes \mathbb{G}_{a,S}, W) & \xrightarrow{\bar{\tau}^*} & \mathrm{Hom}_{\mathcal{O}_S}(\mathbb{V}(M), W) & \xrightarrow{i^*} & \mathrm{Hom}_{\mathcal{O}_S}(\mathbb{V}(G), W) \\
 & & \downarrow \epsilon_X & & \downarrow \epsilon & & \downarrow \iota \\
 0 & \longrightarrow & \mathrm{Ext}^1([X \rightarrow 0], W) & \longrightarrow & \mathrm{Ext}^1(M, W) & \longrightarrow & \mathrm{Ext}^1(G, W).
 \end{array}$$

where the upper sequence is obtained from (5) and the lower sequence is obtained from (3). Given a morphism of vector groups $f: X \otimes \mathbb{G}_{a,S} \rightarrow W$, $\epsilon_X(f)$ is the trivial extension of 0 by W together with the morphism $f \circ \tau \circ v: X \rightarrow W$; for a $g: \mathbb{V}(M) \rightarrow W$ the extension $\epsilon(g)$ is the isomorphism class of the push-out with respect to g of the extension $v: X \rightarrow \mathbf{G}^{\natural}$ of M by $\mathbb{V}(M)$. The push-out homomorphism on the right is an isomorphism because of the the universal property of G^{\natural} . By construction the diagram is commutative.

If $v: X \rightarrow \mathbf{G}^{\natural}$ is a universal extension of M , the homomorphism ϵ is an isomorphism and hence also ϵ_X is an isomorphism. This says that $\tau \circ v$ is a universal extension of $[X \rightarrow 0]$. Suppose now that $\tau \circ v$ is a universal extension of $[X \rightarrow 0]$, i.e. ϵ_X is an isomorphism. This implies that ϵ is injective. If i^* is surjective, we can deduce that also ϵ is an isomorphism and hence v is a universal extension of M . In general i^* is surjective Zariski locally on S . As the functor in (2) is a Zariski-sheaf, as well as $U \rightsquigarrow \mathrm{Hom}_{\mathcal{O}_{S|U}}(\mathbb{V}(M)|_U, W|_U)$, we conclude that ϵ is an isomorphism. \square

Remark 2.5 Observe that if $v: X \rightarrow \mathbf{G}^{\natural}$ is a universal extension of M and $f: X \rightarrow \mathbb{V}(M)$ is a homomorphism that factors through $\mathbb{V}(G)$ then $v + f: X \rightarrow \mathbf{G}^{\natural}$ is a universal extension too. However, $v + f$ is isomorphic to v , as extension of M by $\mathbb{V}(M)$, if and only if $f = 0$ because the extension \mathbf{G}^{\natural} admits no non-trivial automorphisms.

3 \natural -structures

In order to define Deligne's pairing for the de Rham realizations of 1-motives, we need to recall first some definitions and results on \natural -extensions and \natural -bixtensions. Proposition 3.8 of this section is the key result that permits to generalize [5], 10.2.7.4.

3.1 Some definitions

Let S be a fixed scheme, Z an S -scheme, G, Y_1, Y_2, H_1, H_2 commutative S -group schemes with G smooth. As usual G_Z means $G \times_S Z$. Denote by $\Delta^1(Z)$ the first infinitesimal neighborhood of the diagonal $Z \rightarrow Z \times_S Z$ and by $p_j: \Delta^1(Z) \rightarrow Z$, $j = 1, 2$, the morphisms induced by the usual projections $p_j: Z \times_S Z \rightarrow Z$.

Definition 3.1 ([9]) A \mathfrak{h} - G -torsor on Z is a torsor (for the étale topology¹) P on Z under G_Z endowed with an integrable connection, i.e. an isomorphism $\nabla: p_1^*P \rightarrow p_2^*P$ of $G_{\Delta^1(Z)}$ -torsors which restricts to the identity on Z and has zero curvature.

The trivial \mathfrak{h} - G -torsor is the trivial torsor G_Z endowed with the trivial connection ∇^0 , i.e. the identity on $G_{\Delta^1(Z)}$. A trivialization of a \mathfrak{h} - G -torsor (P, ∇) is a section $s: Z \rightarrow P$ such that the induced isomorphism $\varphi_s: (P, \nabla) \rightarrow (G_Z, \nabla^0)$ is horizontal. Observe that given a trivialization s of a torsor P on Z under G there is a unique possible \mathfrak{h} -structure that makes φ_s horizontal. We describe in details a case that will be needed later.

Example 3.2 Let $Z = \mathbb{G}_{m,S} = G$, $P = G \times Z$ with the trivialization $s: Z \rightarrow P$, $b \mapsto (b^n, b)$. We have an isomorphism $\varphi_s: P \rightarrow \mathbb{G}_{m,S}^2$, $(x, y) \mapsto (x/y^n, y)$ such that $\varphi_s \circ s$ is the usual trivialization $b \mapsto (1, b)$. Consider a connection ∇ on P given by a global differential ω on Z . There is a unique possible choice of ∇ that makes φ_s horizontal with respect to ∇ on P and the trivial connection ∇^0 on $\mathbb{G}_{m,S}^2$. More precisely, let t (resp. z) be the parameter of G (resp. Z); the isomorphism φ_s induces an isomorphism of algebras such that $\varphi_s^*(z) = z$ and $\varphi_s^*(t) = t/z^n$, hence an isomorphism $\varphi_s^*: \mathcal{O}_Z \rightarrow \mathcal{O}_Z$, $a \mapsto a/z^n$. The horizontality condition says that the induced connections on sheaves (see [9], I, 3.1.2) $\nabla, \nabla^0: \mathcal{O}_Z \rightarrow \Omega_{Z/S}^1$ satisfy

$$0 = \varphi_s^*(\nabla^0(1)) = \nabla(\varphi_s^*(1)) = \nabla(1/z^n) = -ndz/z^{n+1} + (1/z^n)w$$

and hence $w = ndz/z$.

Let in the following Z be a group scheme over S and denote by $\mu_Z: Z \times_S Z \rightarrow Z$ its group law.

Definition 3.3 ([9]) A \mathfrak{h} -extension of Z by G is a \mathfrak{h} - G -torsor (P, ∇) on Z where P is extension of Z by G and the usual morphism

$$\nu: p_1^*P + p_2^*P \rightarrow \mu_Z^*P$$

is horizontal.

Two \mathfrak{h} -extensions (P_i, ∇_i) , $i = 1, 2$, of Z by G are isomorphic if there exists an isomorphism (of extensions) $\phi: P_1 \rightarrow P_2$ that is horizontal. The trivial \mathfrak{h} -extension of Z by G is the trivial extension $Z \times_S G$ equipped with the trivial connection ∇^0 . A trivialization of a \mathfrak{h} -extension (P, ∇) is a section $s: Z \rightarrow P$ that provides an isomorphism of (P, ∇) with the trivial \mathfrak{h} -extension. One denotes by $\text{Ext}^{\mathfrak{h}}(Z, G)$ the group of isomorphism classes of \mathfrak{h} -extensions of Z by G . We have an exact sequence (cf. [9], II 4.2)

$$\text{Hom}(Z, G) \longrightarrow \Gamma(\omega_Z \otimes \underline{\text{Lie}}(G)) \longrightarrow \text{Ext}^{\mathfrak{h}}(Z, G) \xrightarrow{f} \text{Ext}^1(Z, G). \quad (16)$$

Following [4], 0.2, it is easy to describe \mathfrak{h} -extensions of Z by $\mathbb{G}_{m,S}$.

Proposition 3.4 *Let Z be a commutative group scheme over S and E an extension of Z by $\mathbb{G}_{m,S}$. There is a one-to-one correspondence between connections ∇ on E making (E, ∇) a \mathfrak{h} -extension of Z by $\mathbb{G}_{m,S}$ and normal² invariant differentials on E .*

Proof. It is known (see [4], 0.2.1) that there is a one-to-one correspondence between connections on E and global differentials on E that pull back to dz/z and are invariant under the action of $\mathbb{G}_{m,S}$. Now, the horizontality condition in Definition 3.3 requires the global differential of E to be invariant. \square

We recall now some definitions from [5].

¹ Cf. [9], I, 3.1. This hypothesis is needed to define the curvature form of a connection by descent.

² Denote by z the standard parameter of $\mathbb{G}_{m,S}$. An invariant differential on E is said to be normal if it pulls back to dz/z on $\mathbb{G}_{m,S}$.

Definition 3.5 Let P be a biextension of (H_1, H_2) by G and consider the usual morphisms

$$\begin{aligned} \nu_1: p_{13}^*P + p_{23}^*P &\rightarrow (\mu_1 \times \text{id})^*P, & \text{on } H_1 \times_S H_1 \times_S H_2, \\ \nu_2: p_{12}^*P + p_{13}^*P &\rightarrow (\text{id} \times \mu_2)^*P & \text{on } H_1 \times_S H_2 \times_S H_2. \end{aligned}$$

Here p_{ij} are the obvious projections and μ_i is the group law on H_i . A \natural -structure on P is a connection ∇ on the G -torsor P over $H_1 \times H_2$ such that ν_1, ν_2 are horizontal. We will also say that (P, ∇) is a \natural -biextension of (H_1, H_2) by G .

A trivialization of a \natural -biextension (P, ∇) is a horizontal isomorphism of P with the trivial biextension endowed with the trivial connection.

Definition 3.6 Let P be a biextension of (H_1, H_2) by G . A \natural -1-structure on P is a connection on P such that P becomes a \natural -extension³ of H_{1, H_2} by G_{H_2} with ν_2 horizontal. A \natural -2-structure on P is a connection on P such that P becomes a \natural -extension of H_{2, H_1} by G_{H_1} with ν_1 horizontal.

