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Closed differential forms on moduli spaces of sheaves

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Abstract: *Let X be a smooth projective variety, and let $\mathcal{M}$ be a moduli space of stable sheaves on X. For any flat family E of coherent sheaves on X parametrized by a smooth scheme Y, and for any integer m, with $1 \leq m \leq \dim X$, we construct a closed differential form $\Omega = \Omega_E$ on Y with values in $H^m(X, \mathcal{O}_X)$. By using the vector-valued differential form Ω we then prove that the choice of a (non-zero) differential m-form σ on X, $\sigma \in H^0(X, \Omega_X^m)$, determines, in a natural way, a closed differential m-form Ω_σ on $\mathcal{M}$.*

– Introduction

Let X be a smooth projective variety, defined over an algebraically closed field k of characteristic 0, and let $\mathcal{M}$ be a moduli space of stable sheaves on X. It is by now well known that many geometric properties of the moduli space $\mathcal{M}$ are determined by similar geometric properties of the base variety X.

A beautiful example of this general fact was discovered by S. Mukai in [Mu1], where he shows that if X is an abelian or K3 surface (hence it is a symplectic algebraic surface), then the choice of a symplectic structure (i.e., a non-degenerate 2-form) on X determines a symplectic structure on the moduli space $\mathcal{M}$. This result was later generalized by S. Kobayashi [K] to the case of a compact Kähler manifold X with a holomorphic symplectic structure: in this case too, the choice of a symplectic structure on X determines a symplectic structure on the nonsingular part of the moduli space of stable vector bundles on X.

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In [B1] we generalized Mukai's result to the case of Poisson surfaces, by proving that, if the algebraic surface X admits a Poisson structure, then the choice of such a structure on X determines, in a natural way, a Poisson structure on the moduli space $\mathcal{M}$. Similar results also hold for other types of moduli spaces, such as moduli spaces of framed vector bundles [B3], moduli spaces of parabolic bundles [B4] and Hilbert schemes of points of a surface [B2].

In this paper we provide another example showing how geometric structures on X often determine similar geometric structures on moduli spaces of sheaves on X. We prove that, if X admits non-zero differential forms of degree m, then the choice of any such m-form σ determines a closed differential m-form Ω_σ on the moduli space $\mathcal{M}$ of stable sheaves on X.

A particularly interesting special case of this result is obtained by taking X to be a Calabi-Yau n-fold. In this case there is a canonical choice (up to scalar multiples) of a n-form σ on X. This implies that there is also a closed n-form Ω_σ on the moduli space $\mathcal{M}$. This result may be considered as a higher dimensional generalization of Mukai's result in [Mu1].

This paper is organized as follows: in Section 1 we recall some useful results about cup-products and trace maps. Then we introduce the symmetrized trace map and study its graded commutativity properties. This is the main technical tool needed to construct closed differential forms on moduli spaces of stable sheaves on X.

In Section 2 we construct, for any flat family E of coherent sheaves on X parametrized by a smooth scheme Y, and for any integer m, with $1 \leq m \leq \dim X$, a differential form $\Omega = \Omega_E$ on Y with values in $H^m(X, \mathcal{O}_X)$. The main part of this section is then devoted to prove that $d\Omega = 0$ (in order to symplify the exposition this is proven under the additional assumptions that the base field k is the complex field and that E is a family of locally free sheaves).

Finally, in Section 3, we use the vector-valued differential form Ω to construct ordinary (i.e., scalar-valued) differential forms on the moduli space $\mathcal{M}$ of stable sheaves on X. More precisely, we prove that the choice of a (non-zero) differential m-form σ on X, $\sigma \in H^0(X, \Omega^m_X)$, determines, in a natural way, a differential m-form Ω_σ on $\mathcal{M}$, defined by using the vector-valued m-form Ω. Then the closure of Ω, proved in Section 2, immediately implies the closure of the m-form Ω_σ.

The case of a smooth Calabi-Yau n-fold X is especially interesting. In fact, in this case there is a canonical choice (up to scalar multiples) of the n-form σ on X, hence there is also a closed differential n-form Ω_σ on the moduli space of stable sheaves on X. The natural question of the non-degeneracy of Ω_σ is then discussed.

In the last part of this section we provide an example in order to explain how the construction of the differential forms Ω_σ can be generalized to other types of moduli spaces. We analyze in detail the case of moduli spaces of stable framed vector bundles.

1 – Preliminaries

1.1 – Cup-product and trace maps

In this section we shall recall, without proofs, some standard facts about trace maps and cup-products. For more details (and proofs) we refer the reader to [A] or [HL2, p. 217].

Let X be a smooth n-dimensional projective variety over an algebraically closed field k of characteristic 0. In the sequel, whenever we speak of a sheaf on a scheme S we shall always mean a sheaf of $\mathcal{O}_S$-modules.

For any coherent sheaf E on X, the usual trace map

$$\mathrm{tr} : \mathcal{E}nd(E) \to \mathcal{O}_X$$

induces natural maps (also called trace maps and denoted by the same symbol)

$$\mathrm{tr} : \mathrm{Ext}^i(E, E) \to H^i(X, \mathcal{O}_X).$$

For any i and j there is a natural cup-product (or Yoneda composition) map

$$\mathrm{Ext}^i(E, E) \times \mathrm{Ext}^j(E, E) \xrightarrow{\circ} \mathrm{Ext}^{i+j}(E, E)$$

(which can be easily defined in terms of Čech cocycles by replacing E with a finite locally free resolution $E^{\cdot}$), and the composition of cup-product and trace is graded commutative in the following sense: if $\alpha \in \mathrm{Ext}^i(E, E)$ and $\beta \in \mathrm{Ext}^j(E, E)$, then

$$\mathrm{tr}(\alpha \circ \beta) = (-1)^{ij}\,\mathrm{tr}(\beta \circ \alpha) \tag{1.1}$$

as an element of $H^{i+j}(X, \mathcal{O}_X)$ (cf. [HL2, pp. 216-217]).

Analogous maps can also be defined in a relative situation. Let us consider a scheme Y of finite type over k and denote by $p : X \times Y \to X$ and $q : X \times Y \to Y$ the canonical projections.

If E is a Y-flat family of coherent sheaves on X, we shall denote by $\mathcal{E}xt^i_q(E,E)$ the i-th relative Ext-sheaf, i.e., the sheaf on Y associated to the presheaf

$$U \mapsto \mathrm{Ext}^i(E|_{X\times U}, E|_{X\times U}),$$

for any open subset $U \subset Y$.

Since any such E admits a finite locally free resolution, the preceding constructions of the cup-product and the trace maps carry over to this relative situation. More precisely, for any i and j we have a cup-product map

$$\mathcal{E}xt^i_q(E, E) \times \mathcal{E}xt^j_q(E, E) \xrightarrow{\circ} \mathcal{E}xt^{i+j}_q(E, E)$$

and a trace map

$$\mathrm{tr} : \mathcal{E}xt^i_q(E, E) \to R^i q_* \mathcal{O}_{X\times Y} \cong H^i(X, \mathcal{O}_X) \otimes_k \mathcal{O}_Y$$

satisfying (1.1) for any sections α and β of $\mathcal{E}xt^i_q(E, E)$ and $\mathcal{E}xt^j_q(E, E)$, respectively.

1.2 – The symmetrized trace map

Let E be a coherent sheaf on X. For any integer $m \geq 1$ let us consider the "symmetrized composition map"

$$\underbrace{\mathcal{E}nd(E) \times \cdots \times \mathcal{E}nd(E)}_{m} \xrightarrow{S} \mathcal{E}nd(E) \tag{1.2}$$

defined by setting

$$S(\phi_1, \phi_2, \ldots, \phi_m) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \phi_{\sigma(1)} \circ \phi_{\sigma(2)} \circ \cdots \circ \phi_{\sigma(m)},$$

where the sum runs over the group $\mathfrak{S}_m$ of permutations of m elements.

We define the "symmetrized trace", denoted by Str, to be the composition of S with the usual trace map:

$$\operatorname{Str}(\phi_1, \phi_2, \ldots, \phi_m) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \operatorname{tr}(\phi_{\sigma(1)} \circ \phi_{\sigma(2)} \circ \cdots \circ \phi_{\sigma(m)}). \tag{1.3}$$

The map

$$\operatorname{Str} : \mathcal{E}nd(E) \times \cdots \times \mathcal{E}nd(E) \to \mathcal{O}_X \tag{1.4}$$

is totally symmetric and multilinear.

It is easy to prove that, for a general $m \geq 2$,

$$\operatorname{Str}(\phi_1, \phi_2, \ldots, \phi_m) = \frac{1}{(m-1)!} \sum_{\sigma} \operatorname{tr}(\phi_1 \circ \phi_{\sigma(2)} \circ \cdots \circ \phi_{\sigma(m)}), \tag{1.5}$$

where the sum runs over all permutations σ of the set $\{2, 3, \ldots, m\}$.

