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Superstring measure and non-renormalization of the three-point amplitude

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Abstract

We show that a recently conjectured expression for the superstring three-point amplitude, in the framework of the Cacciatori, Dalla Piazza, van Geemen–Grushevsky ansatz for the chiral measure, fails to vanish at three-loop, in contrast with expectations from non-renormalization theorems. Based on analogous two-loop computations, we discuss the possibility of a non-trivial correction to the amplitude and propose a natural candidate for such a contribution. Thanks to a new remarkable identity, it is reasonable to expect that the corrected three-point amplitude vanishes at three-loop, recovering the agreement with non-renormalization theorems.

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In the last years there has been a considerable progress in the conceptual understanding and in derivation of explicit formulas for multiloop superstring amplitudes. In a series of papers [1–6], D’Hoker and Phong have derived from first principles an explicitly gauge independent expression for the 2-loop chiral superstring measure on the moduli space of Riemann surfaces, given by

$$d\mu[\delta] = \theta[\delta](0)^4 \mathcal{E}_6[\delta] d\mu_{\text{Bos}}, \quad (1)$$

where $\delta \in \mathbb{Z}_2^2$ is an even spin structure, $\mathcal{E}_6[\delta]$ is a modular form of weight 6 for the subgroup $\Gamma(2) \subset \text{Sp}(4, \mathbb{Z})$ leaving theta characteristics invariant and $d\mu_{\text{Bos}}$ is the (genus 2) bosonic string measure. Analogous procedures also led them to prove the non-renormalization of the cosmological constant and of the n -point functions, $n \leq 3$, up to 2-loops, as expected by space–

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time supersymmetry arguments [7]. Furthermore, the 4-point amplitude has been computed and checked against the constraints coming from S-duality [8].

Direct computations of higher loop corrections to superstring amplitudes appear, at the moment, out of reach. However, the strong constraints coming from modular invariance and from factorization under degeneration limits, together with the explicit 2-loop expressions, can lead to reliable conjectures on such corrections. This is the point of view adopted, for example, in [9], where a general formula for the higher loop contributions to the 4-point function has been proposed.

In [2,3], D'Hoker and Phong conjectured that the 3-loop chiral superstring measure may be expressed in the same form as (1) for a suitable modular form $\mathcal{E}_6[\delta]$ of weight 6. Such a form is required to fulfill a series of constraints related to holomorphicity, modular invariance and factorization; however, no form has been found satisfying all such conditions. In [10] Cacciatori, Dalla Piazza and van Geemen (CDvG) showed that a series of equivalent constraints can be solved by assuming the more general expression

$$d\mu[\delta] = \mathcal{E}_8[\delta] d\mu_{\text{Bos}}, \tag{2}$$

for the 3-loop chiral measure, where $\mathcal{E}_8[\delta]$ is a modular form of weight 8 (not necessarily divisible by $\theta[\delta](0)^4$). In [11], it has been proved that such constraints admit a unique solution for genus 3, thus ruling out the more restrictive assumption by D'Hoker and Phong. Hereafter, we will drop the subscript in $\mathcal{E}_8[\delta]$.

The CDvG ansatz for the 3-loop measure has been generalized by Grushevsky [12] to a formula defined for every loop. In particular, it has been shown that the D'Hoker and Phong formula for genus 2 and the CDvG ansatz can be re-expressed in terms of modular forms associated to isotropic spaces of theta characteristics, and this leads to a straightforward generalization of \mathcal{E} , solving the constraints for all genera. A possible problem in Grushevsky construction concerns the existence of holomorphic roots of modular forms appearing in the definition of \mathcal{E} for genus $g \geq 5$. Proving that such roots are well-defined seems highly non-trivial and, up to now, it has been shown only for genus 5 [13].

Further consistency checks for chiral measure ansätze may be provided by the non-renormalization of the cosmological constant and of the 1, 2, 3-point functions. In this respect, it is useful to consider more closely the general form of the g -loop contribution to the n -point function in superstring theories. Using the notation of [6], the general n -point amplitude can be expressed as

$$A(k_1, \dots, k_n, \epsilon_1, \dots, \epsilon_n) = \int \prod dp_j^\mu \int_{\mathcal{M}_g} \int_{C^n} \left| \sum_{\delta \text{ even}} \mathcal{B}[\delta](k_i, \epsilon_i, z_i, p_j^\mu) \right|^2,$$

where $k_i, \epsilon_i, i = 1, \dots, n$, are the space–time momenta and polarizations of the external states, $p_j^\mu, j = 1, \dots, g$, are the internal loop momenta, \mathcal{M}_g is the moduli space of Riemann surfaces of genus g , C is the Riemann surface at the corresponding point of \mathcal{M}_g and $z_i, i = 1, \dots, n$, are the insertion position of the vertex operators on C . The sum is over all the even spin structures δ and $\mathcal{B}[\delta](k_i, \epsilon_i, z_i, p_j^\mu)$ are the so-called chiral amplitudes. In principle, upon applying a consistent gauge fixing procedure, the chiral amplitudes $\mathcal{B}[\delta]$ can be obtained by computing suitable CFT correlators of products of vertex operators, supercurrents and stress–energy tensor operators [4]. It is often useful to split the chiral amplitudes into a sum

$$\mathcal{B}[\delta] := \mathcal{B}^c[\delta] + \mathcal{B}^d[\delta].$$

Here, the so-called disconnected part $\mathcal{B}^d[\delta]$ includes all the terms where the Wick contractions between the vertex operators are disconnected from the contractions of the supercurrents and of the stress–energy tensor; the connected part $\mathcal{B}^c[\delta]$ includes all the other terms [4].