Giving a \natural -structure to a biextension is equivalent to giving a \natural -1-structure and a \natural -2-structure.

Definition 3.7 A biextension of complexes $([Y_1 \rightarrow H_1], [Y_2 \rightarrow H_2])$ by G is a biextension P of (H_1, H_2) by G endowed with a trivialization of the pull-back of P to $Y_1 \times_S H_2$ and a trivialization of the pull-back of P to $H_1 \times_S Y_2$ that coincide on $Y_1 \times_S Y_2$.

A \natural -extension of a complex $[u: Y \rightarrow H]$ by a group G is a \natural -extension (P, ∇) of H by G with a trivialization (as \natural -biextension) of the pull-back of (P, ∇) to Y .

A \natural -biextension of complexes $([Y_1 \rightarrow H_1], [Y_2 \rightarrow H_2])$ by G is a \natural -biextension (P, ∇) of (H_1, H_2) by G endowed with a trivialization (as \natural -biextension) of the pull-back of (P, ∇) to $Y_1 \times_S H_2$ and a trivialization of the pull-back of (P, ∇) to $H_1 \times_S Y_2$ that coincide on $Y_1 \times_S Y_2$.

3.2 \natural -structures and biextensions.

It is shown in [9] that the universal extension A^\natural of an abelian scheme over S represents the functor that associates to any S -scheme S' the group of isomorphism classes of \natural -extensions of $A'_{S'}$ by $\mathbb{G}'_{m, S'}$. See also [4], 0.3.1. We will prove in Lemma 5.2 that $G^\natural = \underline{\text{Ext}}^\natural([X' \rightarrow A'], \mathbb{G}_{m, S})$, or the same, that G^\natural represents the pre-sheaf for the flat topology

$$S' \rightsquigarrow \left\{ (g, \nabla), \quad \begin{array}{l} g \in G(S'), \nabla \text{ a } \natural\text{-structure on the extension } \mathcal{P}'_g \text{ of} \\ [X' \rightarrow A'] \text{ by } \mathbb{G}_{m, S'} \text{ associated to } g \end{array} \right\}. \quad (17)$$

Observe that \mathcal{P}'_g is the fibre at g of the Poincaré biextension \mathcal{P}' of $(G, [X' \rightarrow A'])$ by $\mathbb{G}_{m, S}$.

We can generalize the result above to any 1-motive:

Proposition 3.8 Let M be a 1-motive, M' its Cartier dual and \mathcal{P} the Poincaré biextension of (M, M') . The group scheme \mathbf{G}^\natural defined in (15) represents the pre-sheaf for the flat topology

$$\mathcal{E}: S' \rightsquigarrow \left\{ (g, \nabla), \quad \begin{array}{l} g \in G(S'), \nabla \text{ a } \natural\text{-structure on the extension } \mathcal{P}_g \text{ of} \\ M' \text{ by } \mathbb{G}_{m, S'} \text{ associated to } g \end{array} \right\}.$$

Observe that \mathcal{P}_g is the fibre at g of the Poincaré biextension \mathcal{P} of (M, M') . The biextension \mathcal{P} is also the pull-back of \mathcal{P}' (the Poincaré biextension of $(G, [X' \rightarrow A'])$) to (G, M') together with a suitable trivialization on $X \times G'$. Hence \mathcal{P}_g can be seen as the pull-back to M' of \mathcal{P}'_g . In the following we will denote by \mathcal{P} (resp. \mathcal{P}') also the $\mathbb{G}_{m, S}$ -torsor over $G \times G'$ (resp. over $G \times A'$) underlying \mathcal{P} (resp. \mathcal{P}'). In particular, the fibre \mathcal{P}_g at a point $g \in G(S')$ can be read as the pull-back to G' of the fibre of \mathcal{P}' at g :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{G}_{m, S'} & \longrightarrow & \mathcal{P}_g & \longrightarrow & G'_{S'} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{G}_{m, S'} & \longrightarrow & \mathcal{P}'_g & \longrightarrow & A'_{S'} \longrightarrow 0. \end{array} \quad (18)$$

³ Here H_2 is seen as base scheme; ν_1 is automatically horizontal because of Definition 3.3.

Proof. We will construct a canonical isomorphism

$$\Psi: \mathbf{G}^{\natural}(S') \rightarrow \mathcal{E}(S'). \quad (19)$$

It is immediate to show that \mathcal{E} is a sheaf for the flat topology. Hence, we reduce the proof to the case where $S' = S$ is affine and the short exact sequence in (13)

$$0 \longrightarrow \omega_{A'} \longrightarrow \omega_{G'} \xrightarrow[\delta]{\bar{\tau}} \omega_{T'} \longrightarrow 0$$

is split over S . Recall the notations in (15) and that \mathbf{G}^{\natural} is extension of G by $\omega_{G'} = \mathbb{V}(M)$. Let δ be a section of $\bar{\tau}$ and denote by δ also the induced section of $\tau: \mathbf{G}^{\natural} \rightarrow \omega_{T'} = X \otimes \mathbb{G}_{a,S}$. We identify then \mathbf{G}^{\natural} with $G^{\natural} \oplus \omega_{T'}$. An S -valued point a of \mathbf{G}^{\natural} becomes a sum $a - \delta(\tau(a)) \oplus \tau(a)$ where $a - \delta(\tau(a)) \in G^{\natural}(S)$ corresponds to a \natural -structure on $\mathcal{P}'_{\rho(a)}$ via the functor in (17); let $\eta_{A,a}$ be the corresponding normal invariant differential of $\mathcal{P}'_{\rho(a)}$ (cf. Proposition 3.4). Then $\eta_a := \eta_{A,a} + \delta(\tau(a))$ is a normal invariant differential of $\mathcal{P}_{\rho(a)}$ and hence it provides a \natural -structure ∇_{η_a} on $\mathcal{P}_{\rho(a)}$. Observe that the definition of η_a makes sense because of diagram (18). We define then $\Psi(a) = (\rho(a), \nabla_{\eta_a})$.

For the injectivity of Ψ , let a, b be two S -valued points of \mathbf{G}^{\natural} such that $(\rho(a), \nabla_{\eta_a}) = (\rho(b), \nabla_{\eta_b})$. Then $\rho(a) = \rho(b)$ and $\eta_a = \eta_b$. Define $\omega := a - b \in \omega_{G'}(S) = (\ker \rho)(S)$. It holds

$$\begin{aligned} \eta_a &= \eta_{A,a} + \delta(\tau(a)) = \eta_{A,b} + \omega - \delta(\tau(\omega)) + \delta(\tau(b)) + \delta(\tau(\omega)) \\ &= \eta_{A,b} + \delta(\tau(b)) + \omega = \eta_b + \omega, \end{aligned}$$

because

$$a - \delta(\tau(a)) - b + \delta(\tau(b)) = \omega - \delta(\tau(\omega)).$$

Now, $\eta_a = \eta_b$ implies $\omega = 0$ and hence $a = b$.

For the surjectivity of Ψ , as S is affine, we may assume that the homomorphism $\rho: \mathbf{G}^{\natural} \rightarrow G$ is surjective on S -valued points. Consider then a pair $(\rho(a), \nabla_{\eta}) \in \mathcal{E}(S)$ with a an S -valued point of \mathbf{G}^{\natural} and η a normal invariant differential of $\mathcal{P}_{\rho(a)}$. We defined $\Psi(a) = (\rho(a), \eta_a) \in \mathcal{E}(S)$. Now, as both η and η_a are normal invariant differentials of $\mathcal{P}_{\rho(a)}$ (i.e. they restrict to dz/z on $\mathbb{G}_{m,S}$) the differential $\eta - \eta_a$ equals ω for a suitable $\omega \in \omega_{G'}(S)$. Define $b := a + \omega$. It holds $\rho(a) = \rho(b)$ and $\eta_b = \eta_a + \omega = \eta$.

It is also immediate to check that the homomorphism Ψ does not depend on the choice of the section δ . \square

The definition of Deligne's pairing for the de Rham realizations of 1-motives over a field uses the fact that the pull-back of a biextension of 1-motives (M_1, M_2) by $\mathbb{G}_{m,S}$ to the universal extensions $(M_1^{\natural}, M_2^{\natural})$ admits a canonical \natural -structure. The case of Poincaré biextensions can be deduced from Theorem 3.10. However, we prove it separately because we will use in the next section the explicit description of the canonical \natural -structure contained in the proof.

Proposition 3.9 *Let \mathcal{P}^{\natural} be the pull-back to $(M^{\natural}, M'^{\natural})$ of the Poincaré biextension \mathcal{P} of (M, M') by $\mathbb{G}_{m,S}$. It admits a canonical \natural -structure, i.e. there is a canonical connection on the underlying torsor that makes \mathcal{P}^{\natural} a \natural -biextension of $(M^{\natural}, M'^{\natural})$ by $\mathbb{G}_{m,S}$. This is the unique \natural -structure on \mathcal{P}^{\natural} if $\text{Hom}(G^{\natural}, \mathbb{G}_a) = 0 = \text{Hom}(G'^{\natural}, \mathbb{G}_a)$.*

Proof. Denote by \mathcal{P}_{ρ} the pull-back of \mathcal{P} to (M^{\natural}, M') as well its associated $\mathbb{G}_{m,S}$ -torsor on $\mathbf{G}^{\natural} \times G'$. By Proposition 3.8 the identity map on \mathbf{G}^{\natural} provides a \natural -structure ∇_2 on \mathcal{P}_{ρ} (viewed as extension of M' by the multiplicative group over \mathbf{G}^{\natural}). To check the horizontality condition on ν_1 (see Definition 3.6) one uses the isomorphism Ψ in the proof of Proposition 3.8. Indeed, the pull-back via

$$p_{13}: \mathbf{G}^{\natural} \times \mathbf{G}^{\natural} \times G' \rightarrow \mathbf{G}^{\natural} \times G' \quad (\text{resp. } p_{23}, \text{ resp. } \mu_{\mathbf{G}^{\natural}} \times \text{id}_{G'})$$

of $(\mathcal{P}_{\rho}, \nabla_2)$ is the image via Ψ of the $\mathbf{G}^{\natural} \times \mathbf{G}^{\natural}$ -valued point $\rho \circ p_1$ of G (resp. $\rho \circ p_2$, resp. $\rho \circ \mu_{\mathbf{G}^{\natural}}$) and it holds $p_1 + p_2 = \mu_{\mathbf{G}^{\natural}}$. Changing the role of M and M' we get a \natural -1-structure of \mathcal{P}^{\natural} .