The symmetrized trace map (1.4) induces a map, also denoted by Str,

$$\operatorname{Str} : \operatorname{Ext}^{i_1}(E,E) \times \cdots \times \operatorname{Ext}^{i_m}(E,E) \to H^{i_1+\cdots+i_m}(X, \mathcal{O}_X). \tag{1.6}$$

This map satisfies a kind of graded commutativity property similar to the one stated in (1.1).

Proposition 1.1. *Let $\phi_h \in \operatorname{Ext}^{i_h}(E,E)$, for $h = 1, \ldots, m$. For any integer p, with $1 \leq p \leq m-1$, we have*:

$$\operatorname{Str}(\phi_1, \ldots, \phi_p, \phi_{p+1}, \ldots, \phi_m) = (-1)^{i_p i_{p+1}} \operatorname{Str}(\phi_1, \ldots, \phi_{p+1}, \phi_p, \ldots, \phi_m),$$

i.e., whenever we mutually exchange two adjacent elements ϕ_p and ϕ_{p+1}, the value of Str *acquires the factor $(-1)^{\deg(\phi_p)\deg(\phi_{p+1})} = (-1)^{i_p i_{p+1}}$.*

Proof. Let $E^\cdot$ be a finite locally free resolution of E and set $A^\cdot = \mathrm{Hom}^\cdot(E^\cdot, E^\cdot)$. Then we have $\mathrm{Ext}^i(E,E) = H^i(A^\cdot)$, the i-th hypercohomology group of the complex $A^\cdot$.

The symmetrized trace map

$$\mathrm{Str} : H^{i_1}(A^\cdot) \otimes \cdots \otimes H^{i_m}(A^\cdot) \to H^{i_1+\cdots+i_m}(X, \mathcal{O}_X)$$

is the composition of the following three maps: first, the multiplication

$$\mu : H^{i_1}(A^\cdot) \otimes \cdots \otimes H^{i_m}(A^\cdot) \to H^{i_1+\cdots+i_m}(A^\cdot \otimes \cdots \otimes A^\cdot), \tag{1.7}$$

then the symmetrized composition map

$$S : H^{i_1+\cdots+i_m}(A^\cdot \otimes \cdots \otimes A^\cdot) \to H^{i_1+\cdots+i_m}(A^\cdot) \tag{1.8}$$

and finally the usual trace map

$$\mathrm{tr} : H^{i_1+\cdots+i_m}(A^\cdot) \to H^{i_1+\cdots+i_m}(X, \mathcal{O}_X). \tag{1.9}$$

Let us denote by $\pi = \pi_{p,p+1}$ the twist operator that exchanges the factors at places p and $p+1$:

$$\begin{aligned} \pi : H^{i_1}(A^\cdot) \otimes \cdots \otimes H^{i_p}(A^\cdot) \otimes H^{i_{p+1}}(A^\cdot) \otimes \cdots \otimes H^{i_m}(A^\cdot) \to \\ \to H^{i_1}(A^\cdot) \otimes \cdots \otimes H^{i_{p+1}}(A^\cdot) \otimes H^{i_p}(A^\cdot) \otimes \cdots \otimes H^{i_m}(A^\cdot). \end{aligned}$$

We have

$$\pi(\alpha_1 \otimes \cdots \otimes \alpha_p \otimes \alpha_{p+1} \otimes \cdots \otimes \alpha_m) = (-1)^{i_p i_{p+1}} \alpha_1 \otimes \cdots \otimes \alpha_{p+1} \otimes \alpha_p \otimes \cdots \otimes \alpha_m,$$

where $i_p = \deg(\alpha_p)$ and $i_{p+1} = \deg(\alpha_{p+1})$. Then the following diagram is commutative:

$$\begin{array}{ccc} H^{i_1}(A^\cdot) \otimes \cdots \otimes H^{i_m}(A^\cdot) & \xrightarrow{\ \mu\ } & H^{i_1+\cdots+i_m}(A^\cdot \otimes \cdots \otimes A^\cdot) \\ \big\downarrow \pi_{p,p+1} & & \big\downarrow H(\pi_{p,p+1}) \\ H^{i_1}(A^\cdot) \otimes \cdots \otimes H^{i_m}(A^\cdot) & \xrightarrow{\ \mu\ } & H^{i_1+\cdots+i_m}(A^\cdot \otimes \cdots \otimes A^\cdot). \end{array}$$

By composing with the maps S and tr of (1.8) and (1.9), it follows that $\mathrm{Str} = \mathrm{Str} \circ \pi_{p,p+1}$, which is precisely what we had to prove.

In the sequel we shall be interested in a special case of (1.6). By taking all i_h equal to 1, we get the map

$$\mathrm{Str} : \underbrace{\mathrm{Ext}^1(E,E) \times \cdots \times \mathrm{Ext}^1(E,E)}_{m} \to H^m(X, \mathcal{O}_X),$$

satisfying

$$\mathrm{Str}(\phi_1, \ldots, \phi_p, \phi_{p+1}, \ldots, \phi_m) = -\,\mathrm{Str}(\phi_1, \ldots, \phi_{p+1}, \phi_p, \ldots, \phi_m),$$

for every $p \in [1, m-1]$.

COROLLARY 1.2. *For any $m \geq 1$, the map*

$$\mathrm{Str} : \underbrace{\mathrm{Ext}^1(E,E) \times \cdots \times \mathrm{Ext}^1(E,E)}_{m} \to H^m(X, \mathcal{O}_X)$$

is alternating, i.e., for any permutation σ of $\{1, 2, \ldots, m\}$, we have:

$$\mathrm{Str}(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(m)}) = \mathrm{sgn}(\sigma)\,\mathrm{Str}(\alpha_1, \alpha_2, \ldots, \alpha_m).$$

Let now Y be a scheme of finite type over k and let E be a Y-flat family of coherent sheaves on X. Since any such E admits a finite resolution by locally free sheaves, the preceding constructions can be generalized to this relative situation (exactly as in the case of the usual trace map, described in Section 1.1). We leave the details to the reader and just state the relative version of Corollary 1.2:

COROLLARY 1.3. *Let E be a Y-flat family of coherent sheaves on X. For any $m \geq 1$, the map*

$$\begin{aligned} \mathrm{Str} : \underbrace{\mathcal{E}xt^1_q(E,E) \times \cdots \times \mathcal{E}xt^1_q(E,E)}_{m} &\to R^m q_*(\mathcal{O}_{X\times Y}) \cong \\ &\cong H^m(X, \mathcal{O}_X) \otimes_k \mathcal{O}_Y \end{aligned} \tag{1.10}$$

is alternating, i.e., for any permutation σ of $\{1, 2, \ldots, m\}$, we have:

$$\mathrm{Str}(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(m)}) = \mathrm{sgn}(\sigma)\,\mathrm{Str}(\alpha_1, \alpha_2, \ldots, \alpha_m).$$

2 – Vector-valued differential forms

Let Y be a *smooth* scheme of finite type over k and E a Y-flat family of coherent sheaves on X. In this section we shall define, for any such E and any integer m, with $1 \leq m \leq n = \dim X$, a vector-valued m-form on Y (more precisely, a differential form of degree m on Y with values in $H^m(X, \mathcal{O}_X)$). Then we shall prove that these differential forms are closed.

Let us begin by recalling that, for any sheaf F on X, the set of isomorphism classes of infinitesimal deformations of F is canonically identified with $\mathrm{Ext}^1(F, F)$. It follows that, for any family E of coherent sheaves on X parametrized by Y, there is a map, known as the Kodaira-Spencer map,

$$\rho : TY \to \mathcal{E}xt^1_q(E, E), \tag{2.1}$$

that sends a tangent vector $v \in T_yY$ to the class $\rho(v) \in \mathrm{Ext}^1(E_y, E_y)$ corresponding to the infinitesimal deformation of the sheaf E_y along the direction of v.

Now, for any m as above, we define an $H^m(X, \mathcal{O}_X)$-valued differential m-form $\Omega = \Omega_E$ on Y by setting

$$\Omega : \underbrace{TY \times \cdots \times TY}_{m} \to \mathcal{E}xt^1_q(E, E) \times \cdots \times \mathcal{E}xt^1_q(E, E) \to H^m(X, \mathcal{O}_X) \otimes_k \mathcal{O}_Y,$$

where the first map is induced by the Kodaira-Spencer map (2.1), and the second one is the symmetrized trace map (1.10). In other words, we set

$$\Omega(v_1, \dots, v_m) = \mathrm{Str}(\rho(v_1), \dots, \rho(v_m)), \tag{2.2}$$

for any sections $v_1, \dots, v_m$ of the tangent bundle TY. It follows from Corollary 1.3 that Ω is a vector-valued differential form of degree m.

