On Differential Games with Long-Time-Average Cost*

Martino Bardi
Dipartimento di Matematica Pura ed Applicata
Università di Padova
via Trieste 63, 35121 Padova, Italy
bardi@math.unipd.it

Abstract

The paper deals with the ergodicity of deterministic zero-sum differential games with long-time-average cost. Some new sufficient conditions are given, as well as a class of games that are not ergodic. In particular, we settle the issue of ergodicity for the simple games whose associated Isaacs equation is a convex-concave eikonal equation.

Key words. Ergodic control, differential games, viscosity solutions, Hamilton-Jacobi-Isaacs equations.

AMS Subject Classifications. Primary 49N70; Secondary 37A99, 49L25, 91A23.

Introduction

We consider a nonlinear system in $\mathbb{R}^m$ controlled by two players

$$\dot{y}(t) = f(y(t), a(t), b(t)), \qquad y(0) = x, \quad a(t) \in A, \quad b(t) \in B, \qquad (1)$$

and we denote with $y_x(\cdot)$ the trajectory starting at x. We are also given a bounded, uniformly continuous running cost l, and we are interested in the payoffs associated to the *long-time-average cost* (briefly, LTAC), namely:

$$J^{\infty}(x, a(\cdot), b(\cdot)) := \limsup_{T\to\infty} \frac{1}{T}\int_0^T l(y_x(t), a(t), b(t))\, dt,$$

$$J_{\infty}(x, a(\cdot), b(\cdot)) := \liminf_{T\to\infty} \frac{1}{T}\int_0^T l(y_x(t), a(t), b(t))\, dt.$$

We denote with $u - \text{val}\, J^{\infty}(x)$ (respectively, $l - \text{val}\, J_{\infty}(x)$) the upper value of the zero-sum game with payoff J^{∞} (respectively, the lower value of the game

*This work has been partially supported by funds from M.I.U.R., project "Viscosity, metric, and control theoretic methods for nonlinear partial differential equations", and G.N.A.M.P.A., project "Equazioni alle derivate parziali e teoria del controllo".

with payoff J_∞) which the 1st player $a(\cdot)$ wants to minimize while the 2nd player $b(\cdot)$ wants to maximize, and the values are in the sense of Varaiya-Roxin-Elliott-Kalton. We look for conditions under which

$$u - \text{val}\, J^\infty(x) = l - \text{val}\, J_\infty(x) = \lambda \quad \forall\, x,$$

for some constant λ, a property that was called *ergodicity of the LTAC game* in [3]. The terminology is motivated by the analogy with classical ergodic control theory, see, e.g., [30,14,28,9,25,6,7,2]. Similar problems were studied for some games by Fleming and McEneaney [21] in the context of risk-sensitive control, by Carlson and Haurie [16] within the turnpike theory, and by Kushner [29] for controlled nondegenerate diffusion processes. There is a large literature on related problems for discrete-time games; see the survey by Sorin [35].

More recently, several sufficient conditions for the ergodicity of the LTAC game were given by Ghosh and Rao [24] and Alvarez and the author [3]. Among other things, these papers clarified the connections with the solvability of the stationary Hamilton-Jacobi-Isaacs equation associated to the problem and with the long-time behavior of the value functions of the finite horizon games with the same running cost. In particular, under the classical Isaacs' condition

$$\min_{b\in B}\max_{a\in A}\{-f(y,a,b)\cdot p - l(y,a,b)\} = \max_{a\in A}\min_{b\in B}\{-f(y,a,b)\cdot p - l(y,a,b)\} \quad (2)$$

for all $y, p \in \mathbb{R}^m$, the LTAC game is ergodic with value λ if the viscosity solution $u(t,x)$ of the evolutive Hamilton-Jacobi-Isaacs equation

$$\frac{\partial u}{\partial t} + \min_{b\in B}\max_{a\in A}\{-f(y,a,b)\cdot D_x u - l(y,a,b)\} = 0, \qquad u(0,x) = 0,$$

satisfies

$$\lim_{t\to+\infty} \frac{u(t,x)}{t} = \lambda, \qquad \text{locally uniformly in } x,$$

a property called *ergodicity of the lower game*. However, the results of the quoted papers do not give much information on some very simple games such as:

$$\begin{cases} \dot{y}^A(t) = a(t), & y^A(0) = x^A \in \mathbb{R}^{m/2}, \quad |a(t)| \le 1, \\ \dot{y}^B(t) = b(t), & y^B(0) = x^B \in \mathbb{R}^{m/2}, \quad |b(t)| \le \gamma, \end{cases} \quad (3)$$

with running cost $l = l(y^A, y^B)$ independent of the controls and $\mathbb{Z}^m$-periodic. This is related to the asymptotic behavior of the solution to the *convex-concave eikonal equation*

$$u_t + |D_{x^A} u| - \gamma |D_{x^B} u| = l(x^A, x^B), \quad u(0, x^A, x^B) = 0,$$

where $D_{x^A} u, D_{x^B} u$ denote, respectively, the gradient of u with respect to the x^A and the x^B variables. >From [3] we can only say that the lower game and the LTAC game are ergodic if l has a saddle, namely

$$\min_{x^A}\max_{x^B} l(x^A, x^B) = \max_{x^B}\min_{x^A} l(x^A, x^B) =: \bar{l},$$

and then the ergodic value is $\lambda = \bar{l}$. Nothing seems to be known if, for instance, $l(x^A, x^B) = n(x^A - x^B)$.