To show the uniqueness result it is sufficient to show that any \natural -structure ∇ on the trivial biextension of $(M^{\natural}, M'^{\natural})$ by $\mathbb{G}_{m,S}$ is trivial (cf. [5], 10.2.7.4.). We are considering the trivial $\mathbb{G}_{m,S}$ -torsor on $\mathbf{G}^{\natural} \times G'$ with a connection ∇ such that the morphisms ν_1, ν_2 in Definition (3.5) are horizontal and the pull-back of ∇ to $\mathbf{G}^{\natural} \times X'$ is trivial as well as the pull-back to $X \times \mathbf{G}^{\natural}$. The connection ∇ is determined by giving a global differential

$\omega = \omega_1 + \omega_2$ on $\mathbf{G}^{\natural} \times \mathbf{G}'^{\natural}$ where the ω_i depends on the \natural - i -structure associated to ∇ . Recall now that ω_1 has to be a global invariant differential on $\mathbf{G}_{\mathbf{G}'^{\natural}}^{\natural}$. We may work Zariski locally on S and then assume that the sheaf of differential forms of \mathbf{G}^{\natural} over S is free. We can then write $\omega_1 = \sum_j F_j \omega_{1j}$ with $\{\omega_{1,j}\}_j$ (the pull-back) of a free basis of invariant differentials of \mathbf{G}^{\natural} and F_j (the pull-back) of a global section of \mathbf{G}'^{\natural} . The condition on ν_2 requires that F_j is additive, i.e. it corresponds to a homomorphism $\mathbf{G}'^{\natural} \rightarrow \mathbb{G}_{a,S}$. However \mathbf{G}'^{\natural} is extension of $X' \otimes \mathbb{G}_{a,S}$ by G'^{\natural} and then by hypothesis F_j comes from an additive global section of the vector group $X' \otimes \mathbb{G}_{a,S}$. It is clear that the pull-back of ω_1 to $X \times \mathbf{G}'^{\natural}$ is trivial because X is étale. Moreover, the condition that the pull-back of ω_1 to $\mathbf{G}^{\natural} \times X'$ has to be trivial implies that $F_j = 0$. Hence $\omega_1 = 0$. In the same way one sees that $\omega_2 = 0$. \square

More generally:

Theorem 3.10 *Let $M_i = [u_i: X_i \rightarrow G_i], i = 1, 2$, be two 1-motives, \mathcal{P} a biextension of (M_1, M_2) by $\mathbb{G}_{m,S}$ and \mathcal{P}^{\natural} its pull-back to $(M_1^{\natural}, M_2^{\natural})$. Then \mathcal{P}^{\natural} admits a canonical \natural -structure. This is the unique \natural -structure on \mathcal{P}^{\natural} if $\text{Hom}(G_i^{\natural}, \mathbb{G}_a) = 0$.*

Proof. This is essentially Deligne's proof in [5], 10.2.7.4. The uniqueness result can be proved as in the previous Proposition. For the existence, observe that [7] VIII, 3.5, implies that the pull-back homomorphism

$$\text{Biext}^1(G_1, [X_2 \rightarrow A_2]; \mathbb{G}_{m,S}) \longrightarrow \text{Biext}^1(G_1, M_2; \mathbb{G}_{m,S})$$

is indeed an isomorphism. Hence \mathcal{P} is the pull-back of a biextension $\tilde{\mathcal{P}}$ of $(G_1, [X_2 \rightarrow A_2])$ by $\mathbb{G}_{m,S}$. Moreover, $\tilde{\mathcal{P}}$ provides a homomorphism

$$\psi: G_1 \rightarrow \underline{\text{Ext}}^1([X_2 \rightarrow A_2], \mathbb{G}_{m,S}) = G'_2$$

(cf. [7], VIII 1.1.4) and $\tilde{\mathcal{P}}$ is the pull-back via $\psi \times \text{id}$ of the Poincaré biextension of $(G'_2, [X_2 \rightarrow A_2])$. We define now an S -group scheme

$$C := \mathbf{G}'^{\natural}_2 \times_{G'_2} G_1$$

via the usual homomorphism $\mathbf{G}'^{\natural}_2 \rightarrow G'_2$ and ψ . The group C is extension of G_1 by ω_{G_2} . Using Proposition 3.8 one shows that

$$C(S') = \left\{ (g, \nabla), \begin{array}{l} g \in G_1(S'), \nabla \text{ a } \natural\text{-structure on the corresponding} \\ \text{extension } \mathcal{P}_g \text{ of } M_2 \text{ by } \mathbb{G}_{m,S'} \end{array} \right\}. \quad (20)$$

Define now a homomorphism $u_C: X_1 \rightarrow C$, as $u_C(x) = (u_1(x), \nabla^0)$ where ∇^0 denotes the trivial connection on $\mathcal{P}_{u_1(x)}$. Observe that, by definition of biextensions of complexes, the pull-back of \mathcal{P} to $X_1 \times G_2$ is isomorphic to the trivial biextension. In this way $u_C: X_1 \rightarrow C$ becomes an extension of M_1 by the vector group ω_{G_2} . Using the universal property of the universal extension $M_1^{\natural} = [X_1 \rightarrow \mathbf{G}_1^{\natural}]$ of M_1 , $u_C: X_1 \rightarrow C$ is the push-out of M_1^{\natural} for a suitable homomorphism $\omega_{G_1} \rightarrow \omega_{G_2}$. Denote by Γ the induced homomorphism $\mathbf{G}_1^{\natural} \rightarrow C$. It is clear that the image via Γ of the identity of \mathbf{G}_1^{\natural} provides a \mathbf{G}_1^{\natural} -valued point of C that corresponds, because of (20), to a \natural -2-structure on the pull-back of \mathcal{P} to $(\mathbf{G}_1^{\natural}, M_2)$ and hence on \mathcal{P}^{\natural} . In a similar way, one gets a \natural -1-structure on \mathcal{P}^{\natural} and hence the canonical \natural -structure we are looking for. \square

Remark 3.11 The uniqueness result in [5], 10.2.7.4 (see also the proof of Propositions 3.9) depends on the fact that $\text{Hom}(G^{\natural}, \mathbb{G}_a) = 0$. This is not true in general. Indeed $\text{Hom}(G^{\natural}, \mathbb{G}_a)$ is the kernel of the push-out homomorphism $\text{Hom}(\omega_{A'}, \mathbb{G}_a) \rightarrow \text{Ext}^1(G, \mathbb{G}_a)$. This map is an epimorphism because of the universal property of universal extensions (cf. (1)). It can not be an isomorphism when $\text{Hom}(\omega_{A'}, \mathbb{G}_a)$ is bigger than $\text{Hom}_{\mathcal{O}_S}(\omega_{A'}, \mathbb{G}_a)$. As an example, over a field k of characteristic $p > 0$, for $\omega_{A'} = \mathbb{G}_a$, the homomorphisms of k -group schemes $\mathbb{G}_a \rightarrow \mathbb{G}_a$ correspond to polynomials of the type $\sum_i a_i x^{p^i}$, $a_i \in k$, while the homomorphisms of vector groups $\mathbb{G}_a \rightarrow \mathbb{G}_a$ correspond to linear polynomials ax , $a \in k$.

4 Deligne's pairing Φ

Let M_1, M_2 be two S -1-motives, \mathcal{P} a biextension of (M_1, M_2) by $\mathbb{G}_{m,S}$ and \mathcal{P}^\natural the pull-back of \mathcal{P} as biextension of $(M_1^\natural, M_2^\natural)$ by $\mathbb{G}_{m,S}$. Following Deligne, denote $\underline{\mathrm{Lie}}(\mathbf{G}_i^\natural)$ by $\mathrm{T}_{\mathrm{dR}}(M_i)$. We know from Theorem 3.10 that \mathcal{P}^\natural admits a canonical \natural -structure. Hence \mathcal{P}^\natural is equipped with a canonical connection ∇ . Consider now the curvature form of ∇ (see, for example, [9] I, 3.1.4). It is an invariant 2-form on $\mathbf{G}_1^\natural \times \mathbf{G}_2^\natural$; hence it gives an alternating pairing R on

$$\underline{\mathrm{Lie}}(\mathbf{G}_1^\natural \times \mathbf{G}_2^\natural) = \underline{\mathrm{Lie}}(\mathbf{G}_1^\natural) \oplus \underline{\mathrm{Lie}}(\mathbf{G}_2^\natural) = \mathrm{T}_{\mathrm{dR}}(M_1) \oplus \mathrm{T}_{\mathrm{dR}}(M_2)$$

with values in $\underline{\mathrm{Lie}}(\mathbb{G}_{m,S})$. As the restrictions of R to $\underline{\mathrm{Lie}}(\mathbf{G}_i^\natural)$, $i = 1, 2$, are trivial it holds

$$R(g_1 + g_2, g'_1 + g'_2) = \Phi(g_1, g'_2) - \Phi(g_2, g'_1)$$

with $\Phi: \mathrm{T}_{\mathrm{dR}}(M_1) \otimes \mathrm{T}_{\mathrm{dR}}(M_2) \rightarrow \underline{\mathrm{Lie}}(\mathbb{G}_{m,S})$ a bilinear map. We will show in this section that Φ is a non-degenerate pairing when M_1, M_2 are Cartier duals and \mathcal{P} is the Poincaré biextension. This result has been proved by Deligne for $S = \mathrm{Spec}(\mathbb{C})$ and by Coleman for abelian schemes and S flat over \mathbb{Z} . See also [6], V §4. Both proofs are based on the comparison with another perfect pairing and do not work in our general case.