Remark 2.1. Let E be a Y-flat family of sheaves on X and L be a line bundle on Y. We can define another Y-flat family of sheaves E' on X by setting $E' = E \otimes q^*(L)$. These two families of sheaves may be considered as equivalent because, for every closed point $y \in Y$, the sheaves E_y and E'_y on X are isomorphic. Under these hypotheses, the differential m-forms Ω_E and $\Omega_{E'}$ are equal.

Remark 2.2. Let us observe that in the definition of Ω_E we do not use directly the sheaf E, but rather the sheaf $\mathcal{E}xt^1_q(E, E)$. This is very important because in most interesting applications, when we take as Y a suitable moduli space of stable sheaves on X, a global universal family E does not exist (at least as an ordinary sheaf), but the sheaf $\mathcal{E}xt^1_q(E, E)$ on Y is, nevertheless, well defined (cf. Remark 3.1). It follows that our definition of the differential form Ω_E remains valid also in these more general situations.

The rest of this section will be devoted to the proof that, for any E and any integer m, $d\Omega_E = 0$.

In order to try to simplify the exposition as far as possible, we shall assume, from now on, that k is the complex field $\mathbb{C}$, even if our proof works, with only minor modifications, for any algebraically closed field k of characteristic 0.

So let X be a smooth n-dimensional complex projective variety and let Y be a smooth complex variety of dimension N. We shall use the complex analytic topology on the complex manifolds associated to the algebraic varieties X and Y.

Let E be a Y-flat family of sheaves on X. We shall assume that E is locally free of rank r. Let $\Omega = \Omega_E$ be the m-form on Y defined above.

First of all, we remark that, since the closure of Ω is a local property on Y, we may freely replace Y be an open neighborhood of any one of its points, hence we may assume that a local system of holomorphic coordinates $y = (y_i)_{i=1,\dots,N}$ is given on Y.

Now, by eventually replacing Y with a smaller open subset, if necessary, we may find an open covering $(U_i)_{i\in I}$ of X such that the restriction of the vector bundle E to $U_i \times Y$ is trivial. Let us denote by

$$f_i : E|_{U_i\times Y} \xrightarrow{\sim} (U_i \times Y) \times \mathbb{C}^r$$

the trivialization isomorphisms. On the intersection $U_{ij} = U_i \cap U_j$ of two open subsets, we have the isomorphism

$$g_{ij} : (U_{ij} \times Y) \times \mathbb{C}^r \xrightarrow{\sim} (U_{ij} \times Y) \times \mathbb{C}^r$$

defined by setting $g_{ij} = f_i|_{U_{ij}\times Y} \circ f_j^{-1}|_{U_{ij}\times Y}$. For any point $(x, y) \in U_{ij} \times Y$ and any $v \in \mathbb{C}^r$, we have

$$g_{ij} : ((x, y), v) \mapsto ((x, y), \tilde{g}_{ij}(x, y)\, v),$$

where

$$\tilde{g}_{ij} : U_{ij} \times Y \to \mathrm{GL}(r, \mathbb{C})$$

is a holomorphic function. In the sequel, by abuse of notation, we shall identify g_{ij} with $\tilde{g}_{ij}$. The functions g_{ij} are called the transition functions of the vector bundle E (with respect to the given open covering).

The transition functions $(g_{ij})_{i,j\in I}$ satisfy the usual cocycle identities

$$g_{ii} = 1, \qquad g_{ji} = g_{ij}^{-1}, \qquad g_{ij} \circ g_{jk} = g_{ik}, \tag{2.3}$$

on the intersection of three open subsets $U_{ijk} = U_i \cap U_j \cap U_k$.

Now we can give an explicit description of the Kodaira-Spencer map

$$\rho : TY \to \mathcal{E}xt^1_q(E, E) = R^1 q_* \mathcal{E}nd(E).$$

At any point $y \in Y$ the derivations $\partial_\alpha = \frac{\partial}{\partial y_\alpha}$, for $\alpha = 1, \dots, N$, form a basis of the tangent space $T_y Y$. If we consider the vector bundle E on $X \times Y$ as a family of vector bundles $\{E_y\}_{y \in Y}$ on X varying holomorphically with $y \in Y$, the infinitesimal deformation of the vector bundle E_y corresponding to the tangent vector ∂_α is given by $\frac{\partial E_y}{\partial y_\alpha}$. To give a meaning to this symbol, let us consider the transition functions $\{g_{ij}(\cdot, y)\}_{i,j \in I}$ of the vector bundle E_y on X. Then the infinitesimal deformation $\frac{\partial E_y}{\partial y_\alpha}$ is given (by definition) by the collection of functions $\{\partial_\alpha g_{ij}(\cdot, y)\}_{i,j \in I}$.

By deriving the (multiplicative) cocycle identities (2.3), we obtain the following (additive) cocycle identities for the functions $\partial_\alpha g_{ij}(\cdot, y)$:

$$\text{(2.4)} \qquad \partial_\alpha g_{ji} = -g_{ij}^{-1} \circ (\partial_\alpha g_{ij}) \circ g_{ij}^{-1}, \quad (\partial_\alpha g_{ij}) \circ g_{jk} + g_{ij} \circ (\partial_\alpha g_{jk}) = \partial_\alpha g_{ik}.$$

By recalling that $g_{ij} = f_i \circ f_j^{-1}$, these identities can be rewritten as

$$f_j^{-1}(\partial_\alpha g_{ji}) f_i = -f_i^{-1}(\partial_\alpha g_{ij}) f_j, \quad f_i^{-1}(\partial_\alpha g_{ij}) f_j + f_j^{-1}(\partial_\alpha g_{jk}) f_k = f_i^{-1}(\partial_\alpha g_{ik}) f_k.$$

Finally, if we set

$$\eta_{ij}^\alpha = f_i^{-1}(\partial_\alpha g_{ij}) f_j,$$

we obtain a collection of sections $\eta_{ij}^\alpha(\cdot, y) \in \Gamma(U_{ij}, \mathcal{E}nd(E_y))$ satisfying the usual (additive) cocycle relations

$$\text{(2.5)} \qquad \eta_{ji}^\alpha = -\eta_{ij}^\alpha, \quad \eta_{ij}^\alpha + \eta_{jk}^\alpha = \eta_{ik}^\alpha.$$

The cohomology class $\overline{\eta}^\alpha \in H^1(X, \mathcal{E}nd(E_y))$ represented by the Čech 1-cocycle $\{\eta_{ij}^\alpha\}_{i,j \in I}$ is the image of the tangent vector $\partial_\alpha \in T_y Y$ by the Kodaira-Spencer map

$$\rho : T_y Y \to (R^1 q_* \mathcal{E}nd(E))_y = H^1(X, \mathcal{E}nd(E_y)).$$

Remark 2.3. Other equivalent representations of the infinitesimal deformation $\frac{\partial E_y}{\partial y_\alpha}$ are possible. For instance, in [Mu2], Mukai represents the cohomology class $\rho(\partial_\alpha) \in H^1(X, \mathcal{E}nd(E_y))$ by using the 1-cocycle $\{a_{ij}\}_{i,j \in I}$ defined by setting

$$a_{ij} = g_{ij}^{-1} \partial_\alpha g_{ij}.$$

This is equivalent to our representation, except for the fact that the a_{ij}'s satisfy a kind of twisted cocycle identities, given by

$$a_{ji} = -g_{ij} a_{ij} g_{ij}^{-1}, \qquad g_{jk}^{-1} a_{ij} g_{jk} + a_{jk} = a_{ik}.$$

(See [Mu2, Section 3, p. 151]).

We are now able to give an explicit description of the m-form $\Omega = \Omega_E$ on Y in terms of Čech cocycles.

For any closed point $y \in Y$ and any $\partial_\alpha \in T_yY$ we have set $\eta_{ij}^\alpha = f_i^{-1}\,\partial_\alpha g_{ij}\, f_j$, and we have seen that the 1-cocycle $\{\eta_{ij}^\alpha\}_{i,j\in I}$ represents the cohomology class $\bar\eta^\alpha = \rho(\partial_\alpha) \in H^1(X, \mathcal{E}nd(E_y))$. It follows from the definition (2.2) of Ω that, for any $\alpha_1, \dots, \alpha_m \in [1, N]$, we have:

$$\Omega(\partial_{\alpha_1}, \dots, \partial_{\alpha_m}) = \mathrm{Str}(\bar\eta^{\alpha_1}, \dots, \bar\eta^{\alpha_m}).$$

We shall now recall the standard expression of the cup-product in terms of Čech cocycles:

LEMMA 2.4. *Let $\{\psi_{ij}^h\}_{i,j\in I}$, for $h = 1, \dots, m$, be Čech 1-cocycles representing the cohomology classes $\bar\psi^h \in H^1(X, \mathcal{E}nd(F))$, for some locally free sheaf F on X. Then the cup-product (or Yoneda composition)*

$$\bar\psi^1 \circ \bar\psi^2 \circ \cdots \circ \bar\psi^m \in H^m(X, \mathcal{E}nd(F))$$

is the cohomology class represented by the Čech m-cocycle

$$\{\psi_{i_1 i_2}^1 \circ \psi_{i_2 i_3}^2 \circ \cdots \circ \psi_{i_m i_{m+1}}^m\}_{i_1,\dots,i_{m+1}\in I}.$$