It is worth noticing that $\mathcal{B}^c[\delta]$ is a sum of a huge number of terms and that, in general, each of these terms strongly depends on the details of the gauge fixing; only the whole sum is gauge independent. For the 2-loop n -point functions with $n \leq 3$ (but, for example, not for $n = 4$), all the different contributions to $\sum_{\delta} \mathcal{B}^c[\delta]$ simply cancel each other, just giving $\sum_{\delta} \mathcal{B}^c[\delta] = 0$. It follows that the non-renormalization theorems at 2-loops are implemented by $\sum_{\delta} \mathcal{B}^c[\delta]$ and $\sum_{\delta} \mathcal{B}^d[\delta]$ vanishing separately. For arbitrary genus $g > 2$, a general consistent gauge fixing procedure is not known (even though several steps of the genus 2 derivation generalize to genus 3). As a consequence, the precise mechanism for the cancelation of the gauge fixing ambiguities among such terms is not understood in full detail. It is reasonable, therefore, to assume that, in analogy with the genus 2 case, all the ambiguous terms exactly cancel giving

$$\sum_{\delta \text{ even}} \mathcal{B}^c[\delta] = 0, \tag{3}$$

for $n \leq 3$ and for arbitrary genus.

Under the assumption (3), the non-renormalization theorems are equivalent to the vanishing of the disconnected part $\mathcal{B}^d[\delta]$ of the chiral amplitude, which can be easily expressed in terms of the chiral superstring measure. In fact, given the ansatz (2), it can be proved that the vanishing of the n -point function is equivalent to

$$A_n(z_1, \dots, z_n) := \sum_{\delta \text{ even}} \mathcal{E}[\delta] \mathcal{A}[\delta](z_1, \dots, z_n) = 0, \tag{4}$$

where $\mathcal{A}[\delta](z_1, \dots, z_n)$ can be derived from the Wick contractions of the n vertex operators, inserted at the points z_1, \dots, z_n of the Riemann surface. More precisely, the 0-point function is obtained by setting $\mathcal{A}[\delta] = 1$, so that the non-renormalization theorem for the cosmological constant is just equivalent to

$$A_0 \equiv \sum_{\delta \text{ even}} \mathcal{E}[\delta] = 0. \tag{5}$$

Notice that (3) trivially holds for $n = 0$; the vanishing of the cosmological constant is, therefore, the most reliable check of the correctness of the chiral superstring measure ansätze. The 1-point function vanishes automatically under the assumption (3), whereas the non-renormalization of the 2- and 3-point amplitudes corresponds to

$$A_2(a, b) \equiv \sum_{\delta \text{ even}} \mathcal{E}[\delta] S_{\delta}(a, b)^2 = 0, \tag{6}$$

$$A_3(a, b, c) \equiv \sum_{\delta \text{ even}} \mathcal{E}[\delta] S_{\delta}(a, b) S_{\delta}(b, c) S_{\delta}(c, a) = 0, \tag{7}$$

respectively, where a, b, c are arbitrary points of the genus g Riemann surface C and $S_{\delta}(a, b)$ is the Szegő kernel. The identity (5) has been proved for the CDvG–Grushevsky (CDvG–G) ansatz at genus 3 [10] and 4 [12]. Remarkably, for genus 4, A_0 corresponds to a non-zero Siegel modular form of weight 8 (the Schottky form), which vanishes only on the locus of Jacobians of Riemann surfaces. A strong argument for the identities (6) and (7) to hold on the hyperelliptic locus for any genus has been given by Morozov in [14], whereas in [15] Grushevsky and Salvati Manni proved (6) for genus 3.

In this paper, we will prove that (7) does not hold for any non-hyperelliptic Riemann surface of genus 3. More precisely, one of the main results of the paper is the following theorem

Theorem 1. *Let C be a Riemann surface of genus three. Then, $A_3(a, b, c) = 0$ for all $a, b, c \in C$ if and only if C is hyperelliptic.*

In particular, we will prove that the remarkable identity

$$A_3(p_1, p_2, p_3) d\mu_{\text{Bos}} = \frac{c}{c_3} \det \omega_i(p_j) \prod_{i \leq j} d\tau_{ij}, \tag{8}$$

where $c_3 := 2^6 \pi^{18}$ and $c \in \mathbb{C}$ is a non-vanishing constant, holds for genus 3, so giving a simple expression for A_3 . Moreover, we discuss the possibility of a further non-vanishing contribution to the three-point function coming from the connected part of the chiral amplitude and give some arguments suggesting that the natural candidate for such a term should exhibit the same structure of the right-hand side of (8). In this respect, (8) can be interpreted as the statement of the non-renormalization of the three-point function at three-loop.