Let in the following $M = [u: X \rightarrow G]$, $M' = [u': X' \rightarrow G']$ be Cartier duals, \mathcal{P} the Poincaré biextension of (M, M') and

$$\Phi: \mathrm{T}_{\mathrm{dR}}(M) \otimes \mathrm{T}_{\mathrm{dR}}(M') \rightarrow \underline{\mathrm{Lie}}(\mathbb{G}_{m,S}) \quad (21)$$

Deligne's pairing. Recall that we have vectorial extensions of M and M'

$$0 \longrightarrow \omega_{G'} \xrightarrow{i} M^\natural \xrightarrow{\rho} M \longrightarrow 0, \quad 0 \longrightarrow \omega_G \xrightarrow{i'} M'^\natural \xrightarrow{\rho'} M' \longrightarrow 0, \quad (22)$$

with $M^\natural = [X \rightarrow \mathbf{G}^\natural]$, $M'^\natural = [X' \rightarrow \mathbf{G}'^\natural]$ the universal extensions of M, M' . Recall that \mathcal{P}^\natural denotes the pull-back of \mathcal{P} to $(M^\natural, M'^\natural)$ endowed with its canonical \natural -structure. We showed in Proposition 3.9 that \mathcal{P}^\natural is the sum of $(\mathcal{P}_\rho, \nabla_2)$ and $(\mathcal{P}_{\rho'}, \nabla_1)$ (after suitable pull-backs) where $(\mathcal{P}_\rho, \nabla_2)$ is the \natural -extension of M' by the multiplicative group over \mathbf{G}^\natural that corresponds to the identity map on \mathbf{G}^\natural via the isomorphism Ψ in (19). Similarly for $(\mathcal{P}_{\rho'}, \nabla_1)$.

Lemma 4.1 *Let $\alpha_{G'}$ be the invariant differential of G' over $\omega_{G'}$ that corresponds to the identity map on $\omega_{G'}$. The restriction of $(\mathcal{P}_\rho, \nabla_2)$ to $\omega_{G'}$ via $i: \omega_{G'} \rightarrow \mathbf{G}^\natural$ in (22) is isomorphic to the trivial extension of M' by the multiplicative group over $\omega_{G'}$ equipped with the connection associated to $\alpha_{G'}$.*

Proof. (See also [4], Lemma 2.0 for the case $M = [0 \rightarrow A]$.) Recall that we have the following arrows

$$\mathbf{G}^\natural(\mathbf{G}^\natural) \xrightarrow{F} \mathbf{G}^\natural(\omega_{G'}) \xleftarrow{H} \omega_{G'}(\omega_{G'})$$

where the $F(f) = f \circ i$ and $H(h) = i \circ h$. In terms of \natural -extensions of M' by the multiplicative group, the homomorphism F is the base-change via i , while H associates to a differential η the trivial extension of M' by the multiplicative group over $\omega_{G'}$ endowed with the connection associated to η . As $f(\mathrm{id}) = i = H(\mathrm{id})$, the restriction of $(\mathcal{P}_\rho, \nabla_2)$ to $\omega_{G'}$ is isomorphic to the trivial extension of M' by the multiplicative group over $\omega_{G'}$ equipped with the connection associated to $\alpha_{G'}$. \square

Changing the role of M and M' , denote by α_G the invariant differential of G over ω_G that corresponds to the identity map on ω_G . The restriction $(\mathcal{P}_{\rho'}, \nabla_1)$ to ω_G is isomorphic to the trivial extension of M' by the multiplicative group over ω_G equipped with the connection associated to α_G .

In order to study the curvature forms of the connections ∇_i we start considering the curvatures of α_G and $\alpha_{G'}$. We will use in the following the same notation for a locally free sheaf and its associated vector group.

Lemma 4.2 *The curvature of α_G provides a perfect pairing*

$$d\alpha_G: \omega_G \otimes \underline{\mathrm{Lie}}(G) \longrightarrow \underline{\mathrm{Lie}}(\mathbb{G}_{m,S})$$

that is the usual duality.

Proof. We may work locally. Let $\omega_1, \dots, \omega_g$ be a basis of invariant differentials of G . We have $\omega_G = \text{Spec}(\mathcal{O}_S[x_1, \dots, x_n])$ where x_i is the basis of $\underline{\text{Lie}}(G)$ dual to $(\omega_i)_i$. An S' -valued point of ω_G corresponds to a g -tuple $(a_i)_i \in \Gamma(\mathcal{O}_{S'}, S')^g$, hence to the invariant differential $\sum_i a_i \omega_i$ of G over S' . Therefore $\alpha_G = \sum_i x_i \omega_i$. In particular, its curvature form $d\alpha_G = \sum_i dx_i \wedge \omega_i$ provides a pairing

$$\underline{\text{Lie}}(\omega_G) \otimes \underline{\text{Lie}}(G) \rightarrow \mathcal{O}_S$$

that is the usual duality, once identified $\underline{\text{Lie}}(\omega_G)$ with ω_G . \square

Theorem 4.3 *Let M be an S -1-motive. Then Deligne's pairing in (21) is perfect.*

Proof. Observe that the biextension $\mathcal{P}_{\rho'}$ can be defined also as the pull-back of the Poincaré biextension of $([X \rightarrow A], G')$ to $(M, \mathbf{G}'^{\natural})$ and the biextension \mathcal{P}^{\natural} is the pull-back of $\mathcal{P}_{\rho'}$ via

$$(\rho \times \text{id}): M^{\natural} \times \mathbf{G}'^{\natural} \rightarrow M \times \mathbf{G}'^{\natural}$$

together with a suitable trivialization on $\mathbf{G}^{\natural} \times X'$. Furthermore, we have an exact sequence of \mathbf{G}'^{\natural} -group schemes

$$0 \longrightarrow \omega_{G'} \times_S \mathbf{G}'^{\natural} \longrightarrow \mathcal{P}^{\natural} \longrightarrow \mathcal{P}_{\rho'} \longrightarrow 0.$$

This assures that, after pull-back to $\omega_{G'} \times_S \mathbf{G}'^{\natural}$, the \natural -1-structure of \mathcal{P}^{\natural} is the trivial connection because it comes from the connection ∇_1 on $\mathcal{P}_{\rho'}$. Lemma 4.1 implies that after pull-back to $\omega_{G'} \times \mathbf{G}'^{\natural}$, the \natural -2-structure of \mathcal{P}^{\natural} is the connection associated to the invariant differential $\rho'^* \alpha_{G'}$ of \mathbf{G}'^{\natural} . Hence, the restriction of the curvature form of ∇ to

$$\underline{\text{Lie}}(\omega_{G'}) \otimes \underline{\text{Lie}}(\mathbf{G}'^{\natural}) = \omega_{G'} \otimes \underline{\text{Lie}}(\mathbf{G}'^{\natural})$$

is $d(\rho'^* \alpha_{G'})$. This says that the homomorphisms ι and g' in the following sequences of Lie algebras deduced from (22)

$$0 \longrightarrow \omega_{G'} \xrightarrow{\iota} \underline{\text{Lie}}(\mathbf{G}^{\natural}) \xrightarrow{g} \underline{\text{Lie}}(G) \longrightarrow 0,$$

$$0 \longleftarrow \underline{\text{Lie}}(G') \xleftarrow{g'} \underline{\text{Lie}}(\mathbf{G}'^{\natural}) \xleftarrow{\iota'} \omega_G \longleftarrow 0$$

are transposes of each other with respect to the pairing $d\alpha_{G'}$ and Φ . Changing the role of M and M' , we get that ι' and g are transposes of each other with respect to $d\alpha_G$ and Φ . The perfectness of $d\alpha_G$ and $d\alpha_{G'}$ was proved in Lemma 4.2. Hence also Φ is perfect. \square

Example 4.4 Case $A = A' = 0$ and T, T' split. In this case it is possible to give an explicit description of Deligne's pairing. Let M be of the form $[u: \mathbb{Z}^r \rightarrow \mathbb{G}_{m,S}^d]$. Then M' is of the form $[u': \mathbb{Z}^d \rightarrow \mathbb{G}_{m,S}^r]$ and

$$M^{\natural} = [(\iota, u): \mathbb{Z}^r \rightarrow \mathbb{G}_{a,S}^r \times \mathbb{G}_{m,S}^d], \quad M'^{\natural} = [(\iota', u'): \mathbb{Z}^d \rightarrow \mathbb{G}_{a,S}^d \times \mathbb{G}_{m,S}^r],$$

where we write $\mathbb{G}_{a,S}^i$ in place of $\omega_{\mathbb{G}_{m,S}^i}$, for $i = r, d$, and ι (resp. ι') sends an r -tuple (resp. a d -tuple) \mathbf{n} to \mathbf{n} .