From this result, and the definition (1.3) of the map Str, we obtain the following explicit expression of Ω:

$$\begin{aligned}\Omega(\partial_{\alpha_1}, \dots, \partial_{\alpha_m}) &= \mathrm{Str}(\bar\eta^{\alpha_1}, \dots, \bar\eta^{\alpha_m}) = \\ &= \frac{1}{m!} \sum_{\sigma\in\mathfrak{S}_m} \Big\{ \mathrm{tr}(\eta_{i_1 i_2}^{\alpha_{\sigma(1)}} \circ \eta_{i_2 i_3}^{\alpha_{\sigma(2)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}}) \Big\}_{i_1, i_2, \dots, i_{m+1}}.\end{aligned}$$

To simplify the notation, we introduce a multiindex $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$. Then, for every permutation $\sigma \in \mathfrak{S}_m$, we set

$$\sigma(\boldsymbol{\alpha}) = (\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \dots, \alpha_{\sigma(m)})$$

and

$$\boldsymbol{\eta}_{i_1, i_2, \dots, i_{m+1}}^{\sigma(\boldsymbol{\alpha})} = \eta_{i_1 i_2}^{\alpha_{\sigma(1)}} \circ \eta_{i_2 i_3}^{\alpha_{\sigma(2)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}}.$$

We shall also write $\boldsymbol{\partial_\alpha} = (\partial_{\alpha_1}, \partial_{\alpha_2}, \dots, \partial_{\alpha_m})$ and $\mathbf{dy}_{\boldsymbol{\alpha}} = dy_{\alpha_1} \wedge dy_{\alpha_2} \wedge \cdots \wedge dy_{\alpha_m}$.

With these notations we may write

$$\Omega(\partial_{\alpha_1}, \dots, \partial_{\alpha_m}) = \Omega(\boldsymbol{\partial_\alpha}) = \frac{1}{m!} \sum_{\sigma\in\mathfrak{S}_m} \Big\{ \mathrm{tr}(\boldsymbol{\eta}_{i_1, i_2, \dots, i_{m+1}}^{\sigma(\boldsymbol{\alpha})}) \Big\}_{i_1, i_2, \dots, i_{m+1}}.$$

Since the derivations ∂_α, for $\alpha = 1, \ldots, N$, are a basis of the tangent space T_yY, we have:

$$
\begin{aligned}
\Omega &= \sum_{\alpha_1 < \cdots < \alpha_m} \Omega(\partial_{\alpha_1}, \ldots, \partial_{\alpha_m})\, dy_{\alpha_1} \wedge \cdots \wedge dy_{\alpha_m} = \\
&= \frac{1}{m!} \sum_{\alpha_1, \ldots, \alpha_m} \Omega(\partial_{\alpha_1}, \ldots, \partial_{\alpha_m})\, dy_{\alpha_1} \wedge \cdots \wedge dy_{\alpha_m} = \\
&= \frac{1}{m!} \sum_{\boldsymbol{\alpha}} \Omega(\boldsymbol{\partial_\alpha})\, \mathbf{dy_{\boldsymbol{\alpha}}} = \\
&= \frac{1}{(m!)^2} \sum_{\boldsymbol{\alpha}} \sum_{\sigma \in \mathfrak{S}_m} \Big\{ \operatorname{tr}(\boldsymbol{\eta}^{\sigma(\boldsymbol{\alpha})}_{i_1, i_2, \ldots, i_{m+1}}) \Big\}_{i_1, i_2, \ldots, i_{m+1}} \mathbf{dy_{\boldsymbol{\alpha}}}.
\end{aligned} \tag{2.6}
$$

An equivalent expression of Ω in terms of the transition functions g_{ij} can be given (cf. [Mu2, p. 154]). This will be useful in order to simplify the computation of $d\Omega$.

In fact, by recalling that $\eta^\alpha_{ij} = f_i^{-1} \partial_\alpha g_{ij} f_j$, we have:

$$
\begin{aligned}
\operatorname{tr}(\boldsymbol{\eta}^{\sigma(\boldsymbol{\alpha})}_{i_1, i_2, \ldots, i_{m+1}}) &= \operatorname{tr}(\eta^{\alpha_{\sigma(1)}}_{i_1 i_2} \circ \eta^{\alpha_{\sigma(2)}}_{i_2 i_3} \circ \cdots \circ \eta^{\alpha_{\sigma(m)}}_{i_m i_{m+1}}) = \\
&= \operatorname{tr}\Big(f_{i_1}^{-1} \frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \frac{\partial g_{i_2 i_3}}{\partial y_{\alpha_{\sigma(2)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} f_{i_{m+1}} \Big),
\end{aligned}
$$

and, since $\operatorname{tr}(\phi) = \operatorname{tr}(\psi \phi \psi^{-1})$, we also have:

$$
\begin{aligned}
\operatorname{tr}\Big(f_{i_1}^{-1} \frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} f_{i_{m+1}} \Big) &= \operatorname{tr}\Big(\frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} f_{i_{m+1}} f_{i_1}^{-1} \Big) = \\
&= \operatorname{tr}\Big(\frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} g_{i_{m+1} i_1} \Big).
\end{aligned}
$$

It follows that:

$$
\Omega = \frac{1}{(m!)^2} \sum_{\boldsymbol{\alpha}} \sum_{\sigma \in \mathfrak{S}_m} \Big\{ \operatorname{tr}\Big(\frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} g_{i_{m+1} i_1} \Big) \Big\}_{i_1, i_2, \ldots, i_{m+1}} \mathbf{dy_{\boldsymbol{\alpha}}}. \tag{2.7}
$$

We shall now compute the exterior differential of Ω.

Lemma 2.5. *For Ω as above, we have*

$$
d\Omega = c \sum_{\boldsymbol{\alpha}} \sum_{\sigma \in \mathfrak{S}_m} \Big\{ \operatorname{tr}\Big(\frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} \frac{\partial g_{i_{m+1} i_1}}{\partial y_{\alpha_{m+1}}} \Big) \Big\}_{i_1, i_2, \ldots, i_{m+1}} \mathbf{dy_{\boldsymbol{\alpha}}},
$$

where $c = \frac{(-1)^m}{(m!)^2}$ and where the first sum runs now over all multiindices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{m+1})$.

Proof. By taking the exterior differential of the expression (2.7), we obtain

$$d\Omega = \frac{1}{(m!)^2} \sum_{\boldsymbol{\alpha}} \sum_{\sigma \in \mathfrak{S}_m} \Big\{ \sum_{\alpha_{m+1}} \frac{\partial}{\partial y_{\alpha_{m+1}}} \operatorname{tr}\Big(\frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} g_{i_{m+1} i_1} \Big) \Big\} dy_{\alpha_{m+1}} \wedge \mathbf{dy}_{\boldsymbol{\alpha}},$$

where we recall that $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$.

The sum over α_{m+1} followed by the sum over all multiindices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is equivalent to a sum over all multiindices (still denoted by the same symbol) $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{m+1})$, and since $dy_{\alpha_{m+1}} \wedge \mathbf{dy}_{\boldsymbol{\alpha}} = dy_{\alpha_{m+1}} \wedge dy_{\alpha_1} \wedge \cdots \wedge dy_{\alpha_m} = (-1)^m dy_{\alpha_1} \wedge \cdots \wedge dy_{\alpha_m} \wedge dy_{\alpha_{m+1}}$, we can write

$$d\Omega = c \sum_{\boldsymbol{\alpha}} \sum_{\sigma \in \mathfrak{S}_m} \left\{ \frac{\partial}{\partial y_{\alpha_{m+1}}} \operatorname{tr}\left(\frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} g_{i_{m+1} i_1} \right) \right\} \mathbf{dy}_{\boldsymbol{\alpha}}.$$