In the present paper, we present some new conditions for ergodicity and a class of non-ergodic differential games. The sufficient conditions for ergodicity assume some form of controllability of each player on some state variables. Different from the controllability conditions in [3], they depend on the running cost l, that is assumed independent of the controls, and give an explicit formula for the ergodic value λ in terms of l. The result of non-ergodicity holds for systems of the form

$$\begin{cases} \dot{y}^A(t) = g(y(t), a(t)), & y^A(0) = x^A \in \mathbb{R}^{m/2}, \quad a(t) \in A, \\ \dot{y}^B(t) = g(y(t), b(t)), & y^B(0) = x^B \in \mathbb{R}^{m/2}, \quad b(t) \in B, \end{cases} \tag{4}$$

with $A = B$, and running cost $l(x) = n(x^A - x^B) + h(x^A, x^B)$ with a smallness assumption on h. As a special case we settle the issue of the game (3) with the running cost $l(x) = n(x^A - x^B)$ and of the convex-concave eikonal equation: it is ergodic if and only if $\gamma \neq 1$.

Undiscounted infinite horizon control problems arise in many applications to economics and engineering; see [17,14,28] and [16,21,35] for games. Our additional motivation is that ergodicity plays a crucial role in the theory of singular perturbation problems for the dimension reduction of multiple-scale systems [27,14,28,23,36,26,32] and for the homogenization in oscillating media [31,19,5]. A general principle emerging in the papers [8,1,2,4] is that an appropriate form of ergodicity of the fast variables (for frozen slow variables) ensures the convergence of the singular perturbation problem, in a suitable sense. The explicit applications of the results of the present paper to singular perturbations will be presented in a future article.

The paper is organized as follows. Section 1 recalls some definitions and known results. Section 2 gives two different sets of sufficient conditions for the ergodicity of the finite horizon games. Section 3 presents the non-ergodic games. Section 4 applies the preceding results to a slight generalization of the system (3) and of the convex-concave eikonal equation.

1 Definitions and preliminary results

About the system (1) and the cost we assume throughout the paper that $f : \mathbb{R}^m \times A \times B \mapsto \mathbb{R}^m$ and $l : \mathbb{R}^m \times A \times B \mapsto \mathbb{R}$ are continuous and bounded, A and B are compact metric spaces, and f is Lipschitz continuous in x uniformly in a, b.

We consider the cost functional

$$J(T, x) = J(T, x, a(\cdot), b(\cdot)) := \frac{1}{T} \int_0^T l(y_x(t), a(t), b(t))\, dt,$$

where $y_x(\cdot)$ is the trajectory corresponding to $a(\cdot)$ and $b(\cdot)$. We denote with $\mathcal{A}$ and $\mathcal{B}$, respectively, the sets of open-loop (measurable) controls for the first and second player, and with Γ and Δ, respectively, the sets of nonanticipating strategies for the first and the second player; see, e.g., [18,20,9] for the precise definition. Following Elliott and Kalton [18], we define the upper and lower values for the finite horizon game with average cost:

$$u - \operatorname{val} J(T,x) := \sup_{\beta\in\Delta} \inf_{a\in\mathcal{A}} J(T,x,a,\beta[a]),$$

$$l - \operatorname{val} J(T,x) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} J(T,x,\alpha[b],b).$$

The player using nonanticipating strategies has an information advantage with respect to the other, so the inequality $l - \operatorname{val} J(T,x) \le u - \operatorname{val} J(T,x)$ holds; see [18,20,9]. Moreover, all other reasonable notion of value are between $l - \operatorname{val} J$ and $u - \operatorname{val} J$; see [18] or Chapter 8 of [9] for a discussion. Therefore, when the game has a value, i.e., $l - \operatorname{val} J = u - \operatorname{val} J$, all notions of value coincide. For the LTAC game we define:

$$u - \operatorname{val} J^\infty(x) := \sup_{\beta\in\Delta} \inf_{a\in\mathcal{A}} \limsup_{T\to\infty} J(T,x,a,\beta[a]),$$

$$l - \operatorname{val} J_\infty(x) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} \liminf_{T\to\infty} J(T,x,\alpha[b],b).$$

Note that we chose $\limsup_{T\to\infty}$ for the upper value and $\liminf_{T\to\infty}$ for the lower value, so we expect again that any other definition of ergodic value falls between them.

We say that the *the lower game is (locally uniformly) ergodic* if the long time limit of the finite horizon value exists, locally uniformly in x, and it is constant, i.e.,

$$l - \operatorname{val} J(T,\cdot) \to \lambda \quad \text{as } T\to\infty \text{ locally uniformly in } \mathbb{R}^m.$$

Similarly, *the upper game is ergodic* if

$$u - \operatorname{val} J(T,\cdot) \to \Lambda \quad \text{as } T\to\infty \text{ locally uniformly in } \mathbb{R}^m.$$

The next result gives the precise connection between these properties and the LTAC game.

Theorem 1.1. *[21,3] If the lower game is ergodic, then*

$$l - \operatorname{val} J_\infty(x) = \lim_{T\to\infty} l - \operatorname{val} J(T,x) = \lambda \quad \forall x\in\mathbb{R}^m; \tag{5}$$

if the upper game is ergodic, then

$$u - \operatorname{val} J^\infty(x) = \lim_{T\to\infty} u - \operatorname{val} J(T,x) = \Lambda \quad \forall x\in\mathbb{R}^m. \tag{6}$$

If the classical Isaacs' condition (2) holds then the finite horizon game has a value, which we denote with $\operatorname{val} J(T,x)$; see [20,9]. Therefore, we immediately get the following consequence of Theorem 1.1.