Suppose $r = 1, d = 0$. The pull-back of \mathcal{P} to the universal extensions $(M^{\natural}, M'^{\natural})$ is the trivial biextension of $(\mathbb{G}_{a,S}, \mathbb{G}_{m,S})$ by $\mathbb{G}_{m,S}$ together with two trivializations

$$\begin{aligned} \tau_1: \quad \mathbb{Z} \times \mathbb{G}_{m,S} &\rightarrow \mathcal{P}^{\natural} = \mathbb{G}_{a,S} \times \mathbb{G}_{m,S} \times \mathbb{G}_{m,S}, & (n, b) &\mapsto (n, b, b^n), \\ \tau_2: \quad \mathbb{G}_{a,S} \times 0 &\rightarrow \mathcal{P}^{\natural} = \mathbb{G}_{a,S} \times \mathbb{G}_{m,S} \times \mathbb{G}_{m,S}, & (a, 0) &\mapsto (a, 1, 1). \end{aligned}$$

that coincide on $\mathbb{Z} \times 0$. Observe that the above biextension of complexes is *not* trivial. In order to describe the canonical \natural -structure on \mathcal{P}^{\natural} , we start constructing the global differential ω on $\mathbb{G}_{a,S} \times \mathbb{G}_{m,S}$ associated to a connection on \mathcal{P}^{\natural} . Let $\mathbb{G}_{a,S} = \text{Spec}(\mathcal{O}_S[x])$, $\mathbb{G}_{m,S} = \text{Spec}(\mathcal{O}_S[t, 1/t])$ and $dx, dt/t$ be the usual invariant differentials; it will be

$$\omega = f(x, t)dx + g(x, t)dt/t \quad \text{with} \quad f, g \in \Gamma(S, \mathcal{O}_S)[x, t, 1/t].$$

Recall that ν_1, ν_2 in Definition 3.5 are asked to be horizontal. An easy computation shows that necessarily $f = 0$ and g is “additive” in x and does not depend on t . Observe that over a field of characteristic zero $g(x) = ax$, while over a field of characteristic p , we could a priori have the case $g(x) = x^p$. Hence

$$\omega = g(x)dt/t, \quad g(x) \in \Gamma(S, \mathcal{O}_S)[x].$$

We use now the hypothesis that the pull-back of \mathcal{P}^{\natural} to $\mathbb{Z} \times \mathbb{G}_{m,S}$ is isomorphic to the trivial \natural -biextension. The trivialization τ_1 restricted to $\{n\} \times \mathbb{G}_{m,S}$ is $(n, b) \rightarrow (n, b, b^n)$ and $\mathcal{P}^{\natural}_{\{n\} \times \mathbb{G}_{m,S}}$ has to be isomorphic to the trivial \natural -extension of $\mathbb{G}_{m,S}$ by itself. The pull-back of ω to $\mathbb{G}_{m,S}$ is $g(n)dt/t$ and this has to equal ndt/t (see Example 3.2). Therefore $g(x) = x$, the canonical \natural -structure on \mathcal{P}^{\natural} is given by the differential xdt/t and its curvature by $dx \wedge dt/t$. In particular, Deligne's pairing

$$\Phi: \underline{\mathrm{Lie}}(\mathbb{G}_{a,S}) \otimes \underline{\mathrm{Lie}}(\mathbb{G}_{m,S}) \rightarrow \underline{\mathrm{Lie}}(\mathbb{G}_{m,S})$$

is non degenerate.

In the general situation one proceed in a similar way and the global differential associated to the canonical \natural -structure on \mathcal{P}^{\natural} is of the form

$$\omega = \sum_{i=1}^r x_i dz_i/z_i + \sum_{j=1}^d y_j dt_j/t_j$$

where the parameters x_i (resp. y_i) refer to $\mathbb{G}_{a,S}^r$ (resp. to $\mathbb{G}_{a,S}^d$) and the z_i (resp. t_j) refer to $\mathbb{G}_{m,S}^r$ (resp. to $\mathbb{G}_{m,S}^d$). Again the curvature gives rise to a non degenerate pairing whose matrix is built of two blocks, each one involving a torus and its group of characters.

Remark 4.5 As one expects, Deligne's pairing is compatible with weight filtration. To see this fact we have to work locally because, in general, we have no canonical morphisms $\mathbf{G}^{\natural} \rightarrow A^{\natural}$. We assume then that there exist sections δ of $\bar{\tau}: \omega_{G'} \rightarrow \omega_{T'}$ and δ' of $\bar{\tau}': \omega_G \rightarrow \omega_T$ as in the proof of Proposition 3.8. We get then a morphism $\mathbf{G}^{\natural} \xrightarrow{\mathrm{id} - \delta\tau} \mathbf{G}^{\natural} \rightarrow A^{\natural}$ and similar for \mathbf{G}'^{\natural} so that we have an extension

$$0 \longrightarrow T \times T' \times \omega_{T'} \times \omega_T \longrightarrow \mathbf{G}^{\natural} \times \mathbf{G}'^{\natural} \longrightarrow A^{\natural} \times A'^{\natural} \longrightarrow 0. \quad (23)$$

Recall now that Deligne's pairing Φ for M, M' is defined via the canonical \natural -structure ∇ on \mathcal{P}^{\natural} , the pull-back of the Poincaré biextension of (M, M') to $(M^{\natural}, M'^{\natural})$. Let now $\mathcal{P}^{A^{\natural}}$ be the pull-back of the Poincaré biextension of A, A' to $A^{\natural}, A'^{\natural}$ and ∇^A its canonical \natural -structure. We denote by $(\mathcal{P}^{A^{\natural}}, \nabla^A)$ also its pull-back to $\mathbf{G}^{\natural} \times \mathbf{G}'^{\natural}$. With the notations of Lemma 4.1, let $\delta^* \alpha_{T'}$ be the invariant differential of G' over \mathbf{G}^{\natural} associated to $\delta\tau \in \omega_{G'}(\mathbf{G}^{\natural})$ and also its pull-back to \mathbf{G}'^{\natural} . Similarly for $\delta'^* \alpha_T$. Following the constructions in the proofs of Propositions 3.8, 3.9, for $\mathrm{id}_{\mathbf{G}^{\natural}} = \mathrm{id}_{\mathbf{G}^{\natural}} - \delta\tau \oplus \delta\tau$, and the “dual” one, we get that

$$(\mathcal{P}^{\natural}, \nabla) = (\mathcal{P}^{A^{\natural}}, \nabla^A) + (0, \delta^* \alpha_{T'} + \delta'^* \alpha_T)$$

as \natural -biextension of $(\mathbf{G}^{\natural}, \mathbf{G}'^{\natural})$ by $\mathbb{G}_{m,S}$. From this decomposition and the sequence of Lie algebras of (23) we get that Deligne's pairing for M, M' lies in the middle of a diagram where on the right we have Deligne's pairing for A, A' and on the left, after re-ordering the summands, we have Deligne's pairing for $[X \rightarrow T], [X' \rightarrow T']$. The latter pairing has locally the concrete description given in the previous example.

5 \natural -extensions of 1-motives

This section contains results on \natural -extensions of a 1-motive by the multiplicative group that generalize what happens in the classical case of abelian schemes. In particular, we show that \natural -extensions are no longer sufficient to describe universal extensions of 1-motives.

Let notations be as in section 2. Recall that A' represents the functor $S' \rightsquigarrow \mathrm{Ext}^1(A_{S'}, \mathbb{G}_{m,S'})$.

Theorem 5.1 ([9]) *The universal extension A'^{\natural} of the abelian scheme A' over S represents the functor $\mathcal{F}_A: S' \rightsquigarrow \mathrm{Ext}^{\natural}(A_{S'}, \mathbb{G}_{m,S'})$.*

As a consequence we can interpret S' -valued points of A'^{\natural} as \natural -extensions of A by \mathbb{G}_m over S' .

Lemma 5.2 *The pre-sheaf for the flat topology $S' \rightsquigarrow \text{Ext}^{\natural}([X_{S'} \rightarrow A_{S'}], \mathbb{G}_{m,S'})$ is a sheaf represented by G'^{\natural} , the universal extension of G' .*

Proof. It is known (cf. [1], 2.2.1) that $G'^{\natural} = A'^{\natural} \times_{A'} G'$. In particular

$$G'^{\natural}(S') = \{(x, y) \in A'^{\natural}(S') \times G'(S') \text{ inducing the same } S'\text{-valued point on } A'\}.$$

One checks immediately that $\text{Ext}^{\natural}([X_{S'} \rightarrow A_{S'}], \mathbb{G}_{m,S'})$ is the group

$$\left\{ (x, y) \in \begin{array}{l} \text{Ext}^{\natural}(A_{S'}, \mathbb{G}_{m,S'}) \times \text{Ext}^1([X_{S'} \rightarrow A_{S'}], \mathbb{G}_{m,S'}) \\ \text{inducing the same extension of } A_{S'} \text{ by } \mathbb{G}_{m,S'} \end{array} \right\}.$$

Recalling that G' represents the functor $S' \rightsquigarrow \text{Ext}^1([X_{S'} \rightarrow A_{S'}], \mathbb{G}_{m,S'})$, the conclusion follows from Thm. 5.1. \square

For G a semi-abelian scheme over S with maximal subtorus T , it is no longer true that the pre-sheaf for the flat topology $S' \rightsquigarrow \text{Ext}^{\natural}(G_{S'}, \mathbb{G}_{m,S'})$ is a sheaf. Indeed, for $G = T$ the associated sheaf is trivial. However the functor

$$\mathcal{F}_G : S' \rightsquigarrow \text{Ext}^{\natural}(G_{S'}, \mathbb{G}_{m,S'})$$

is still a sheaf if we restrict to a suitable site.

Lemma 5.3 *Suppose S flat over \mathbb{Z} . Let $(E_1, \nabla_1), (E_2, \nabla_2)$ be \natural -extensions of the semi-abelian scheme G by $\mathbb{G}_{m,S}$. Suppose given two horizontal isomorphisms of extensions $f, g : E_1 \rightarrow E_2$. Then $f = g$.*

Proof. The result is trivially true if $G = A$ because $g^{-1}f$ (resp. $f^{-1}g$) is an automorphism of the extension E_1 (resp. E_2) and hence it coincides with the identity map.