Now, to complete the proof, it suffices to observe that when we compute the partial derivative

$$\frac{\partial}{\partial y_{\alpha_{m+1}}} \operatorname{tr}\left(\frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} g_{i_{m+1} i_1} \right)$$

we obtain the term

$$\operatorname{tr}\left(\frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} \frac{\partial g_{i_{m+1} i_1}}{\partial y_{\alpha_{m+1}}} \right)$$

plus other terms involving second-order partial derivatives of the transition functions g_{ij}. But when we finally sum over all multiindices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{m+1})$, for every term of the form

$$\operatorname{tr}\left(\cdots \frac{\partial^2 g_{i_h i_{h+1}}}{\partial y_\mu \partial y_\nu} \cdots g_{i_{m+1} i_1} \right) dy_{\alpha_1} \wedge \cdots \wedge dy_\nu \wedge \cdots \wedge dy_{\alpha_m} \wedge dy_\mu$$

there is another term equal to

$$\operatorname{tr}\left(\cdots \frac{\partial^2 g_{i_h i_{h+1}}}{\partial y_\nu \partial y_\mu} \cdots g_{i_{m+1} i_1} \right) dy_{\alpha_1} \wedge \cdots \wedge dy_\mu \wedge \cdots \wedge dy_{\alpha_m} \wedge dy_\nu,$$

and these two terms add to zero because

$$\frac{\partial^2 g_{i_h i_{h+1}}}{\partial y_\mu \partial y_\nu} = \frac{\partial^2 g_{i_h i_{h+1}}}{\partial y_\nu \partial y_\mu}$$

while

$$dy_{\alpha_1} \wedge \cdots \wedge dy_\nu \wedge \cdots \wedge dy_{\alpha_m} \wedge dy_\mu = -dy_{\alpha_1} \wedge \cdots \wedge dy_\mu \wedge \cdots \wedge dy_{\alpha_m} \wedge dy_\nu.$$

Having computed $d\Omega$ in terms of the partial derivatives of the trivialization functions g_{ij}'s, we can now switch back to the representation in terms of the Čech cocycles $\eta_{ij}^{\alpha} = f_i^{-1}(\partial_\alpha g_{ij})f_j$ (this is convenient essentially because the cocycle identities (2.5) for the η_{ij}^{α}'s are simpler then the analogous cocycle identities (2.4) for the functions $\partial_\alpha g_{ij}$'s).

Since

$$\begin{aligned}\operatorname{tr}(\eta_{i_1 i_2}^{\alpha_{\sigma(1)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}} \circ \eta_{i_{m+1} i_1}^{\alpha_{m+1}}) &= \operatorname{tr}\Big(f_{i_1}^{-1} \frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} \frac{\partial g_{i_{m+1} i_1}}{\partial y_{\alpha_{m+1}}} f_{i_1}\Big) = \\ &= \operatorname{tr}\Big(\frac{\partial g_{i_1 i_2}}{\partial y_{\alpha_{\sigma(1)}}} \cdots \frac{\partial g_{i_m i_{m+1}}}{\partial y_{\alpha_{\sigma(m)}}} \frac{\partial g_{i_{m+1} i_1}}{\partial y_{\alpha_{m+1}}}\Big),\end{aligned}$$

we have

$$d\Omega = c \sum_{\boldsymbol{\alpha}} \sum_{\sigma \in \mathfrak{S}_m} \Big\{ \operatorname{tr}(\eta_{i_1 i_2}^{\alpha_{\sigma(1)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}} \circ \eta_{i_{m+1} i_1}^{\alpha_{m+1}}) \Big\} \mathbf{dy}_{\boldsymbol{\alpha}}. \tag{2.8}$$

Now, it follows from the cocycle relations (2.5) that, on $U_{i_1 i_2 \cdots i_{m+1}} = U_{i_1} \cap \cdots \cap U_{i_{m+1}}$, we have

$$\eta_{i_{m+1} i_1}^{\alpha_{m+1}} = -\eta_{i_1 i_{m+1}}^{\alpha_{m+1}} = -(\eta_{i_1 i_2}^{\alpha_{m+1}} + \eta_{i_2 i_3}^{\alpha_{m+1}} + \cdots + \eta_{i_m i_{m+1}}^{\alpha_{m+1}}).$$

By inserting this expression into equation (2.8), we have

$$d\Omega = -c \sum_{\boldsymbol{\alpha}} \sum_{\sigma \in \mathfrak{S}_m} \sum_{k=1}^{m} \Big\{ \operatorname{tr}(\eta_{i_1 i_2}^{\alpha_{\sigma(1)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}} \circ \eta_{i_k i_{k+1}}^{\alpha_{m+1}}) \Big\} \mathbf{dy}_{\boldsymbol{\alpha}}.$$

Finally, by exchanging the order of summations, we can write

$$d\Omega = -c \sum_{k=1}^{m} A_k, \tag{2.9}$$

where

$$A_k = \sum_{\boldsymbol{\alpha}} \sum_{\sigma \in \mathfrak{S}_m} \Big\{ \operatorname{tr}(\eta_{i_1 i_2}^{\alpha_{\sigma(1)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}} \circ \eta_{i_k i_{k+1}}^{\alpha_{m+1}}) \Big\} \mathbf{dy}_{\boldsymbol{\alpha}}.$$

LEMMA 2.6. *For $2 \le k \le m-1$, we have $A_k = 0$.*

PROOF. Let us fix $k \in [2, m-1]$. In the expansion of A_k, for every multiindex $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{m+1})$ there is a multiindex $\boldsymbol{\beta} = \boldsymbol{\beta}_{r,s} = (\beta_1, \ldots, \beta_{m+1})$ differing from $\boldsymbol{\alpha}$ only for the exchange of two elements, at places r and s with $1 \leq r < s \leq m$; $\beta_r = \alpha_s$, $\beta_s = \alpha_r$, and $\beta_j = \alpha_j$ for $j \neq r, s$. Note that we have:

$$\begin{aligned}\mathbf{dy}_{\boldsymbol{\beta}} &= dy_{\beta_1} \wedge \cdots \wedge dy_{\beta_r} \wedge \cdots \wedge dy_{\beta_s} \wedge \cdots \wedge dy_{\beta_{m+1}} = \\ &= dy_{\alpha_1} \wedge \cdots \wedge dy_{\alpha_s} \wedge \cdots \wedge dy_{\alpha_r} \wedge \cdots \wedge dy_{\alpha_{m+1}} = \\ &= -dy_{\alpha_1} \wedge \cdots \wedge dy_{\alpha_r} \wedge \cdots \wedge dy_{\alpha_s} \wedge \cdots \wedge dy_{\alpha_{m+1}} = \\ &= -\mathbf{dy}_{\boldsymbol{\alpha}}.\end{aligned}$$

Now, for any such pair of multiindices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and for any permutation σ of $\{1, 2, \ldots, m\}$, let $\tau = \tau_{r,s}$ be the permutation given by the composition $\tau = \pi_{r,s} \circ \sigma$, where $\pi_{r,s}$ is the permutation that exchanges the elements at places r and s:

$$\pi_{r,s}(1, \ldots, r, \ldots, s, \ldots, m) = (1, \ldots, s, \ldots, r, \ldots, m).$$

It follows that

$$\eta_{i_1 i_2}^{\beta_{\tau(1)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\beta_{\tau(m)}} \circ \eta_{i_k i_{k+1}}^{\beta_{m+1}} = \eta_{i_1 i_2}^{\alpha_{\sigma(1)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}} \circ \eta_{i_k i_{k+1}}^{\alpha_{m+1}},$$

hence the two terms

$$\operatorname{tr}(\eta_{i_1 i_2}^{\alpha_{\sigma(1)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}} \circ \eta_{i_k i_{k+1}}^{\alpha_{m+1}})\, \mathbf{dy}_{\boldsymbol{\alpha}}$$

and

$$\operatorname{tr}(\eta_{i_1 i_2}^{\beta_{\tau(1)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\beta_{\tau(m)}} \circ \eta_{i_k i_{k+1}}^{\beta_{m+1}})\, \mathbf{dy}_{\boldsymbol{\beta}}$$

add to zero. Since all terms in the expansion of A_k can be paired in this way, we conclude that $A_k = 0$.

Now, it remains to consider the two terms A_1 and A_m.

LEMMA 2.7. *We have $A_1 = -A_m$.*

PROOF. Let us consider

$$A_1 = \sum_{\boldsymbol{\alpha}} \sum_{\sigma \in \mathfrak{S}_m} \left\{ \operatorname{tr}(\eta_{i_1 i_2}^{\alpha_{\sigma(1)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}} \circ \eta_{i_1 i_2}^{\alpha_{m+1}}) \right\} \mathbf{dy}_{\boldsymbol{\alpha}}.$$

By recalling the usual symmetry property of the trace, we may write

$$\operatorname{tr}(\eta_{i_1 i_2}^{\alpha_{\sigma(1)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}} \circ \eta_{i_1 i_2}^{\alpha_{m+1}}) = \operatorname{tr}\left((\eta_{i_1 i_2}^{\alpha_{m+1}} \circ \eta_{i_1 i_2}^{\alpha_{\sigma(1)}}) \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}}\right),$$

hence, if we set $\xi_{ij}^{\alpha,\sigma} = \eta_{ij}^{\alpha_{m+1}} \circ \eta_{ij}^{\alpha_{\sigma(1)}} \in \Gamma(U_{ij}, \mathcal{E}nd(E))$, we have

$$A_1 = \sum_{\boldsymbol{\alpha}} \sum_{\sigma \in \mathfrak{S}_m} \left\{ \operatorname{tr}(\xi_{i_1 i_2}^{\alpha,\sigma} \circ \eta_{i_2 i_3}^{\alpha_{\sigma(2)}} \circ \cdots \circ \eta_{i_m i_{m+1}}^{\alpha_{\sigma(m)}}) \right\} \mathbf{dy}_{\boldsymbol{\alpha}}.$$