Corollary 1.1. *Assume* (2) *and that either the lower or the upper game is ergodic. Then the LTAC game is ergodic, i.e.,*

$$l - \text{val}\, J_\infty(x) = u - \text{val}\, J^\infty(x) = \lim_{T\to\infty} \text{val}\, J(T,x) = \lambda, \quad \forall x \in \mathbb{R}^m.$$

Remark 1.1. The ergodic value can also be characterized as the limit as $\delta \to 0$ of δw_δ where w_δ solves

$$\delta w_\delta + \min_b \max_a \{-f(y,a,b)\cdot Dw_\delta - l(y,a,b)\} = 0, \quad \text{in } \mathbb{R}^m,$$

and as the unique constant λ such that there exists a solution of

$$\lambda + \min_b \max_a \{-f(y,a,b)\cdot D\chi - l(y,a,b)\} = 0, \quad \text{in } \mathbb{R}^m;$$

see [2,3,24] for the precise statements. We will not use these properties in the present paper.

2 Sufficient conditions of ergodicity

In this section we prove two results on the ergodicity of the LTAC games. Both make controllability assumptions on at least one of the players, but they are weaker than those of Theorem 2.2 in [3]. On the other hand, here we assume the running cost $l = l(y)$ depends only on the state variables and the controllability assumptions are designed to get as value of the LTAC game a number depending explicitly on l. In the first result this is either $\min l$ or $\max l$.

We denote with $\mathcal{KL}$ the class of continuous functions $\eta : [0,+\infty)\times[0,+\infty) \to [0,+\infty)$ strictly increasing in the first variable, strictly decreasing in the second variable, and satisfying

$$\eta(0,t) = 0 \quad \forall t \geq 0, \qquad \lim_{t\to+\infty} \eta(r,t) = 0 \quad \forall r \geq 0. \tag{7}$$

Given a closed target $\mathcal{T} \subseteq \mathbb{R}^m$, we say that the system (1) is *(uniformly) asymptotically controllable to $\mathcal{T}$ in the mean by the first player* if the following holds: there exists a function $\eta \in \mathcal{KL}$ and for all $x \in \mathbb{R}^m$, there is a strategy $\widetilde{\alpha} \in \Gamma$ such that:

$$\frac{1}{T}\int_0^T \text{dist}(y_x(t),\mathcal{T})\,dt \leq \eta(\|x\|,T), \ \forall b \in \mathcal{B}, \tag{8}$$

where $y_x(\cdot)$ is the trajectory corresponding to the strategy $\widetilde{\alpha}$ and the control function b, i.e., it solves

$$\dot{y}(t) = f(y(t), \widetilde{\alpha}[b](t), b(t)), \qquad y(0) = x. \tag{9}$$

Here $\|x\| := |x|$ in the general case, whereas when the state space is the m-dimensional torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ (i.e., all data are $\mathbb{Z}^m$-periodic)

$$\|x\| := \min_{k\in\mathbb{Z}^m} |x-k|,$$

and $\mathrm{dist}(z, \mathcal{T}) := \inf_{w\in\mathcal{T}} \|z - w\|$. The condition (8) means that the first player can drive asymptotically the state $y(t)$ near the target $\mathcal{T}$, in the sense that the average distance tends to 0, uniformly with respect to x and the control of the other player b.

Symmetrically, we say that the system (1) is *(uniformly) asymptotically controllable to $\mathcal{T}$ by the second player* if for all $x \in \mathbb{R}^m$, there is a strategy $\widetilde{\beta} \in \Delta$ such that:

$$\frac{1}{T}\int_0^T \mathrm{dist}(y_x(t), \mathcal{T})\, dt \le \eta(\|x\|, T),\ \forall a \in \mathcal{A},$$

where $y_x(\cdot)$ is the trajectory corresponding to the strategy $\widetilde{\beta}$ and the control function a, i.e., it solves

$$\dot{y}(t) = f(y(t), a(t), \widetilde{\beta}[a](t)), \qquad y(0) = x.$$

In the next result we will use as target either the set

$$\text{argmin}\, l := \{y \in \mathbb{R}^m \ :\ l(y) = \min l\}$$

or the set

$$\text{argmax}\, l := \{y \in \mathbb{R}^m \ :\ l(y) = \max l\}.$$

Proposition 2.1. *Assume the running cost is uniformly continuous and independent of the controls, i.e., $l = l(y)$.*
If the system (1) *is asymptotically controllable to $\mathcal{T} = \mathrm{argmin}\ l$ in the mean by the first player, then the lower game is ergodic with value $\lambda = \min l$.*
If the system (1) *is asymptotically controllable to $\mathcal{T} =$ argmax l in the mean by the second player, then the upper game is ergodic with value $\lambda = \max l$.*

Proof. We prove only the first statement because the proof of the second is analogous. Set $v(T, x) := l - \text{val}\ J(T, x)$.