The paper is organized as follows. In Section 1 we introduce the mathematical background on Riemann surfaces and theta functions needed for the later construction. In Section 2 we give a necessary and sufficient condition for $A_3(a, b, c)$ to vanish identically and we find that such a condition is trivially satisfied for $g = 2$, whereas, for genus 3, it is fulfilled only for hyperelliptic surfaces. In Section 3, a strikingly simple formula for $A(a, b, c)$ is provided for genus 3, which does not include any summation over spin characteristics; Eq. (8) is an immediate consequence of such a formula. In Section 4, we discuss how the non-renormalization theorems could be implemented in the framework of the CDvG–G ansätze for the chiral superstring measures in view of our results.

1. Theta functions and Riemann surfaces

In this section, we provide the basic background on theta functions and Riemann surfaces necessary for the subsequent derivations. We refer to [16–18] for proofs and further details.

Let \mathfrak{H}_g denote the Siegel upper half-space, i.e. the space of $g \times g$ complex symmetric matrices with positive definite imaginary part

$$\mathfrak{H}_g := \{ \tau \in M_{g \times g}(\mathbb{C}) \mid {}^t \tau = \tau, \text{Im } \tau > 0 \}.$$

Let $\text{Sp}(2g, \mathbb{Z})$ be the symplectic modular group, i.e. the group of $2g \times 2g$ complex matrices $M := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are $g \times g$ blocks satisfying

$${}^t AC = {}^t CA, \quad {}^t BD = {}^t DB, \quad {}^t DA - {}^t BC = \mathbb{I}_g.$$

Let us define the action of $\text{Sp}(2g, \mathbb{Z})$ on $\mathbb{C}^g \times \mathfrak{H}_g$ by

$$(M \cdot z, M \cdot \tau) := ({}^t(C\tau + D)^{-1}z, (A\tau + B)(C\tau + D)^{-1}), \tag{9}$$

where $M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$ and $(z, \tau) \in \mathbb{C}^g \times \mathfrak{H}_g$.

Let $\mathbb{Z}_2 := \mathbb{Z}/(2\mathbb{Z})$ be the additive group with elements $\{0, 1\}$. For each $\delta', \delta'' \in \mathbb{Z}_2^g$, the theta function $\theta[\delta] \equiv \theta \left[\begin{smallmatrix} \delta' \\ \delta'' \end{smallmatrix} \right] : \mathbb{C}^g \times \mathfrak{H}_g \rightarrow \mathbb{C}$ with characteristics $[\delta] \equiv \left[\begin{smallmatrix} \delta' \\ \delta'' \end{smallmatrix} \right]$ is defined by

$$\theta[\delta](z, \tau) := \sum_{k \in \mathbb{Z}^g} \exp \pi i \left[{}^t \left(k + \frac{\delta'}{2} \right) \tau \left(k + \frac{\delta'}{2} \right) + 2 \left(k + \frac{\delta'}{2} \right) \left(z + \frac{\delta''}{2} \right) \right],$$

where $(z, \tau) \in \mathbb{C}^g \times \mathfrak{H}_g$. For each fixed τ , $\theta[\delta](z, \tau)$ is an even or odd function on \mathbb{C}^g depending whether $(-1)^{\delta' \cdot \delta''}$ is $+1$ or -1 , respectively. Correspondingly, there are $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd theta characteristics. Under translations $z \mapsto z + \lambda$, $z \in \mathbb{C}^g$, $\lambda \in \mathbb{Z}^g + \tau\mathbb{Z}^g \subset \mathbb{C}^g$, theta functions get multiplied by a nowhere vanishing factor

$$\theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (z + n + \tau m, \tau) = e^{-\pi i {}^t m \tau m - 2\pi i {}^t m z + \pi i ({}^t \delta' n - {}^t \delta'' m)} \theta \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (z, \tau),$$

$m, n \in \mathbb{Z}^g$. It follows that, for any fixed τ , the theta functions can be seen as sections of line bundles on the complex torus $A_\tau := \mathbb{C}^g / (\mathbb{Z}^g + \tau\mathbb{Z}^g)$, with a well defined divisor on A_τ . We denote by Θ the divisor of $\theta(z) \equiv \theta[0](z, \tau) \equiv \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau)$ and by $\text{Sing } \Theta$ its singular locus, i.e. the locus of points at which $\theta(z)$ and all its first partial derivatives vanish.

The second order theta functions are defined by

$$\Theta[\epsilon](z, \tau) := \theta \begin{bmatrix} \epsilon \\ 0 \end{bmatrix} (2z, 2\tau),$$

for all $\epsilon \in \mathbb{Z}^g$. They are a basis for $H^0(A_\tau, \mathcal{O}(2\Theta))$ and are related to the first order theta functions by the Riemann bilinear identities

$$\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (z_1 + z_2, \tau) \theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (z_1 - z_2, \tau) = \sum_{\sigma \in \mathbb{Z}^g} (-1)^{\delta \cdot \sigma} \Theta[\sigma](z_1, \tau) \Theta[\sigma + \epsilon](z_2, \tau), \quad (10)$$

for all $z_1, z_2 \in \mathbb{C}^g$, $\epsilon, \delta \in \mathbb{Z}_2^g$.