Now, from the skew-symmetry of the map Str in (1.10) (see Corollary 1.3), it follows that

$$\sum_{\sigma\in\mathfrak{S}_m} \operatorname{tr}(\xi^{\alpha,\sigma}_{i_1 i_2} \circ \eta^{\alpha_{\sigma(2)}}_{i_2 i_3} \circ \cdots \circ \eta^{\alpha_{\sigma(m)}}_{i_m i_{m+1}}) = -\sum_{\sigma\in\mathfrak{S}_m} \operatorname{tr}(\eta^{\alpha_{\sigma(m)}}_{i_1 i_2} \circ \eta^{\alpha_{\sigma(2)}}_{i_2 i_3} \circ \cdots \circ \xi^{\alpha,\sigma}_{i_m i_{m+1}}),$$

hence we have

$$A_1 = -\sum_{\boldsymbol{\alpha}} \sum_{\sigma\in\mathfrak{S}_m} \left\{ \operatorname{tr}\left(\eta^{\alpha_{\sigma(m)}}_{i_1 i_2} \circ \eta^{\alpha_{\sigma(2)}}_{i_2 i_3} \circ \cdots \circ (\eta^{\alpha_{m+1}} \circ \eta^{\alpha_{\sigma(1)}})_{i_m i_{m+1}}\right)\right\} \mathbf{dy}_{\boldsymbol{\alpha}}.$$

On the other hand, we have

$$A_m = \sum_{\boldsymbol{\beta}} \sum_{\tau\in\mathfrak{S}_m} \left\{ \operatorname{tr}\left(\eta^{\beta_{\tau(1)}}_{i_1 i_2} \circ \eta^{\beta_{\tau(2)}}_{i_2 i_3} \circ \cdots \circ (\eta^{\beta_{\tau(m)}} \circ \eta^{\beta_{m+1}})_{i_m i_{m+1}}\right)\right\} \mathbf{dy}_{\boldsymbol{\beta}}.$$

Now, for every multiindex $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{m+1})$ and every permutation $\sigma \in \mathfrak{S}_m$ there is a multiindex $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{m+1})$ and a permutation $\tau \in \mathfrak{S}_m$ such that

$$\beta_{\tau(1)} = \alpha_{\sigma(m)}, \beta_{\tau(2)} = \alpha_{\sigma(2)}, \ldots, \beta_{\tau(m-1)} = \alpha_{\sigma(m-1)}, \beta_{\tau(m)} = \alpha_{m+1}, \beta_{m+1} = \alpha_{\sigma(1)}.$$

With this choice of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, σ and τ, we have

$$\begin{aligned}\mathbf{dy}_{\boldsymbol{\beta}} &= dy_{\beta_1} \wedge dy_{\beta_2} \wedge \cdots \wedge dy_{\beta_{m-1}} \wedge dy_{\beta_m} \wedge dy_{\beta_{m+1}} = \\ &= dy_{\alpha_m} \wedge dy_{\alpha_2} \wedge \cdots \wedge dy_{\alpha_{m-1}} \wedge dy_{\alpha_{m+1}} \wedge dy_{\alpha_1} = \\ &= (-1)^{2m-2} dy_{\alpha_1} \wedge dy_{\alpha_2} \wedge \cdots \wedge dy_{\alpha_{m+1}} = \\ &= \mathbf{dy}_{\boldsymbol{\alpha}}.\end{aligned}$$

It follows that $A_1 = -A_m$.

Now, from equation (2.9) and Lemmas 2.6 and 2.7, we obtain the following result:

THEOREM 2.8. *For any Y-flat family E of locally free sheaves on X and any integer m, with $1 \le m \le \dim X$, the $H^m(X, \mathcal{O}_X)$-valued m-form $\Omega = \Omega_E$ on Y is closed, i.e., $d\Omega = 0$.*

3 – Differential Forms on Moduli Spaces

In this section we shall apply the results of Section 2 to the construction of closed holomorphic differential forms on moduli spaces of sheaves on a smooth complex projective variety X (we choose to work over the complex field because Theorem 2.8 was proved under the assumption $k = \mathbb{C}$, but everything we shall say holds true for any algebraically closed field k of characteristic 0).

So, let X be a smooth n-dimensional complex projective variety, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X and let $\mathcal{M}$ be a moduli space of stable sheaves on X (with some fixed moduli data).

Remark 3.1. On the moduli space $\mathcal{M}$ there does not exist, in general, a universal family of sheaves $\mathcal{E}$, not even locally in the Zariski topology. In any case, a universal family $\mathcal{E}$ on $\mathcal{M}$ exists locally in the complex analytic topology (or in the étale topology, if we are working over an algebraically closed field k of characteristic zero) [S, Theorem 1.21]. As noted in Remark 2.1, these local universal families are not uniquely determined, in fact they are defined only up to tensoring with the pull-back of a line bundle on $\mathcal{M}$. In general, these ambiguities prevent the local universal families to glue together to a globally defined one (see [Ma, Theorem 6.11] or [HL2, Section 4.6] for numerical conditions ensuring the existence of a global universal family on $\mathcal{M}$). On the other hand, when we consider the relative Ext-sheaves $\mathcal{E}xt^i_q(\mathcal{E},\mathcal{E})$ (or the sheaf $\mathcal{E}nd(\mathcal{E})$), these ambiguities disappear, and these locally defined sheaves glue together to a globally defined one on $\mathcal{M}$. For this reason, we shall abuse the notation and write $\mathcal{E}xt^i_q(\mathcal{E},\mathcal{E})$ (resp. $\mathcal{E}nd(\mathcal{E})$) even if the universal family $\mathcal{E}$ does not exist on $\mathcal{M}$.

Since the moduli space $\mathcal{M}$ is, in general, not smooth, we shall denote by $\mathcal{M}^{sm}$ its smooth locus. Analogously, we shall denote by $\mathcal{M}_{lf}$ the open subscheme of $\mathcal{M}$ parametrizing isomorphism classes of locally free sheaves and by $\mathcal{M}^{sm}_{lf}$ the smooth part of it.

For any $E \in \mathcal{M}^{sm}$ the Kodaira-Spencer map (2.1) gives a natural isomorphism

$$T_E\mathcal{M}^{sm} \cong \operatorname{Ext}^1(E,E). \tag{3.1}$$

If $E \in \mathcal{M}^{sm}_{lf}$, we also have

$$T_E\mathcal{M}^{sm}_{lf} \cong H^1(X,\mathcal{E}nd(E)), \tag{3.2}$$

because, for a locally free sheaf E, there are canonical isomorphisms

$$\operatorname{Ext}^i(E,E) \cong H^i(X,\mathcal{E}nd(E)).$$

The global versions of the Kodaira-Spencer isomorphisms (3.1) and (3.2) provide natural isomorphisms

$$T\mathcal{M}^{sm} \cong \mathcal{E}xt^1_q(\mathcal{E},\mathcal{E}),$$

and

$$T\mathcal{M}^{sm}_{lf} \cong R^1q_*\mathcal{E}nd(\mathcal{E}).$$

We can now apply the results of the preceding section to construct natural holomorphic differential forms on $\mathcal{M}^{sm}$.

More precisely, by setting $Y = \mathcal{M}^{sm}$ and denoting by $\mathcal{E}$ a locally defined universal family on Y (cf. also Remark 2.2), we have, for any m with $1 \leq m \leq n = \dim X$, a vector-valued m-form

$$\Omega : T\mathcal{M}^{sm} \times \cdots \times T\mathcal{M}^{sm} \to H^m(X, \mathcal{O}_X) \otimes_k \mathcal{O}_{\mathcal{M}^{sm}}. \tag{3.3}$$

Let us now assume that there exists a holomorphic m-form σ on X, $\sigma \in H^0(X, \Omega_X^m)$. The multiplication by σ defines a map

$$H^m(X, \mathcal{O}_X) \xrightarrow{\sigma} H^m(X, \Omega_X^m). \tag{3.4}$$

Finally, if we denote by $\eta_X \in H^1(X, \Omega_X^1)$ the cohomology class of the polarization $\mathcal{O}_X(1)$ (the cohomology class of the Kähler $(1,1)$-form on X), we have a map

$$H^m(X, \Omega_X^m) \xrightarrow{\eta_X^{n-m}} H^n(X, \Omega_X^n) \cong \mathbb{C}. \tag{3.5}$$

By composing the vector-valued differential form Ω with the maps (3.4) and (3.5), we obtain an ordinary (scalar-valued) m-form, which we denote by Ω_σ:

$$\Omega_\sigma : T\mathcal{M}^{sm} \times \cdots \times T\mathcal{M}^{sm} \to \mathcal{O}_{\mathcal{M}^{sm}}.$$

Since, by Theorem 2.8, the restriction of Ω to the open subset $\mathcal{M}_{lf}^{sm}$ parametrizing locally free sheaves is closed, it follows that the restriction of Ω_σ to $\mathcal{M}_{lf}^{sm}$ is a closed homolorphic m-form.