Fix x and consider the strategy $\widetilde{\alpha} \in \Gamma$ from the asymptotic controllability assumption. If $y_x(\cdot) = y_x(\cdot, b)$ is the corresponding trajectory and $z(t)$ is its projection on the target, i.e.,

$$\mathrm{dist}(y_x(t), \mathcal{T}) = \|y_x(t) - z(t)\|, \quad z(t) \in \mathcal{T},$$

then the choice $\mathcal{T} = \mathrm{argmin}\ l$ gives

$$l(y_x(t)) \le \omega_l(\|y_x(t) - z(t)\|) + l(z(t)) = \omega_l(\mathrm{dist}(y_x(t), \mathcal{T})) + \min l,$$

where ω_l is the modulus of continuity of l. We recall that ω_l is defined by

$$|l(x) - l(y)| \le \omega_l(\|x - y\|),\ \forall x, y \in \mathbb{R}^m, \quad \lim_{r\to 0} \omega_l(r) = 0,$$

and it is not restrictive to assume its concavity. Therefore, Jensen's inequality and (8) imply, for all $b \in \mathcal{B}$,

$$\frac{1}{T}\int_0^T \omega_l(\mathrm{dist}(y_x(t), \mathcal{T}))\,dt \le \omega_l(\eta(\|x\|, T)).$$

Then:

$$v(T,x) \le \sup_{b\in\mathcal{B}} \frac{1}{T}\int_0^T l(y_x(t))\,dt \le \omega_l(\eta(\|x\|, T)) + \min l.$$

On the other hand, $v(T,x) \ge \min l$ by definition, thus

$$\lim_{T\to\infty} v(T,x) = \min l,$$

uniformly in x for $\|x\|$ bounded. □

An immediate consequence of this proposition and of Corollary 1.1 is the following.

Corollary 2.1. *Assume the Isaacs' condition* (2) *and that the system* (1) *is asymptotically controllable either to* $\operatorname{argmin} l$ *by the first player or to* $\operatorname{argmax} l$ *by the second player. Then the LTAC game is ergodic, i.e.,*

$$l - \mathrm{val}\, J_\infty(x) = u - \mathrm{val}\, J^\infty(x) = \lim_{T\to\infty} \mathrm{val}\, J(T,x) = \lambda, \quad \forall x \in \mathbb{R}^m.$$

Moreover, $\lambda = \min l$ *in the former case and* $\lambda = \max l$ *in the latter.*

Remark 2.1. The main improvement of this result with respect to Corollary 2.1 in [3] is that here we assume only the controllability in the mean to a target, instead of the bounded-time controllability to each point of the state space. On the other hand, here we must assume the independence of l from the controls a, b.

Remark 2.2. A sufficient condition for the asymptotic controllability in the mean is that the system (1) be *locally bounded-time controllable to* $\mathcal{T}$ *by the first player*, i.e., for each x there exist $S(\|x\|) > 0$ and a strategy $\widetilde{\alpha} \in \Gamma$ such that for all control functions $b \in \mathcal{B}$ there is a time $t^\# = t^\#(x, \widetilde{\alpha}, b, \mathcal{T})$ with the properties

$$t^\# \le S(\|x\|) \text{ and } y_x(t) \in \mathcal{T} \text{ for all } t \ge t^\#.$$

In other words, the first player can drive the system from any initial position x to some point of the target $\mathcal{T}$ within a time that is uniformly bounded for bounded x, and keep it forever on $\mathcal{T}$, for all possible behaviors of the second player. The proof of Proposition 2.1 shows that this strategy is optimal for the first player. This kind of behavior is called a *turnpike*; see [17,16].

The sufficient condition described above can be better studied by splitting it in two: reaching $\mathcal{T}$ and remaining in $\mathcal{T}$ afterwards. The first amounts to the local boundedness of the lower value of the generalized pursuit-evasion game with target $\mathcal{T}$. This occurs if such value function is finite and continuous, and a sufficient condition for it is the existence of a continuous supersolution U of the Isaacs equation for minimum-time problems

$$\min_{b\in B}\max_{a\in A}\{-f(y,a,b)\cdot DU\}\geq 1 \qquad \text{in } \mathbb{R}^m\setminus\mathcal{T},$$

such that $U=0$ on $\mathcal{T}$; see [12,33]. As for the second property, it is the viability of $\mathcal{T}$ by the first player against the second. This is well understood and has explicit characterizations; see [15,11].

Remark 2.3. Another sufficient condition for the asymptotic controllability in the mean is that the system (1) be *worst-case stabilizable to* $\mathcal{T}$ *by the first player*, i.e., there exists $\kappa\in\mathcal{KL}$ and for each x there exists a strategy $\widetilde{\alpha}\in\Gamma$ such that:

$$\operatorname{dist}(y_x(t),\mathcal{T})\leq\kappa(\|x\|,t), \qquad \forall b\in\mathcal{B},\ \forall t\geq 0, \tag{10}$$

where $y_x(\cdot)$ is the trajectory corresponding to the strategy $\widetilde{\alpha}$ and the control function b. In fact, it is enough to take

$$\eta(r,T)=\frac{1}{T}\int_0^T\kappa(r,t)\,dt.$$

This property was studied by Soravia [33,34] and the author and Cesaroni [10]. They characterized it in terms of the existence of a Lyapunov pair, that is, a lower semicontinuous W, continuous at $\partial\mathcal{T}$ and proper, and a Lipschitz h, both positive off $\mathcal{T}$ and null on $\mathcal{T}$, such that:

$$\min_{b\in B}\max_{a\in A}\{-f(y,a,b)\cdot DW\}\geq h(x) \qquad \text{in } \mathbb{R}^m,$$

in the viscosity sense. Related notions are known in the context of robust control [13,22].