Let us define the action of $\text{Sp}(2g, \mathbb{Z})$ on \mathbb{Z}_2^{2g} by

$$M \cdot \delta \equiv M \cdot \begin{pmatrix} \delta' \\ \delta'' \end{pmatrix} := \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \delta' \\ \delta'' \end{pmatrix} + \begin{pmatrix} \text{diag}(C {}^t D) \\ \text{diag}(A {}^t B) \end{pmatrix} \pmod{2}. \quad (11)$$

Theta characteristics are invariant under the action of the subgroup $\Gamma(2) \subset \text{Sp}(2g, \mathbb{Z})$, where

$$\Gamma(n) := \{M \in \text{Sp}(2g, \mathbb{Z}) \mid M = \mathbb{I}_{2g} \pmod{n}\},$$

is the subgroup of elements of $\text{Sp}(2g, \mathbb{Z})$ congruent to the $2g \times 2g$ identity matrix mod n . The action of $\text{Sp}(2g, \mathbb{Z})$ on \mathbb{Z}_2^{2g} factorizes through the action of $\text{Sp}(2g, \mathbb{Z}_2) \equiv \text{Sp}(2g, \mathbb{Z}) / \Gamma(2)$. Symplectic transformations preserve the parity of the characteristics and, for any two $\delta, \epsilon \in \mathbb{Z}_2^{2g}$ of the same parity, there exists $M \in \text{Sp}(2g, \mathbb{Z}_2)$ such that $\epsilon = M \cdot \delta$.

A (Siegel) modular form f of weight $k \in \mathbb{Z}$ for a subgroup $\Gamma \in \text{Sp}(2g, \mathbb{Z})$ is a holomorphic function on \mathfrak{H}_g such that

$$f(M \cdot \tau) = \det(C\tau + D)^k f(\tau),$$

for all $M \in \Gamma$. A condition of regularity is also required for $g = 1$, but it is automatically satisfied for $g > 1$.

The modular transformation of the theta function is given by

$$\theta[M \cdot \delta](M \cdot z, M \cdot \tau) = \kappa(M) \det(C\tau + D)^{\frac{1}{2}} e^{\pi i [\phi[\delta](M) + {}^t z (C\tau + D)^{-1} C z]} \theta[\delta](z, \tau), \quad (12)$$

where $\kappa(M)$ is an eighth root of 1 depending on M and

$$4\phi \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} (M) := ({}^t \delta' {}^t \delta'') \begin{pmatrix} -{}^t B D & {}^t B C \\ {}^t B C & -{}^t A C \end{pmatrix} \begin{pmatrix} \delta' \\ \delta'' \end{pmatrix} + 2 \text{diag}(A {}^t B) \cdot (D\delta' - C\delta'').$$

Powers of theta constants $\theta[\delta](\tau) \equiv \theta[\delta](0, \tau)$ are the basic building blocks for modular forms, at least for low genera.

Let C be a Riemann surface of genus $g > 1$. The choice of a marking for C provides a set of generators $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ for the first homology group $H_1(C, \mathbb{Z})$ on C , with symplectic intersection matrix, that is

$$\alpha_i \cdot \alpha_j = 0 = \beta_i \cdot \beta_j, \quad \alpha_i \cdot \beta_j = \delta_{ij}, \tag{13}$$

for all $i, j = 1, \dots, g$. The choice of such generators canonically determines a basis $\{\omega_1, \dots, \omega_g\}$ for the space $H^0(K_C)$ of holomorphic 1-differentials on C , with normalized α -periods

$$\oint_{\alpha_i} \omega_j = \delta_{ij},$$

for all $i, j = 1, \dots, g$. The β -periods define the Riemann period matrix of the Riemann surface C

$$\tau_{ij} \equiv \oint_{\beta_i} \omega_j,$$

which is symmetric and with positive-definite imaginary part, so that $\tau \in \mathfrak{H}_g$. By Torelli's theorem, the complex structure of C is completely determined by giving its Riemann period matrix.

By the conditions (13), a general change of marking of C corresponds to a symplectic transformation on the set of generators of $H_1(C, \mathbb{Z})$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} := \begin{pmatrix} D & C \\ B & A \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}), \tag{14}$$

under which

$$(\omega_1, \dots, \omega_g) \mapsto (\tilde{\omega}_1, \dots, \tilde{\omega}_g) := (\omega_1, \dots, \omega_g)(C\tau + D)^{-1}, \tag{15}$$

whereas $\tau \mapsto \tilde{\tau} := M \cdot \tau$ transforms as in (9).

The complex torus $J_C := \mathbb{C}^g / (\mathbb{Z}^g + \tau\mathbb{Z}^g)$ associated to the Riemann period matrix of C is called the Jacobian torus of C . For a fixed base-point $p_0 \in C$, let $I: C \rightarrow J_C$ denote the Abel–Jacobi map, defined by

$$p \mapsto I(p) := {}^t \left(\int_{p_0}^p \omega_1, \dots, \int_{p_0}^p \omega_g \right) \in J_C.$$

Note that different choices of the path of integration from p_0 to p correspond, by the formula above, to points in \mathbb{C}^g differing by elements in the lattice $\mathbb{Z}^g + \tau\mathbb{Z}^g$, so that I is well-defined only on $\mathbb{C}^g / (\mathbb{Z}^g + \tau\mathbb{Z}^g)$. The Abel–Jacobi map extends to a map from the Abelian group of divisors on C to J_C by

$$I\left(\sum_i p_i - \sum_i q_i\right) := \sum_i I(p_i) - \sum_i I(q_i).$$

Such a map is independent of the base point p_0 when restricted to zero degree divisors. In the following, when no confusion is possible, we will identify such zero degree divisors with their image in J_C through I . In particular, we will omit I when considering the theta functions on the

Jacobian evaluated at (the image of) some zero degree divisor on C . Furthermore, the argument τ for theta functions associated to a marked Riemann surface will be understood.