Suppose now that the abelian part A is trivial and $G = T$. We may work (fppf) locally on S and then suppose that both E_i are the trivial extension $E^0 = \mathbb{G}_{m,S} \times_S T$. The isomorphisms f, g correspond, respectively, to characters $a, b : T \rightarrow \mathbb{G}_{m,S}$. Let $dz/z + \omega_i$ be the normal invariant differential on E^0 associated to the connection $\nabla_i, i = 1, 2$, where ω_i are invariant differentials of T . The horizontality condition says that $f^*\nabla_2 = g^*\nabla_2 = \nabla_1$. Hence

$$dz/z + da/a + \omega_2 = dz/z + db/b + \omega_2 = dz/z + \omega_1, \quad \text{where } da/a := a^*(dz/z).$$

Therefore $da/a = db/b$ and this implies $a = b$ because of the hypothesis on S .⁴

In the general situation let E_{iT} be the pull-back of E_i via $T \rightarrow G$. It is clear that we have exact sequences

$$0 \longrightarrow E_{iT} \longrightarrow E_i \longrightarrow A \longrightarrow 0.$$

The isomorphisms f, g induce isomorphisms of tori $f_T, g_T : E_{1T} \rightarrow E_{2T}$ and $f_T = g_T$ because of what we explained above. Hence $g_T^{-1}f_T = \text{id}_{E_{1T}}$ and $g^{-1}f$ is an automorphism of the extension E_1 that necessarily coincides with the identity of E_1 because there exist no non-trivial homomorphisms of A to E_{1T} . \square

Let S be a scheme flat over \mathbb{Z} , \mathbf{Sch}/S the category of S -schemes and \mathbf{Fl}/S the full subcategory of \mathbf{Sch}/S consisting of those S -schemes flat over \mathbb{Z} . Observe that if $S = \text{Spec}(k)$ with k a field of characteristic 0, all S -schemes are flat over \mathbb{Z} , hence, \mathbf{Fl}/S and \mathbf{Sch}/S coincide. More generally, this is true if the following hypothesis holds:

(*) *All residue fields of S have characteristic 0.*

We prove now that \mathcal{F}_G is a sheaf on the site $(\mathbf{Fl}/S)_{\text{fl}}$.

Proposition 5.4 *Let G be a semi-abelian scheme over S and suppose S flat over \mathbb{Z} . Then the functor*

$$\mathcal{F}_G : S' \rightsquigarrow \text{Ext}^{\natural}(G_{S'}, \mathbb{G}_{m,S'})$$

is a sheaf on $(\mathbf{Fl}/S)_{\text{fl}}$.

⁴ This proof does not work in positive characteristic p . For example, given characters a, b of T one has $da/a = d(ab^p)/ab^p$.

Proof. We start showing that it is a separated pre-sheaf. Let (E, ∇) be a \mathfrak{h} -extension of G by the multiplicative group over S' . Suppose it trivializes over a covering $\{S'_j\}_j$ of S' . Hence for any index j we have an isomorphism φ_j of $(E_{S'_j}, \nabla)$ with the trivial \mathfrak{h} -extension (E^0, ∇^0) of $G_{S'_j}$ by \mathbb{G}_{m, S'_j} . Moreover, $\varphi_i^{-1}\varphi_j, \varphi_j^{-1}\varphi_i$ are horizontal automorphisms of E over $S'_{ij} := S'_i \times_{S'} S'_j$. By the previous lemma we conclude that $\varphi_j = \varphi_i$ over S'_{ij} and hence these isomorphisms descend to a $\varphi: E \rightarrow \mathbb{G}_{m, S'} \times_{S'} G$ and E is isomorphic to the trivial extension of $G_{S'}$ by $\mathbb{G}_{m, S'}$. Now, ∇ corresponds to a global invariant differential ω on G over S' . By hypothesis $\omega = da_j/a_j$ over S'_j for a suitable homomorphism $a_j: G_{S'_j} \rightarrow \mathbb{G}_{m, S'_j}$ with $da_j/a_j = da_i/a_i$ over S'_{ij} . Hence the a_i provides a homomorphism $a: G_{S'} \rightarrow \mathbb{G}_{m, S'}$ and $\omega = da/a$.

We finish the proof invoking [8] II, 1.5. First we show that \mathcal{F}_G is a sheaf for the Zariski topology and then that

$$\mathcal{F}_G(U) \longrightarrow \mathcal{F}_G(U') \rightrightarrows \mathcal{F}_G(U' \times_U U')$$

is exact with U, U' both affine and U' flat over U .

Let $\{S'_j\}_j$ be a Zariski-covering of S' and let (E_j, ∇_j) be \mathfrak{h} -extensions over S'_j with isomorphisms φ_{ij} between (E_i, ∇_i) and (E_j, ∇_j) over S'_{ij} . Thanks to Lemma 5.3 the φ_{ij} satisfy the usual cocycle condition and hence the E_j glue together providing an extension E of $G_{S'}$ by $\mathbb{G}_{m, S'}$. The \mathfrak{h} -structure can be defined locally on S' and hence we are done.

Suppose now U affine and let U' be an affine scheme faithfully flat and locally of finite type over U . Let $(E_{U'}, \nabla')$ be a \mathfrak{h} -extension over U' that provides isomorphic \mathfrak{h} -extensions on $U' \times_U U'$ via the projection morphisms. Again, the cocycle condition is satisfied because of Lemma 5.3 and the effectiveness of descent data in the affine case permits to conclude that $E_{U'}$ descends to an extension E of G_U by $\mathbb{G}_{m, U}$. Because of the affine hypothesis, E admits a \mathfrak{h} -structure. Hence we are reduced to see that the \mathfrak{h} -structure descends in the case when E is the trivial extension. But this is obvious because ω_G is a sheaf. \square

It is an easy consequence of the above proposition that

Corollary 5.5 *Let S be a scheme flat over \mathbb{Z} . Then $\mathcal{F}_M: S' \rightsquigarrow \text{Ext}^{\mathfrak{h}}(M_{S'}, \mathbb{G}_{m, S'})$ is a sheaf on $(\mathbf{Fl}/S)_{\mathfrak{h}}$.*

As we have already remarked, the functor

$$(\mathbf{Sch}/S)^0 \longrightarrow (\mathbf{Sets}), \quad S' \mapsto \text{Ext}^{\mathfrak{h}}(M'_{S'}, \mathbb{G}_{m, S'})$$

is not, in general, a sheaf for the flat topology. Let denote by $\underline{\text{Ext}}^{\mathfrak{h}}(M, \mathbb{G}_{m, S})$ the associated sheaf. Its restriction to $(\mathbf{Fl}/S)_{\mathfrak{h}}$ is the sheaf \mathcal{F}_M in Corollary 5.5.

Let in the following use the short notation $\mathcal{H}(M)$ for $\underline{\text{Hom}}(M, \mathbb{G}_{m, S})$ and $\mathcal{H}^{\nabla}(M)$ for $\underline{\text{Hom}}^{\nabla}(M, \mathbb{G}_{m, S}) = \ker(\mathcal{H}(M) \rightarrow \omega_G)$.

The sheaf $\underline{\text{Ext}}^{\mathfrak{h}}(M, \mathbb{G}_{m, S})$ fits in the following exact sequence

$$0 \longrightarrow \mathcal{H}^{\nabla}(M) \longrightarrow \mathcal{H}(M) \longrightarrow \omega_G \xrightarrow{j} \underline{\text{Ext}}^{\mathfrak{h}}(M, \mathbb{G}_{m, S}) \longrightarrow \underline{\text{Ext}}^1(M, \mathbb{G}_{m, S}) \quad (24)$$

that generalizes the one in (16) and the one in [9], II.4.2. The exactness on the left is assured by definition of the first sheaf, while the map on the right is an epimorphism because of the commutativity of the following diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \omega_A & \rightarrow & G'^{\mathfrak{h}} = \underline{\text{Ext}}^{\mathfrak{h}}([X \rightarrow A], \mathbb{G}_{m, S}) & \rightarrow & G' = \underline{\text{Ext}}^1([X \rightarrow A], \mathbb{G}_{m, S}) \rightarrow 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow \gamma \\ & & \omega_G & \xrightarrow{\bar{\iota}} & \underline{\text{Ext}}^{\mathfrak{h}}(M, \mathbb{G}_{m, S}) & \longrightarrow & \underline{\text{Ext}}^1(M, \mathbb{G}_{m, S}) \end{array} \quad (25)$$

where the upper sequence is the one describing $G'^{\mathfrak{h}}$ as universal extension of G' by $\mathbb{V}(G') = \omega_A$ and γ, α are the pull-back homomorphisms.

The remaining part of this section is devoted to prove the following result:

Proposition 5.6 *Let $M = [u: X \rightarrow G]$ be an S -1-motive. The sequence*

$$0 \longrightarrow \frac{\mathcal{H}^{\nabla}(T)}{\mathcal{H}^{\nabla}(T) \cap \mathcal{H}(G)} \xrightarrow{\bar{\delta}} \underline{\text{Ext}}^{\mathfrak{h}}(A, \mathbb{G}_{m, S}) \xrightarrow{\alpha} \underline{\text{Ext}}^{\mathfrak{h}}(G, \mathbb{G}_{m, S}) \xrightarrow{\beta} \underline{\text{Ext}}^{\mathfrak{h}}(T, \mathbb{G}_{m, S}) \longrightarrow 0$$

is exact, where α, β are the usual pull-back homomorphisms, $\mathcal{H}^\nabla(T) \cap \mathcal{H}(G)$ denotes the pull-back of $\mathcal{H}^\nabla(T)$ to $\mathcal{H}(G)$ via the monomorphism $\underline{\text{Hom}}(G, \mathbb{G}_{m,S}) \rightarrow \underline{\text{Hom}}(T, \mathbb{G}_{m,S})$.