We can summarize these results as follows:

THEOREM 3.2. *For any holomorphic m-form σ on the complex projective variety X there is a holomorphic m-form Ω_σ on the smooth locus $\mathcal{M}^{sm}$ of the moduli space of stable sheaves on X. The restriction of Ω_σ to the smooth locus $\mathcal{M}_{lf}^{sm}$ of the moduli space of stable vector bundles on X is closed.*

Actually this result also holds with no assumption on the smoothness of the moduli space $\mathcal{M}$ or on the locally freeness of the stable sheaves. The method of proof is, however, completely different, and will be described in a different paper.

Let us describe now some particularly interesting special cases of Theorem 3.2.

Let us take $m = n = \dim X$ and assume that there exists a non-zero section σ of the canonical line bundle $K_X = \Omega_X^n$. In this case the map (3.5) is the identity, hence the n-form Ω_σ is given by the composition of Ω in (3.3) with the map

$$H^n(X, \mathcal{O}_X) \xrightarrow{\sigma} H^n(X, K_X) \cong \mathbb{C}.$$

Even more interesting is the case when the canonical line bundle of X is trivial, i.e., when X is a smooth Calabi-Yau n-fold. In fact, in this case there is a canonical choice (up to scalars) of the n-form σ on X, namely $\sigma = 1 \in H^0(X, K_X) \cong \mathbb{C}$, hence there is also a n-form Ω_σ on the moduli space $\mathcal{M}^{sm}$.

The natural question that arises at this point is to know under what conditions the canonical n-form Ω_σ on the moduli space $\mathcal{M}^{sm}$ is non-degenerate. We recall that, for $n = 2$, i.e., when X is an abelian or a K3 surface, this is always the case [Mu1]. For $n \geq 3$, on the other hand, there is no hope that Ω_σ be always non-degenerate; in fact there are examples of moduli spaces of stable sheaves (even stable vector bundles) on a smooth Calabi-Yau n-fold that are isomorphic to projective spaces.

We do not know the answer to this question but, in order to investigate the non-degeneracy of the n-form Ω_σ, when X is a Calabi-Yau n-fold or in the more general case of a smooth projective variety X with an effective canonical divisor, it may be helpful to use the following algebraic result (whose proof is elementary):

Proposition 3.3. *Let V be a finite dimensional k-vector space and*

$$\omega : \underbrace{V \times \cdots \times V}_{m} \to k$$

be an alternating, or symmetric, multilinear form. Let us define

$$\tilde{\omega} : \underbrace{V \times \cdots \times V}_{m-1} \to V^*$$

by setting

$$\langle v_1, \tilde{\omega}(v_2, \dots, v_m) \rangle = \omega(v_1, v_2, \dots, v_m),$$

for any $v_1, \dots, v_m \in V$. Then the transpose of $\tilde{\omega}$ is the map

$$\tilde{\omega}^t : V \to \underbrace{V^* \times \cdots \times V^*}_{m-1}$$

given by

$$\langle \tilde{\omega}^t(v_1), (v_2, \dots, v_m) \rangle = \omega(v_1, v_2, \dots, v_m),$$

for any $v_1, \dots, v_m \in V$, and we have

$$\mathrm{Ker}(\omega) = \mathrm{Ker}(\tilde{\omega}^t) = (\mathrm{Im}(\tilde{\omega}))^\perp,$$

where

$$\mathrm{Ker}(\omega) = \{v \in V \mid \omega(v, v_2, \dots, v_m) = 0, \forall v_2, \dots, v_m \in V\}.$$

Hence ω is non-degenerate if and only if $\tilde{\omega}$ is surjective or, equivalently, if and only if $\tilde{\omega}^t$ is injective.

In order to apply this result to our situation let us prove the following lemma (inspired by a similar result in [T]):

LEMMA 3.4. *Let* $m = n = \dim X$ *and let* $\sigma \in H^0(X, K_X)$, *with* $\sigma \neq 0$. *Let* $E \in \mathcal{M}^{sm}$ *and, using the notations of the preceding proposition, let us set* $V = \mathrm{Ext}^1(E,E)$ *and* $\omega = \Omega_\sigma(E)$. *Then, by Serre duality, we have* $V^* \cong \mathrm{Ext}^{n-1}(E, E \otimes K_X)$, *and the map*

$$\tilde{\omega} : \underbrace{\mathrm{Ext}^1(E,E) \times \cdots \times \mathrm{Ext}^1(E,E)}_{n-1} \to \mathrm{Ext}^{n-1}(E, E \otimes K_X) \tag{3.6}$$

is the composition of the map

$$S : \mathrm{Ext}^1(E,E) \times \cdots \times \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^{n-1}(E,E)$$

induced by the symmetrized composition map (1.2), *with the map*

$$\mathrm{Ext}^{n-1}(E,E) \xrightarrow{\sigma} \mathrm{Ext}^{n-1}(E, E \otimes K_X)$$

given by the multiplication by $\sigma \in H^0(X, K_X)$.

PROOF. The duality between $V = \mathrm{Ext}^1(E,E)$ and $V^* = \mathrm{Ext}^{n-1}(E, E \otimes K_X)$ is given by

$$\langle \phi, \phi^* \rangle = \mathrm{tr}(\phi \circ \phi^*) \in H^n(X, K_X) \cong \mathbb{C},$$

for any $\phi \in V$ and $\phi^* \in V^*$.

Let us now take $\phi_1, \ldots, \phi_n \in V$. By recalling the definition of Ω_σ, we have:

$$\Omega_\sigma(\phi_1, \ldots, \phi_n) = \sigma \ \mathrm{Str}(\phi_1, \ldots, \phi_n) = \sigma \, \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} \mathrm{tr}(\phi_{\tau(1)} \circ \phi_{\tau(2)} \circ \cdots \circ \phi_{\tau(n)}).$$

By recalling the expression for the map Str given in (1.5), we can also write

$$\Omega_\sigma(\phi_1, \ldots, \phi_n) = \sigma \, \frac{1}{(n-1)!} \sum_{\tau} \mathrm{tr}(\phi_1 \circ \phi_{\tau(2)} \circ \cdots \circ \phi_{\tau(n)}), \tag{3.7}$$

where now the sum runs over all permutations τ of the set $\{2, 3, \ldots, n\}$.

Let us now set $\phi^* = \tilde{\omega}(\phi_2, \ldots, \phi_n)$. By recalling the definition of $\tilde{\omega}$, we have

$$\langle \phi_1, \phi^* \rangle = \omega(\phi_1, \phi_2, \ldots, \phi_n) = \Omega_\sigma(\phi_1, \ldots, \phi_n).$$

On the other hand, by the expression of Serre duality given above, we have

$$\langle \phi_1, \phi^* \rangle = \mathrm{tr}(\phi_1 \circ \phi^*). \tag{3.8}$$

Now, by comparing (3.8) with the expression in (3.7), we find that

$$\phi^* = \sigma \, \frac{1}{(n-1)!} \sum_{\tau} \phi_{\tau(2)} \circ \cdots \circ \phi_{\tau(n)} = \sigma \, S(\phi_2, \ldots, \phi_n).$$

It seems difficult to investigate, in general, the surjectivity of the map $\tilde{\omega}$ in (3.6). Obviously, a necessary condition is that the map

$$\operatorname{Ext}^{n-1}(E,E) \xrightarrow{\sigma} \operatorname{Ext}^{n-1}(E, E \otimes K_X) \tag{3.9}$$

be surjective. This is equivalent to requiring that the transpose of this map, i.e., the map

$$\operatorname{Ext}^1(E,E) \xrightarrow{\sigma} \operatorname{Ext}^1(E, E \otimes K_X),$$

be injective. By applying the functor $\operatorname{Hom}(E,\cdot)$ to the standard exact sequence

$$0 \longrightarrow E \xrightarrow{\sigma} E \otimes K_X \longrightarrow E \otimes K_X|_D \longrightarrow 0,$$

where $D \in |K_X|$ is the divisor defined by σ, we see that the map above fits into the long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(E,E) &\xrightarrow{\sigma} \operatorname{Hom}(E, E\otimes K_X) \to \operatorname{Hom}(E, E \otimes K_X|_D) \to \\ \to \operatorname{Ext}^1(E,E) &\xrightarrow{\sigma} \operatorname{Ext}^1(E, E \otimes K_X). \end{aligned}$$

From the stability of E it follows that $\operatorname{Hom}(E,E) \cong \mathbb{C}$, but it is difficult to get informations on $\operatorname{Hom}(E, E\otimes K_X)$ and $\operatorname{Hom}(E, E\otimes K_X|_D)$, in general. Obviously, this problem simply disappears when X is Calabi-Yau, i.e., when $K_X \cong \mathcal{O}_X$ and $\sigma = 1$ (in this case the map (3.9) is the identity).