Given a Riemann surface C with marking, one can canonically associate to each theta characteristic $\delta \in \mathbb{Z}_2^{2g}$ a spin structure, correspondent to a line bundle L_δ on C such that $L_\delta^2 \cong K_C$, with K_C the canonical line bundle on C . Such a correspondence can be defined as follows. Let δ be a non-singular theta characteristic (that is, such that at least one among $\theta[\delta](z)$ and its first partial derivatives does not vanish at $z = 0$) and, for an arbitrary $y \in C$, set $f_{\delta,y}(x) := \theta[\delta](x - y)$. By the Riemann vanishing theorem [18], the divisor $2(f_{\delta,y}) - 2y$ is linearly equivalent to the canonical divisor, so that $(f_{\delta,y}) - y$ defines the divisor class of a spin bundle, that we denote by L_δ . It can be proved that such a divisor class $[(f_{\delta,y}) - y]$, and thus also L_δ , is independent of $y \in C$, so that for each marked Riemann surface we have a correspondence $\delta \mapsto L_\delta$. By (12), under a change of marking (14), such a correspondence transforms into the map $\delta \mapsto \tilde{L}_\delta$, where

$$\tilde{L}_{M \cdot \delta} = L_\delta. \tag{16}$$

Fix a non-singular odd spin structure $\nu \in \mathbb{Z}_2^{2g}$ and consider the holomorphic 1-differential

$$\sum_{i=1}^g \frac{\partial \theta[\nu](z)}{\partial z_i} \Big|_{z=0} \omega_i.$$

It can be proved that such a 1-differential has $g - 1$ double zeroes and corresponds to the square h_ν^2 of a holomorphic section of the line bundle L_ν .

Let us define the prime form by

$$E(a, b) := \frac{\theta[\nu](b - a)}{h_\nu(a)h_\nu(b)},$$

$a, b \in C$, for an arbitrary non-singular odd spin-structure ν . The prime form is a section of a line bundle on $C \times C$, it is antisymmetric in its arguments and vanishes only on the diagonal $a = b$. Furthermore, it does not depend on the choice of ν .

For each non-singular even characteristic $\delta \in \mathbb{Z}_2^{2g}$, the Szegő kernel is defined by

$$S_\delta(a, b) \equiv S(a, b; L_\delta) := \frac{\theta[\delta](a - b)}{\theta[\delta](0) E(a, b)}.$$

For each fixed $b \in C$, $S_\delta(a, b)$ is the unique meromorphic section of L_δ with a single pole of residue -1 at b and holomorphic elsewhere. Such a characterization implies that, for a fixed spin bundle L , $S(a, b; L)$ is independent of the marking. It follows that, under a change of marking corresponding to (14), by (16) we have

$$\tilde{S}_{M \cdot \delta}(a, b) \equiv S(a, b; \tilde{L}_{M \cdot \delta}) = S(a, b; L_\delta) = S_\delta(a, b), \tag{17}$$

with $M \cdot \delta$ given by (11).

Finally, we denote by

$$\omega_{a-b}(x) := \frac{\partial}{\partial x} \log \frac{E(x, a)}{E(x, b)},$$

$a, b, x \in C$, the Abelian 1-differential of the second kind with single poles on a and b with residue $+1$ and -1 , respectively, holomorphic on $C \setminus \{a, b\}$, and with vanishing α -periods.

2. Proof of Theorem 1

Let $\mathcal{E}[\delta](\tau)$ be the modular form of weight 8 for $\Gamma(2) \subset \mathrm{Sp}(2g, \mathbb{Z})$ defined in [10] for genus 3 and in [12] for arbitrary genus. It satisfies the property

$$\mathcal{E}[M \cdot \delta](M \cdot \tau) = \det(C\tau + D)^8 \mathcal{E}[\delta](\tau), \tag{18}$$

for an arbitrary $M \in \mathrm{Sp}(2g, \mathbb{Z})$.

In terms of $\mathcal{E}[\delta]$ and the Szegő kernel, one can define the sections $A_2(a, b)$ on $C \times C$ and $A_3(a, b, c)$ on $C \times C \times C$ by (6) and (7), respectively. In [15], it has been proved that $A_2(a, b) = 0$ for all $a, b \in C$, where C is an arbitrary Riemann surface of genus 2 or 3. It is useful to recall the main points of such a derivation.

Let C be a marked Riemann surface of genus $g > 1$ and let τ be its Riemann period matrix. Define the function $X: \mathbb{C}^g \rightarrow \mathbb{C}$ by

$$X(z) := \sum_{\delta \text{ even}} \mathcal{E}[\delta] \frac{\theta[\delta](z, \tau)^2}{\theta[\delta](0, \tau)^2}, \tag{19}$$

$z \in \mathbb{C}^g$, corresponding to a section of $|2\Theta|$ on the Jacobian J_C . The restriction of such a section to

$$C - C := \{(a - b) \in J_C, a, b \in C\} \subset J_C,$$

is related to $A_2(a, b)$ by

$$A_2(a, b) = \frac{X(a - b)}{E(a, b)^2}.$$

Thus, $A_2(a, b) = 0$ for all $a, b \in C$ if and only if the restriction of X to $C - C$ vanishes identically. On the other hand, as conjectured in [19] and proved in [20], the space of sections of $|2\Theta|$ vanishing on $C - C$ is

$$\Gamma_{00} := \{f \in H^0(J_C, \mathcal{O}(2\Theta)) \mid f(0) = 0 = \partial_i \partial_j f(0), i, j = 1, \dots, g\},$$

so that the following theorem follows.