The second result of this section concern systems of the form:

$$\begin{cases}\dot{y}^A(t)=f_A(y(t),a(t),b(t)), & y^A(0)=x^A\in\mathbb{R}^{m_A},\\ \dot{y}^B(t)=f_B(y(t),a(t),b(t)), & y^B(0)=x^B\in\mathbb{R}^{m_B},\\ y(t)=(y^A(t),y^B(t)).\end{cases} \tag{11}$$

We will assume the asymptotic controllability by the first player to the target

$$\mathcal{T}^*:=\left\{(z^A,z^B)\in\mathbb{R}^m\,:\,l(z^A,z^B)\leq\max_{y^B}\min_{y^A}l(y^A,y^B)\right\}. \tag{12}$$

Moreover, we will assume for some closed set $\mathcal{T}_B \subseteq \mathbb{R}^{m_B}$ that the state variables y^B are *(uniformly) asymptotically controllable to* $\mathcal{T}_B$ *in the mean by the second player* in the following sense: there exists a function $\eta : [0,\infty) \to [0,\infty)$ satisfying (7) and for all $x \in \mathbb{R}^m$ there is a strategy $\widetilde{\beta} \in \Delta$ such that:

$$\frac{1}{T}\int_0^T \operatorname{dist}(y_x^B(t), \mathcal{T}_B)\, dt \le \eta(\|x\|, T), \ \forall a \in \mathcal{A}. \tag{13}$$

Proposition 2.2. *Assume the system* (1) *is of the form* (11)*,* $l = l(y^A, y^B)$*, and* (2) *holds. Suppose also that the system is asymptotically controllable in the mean by the first player to* $\mathcal{T}^*$ *and the state variables* y^B *are asymptotically controllable in the mean by the second player to:*

$$\begin{aligned}\mathcal{T}_B &= \operatorname{argmax}\ \min_{y^A} l(y^A, \cdot)\\ &:= \left\{ z^B \in \mathbb{R}^{m_B} \ :\ \min_{y^A} l(y^A, z^B) = \max_{y^B}\min_{y^A} l(y^A, y^B)\right\}.\end{aligned} \tag{14}$$

Then the LTAC game is ergodic and its value is:

$$l - \operatorname{val} J_\infty(x) = u - \operatorname{val} J^\infty(x) = \lambda := \max_{y^B}\min_{y^A} l(y^A, y^B), \quad \forall x \in \mathbb{R}^m. \tag{15}$$

Proof. The Isaacs conditions (2) implies the existence of the value of the finite horizon games and we set $v(T,x) := l - \text{val } J(T,x) = u - \text{val } J(T,x)$. By repeating the proof of Proposition 2.1 with the target $\mathcal{T}^*$ we obtain:

$$\begin{aligned} l(y_x(t)) &\le \omega_l(\|y_x(t) - z(t)\|) + l(z(t))\\ &\le \omega_l(\operatorname{dist}(y_x(t), \mathcal{T})) + \max_{y^B}\min_{y^A} l(y^A, y^B), \end{aligned} \tag{16}$$

and then

$$v(T,x) \le \omega_l(\eta(\|x\|, T)) + \lambda.$$

To get the opposite inequality fix x and consider the strategy $\widetilde{b} \in \Delta$ from the asymptotic controllability assumption on the y^B variables. Let $y_x(\cdot) = y_x(\cdot, a)$ be the corresponding trajectory and $z^B(t)$ the projection of its component $y_x^B(t)$ on the target $\mathcal{T}_B$, i.e.,

$$\operatorname{dist}(y_x^B(t), \mathcal{T}_B) = \|y_x^B(t) - z^B(t)\|, \quad z^B(t) \in \mathcal{T}_B.$$

Then the definition of $\mathcal{T}_B$ gives

$$\begin{aligned} l(y_x(t)) &\ge l(y_x^A(t), z^B(t)) - \omega_l(\|y_x^B(t) - z^B(t)\|)\\ &\ge \max_{y^B}\min_{y^A} l(y^A, y^B) - \omega_l(\operatorname{dist}(y_x^B(t), \mathcal{T}_B)), \end{aligned} \tag{17}$$

where the modulus of continuity ω_l is defined by (2). The concavity of ω_l, Jensen's inequality, and (13) imply, for all $a \in \mathcal{A}$,

$$\frac{1}{T}\int_0^T \omega_l(\mathrm{dist}(y_x^B(t), \mathcal{T}_B))\,dt \le \omega_l(\eta(\|x\|, T)).$$

Finally, the definition of upper value gives

$$v(T,x) \ge \inf_{a\in\mathcal{A}} \frac{1}{T}\int_0^T l(y_x(t))\,dt \ge \lambda - \omega_l(\eta(\|x\|, T)),$$

and therefore $\lim_{T\to\infty} v(T,x) = \lambda$ uniformly in x, for $\|x\|$ bounded. □

Remark 2.4. The main differences of this result with respect to Proposition 2.2 in [3] is that here we do not assume that the cost l has a saddle and we make different controllability assumptions that give an advantage to the first player.

Remark 2.5. By exchanging the assumptions on the two players and replacing maxmin with minmax in the targets, it is easy to give a symmetric result where the value of the LTAC game is:

$$\lambda = \min_{y^A\in\mathbb{R}^{m_A}} \max_{y^B\in\mathbb{R}^{m_B}} l(y^A, y^B).$$

Remark 2.6. No controllability assumption is necessary for ergodicity, and in general, the ergodic value λ can be any number between $\min l$ and $\max l$. For instance, by a classical result of Jacobi (see, e.g., [7,2]), the system $\dot y(t) = \xi$ with $\xi\cdot k \neq 0$ for all $k\in\mathbb{Z}^m$ is uniformly ergodic for all $\mathbb{Z}^m$-periodic l, and the ergodic value is $\lambda = \int_{[0,1]^m} l(y)\,dy$.

Remark 2.7. The results of this section can be extended to control systems driven by stochastic differential equations, as in Sec. 4 of [3]. We postpone this to a future paper.