Remark 3.5. If X is a smooth Calabi-Yau n-fold, it may happen that, for a suitable choice of moduli data, the corresponding moduli space $\mathcal{M}$ of stable sheaves on X has an irreducible component Y which is smooth, projective and of dimension n. Under these hypotheses, if the restriction of the canonical n-form Ω_σ to Y is non-degenerate, then Y will be a Calabi-Yau n-fold.

An example of this situation can be found in [T, Theorem 4.23]. In this case $n = 3$ and the Calabi-Yau 3-fold X is a K3 fibration over $\mathbb{P}^1$. The moduli space $\mathcal{M}$ is a relative moduli space of stable sheaves on X supported on the fibers (with suitable moduli data). The claim is that $\mathcal{M}$ is again a Calabi-Yau 3-fold. To prove this result, Thomas explicitly constructs a holomorphic 3-form on the moduli space $\mathcal{M}$ and shows that it is non-degenerate.

A similar (and more general) problem has been investigated by T. Bridgeland and A. Maciocia, under the additional assumptions that X is a flat Calabi-Yau fibration over a base S, with fibers of dimension ≤ 2, and $\mathcal{M}$ is a relative moduli space of stable sheaves supported on the fibers of $\pi : X \to S$. We refer to [BM] for details.

Remark 3.6 K. Yoshioka has constructed in [Y] moduli spaces of stable twisted sheaves on a smooth complex projective variety X. These are quasi-projective schemes and can be compactified, in the usual way, by adding S-equivalence classes of semistable twisted sheaves. In greater generality, moduli spaces of twisted sheaves have also been constructed by M. Lieblich in [Li], using the language of algebraic stacks. In any case, it turns out that the tangent space to such a moduli space at a point corresponding to a twisted sheaf E is canonically identified with $\mathrm{Ext}^1(E, E)$. From this fact it should follow immediately that our construction of closed differential forms Ω_σ on moduli spaces of stable sheaves can be generalized, in a straightforward way, to moduli spaces of stable twisted sheaves. When X is a K3 surface, this is explicitly proven in [Y]; in this case, the moduli space of stable twisted sheaves on X has a canonical symplectic structure (just as in the untwisted case [Mu1]).

We end this section with an example explaining how the construction of the differential forms Ω_σ can be adapted to other moduli spaces. We describe the case of moduli spaces of framed bundles.

3.1 – Moduli spaces of framed vector bundles

Let us now see how the construction of the differential forms Ω_σ can be extended to the case of moduli spaces of framed vector bundles.

Let X be a smooth n-dimensional complex projective variety with a very ample invertible sheaf $\mathcal{O}_X(1)$, and let $D \subset X$ be a smooth hypersurface. Let us denote by F a fixed vector bundle on D.

Definition 3.7. A framed vector bundle on X is a pair (E, ϕ) consisting of a locally free sheaf E on X and an isomorphism $\phi : E|_D \xrightarrow{\sim} F$.

Moduli spaces of stable framed vector bundles on a smooth projective variety X were constructed in [Lu] and, in a more general context, in [HL1], to which we refer for definitions and results.

Let us denote by $\mathcal{FB}$ the moduli space of stable framed vector bundles on X (with fixed Hilbert polynomial) and by $\mathcal{FB}^{sm}$ its smooth locus. The moduli space $\mathcal{FB}$ is a quasi-projective variety, and it is actually a fine moduli space, i.e., there exists a (global) universal family of framed vector bundles on $\mathcal{FB}$.

Standard infinitesimal deformation theory gives the following result:

Proposition 3.8. *For any $(E, \phi) \in \mathcal{FB}$ there is a canonical identification*

$$T_{(E,\phi)}\mathcal{FB} \cong H^1(X, \mathcal{E}nd(E) \otimes \mathcal{O}_X(-D)),$$

and the obstruction to the smoothness of the moduli space $\mathcal{FB}$ at the point (E, ϕ) lies in $H^2(X, \mathcal{E}nd_0(E) \otimes \mathcal{O}_X(-D))$, where $\mathcal{E}nd_0(E)$ denotes the sheaf of traceless endomorphisms of E.

In the sequel we shall denote $\mathcal{E}nd(E) \otimes \mathcal{O}_X(-D)$ simply by $\mathcal{E}nd(E)(-D)$.

In this situation we can define a $H^m(X, \mathcal{O}_X(-mD))$-valued differential m-form Ω on $\mathcal{FB}^{sm}$ by setting, for any $(E, \phi) \in \mathcal{FB}^{sm}$,

$$\Omega(E,\phi) : H^1(X, \mathcal{E}nd(E)(-D)) \times \cdots \times H^1(X, \mathcal{E}nd(E)(-D)) \to \\ \to H^m(X, \mathcal{O}_X(-mD)),$$

where this map is the composition of the map

$$S : H^1(X, \mathcal{E}nd(E)(-D)) \times \cdots \times H^1(X, \mathcal{E}nd(E)(-D)) \to \\ \to H^m(X, \mathcal{E}nd(E) \otimes \mathcal{O}_X(-mD))$$

induced by the symmetrized composition map, and the usual trace map

$$\mathrm{tr} : H^m(X, \mathcal{E}nd(E)(-mD)) \to H^m(X, \mathcal{O}_X(-mD)).$$

The proof of the closure of the m-form Ω on $\mathcal{FB}^{sm}$ is formally the same as the proof of the closure of the m-form Ω on Y given in Section 2.

Now, to construct from Ω a scalar-valued differential form on $\mathcal{FB}^{sm}$, we just need a section $\sigma' \in H^0(X, \Omega^m_X(mD))$, i.e., a global differential form of degree m on X, with poles bounded by mD. Then, the multiplication by σ' defines a map

$$H^m(X, \mathcal{O}_X(-mD)) \xrightarrow{\sigma'} H^m(X, \Omega^m_X),$$

hence by composing Ω with this map and then with the map η_X^{n-m} of (3.5), we obtain an ordinary (scalar-valued) m-form on $\mathcal{FB}^{sm}$, which we shall denote by $\Omega_{\sigma'}$. In conclusion, we have proved the following result:

THEOREM 3.9. *Let X, D and F be as above. For any meromorphic m-form σ' on X with poles bounded by mD, $\sigma' \in H^0(X, \Omega^m_X(mD))$, there is a closed holomorphic m-form $\Omega_{\sigma'}$ on the smooth locus $\mathcal{FB}^{sm}$ of the moduli space of stable framed vector bundles on X.*

Finally, let us investigate the relations between the differential forms constructed on the moduli spaces $\mathcal{M}^{sm}$ and $\mathcal{FB}^{sm}$.

Let X be as above and let $\sigma \in H^0(X, \Omega^m_X)$. Let D be a smooth hypersurface of X defined by a section $s \in H^0(X, \mathcal{O}_X(D))$. In this case there is an obvious choice for a global section σ' of $\Omega^m_X(mD)$, namely $\sigma' = \sigma\, s^m$.

Let us set

$$\mathcal{FB}^{sm}_0 = \{(E,\phi) \in \mathcal{FB}^{sm} \mid E \text{ is a stable vector bundle}\}.$$

(In general $\mathcal{FB}^{sm}_0 \neq \mathcal{FB}^{sm}$ because there can be framed bundles (E, ϕ) that are stable as framed bundles, but such that E is not stable as a vector bundle).

Then we have a natural map

$$\pi : \mathcal{FB}_0^{sm} \to \mathcal{M}$$

that forgets the framing, i.e., that sends a framed bundle (E, ϕ) to E.

With the natural identifications explained above, the tangent map to π at a point (E, ϕ) is the map

$$H^1(X, \mathcal{E}nd(E)(-D)) \xrightarrow{s} H^1(X, \mathcal{E}nd(E))$$

induced by the multiplication by $s \in H^0(X, \mathcal{O}_X(D))$.

Then we have a commutative diagram

$$\begin{array}{ccc} H^1(X, \mathcal{E}nd(E)(-D)) \times \cdots \times H^1(X, \mathcal{E}nd(E)(-D)) & \xrightarrow{\text{Str}} & H^m(X, \mathcal{O}_X(-mD)) \\ \Big\downarrow{\scriptstyle s\times\cdots\times s} & & \Big\downarrow{\scriptstyle s^m} \\ H^1(X, \mathcal{E}nd(E)) \times \cdots \times H^1(X, \mathcal{E}nd(E)) & \xrightarrow{\text{Str}} & H^m(X, \mathcal{O}_X). \end{array}$$

From this diagram, and from the preceding definitions of the m-forms on the moduli spaces, it is evident that the pull-back by π of the m-form Ω_σ defined on $\mathcal{M}^{sm}$ is equal to the m-form $\Omega_{\sigma'}$ defined on $\mathcal{FB}_0^{sm}$.

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