Theorem 2. (See Grushevsky, Salvati Manni [15].) *The function $A_2(a, b) = 0$ for all $a, b \in C$ if and only if $X \in \Gamma_{00}$.*

For genus $g > 1$, the dimension of Γ_{00} is $2^g - 1 - g(g + 1)/2$; in particular, $\dim \Gamma_{00} = 0$ for $g = 2$ and $\dim \Gamma_{00} = 1$ for $g = 3$. By applying the Riemann bilinear relations, $X(z)$ can be expressed in terms of second order theta functions. In [15], such an expression is used to show that the function $X(z)$ vanishes identically on \mathbb{C}^g for $g = 2$, whereas for $g = 3$ $X(z)$ is a generator of Γ_{00} . By Theorem 2, this proves that $A_2(a, b) \equiv 0$ for $g = 2, 3$.

Let us show that, for an arbitrary marked Riemann surface C of genus $g > 1$, also $A_3(a, b, c)$ can be expressed in terms of $X(z)$. The starting point is the following identity

$$\frac{S_\delta(c, a)S_\delta(b, c)}{S_\delta(a, b)} = \omega_{a-b}(c) + \sum_{i=1}^g \frac{\partial \log \theta[\delta](z)}{\partial z_i} \Big|_{z=a-b} \omega_i(c), \tag{20}$$

$a, b, c \in C$, which is an immediate consequence of [17, formula (38), p. 25]. It is instructive to derive such an identity from the famous Fay's trisecant identity [17], written in the form

$$\frac{\theta[\delta](a+c-b-d)E(a,c)E(b,d)}{\theta[\delta](0)E(a,b)E(a,d)E(c,b)E(c,d)} = S_\delta(a,d)S_\delta(c,b) - S_\delta(a,b)S_\delta(c,d), \quad (21)$$

which holds for arbitrary $a, b, c, d \in C$ and for each non-singular even spin structure δ . By comparing the Laurent expansion of both sides of (21) in the limit $d \rightarrow c$, with respect to some local coordinate centered in c , we obtain an infinite tower of (possibly trivial) identities, one for each order in $(d - c)$. Using $E(c, d)^{-1} = (d - c)^{-1}(1 + O(d - c)^2)$, it is easy to check that the first non-trivial identity is obtained at $O(1)$

$$-\frac{\theta[\delta](a-b)}{\theta[\delta](0)E(a,b)} \frac{d}{dx} \left(\log \theta[\delta](a+c-b-x) + \log \frac{E(b,x)}{E(a,x)} \right) \Big|_{x=c} = S_\delta(a,c)S_\delta(c,b),$$

and (20) follows immediately. (Note that second term on the RHS of (21) is odd in $(d - c)$, so that it does not contribute to $O(1)$.)

By multiplying both sides of (20) by $\mathcal{E}[\delta]S_\delta(a,b)^2$ and summing over all the even spin structures, we obtain

$$\begin{aligned} & \sum_{\delta \text{ even}} \mathcal{E}[\delta]S_\delta(c,a)S_\delta(b,c)S_\delta(a,b) \\ &= \omega_{a-b}(c) \sum_{\delta \text{ even}} \mathcal{E}[\delta]S_\delta(a,b)^2 \\ &+ \frac{1}{E(a,b)^2} \sum_{\delta \text{ even}} \frac{\mathcal{E}[\delta]}{\theta[\delta](0)^2} \theta[\delta](a-b) \sum_{i=1}^g \frac{\partial \theta[\delta](z)}{\partial z_i} \Big|_{z=a-b} \omega_i(c). \end{aligned}$$

By (19), such an identity can be written as

$$A_3(a,b,c) = \frac{1}{E(a,b)^2} \left[\omega_{a-b}(c)X(a-b) + \frac{1}{2} \sum_{i=1}^g \frac{\partial X(z)}{\partial z_i} \Big|_{z=a-b} \omega_i(c) \right]. \quad (22)$$

Since $\omega_{a-b}, \omega_1, \dots, \omega_g$ are linearly independent, the condition that $A_3(a,b,c) = 0$ for all $a, b, c \in C$, is equivalent to the $g + 1$ conditions

$$\begin{aligned} X(a-b) &= 0, \\ \frac{\partial X(z)}{\partial z_i} \Big|_{z=a-b} &= 0, \quad i = 1, \dots, g, \end{aligned}$$

to hold for all $a, b \in C$. Note that the first condition is equivalent to $A_2(a,b) = 0$. It is natural to define the following subspace of Γ_{00}

$$\Gamma_{00}^{(2)} := \{ f \in \Gamma_{00} \mid \text{mult}_{a-b}(f) \geq 2, \forall a, b \in C \}.$$

We have proved the following theorem.