3 Sufficient conditions for non-ergodicity

In this section we give some examples of games that are not ergodic. Let us first recall that a simple reason for non-ergodicity is the unboundedness of the trajectories, as shown in the next example.

Example 3.1. Consider the system $\dot y = y$ and running cost l such that there exist the limits $\lim_{x\to+\infty} l(x) = l_+$ and $\lim_{x\to-\infty} l(x) = l_-$. Then $\mathrm{val}\,J(T,x) = \frac{1}{T}\int_0^T l(xe^t)\,dt$ converges as $t\to+\infty$ to l_+ if $x>0$, to $l(0)$ if $x=0$, and to l_- if $x<0$.
However, also on a compact state space such as $\mathbb{T}^2$, many systems are not ergodic, such as the next simple example.

Example 3.2. In $\mathbb{R}^2$ take the system $\dot{y} = (1,0)$ and l $\mathbb{Z}^2$-periodic. Then $\text{val}\, J(T,x) = \frac{1}{T}\int_0^T l(x_1+t,x_2)\,dt$ converges as $t \to +\infty$ to $\int_{[0,1]^2} l(s,x_2)\,ds$.

The main result of the section is about systems of the form (11) under assumptions that allow the controllability of the variables y^A by the first player and of y^B by the second player as in the ergodic games described in [3]. However, the running cost does not have a saddle and the system is completely fair, in the sense that both groups of variables have the same dynamics. Here are the precise assumptions. Suppose first that the vector field f_A is independent of b, f_B does not depend on a, and l depends only on the state y. Then the Isaacs condition (2) holds and the Hamiltonian takes the split form:

$$\begin{aligned} H(y,p) &:= \min_{b\in B}\max_{a\in A}\{-f(y,a,b)\cdot p - l(y,a,b)\} \\ &= \max_{a\in A}\{-f_A(y,a)\cdot p^A\} + \min_{b\in B}\{-f_B(y,b)\cdot p^B\} - l(y), \quad p=(p^A,p^B). \end{aligned}$$

Assume further that $A = B$, $m_A = m_B = m/2$, and $f_A = f_B =: g$, so the system takes the form (4). Then, if we define the reduced Hamiltonian

$$H_r(y,q) := \max_{a\in A}\{-g(y,a)\cdot q\}, \quad q \in \mathbb{R}^{m/2},$$

the Hamiltonian H becomes

$$H(y,p) = H_r(y,p^A) - H_r(y,-p^B) - l(y), \quad p=(p^A,p^B). \tag{18}$$

We will also take the running cost of the form

$$l(y) = n(y^A - y^B) + h(y^A,y^B), \quad y=(y^A,y^B), \tag{19}$$

and make assumptions of the functions $n : \mathbb{R}^{m/2} \to \mathbb{R}$ and $h : \mathbb{R}^m \to \mathbb{R}$.

Theorem 3.1. *Assume the Hamiltonian H has the form* (18) *with running cost of the form* (19) *and n, h bounded and uniformly continuous. If*

$$\sup h - \inf h < \sup n - \inf n, \tag{20}$$

then the lower and the upper game are not ergodic.

Proof. We explain first the idea in the special case $h \equiv 0$, $n \in C^1$. In this case, $u(t,y) := t\, n(y^A - y^B)$ solves the Hamilton-Jacobi-Isaacs equation:

$$u_t + H_r(y, D_{y^A}u) - H_r(y, -D_{y^B}u) = n(y^A - y^B).$$

If $v(t,y)$ is the value function of the finite horizon game, then $tv(t,y)$ solves the partial differential equation in viscosity sense [20,9] and takes the same initial value 0 at $t=0$. By the uniqueness of the viscosity solution to the Cauchy problem

[9], $tv(t,y) = u(t,y)$. Then $v(t,y) = n(y^A - y^B)$ does not converge to a constant as $t \to \infty$ because n is not constant.

The general case is a perturbation of the preceding one. Take a mollification $n^\varepsilon \in C^1$ of n such that $n^\varepsilon \to n$ as $\varepsilon \to 0$ uniformly in $\mathbb{R}^{m/2}$. Consider $u(t,y) := t\, n^\varepsilon(y^A - y^B) + tc$, for a constant c to be determined. Then:

$$u_t + H_r(y, D_{y^A} u) - H_r(y, -D_{y^B} u) = n^\varepsilon(y^A - y^B) + c \tag{21}$$

and the right-hand side is $\geq n(y^A - y^B) + h(y)$ for $c = \sup h + \delta$, $\delta > 0$, if ε is small enough. Therefore the comparison principle between viscosity sub- and supersolutions [9] gives:

$$v(t,y) \leq n^\varepsilon(y^A - y^B) + c, \quad \forall\, t, y,$$

and for y_1 such that $n(y_1^A - y_1^B)$ is close to $\inf n$ and ε small enough

$$v(t, y_1) \leq \inf n + \sup h + 2\delta. \tag{22}$$

On the other hand, the right-hand side of (21) is $\leq n(y^A - y^B) + h(y)$ for $c = \inf h - \delta$ and ε small enough. Then:

$$v(t,y) \geq n^\varepsilon(y^A - y^B) + c, \quad \forall\, t, y$$

and

$$v(t, y_2) \geq \sup n + \inf h - 2\delta, \tag{23}$$

if $n(y_2^A - y_2^B)$ is close to $\sup n$ and ε is small enough. By condition (20) we can choose δ so that the right-hand side of (22) is smaller than the right-hand side of (23). Then $v(t,y)$ cannot converge to a constant as $t \to \infty$. □