The morphism $\bar{\delta}$ above is induced by

$$\delta: \underline{\text{Hom}}^\nabla(T, \mathbb{G}_{m,S}) \rightarrow \underline{\text{Ext}}^{\mathfrak{h}}(A, \mathbb{G}_{m,S}), \quad x \mapsto [(G_x, \nabla_x)]$$

where G_x is the extension obtained as push-out of G with respect to the character $x: T \rightarrow \mathbb{G}_{m,S}$. The connection ∇_x is induced by the canonical invariant differential of G_x that pulls back to dz/z on $\mathbb{G}_{m,S}$ and to 0 on G .

To characterize the kernel of β we will need the following result:

Lemma 5.7 *Denote by $f_x: G \rightarrow G_x$ the push-out of G with respect to a character $x: T \rightarrow \mathbb{G}_{m,S}$. Define the homomorphism $\sigma_G: G \rightarrow G_x \times_A G$ via f_x and the identity on G .*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \xrightarrow{j} & G & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow x & & \downarrow f_x & & \parallel & & \\ 0 & \longrightarrow & \mathbb{G}_{m,S} & \xrightarrow{\iota} & G_x & \xrightarrow{g} & A & \longrightarrow & 0 \\ & & \parallel & & \uparrow p_{G_x} & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{G}_{m,S} & \xrightarrow{\iota} & G_x \times_A G & \xrightarrow{\sigma_G} & G & \longrightarrow & 0. \end{array}$$

- Given a normal invariant differential η_x on G_x it holds $j^* f_x^*(\eta_x) = dx/x$.
- Given a normal invariant differential η of $G_x \times_A G$ such that $j^* \sigma_G^*(\eta) = dx/x$, there exists a normal invariant differential η_x of G_x such that $\eta = p_{G_x}^*(\eta_x)$.

Proof. The first statement is immediate because

$$j^* f_x^*(\eta_x) = x^* \iota^*(\eta_x) = x^*(dz/z) = dx/x.$$

For the second statement, let $\{S_h\}_h$ be an affine open covering of S . For any h , let $\eta_{x,h}$ be a normal invariant differential of G_x over S_h . The difference $\omega_h = \eta - p_{G_x}^* \eta_{x,h}$ is the pull-back of an invariant differential of G ; moreover, as $p_{G_x} \circ \sigma_G \circ j = f_x \circ j$, it holds

$$j^* \sigma_G^* p_{G_x}^*(\eta_{x,h}) = j^* f_x^*(\eta_{x,h}) = dx/x = j^* \sigma_G^*(\eta).$$

Hence ω_h is indeed the pull-back of a suitable invariant differential $\omega_{A,h}$ of A_{S_h} . Define $\tilde{\eta}_{x,h} := \eta_{x,h} + g^* \omega_{A,h}$. It satisfies $p_{G_x}^*(\tilde{\eta}_{x,h}) = \eta$, at least over S_h . Hence we proved the assertion locally. To show that $\tilde{\eta}_{x,i} = \tilde{\eta}_{x,h}$ on $S_i \cap S_h$ observe that $p_{G_x}^* \tilde{\eta}_{x,i} = p_{G_x}^* \tilde{\eta}_{x,h} = \eta$ on $S_i \cap S_h$ and $p_{G_x}^*: \omega_{G_x} \rightarrow \omega_{G_x \times_A G}$ is injective. Hence the differentials $\tilde{\eta}_{x,i}$ provide a normal invariant differential η_x of G_x such that $\eta = p_{G_x}^* \eta_x$. \square

Proof. (Proposition 5.6) By definition of δ , it is $\alpha \circ \delta = 0$. Moreover, if x is a character in $\mathcal{H}^\nabla(T) \cap \mathcal{H}(G)$ the extension G_x is isomorphic to the trivial one and the pull-back of η to A is zero because it becomes zero on G and $\omega_A \rightarrow \omega_G$ has trivial kernel. Let (E, ∇) be a \mathfrak{h} -extension of A by the multiplicative group. Suppose that its image via α is trivial. Hence we may think E as the push-out G_x of G with respect to a character $x: T \rightarrow \mathbb{G}_{m,S}$ and ∇ as the connection associated to a normal invariant differential η_∇ on G_x . It holds $\omega_{G_x} = \omega_{\mathbb{G}_{m,S}} \times_{\omega_T} \omega_G$. The projection of η_∇ on $\omega_{\mathbb{G}_{m,S}}$ is dz/z and the projection on ω_G is du/u for a suitable homomorphism $u: G \rightarrow \mathbb{G}_{m,S}$ (because the pull-back of (E, ∇) to G is isomorphic to the trivial \mathfrak{h} -extension). Moreover the image of dz/z in ω_T is dx/x and it must coincide with the image of du/u in ω_T . As the character $x/u|_T$ provides an extension isomorphic to G_x we may assume that $dx/x = 0$. Hence (E, ∇) lies in the image of $\bar{\delta}$. To show that $\beta \circ \alpha = 0$, let (E, ∇) be a \mathfrak{h} -extension of A by the multiplicative group and denote by η_∇ the normal invariant differential of E associated to ∇ . Let (E_G, ∇_G) be the pull-back of (E, ∇) to G . Recall that we have an exact sequence

$$0 \longrightarrow \omega_E \xrightarrow{p_E^*} \omega_{E_G} \longrightarrow \omega_T \longrightarrow 0$$

obtained as push-out of

$$0 \longrightarrow \omega_A \longrightarrow \omega_G \longrightarrow \omega_T \longrightarrow 0. \quad (26)$$

Let now (E_T, ∇_T) be the pull-back of (E_G, ∇_G) to T . Clearly E_T is isomorphic to the trivial extension and the image of $p_E^*(\eta_\nabla)$ in ω_T is 0; hence (E_T, ∇_T) is isomorphic to the trivial \mathfrak{h} -extension.

We show now that the monomorphism $\text{coker}(\delta) \rightarrow \ker(\beta)$ is indeed an isomorphism. Suppose given a \mathfrak{h} -extension (E_G, ∇_G) of G by the multiplicative group over an S -scheme S' such that its image via β is trivial. We may assume that E_G is the pull-back of an extension E of A by $\mathbb{G}_{m,S'}$ and that S' is affine. Denote by η_G the normal invariant differential of E_G associated to ∇_G and by $p_E: E_G \rightarrow E$ the projection homomorphism. The pull-back of η_G to T is an invariant differential of type dx/x for x a character of T . As S' is affine, E admits a \mathfrak{h} -structure ∇_E (associated to a normal invariant differential η_E), so that $\omega = \eta_G - p_E^*\eta_E$ is an invariant differential of G and the restriction of ω to T is dx/x . The second statement of Lemma 5.7 asserts that the trivial extension with the connection induced by ω is isomorphic to the pull-back of (G_x, ∇') for a suitable connection ∇' , hence its isomorphism class lies in the image of α . In particular

$$[(E_G, \nabla_G)] = \alpha[(E, \nabla_E) + (G_x, \nabla')]$$

and we get the result.

The exactness on the right can be deduced from (26) and (24) for $M = T, G$. \square

In a similar way one gets the more general statement:

Proposition 5.8 *Let $M = [u: X \rightarrow G]$ be an S -1-motive. The following sequence*

$$\underline{\text{Ext}}^{\mathfrak{h}}([X \rightarrow A], \mathbb{G}_{m,S}) \xrightarrow{\alpha} \underline{\text{Ext}}^{\mathfrak{h}}(M, \mathbb{G}_{m,S}) \xrightarrow{\beta} \underline{\text{Ext}}^{\mathfrak{h}}(T, \mathbb{G}_{m,S}) \longrightarrow 0$$

is exact, where α, β are the usual pull-back morphisms and $\ker(\alpha) = \frac{\mathcal{H}^\nabla(T)}{\mathcal{H}^\nabla(T) \cap \mathcal{H}(M)}$.

If S satisfies the hypothesis $(*)$, α is a monomorphism. More generally, if S is flat over \mathbb{Z} , the restriction of the above sequence to the site $(\mathbf{F1}/S)_{\mathfrak{h}}$ is also exact on the left.

We will see in Corollary 5.10 that $\mathcal{H}^\nabla(T) \cap \mathcal{H}(M) = \mathcal{H}^\nabla(M)$.

5.1 Universal extensions and \mathfrak{h} -extensions.

Recall that the universal extension $A^{\mathfrak{h}}$ of an abelian variety A represents the sheaf $\underline{\text{Ext}}^{\mathfrak{h}}(A', \mathbb{G}_{m,S})$ (cf. [9]). This does not extend to 1-motives in general.

Proposition 5.9 *Let $M = [u: X \rightarrow G]$ be an S -1-motive and $M^{\mathfrak{h}} = [u^{\mathfrak{h}}: X \rightarrow G^{\mathfrak{h}}]$ its universal extension. There is a canonical epimorphism*

$$\psi_M: G^{\mathfrak{h}} \longrightarrow \omega_{T'} \times_{\underline{\text{Ext}}^{\mathfrak{h}}(T', \mathbb{G}_{m,S})} \underline{\text{Ext}}^{\mathfrak{h}}(M', \mathbb{G}_{m,S})$$

whose kernel is $\ker(\alpha') = \frac{\mathcal{H}^\nabla(T')}{\mathcal{H}^\nabla(T') \cap \mathcal{H}(M')}$.