Theorem 3. For an arbitrary Riemann surface C of genus $g > 1$, $A_3(a,b,c) = 0$ for all $a, b, c \in C$ if and only if $X \in \Gamma_{00}^{(2)}$.

The space $\Gamma_{00}^{(2)}$ and other remarkable subspaces of Γ_{00} have been extensively studied in the last few years, in particular because of their relationship with the geometry of the moduli space $\mathcal{S}U_C(2, K)$ of semi-stable bundles of rank 2 with fixed canonical determinant on a smooth projective curve C [21–23]. There is a simple procedure to construct elements of $\Gamma_{00}^{(2)}$. Let $e \in \text{Sing } \Theta$ be a point of the singular locus of the theta function, i.e. such that $\theta(z)$ and all its first derivatives vanish at $z = e$; note that, by the parity of the theta function, also $-e \in \text{Sing } \Theta$. By the Riemann singularity theorem [18], $\theta(a - b + e) = 0$ for all $a, b \in C$. It follows immediately that

$$F_e(z) := \theta(z + e)\theta(z - e) = \sum_{\sigma \in \mathbb{Z}_2^g} \Theta[\sigma](e)\Theta[\sigma](z),$$

is an element of $\Gamma_{00}^{(2)}$. In [22, Theorem 1.1], it has been proved that, for an arbitrary non-hyperelliptic Riemann surface C , $\Gamma_{00}^{(2)}$ is generated by the sections $F_e(z)$ as e varies in $\text{Sing } \Theta$, that is $\Gamma_{00}^{(2)} = \langle F_e \rangle_{e \in \text{Sing } \Theta}$.

Let us consider the consequences of such results for low genera. For genus 2, the identity $X \equiv 0$ proved in [15], together with Theorem 3, implies that $A_3(a, b, c) \equiv 0$, thus reobtaining the result of [4].

For genus 3, in the non-hyperelliptic case, $\text{Sing } \Theta$ is empty, so that $\dim \Gamma_{00}^{(2)} = 0$. On the other hand, in [15] it has been proved that, in this case, $X \neq 0$, so that we conclude that $A_3(a, b, c)$ does not vanish identically on $C \times C \times C$.

The hyperelliptic curves are not considered in [22]. In this case, one has just the weaker inclusion $\langle F_e \rangle_{e \in \text{Sing } \Theta} \subseteq \Gamma_{00}^{(2)}$. On the other hand, if C is hyperelliptic of genus 3, $\text{Sing } \Theta \subset J_C$ consists of a unique point of order 2, corresponding to a singular even spin-structure δ_{sing} . It follows that, in this case, $\Gamma_{00}^{(2)}$ has at least one non-trivial section, namely

$$F_{\delta_{\text{sing}}}(z) := \theta[\delta_{\text{sing}}](z)^2.$$

Since Γ_{00} is 1-dimensional and contains $\Gamma_{00}^{(2)}$ as a subspace, we get $\Gamma_{00} = \Gamma_{00}^{(2)}$ so that $X \in \Gamma_{00}^{(2)}$ and $A_3(a, b, c) = 0$ for all $a, b, c \in C$, as suggested by the arguments in [14]. This concludes the proof of Theorem 1.

More generally, for genus $g \geq 3$, the space $\langle F_e \rangle_{e \in \text{Sing } \Theta}$ has dimension $2^g - \sum_{i=0}^3 \binom{g}{i}$, which, for non-hyperelliptic surfaces, corresponds to the dimension of $\Gamma_{00}^{(2)}$. In particular, for a non-hyperelliptic surface of genus 4, $\dim \Gamma_{00}^{(2)} = 1$. In this case, $\text{Sing } \Theta$ has only two (possibly coincident) points $\pm e$ and the generator of $\Gamma_{00}^{(2)}$ is $\theta(z + e)\theta(z - e)$. It would be interesting (but probably highly non-trivial) to check whether X is proportional to such a section in this case.

3. A simple expression for $A_3(a, b, c)$

Let us show that $A_3(a, b, c)$ admits the alternative expression (8), that will be useful in the following.

First of all, note that, if $A_2(a, b)$ vanishes identically on a Riemann surface C of genus $g > 1$, then $A_3(a, b, c)$ is a holomorphic 1-differential in each variable. Furthermore, it is anti-symmetric under permutation of such variables and, in particular, it must vanish on the diagonals

of $C \times C \times C$. For $g = 3$, this is enough to conclude that

$$A_3(p_1, p_2, p_3) \equiv \sum_{\delta \text{ even}} \mathcal{E}[\delta] S_\delta(p_1, p_2) S_\delta(p_2, p_3) S_\delta(p_3, p_1) = f \det \omega_i(p_j),$$

$p_1, p_2, p_3 \in C$, for some holomorphic function f , independent of p_1, p_2, p_3 , on the Teichmüller space of genus 3 Riemann surfaces with marking. Under a change of marking, corresponding to a transformation (14) for some $M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$, by (15), (17), (18) we get

$$\begin{aligned} A_3(p_1, p_2, p_3) &\mapsto \det(C\tau + D)^8 A_3(p_1, p_2, p_3), \\ \det \omega_i(p_j) &\mapsto \det(C\tau + D)^{-1} \det \omega_i(p_j). \end{aligned}$$