4 An example: the convex-concave eikonal equation

In this section we fix $g : \mathbb{R}^m \to \mathbb{R}$ Lipschitzean and such that $g(y) \geq g_o > 0$, and discuss the ergodicity of the games where the system is

$$\begin{cases} \dot{y}^A(t) = g(y(t))a(t), & y^A(0) = x^A \in \mathbb{R}^{m/2}, \quad |a(t)| \leq 1, \\ \dot{y}^B(t) = g(y(t))b(t), & y^B(0) = x^B \in \mathbb{R}^{m/2}, \quad |b(t)| \leq \gamma, \\ y(t) = (y^A(t), y^B(t)), & \end{cases} \tag{24}$$

for all values of the parameter $\gamma > 0$. For a running cost l independent of the controls, the finite horizon game has a value $\mathrm{val}(t,x) := l - \mathrm{val}\ J(t,x) = u - \mathrm{val}\ J(t,x)$, and $u(t,x) = t\ \mathrm{val}(t,x)$ solves the Hamilton-Jacobi-Isaacs equation

$$u_t + g(x)|D_{x^A} u| - \gamma g(x)|D_{x^B} u| = l(x), \quad u(0,x) = 0, \tag{25}$$

that we call the convex-concave eikonal equation. As in the preceding section we take l of the form

$$l(y) = n(y^A - y^B) + h(y^A, y^B), \quad y = (y^A, y^B).$$

We also need a compact state space, so we assume for simplicity that all data g, n, h are $\mathbb{Z}^m$ periodic. We recall that the paper by Alvarez and the author [3] covers only the case that $n \equiv 0$ and h has a saddle point, and then the value of the LTAC game is $\lambda = \min_{y^A \in \mathbb{R}^{m/2}} \max_{y^B \in \mathbb{R}^{m/2}} h(y^A, y^B) = \max_{y^B \in \mathbb{R}^{m/2}} \min_{y^A \in \mathbb{R}^{m/2}} h(y^A, y^B)$.

Corollary 4.1. *Under the preceding assumptions, the upper, lower, and LTAC game are ergodic under either one of the following conditions:*

i) $\gamma < 1$ *and* $h = h(y^B)$ *is independent of* y^A*, and in this case*

$$\lim_{t \to \infty} v(t, x) = \min n + \max h;$$

ii) $\gamma > 1$ *and* $h = h(y^A)$ *is independent of* y^B*, and in this case*

$$\lim_{t \to \infty} v(t, x) = \max n + \min h.$$

If, instead, $\gamma = 1$ *and*

$$\sup h - \inf h < \sup n - \inf n,$$

then the upper and lower game are not ergodic.

Proof. If $\gamma < 1$, since the dynamics of the y^A and the y^B variables is the same, but the first player can drive y^A at higher speed, for any fixed $z \in \mathbb{R}^{m/2}$ the first player can drive the system from any initial position to $y^A = y^B + z$ in finite time for all controls of the second player. Since $\mathbb{T}^m$ is compact this can be done in a uniformly bounded time. In particular, the system is asymptotically controllable by the first player to the set

$$\mathcal{T}_n := \left\{ (y^A, y^B) \in \mathbb{R}^m \,:\, y^A - y^B \in \operatorname{argmin} n \right\}.$$

If $h \equiv 0$ we can conclude by Proposition 2.1. Note that in this case we do not need the controllability of y^B by the second player.

In the general case, we observe that $\mathcal{T}_n$ is a subset of the target $\mathcal{T}^*$ defined by (12), because here:

$$\max_{y^B \in \mathbb{R}^{m/2}} \min_{y^A \in \mathbb{R}^{m/2}} l(y^A, y^B) = \min n + \max_{y^B \in \mathbb{R}^{m/2}} h(y^B).$$

Therefore the system is asymptotically controllable to $\mathcal{T}^*$ by the first player.

On the other hand, the variables y^B are bounded time controllable by the second player to any point of $\mathbb{R}^{m/2}$; therefore they are also asymptotically controllable to $\mathcal{T}_B$. Then Proposition 2.2 gives the conclusion *i)*.

The statement *ii)* is proved in the same way by reversing the roles of the two players. In this case:

$$\min_{y^A\in\mathbb{R}^{m/2}} \max_{y^B\in\mathbb{R}^{m/2}} l(y^A, y^B) = \max n + \min_{y^A\in\mathbb{R}^{m/2}} h(y^A).$$

Finally, the case $\gamma = 1$ follows immediately from Theorem 3.1. □

REFERENCES

[1] O. Alvarez and M. Bardi, *Viscosity solutions methods for singular perturbations in deterministic and stochastic control*, SIAM J. Control Optim. **40** (2001), 1159–1188.

[2] O. Alvarez and M. Bardi, *Singular perturbations of degenerate parabolic PDEs: a general convergence result*, Arch. Rational Mech. Anal. **170** (2003), 17–61.

[3] O. Alvarez and M. Bardi, *Ergodic problems in differential games*, in "Advances in Dynamic Game Theory", S. Jorgensen, M. Quincampoix, and T.L. Vincent, eds., pp. 131–152, Ann. Internat. Soc. Dynam. Games, vol. 9, Birkhäuser, Boston, 2007.

[4] O. Alvarez and M. Bardi, *Ergodicity, stabilization, and singular perturbations of Bellman-Isaacs equations*, Preprint, University of Padova, 2007, available at http://cpde.iac.rm.cnr.it/ricerca.php.

[5] O. Alvarez, M. Bardi, and C. Marchi, *Multiscale problems and homogenization for second-order Hamilton-Jacobi equations*, J. Diff. Eq. (2007), doi:10.1016/j.jde.2007.05.027.