Proof. By the universal property of the push-out we get from (15) and (25), for G in place of G' , an epimorphism φ_M making the following diagram to commute

$$\begin{array}{ccccccc} 0 & \longrightarrow & \omega_{G'} & \xrightarrow{\iota} & G^{\mathfrak{h}} & \xrightarrow{\rho} & G \longrightarrow 0 \\ & & \parallel & & \downarrow \varphi_M & & \downarrow \gamma \\ & & \omega_{G'} & \xrightarrow{\bar{\iota}} & \underline{\text{Ext}}^{\mathfrak{h}}(M', \mathbb{G}_{m,S}) & \longrightarrow & \underline{\text{Ext}}^1(M', \mathbb{G}_{m,S}) \longrightarrow 0. \end{array} \quad (27)$$

We show that φ_M fits also in the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & G^{\natural} & \longrightarrow & \mathbf{G}^{\natural} & \xrightarrow{\tau} & X \otimes \mathbb{G}_{a,S} = \omega_{T'} \longrightarrow 0 \\
& & \parallel & & \downarrow \varphi_M & & \downarrow j \\
& & \underline{\mathrm{Ext}}^{\natural}([X' \rightarrow A'], \mathbb{G}_{m,S}) & \xrightarrow{\alpha'} & \underline{\mathrm{Ext}}^{\natural}(M', \mathbb{G}_{m,S}) & \xrightarrow{\beta'} & \underline{\mathrm{Ext}}^{\natural}(T', \mathbb{G}_{m,S}) \longrightarrow 0
\end{array} \quad (28)$$

where the upper sequence in the one in (15), the lower one comes from Proposition 5.8, j is the map in (24). To prove that $j \circ \tau = \beta \circ \varphi_M$ one proceeds as follows: We may work locally and suppose that the vertical sequences in (15) are split. Let δ be a section of $\bar{\tau}: \omega_{G'} \rightarrow \omega_{T'}$. Any point in \mathbf{G}^{\natural} may be written as the sum $g + \iota(\delta(\omega))$ with g a point of G^{\natural} and ω a point of $\omega_{T'}$. Now,

$$\begin{aligned}
\beta'(\varphi_M(g + \iota(\delta(\omega)))) &= \beta'(\alpha'(g)) + \beta'(\bar{\iota}(\delta(\omega))) = j(\bar{\tau}(\delta(\omega))) = j(\omega), \\
j(\tau(g + \iota(\delta(\omega)))) &= j(\tau(\iota(\delta(\omega)))) = j(\bar{\tau}(\delta(\omega))) = j(\omega),
\end{aligned}$$

because $\beta' \circ \alpha' = 0$ by Lemma 5.8 and $\beta' \bar{\iota} = j \bar{\tau}$. Diagram (28) assures the existence of ψ_M whose kernel is isomorphic to the kernel of α' . \square

Corollary 5.10 *With notations as above, it holds $\mathcal{H}^{\nabla}(T') \cap \mathcal{H}(M') = \mathcal{H}^{\nabla}(M')$.*

Proof. Comparing all the previous constructions we get a cross of exact sequences

$$\begin{array}{ccccc}
& & \ker \psi_M = \mathcal{H}^{\nabla}(T')/\mathcal{H}^{\nabla}(T') \cap \mathcal{H}(M') & & \\
& & \downarrow & \searrow & \\
\mathcal{H}(M')/\mathcal{H}^{\nabla}(M') = \ker \bar{\iota} & \longrightarrow & \ker \varphi_M & \longrightarrow & \ker \gamma = \mathcal{H}(T')/\mathcal{H}(M') \\
& \searrow & \downarrow & & \\
& & \ker j = \mathcal{H}(T')/\mathcal{H}^{\nabla}(T') & &
\end{array}$$

where the upper diagonal arrow is a monomorphism by construction. Hence also the lower diagonal arrow is a monomorphism and this happens if and only if $\mathcal{H}^{\nabla}(T') \cap \mathcal{H}(M') = \mathcal{H}^{\nabla}(M')$. \square

Corollary 5.11 *Let M be an S -1-motive with S that satisfies hypothesis (*). The group scheme \mathbf{G}^{\natural} in (15) represents the fibre product*

$$\omega_{T'} \times_{\underline{\mathrm{Ext}}^{\natural}(T', \mathbb{G}_{m,S})} \underline{\mathrm{Ext}}^{\natural}(M', \mathbb{G}_{m,S}).$$

Proposition 5.12 *Let M be a 1-motive over S and φ_M the epimorphism defined in (27). Once fixed a universal extension $[u_X^{\natural}: X \rightarrow \omega_{T'}]$ of $[X \rightarrow 0]$ and \mathbf{G}^{\natural} as in (15), there exists a canonical universal extension $M^{\natural} = [u^{\natural}: X \rightarrow \mathbf{G}^{\natural}]$ of M such that $\tau \circ u^{\natural} = u_X^{\natural}$ and the sequence*

$$X \xrightarrow{u^{\natural}} \mathbf{G}^{\natural} \xrightarrow{\varphi_M} \underline{\mathrm{Ext}}^{\natural}(M', \mathbb{G}_{m,S}) \longrightarrow 0$$

is exact. In particular, the kernel of φ_M is isomorphic to $\mathcal{H}(T')/\mathcal{H}^{\nabla}(M')$.

Proof. Uniqueness. Suppose $u_1^{\natural}, u_2^{\natural}$ are universal extensions such that $\varphi_M \circ u_i^{\natural} = 0$ and $\tau \circ u_i^{\natural} = u_X^{\natural}$. Clearly $u_1^{\natural} - u_2^{\natural}$ factors through $\omega_{G'}$; as $\varphi_M \circ (u_1^{\natural} - u_2^{\natural}) = 0$, the morphism $u_1^{\natural} - u_2^{\natural}$ factors through the subsheaf $\ker \iota = \mathcal{H}(M')/\mathcal{H}^{\nabla}(M')$ of $\omega_{G'}$. Furthermore $\tau \circ (u_1^{\natural} - u_2^{\natural}): X \rightarrow \omega_{T'}$ is the zero map. It follows from Corollary 5.10, that the composition $\mathcal{H}(M')/\mathcal{H}^{\nabla}(M') \rightarrow \mathcal{H}(T')/\mathcal{H}^{\nabla}(T') \rightarrow \omega_{T'}$ is a monomorphism; then $u_1^{\natural} = u_2^{\natural}$.

The uniqueness result assures that we can construct u^{\natural} étale locally. We proceed as in [1], 2.3, assuming that $X = \mathcal{H}(T') = \bigoplus_i \mathbb{Z}e_i$. Let δ be a section of $\tau: \omega_{G'} \rightarrow \omega_{T'}$ as in the proof of Proposition 3.8 so that we identify \mathbf{G}^{\natural} with $G^{\natural} \oplus \omega_{T'}$. If $\tilde{u}: X \rightarrow G^{\natural}$ is a lifting of u , $u^{\natural}: X \rightarrow \mathbf{G}^{\natural}$ can then be defined via $u^{\natural}(e_i) = \tilde{u}(e_i) + \delta u_X^{\natural}(e_i)$.

Recall that $u(e_i)$ is the $\mathbb{G}_{m,S}$ -extension $[X \rightarrow G_{e_i}]$ of M' obtained as the push-out of M' with respect to the character e_i . Let $f_i: G \rightarrow G_{e_i}$ be the induced map. The section $u(e_i)$ lifts to a section $\tilde{u}(e_i)$ of $G^{\natural} = \underline{\text{Ext}}^{\natural}(M_A, \mathbb{G}_{m,S})$ as soon as we fix an invariant differential η_i of G_{e_i} ; locally this is always possible. Then $\varphi_M \circ u^{\natural}(e_i)$ corresponds to the trivial $\mathbb{G}_{m,S}$ -extension of M together with the invariant differential $f_i^* \eta_i + \delta u_X^{\natural}(e_i)$. Applying Lemma 5.7 it is immediate to check that η_i can be chosen so that $\varphi_M \circ u^{\natural}(e_i) = 0$. \square

The previous proposition *does not* imply that for any universal extension $v: X \rightarrow \mathbf{G}^{\natural}$ of M it holds $\varphi_M \circ v = 0$. Indeed, we have seen in Remark 2.5 that $v + f$ is also a universal extension for any homomorphism $f: X \rightarrow \omega_{A'} (\rightarrow \omega_{G'} \rightarrow \mathbf{G}^{\natural})$ and clearly $\varphi_M \circ f$ is not trivial in general.

Corollary 5.13 *Let M be an S -1-motive with S that satisfies hypothesis $(*)$. Consider the homomorphism*

$$v: X \rightarrow \mathbf{G}^{\natural} = \omega_{T'} \times_{\underline{\text{Ext}}^{\natural}(T', \mathbb{G}_{m,S})} \underline{\text{Ext}}^{\natural}(M', \mathbb{G}_{m,S})$$

whose projection to the first (resp. second) factor is $x \mapsto dx/x$ (resp. the 0 map). It is a universal extension of M . Moreover, there is an exact sequence

$$0 \longrightarrow X \xrightarrow{v} \mathbf{G}^{\natural} \xrightarrow{\varphi_M} \underline{\text{Ext}}^{\natural}(M', \mathbb{G}_{m,S}) \longrightarrow 0.$$

Observe that the right hand square in (27) is cartesian as soon as $\underline{\text{Hom}}(M', \mathbb{G}_{m,S}) = 0$, for example if $T' = 0$. Under this hypothesis, we could use the homomorphism φ_M in (27) to prove the result in Proposition 3.9, i.e. the existence of a \natural -structure on \mathcal{P}^{\natural} . In the general case however, the homomorphism φ_M loses information because, if we know the class in $\underline{\text{Ext}}^{\natural}(M_{S'}, \mathbb{G}_{m,S'})$ of a \natural -extension (\mathcal{P}_g, ∇) with \mathcal{P}_g the fibre of the Poincaré biextension of (M, M') at $g \in G'(S')$, we can determine ∇ only up to an invariant differential of the type du/u for u a homomorphism of $M'_{S'} \rightarrow \mathbb{G}_{m,S'}$.

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