It follows that f transforms as

$$f \mapsto \det(C\tau + D)^9 f,$$

and thus it corresponds to a Teichmüller modular form of weight 9 and degree 3 [24]. In general, a Teichmüller modular form of weight $d \geq 0$ and degree g is defined as a holomorphic section of $\lambda_1^{\otimes d}$ on the moduli space \mathcal{M}_g of genus g Riemann surfaces; here, λ_1 is the line bundle whose fiber at the point corresponding to the surface C is $\bigwedge^g H^0(K_C)$. By [24], there is a unique (up to a constant), holomorphic section $\mu_{3,9}$ of $\lambda_1^{\otimes 9}$ on \mathcal{M}_3 , so that

$$f = c \mu_{3,9},$$

for some $c \in \mathbb{C}$; Theorem 1 implies that $c \neq 0$. The section $\mu_{3,9}$ vanishes only on the hyperelliptic locus, consistently with Theorem 1, and its square $(\mu_{3,9})^2$ corresponds to the (Siegel) modular form

$$\Psi_{18}(\tau) := \prod_{\delta \text{ even}} \theta[\delta](0, \tau),$$

of weight 18. (More precisely, $(\mu_{3,9})^2$ is the image of Ψ_{18} under the homomorphism, induced by the Torelli map, mapping Siegel modular forms to Teichmüller modular forms [24]; for such a reason, $\mu_{3,9}$ itself is often identified with the parabolic form Ψ_9 , that is a holomorphic square root of Ψ_{18} .)

Furthermore, $\mu_{3,9}$ also appears in the explicit formula for the chiral bosonic string measure for genus 3 [25]

$$d\mu_{\text{Bos}} = \frac{1}{c_3} \frac{\prod_{i \leq j} d\tau_{ij}}{\mu_{3,9}},$$

with $c_3 = 2^6 \pi^{18}$ [3], and the identity (8) follows.

It would be interesting to compute exactly the constant c . This may be done, for example, by using the factorization properties of string amplitudes under degeneration limits.

4. Non-renormalization theorems and chiral measure ansätze

Theorem 1 shows that the CDvG–G ansatz and the assumption (3) are not compatible with the non-renormalization theorems at 3-loops. It is necessary, therefore, to discuss the validity of the assumptions leading to such a result.

For genus 3, upon assuming the form (2), the chiral superstring measure is completely determined by the constraints related to holomorphicity, modular invariance and factorization. Even

though there are no strong first-principle arguments suggesting that (2) holds for genus higher than 2, a posteriori there are several hints that the ansatz should be correct, at least for low genera: the uniqueness of the genus 3 solution for the constraints, the existence of general well-defined solutions at least up to genus 5, the non-renormalization of the cosmological constant at 3 and 4 loops (a result that, as noted in the introduction, is independent of the assumption (3)).

The evidence for the assumption (3) is much weaker. In fact, it does not hold, for example, for the 2-loop contribution to the 4-point function [4]. Therefore, it is reasonable to consider the possibility of a non-vanishing contribution from the connected part of the chiral amplitude. Some hints on the possible structure of such a contribution can be obtained analyzing the 4-point function at 2 loops. In this case, the connected part of the chiral amplitude gives two non-vanishing terms, both corresponding to Wick contractions between the vertex operators and the stress–energy tensor. One of such contributions exactly cancels the disconnected part of the amplitude. The other one has, very schematically, the following structure

$$A_4(p_1, p_2, p_3, p_4) d\mu_{\text{Bos}} = \sum_{I_1 \sqcup I_2 = \{1,2,3,4\}} K_{I_1} \prod_{\substack{i=1,2 \\ j \in I_1}} \det \omega_i(p_j) \prod_{\substack{i=1,2 \\ j \in I_2}} \det \omega_i(p_j) \prod_{i \leq j} d\tau_{ij}, \quad (23)$$

where K_{I_1} is a kinematical factor and the sum is over all the possible ways of splitting the set $\{1, 2, 3, 4\}$ into the disjoint union of sets I_1 and I_2 of two elements. Hence, it seems reasonable for a non-vanishing 3-loop contribution corresponding to Wick contractions between vertex operators and stress tensor to exhibit a structure analogous to the right-hand side of (23).

On the other hand, the most natural generalization of (23) to the case of a 3-loop contribution to the 3-point function, satisfying the fundamental consistency constraints (modular weight 5 and conformal weight 1 in each variable), is precisely the right-hand side of (8) (times a kinematical factor). In this respect, note that the requirements we are imposing on the possible structure of such a contribution are very restrictive. In fact, no consistent generalization of (23) can be defined for the 2-loop contribution to the 3-point function or for the 2- and 3-loop contributions to the 2-point function. The fact that, for such amplitudes, the disconnected part vanishes separately enforces the validity of our analysis. Remarkably, a structure similar to (8) has been proposed in [9] for the higher loop contributions to the 4-point function.

Thus, by the identity (8) and by the arguments above, it is reasonable to conjecture that the contributions from the connected chiral amplitude could exactly cancel the disconnected part, thus giving the expected non-renormalization theorems. It would be very interesting to check such a conjecture by explicitly computing some of the terms coming from the relevant Wick contractions. Unfortunately, even though such a computation should be considerably simpler than a complete first-principles derivation of the amplitudes (for example, we are neglecting the Wick contractions between supercurrents and vertex operators, together with a huge number of subtleties already emerging at genus 2), it is not clear, at the moment, how it should be performed.

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