[6] M. Arisawa, *Ergodic problem for the Hamilton-Jacobi-Bellman equation II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **15** (1998), 1–24.

[7] M. Arisawa and P.-L. Lions, *On ergodic stochastic control*, Comm. Partial Diff. Eq. **23** (1998), 2187–2217.

[8] Z. Artstein and V. Gaitsgory, *The value function of singularly perturbed control systems*, Appl. Math. Optim. **41** (2000), 425–445.

[9] M. Bardi and I. Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, Boston, 1997.

[10] M. Bardi and A. Cesaroni, Almost sure properties of controlled diffusions and worst case properties of deterministic systems, ESAIM Control Optim. Calc. Var., to appear, available at http://cpde.iac.rm.cnr.it/ricerca.php.

[11] M. Bardi and P. Goatin, *Invariant sets for controlled degenerate diffusions: a viscosity solutions approach*, in "Stochastic Analysis, Control, Optimization

and Applications: A Volume in Honor of W.H. Fleming", W.M. McEneaney, G.G. Yin, and Q. Zhang, eds., pp. 191–208, Birkhäuser, Boston, 1999.

[12] M. Bardi and P. Soravia, Hamilton-Jacobi equations with singular boundary conditions on a free boundary and applications to differential games. Trans. Amer. Math. Soc. **325** (1991), 205–229.

[13] T. Başar and P. Bernhard, H^∞*-optimal control and related minimax design problems. A dynamic game approach*, 2nd edition, Birkhäuser, Boston, 1995.

[14] A. Bensoussan, *Perturbation methods in optimal control*, Wiley/Gauthiers-Villars, Chichester, 1988.

[15] P. Cardaliaguet, *A differential game with two players and one target*, SIAM J. Control Optim. **34** (1996), 1441–1460.

[16] D.A. Carlson and A.B. Haurie, *A turnpike theory for infinite horizon open-loop differential games with decoupled controls*, in "New Trends in Dynamic Games and Applications", G.J. Olsder, ed., pp .353–376, Ann. Internat. Soc. Dynam. Games, vol. 3, Birkhäuser Boston, 1995.

[17] D.A. Carlson, A.B. Haurie, and A. Leizarowitz, *Infinite horizon optimal control: Deterministic and stochastic systems*, Springer-Verlag, Berlin, 1991.

[18] R.J. Elliott and N.J. Kalton, *The existence of value in differential games*, Memoirs American Math. Soc., vol. 126. American Mathematical Society, Providence, RI, 1972.

[19] L. Evans, *Periodic homogenisation of certain fully nonlinear partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **120** (1992), 245–265.

[20] L. Evans and P.E. Souganidis, *Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations*, Indiana Univ. Math. J. **33** (1984), 773–797.

[21] W.H. Fleming and W.M. McEneaney, *Risk-sensitive control on an infinite time horizon*, SIAM J. Control Optim. **33** (1995), 1881–1915.

[22] R.A. Freeman and P.V. Kokotovic, *Robust nonlinear control design. State-space and Lyapunov techniques*, Birkäuser, Boston, 1996.

[23] V. Gaitsgory, *Limit Hamilton-Jacobi-Isaacs equations for singularly perturbed zero-sum differential games*, J. Math. Anal. Appl. **202** (1996), 862–899.

[24] M.K. Ghosh and K.S.M. Rao, *Differential Games with Ergodic Payoff* SIAM J. Control Optim. **43** (2005), 2020–2035.

[25] L. Grüne, *On the relation between discounted and average optimal value functions*, J. Diff. Eq. **148** (1998), 65–99.

[26] Y. Kabanov and S. Pergamenshchikov, *Two-scale stochastic systems. Asymptotic analysis and control*, Springer-Verlag, Berlin, 2003.

[27] P.V. Kokotović, H.K. Khalil, and J. O'Reilly, *Singular perturbation methods in control: analysis and design*, Academic Press, London, 1986.

[28] H.J. Kushner, *Weak convergence methods and singularly perturbed stochastic control and filtering problems*, Birkhäuser, Boston, 1990.

[29] H.J. Kushner, *Numerical approximations for stochastic differential games: the ergodic case*, SIAM J. Control Optim. **42** (2004), 1911–1933.

[30] P.-L. Lions, *Neumann type boundary conditions for Hamilton-Jacobi equations*, Duke Math. J. **52** (1985), 793–820.

[31] P.-L. Lions, G. Papanicolaou, and S.R.S. Varadhan, *Homogenization of Hamilton-Jacobi equations*, Unpublished, 1986.

[32] M. Quincampoix and F. Watbled, *Averaging method for discontinuous Mayer's problem of singularly perturbed control systems*, Nonlinear Anal. **54** (2003), 819–837.

[33] P. Soravia, *Pursuit-evasion problems and viscosity solutions of Isaacs equations*, SIAM J. Control Optim. **31** (1993), 604–623.

[34] P. Soravia, *Stability of dynamical systems with competitive controls: the degenerate case*, J. Math. Anal. Appl. **191** (1995), 428–449.

[35] S. Sorin, *New approaches and recent advances in two-person zero-sum repeated games*, in "Advances in Dynamic Games", A.S. Nowak and K. Szajowski, eds., pp. 67–93, Ann. Internat. Soc. Dynam. Games, vol. 7, Birkhäuser, Boston, 2005.

[36] N.N. Subbotina, *Asymptotics for singularly perturbed differential games*, in "Game Theory and Applications", vol. VII, pp. 175–196, Nova Science Publishers, Huntington, NY, 2001.