

## GENERALIZED HESSIANS OF $C^{1,1}$ -FUNCTIONS AND SECOND-ORDER VISCOSITY SUBJETS\*

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**Abstract.** Given a  $C^{1,1}$ -function  $f : U \rightarrow R$  (where  $U \subset R^n$  open), we deal with the question of whether or not at a given point  $x_0 \in U$  there exists a local minorant  $\varphi$  of  $f$  of class  $C^2$  that satisfies  $\varphi(x_0) = f(x_0)$ ,  $D\varphi(x_0) = Df(x_0)$ , and  $D^2\varphi(x_0) \in \mathcal{H}f(x_0)$  (the generalized Hessian of  $f$  at  $x_0$ ). This question is motivated by the second-order viscosity theory of the PDEs, since for nonsmooth functions, an analogous result between subgradients and first-order viscosity subsets is known to hold in every separable Asplund space. In this work we show that the aforementioned second-order result holds true whenever  $\mathcal{H}f(x_0)$  has a minimum with respect to the positive semidefinite cone (thus, in particular, in one dimension), but it fails in two dimensions even for piecewise polynomial functions. We extend this result by introducing a new notion of directional minimum of  $\mathcal{H}f(x_0)$ .

**Key words.** nonsmooth analysis, generalized Hessian, second-order viscosity subset

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**1. Introduction and examples.** Rademacher's theorem asserts that every Lipschitz continuous function  $f : U \rightarrow R^k$ , where  $U$  is a nonempty open subset of  $R^n$ , is almost everywhere (a.e.) differentiable with respect to the Lebesgue measure, thus, in particular, the set  $\text{dom } Df$  of points where the derivative of  $f$  exists is dense in  $U$ . Given any  $x_0 \in U$ , one of the primary notions of standard variational analysis is the notion of *generalized Jacobian*  $Jf(x_0)$  (respectively, *subdifferential*  $\partial f(x_0)$ , if  $k = 1$ , i.e.,  $f : U \rightarrow R$ ) defined as follows:

$$(1) \quad Jf(x_0) := \left\{ q \in R^{k \times n}, q = \lim_{x_n \rightarrow x_0} Df(x_n), x_n \in \text{dom } Df \right\};$$

that is,  $Jf(x_0)$  is the set of all possible limits of derivatives of  $f$  (identified to their Jacobian matrix) at points in  $\text{dom } Df$  converging to  $x_0$  (see, for example, [3], [8], [9], [11]). Note that the fact that the derivatives exist in a dense set together with the boundedness of their norms (by the Lipschitz constant of  $f$ ) guarantee that the generalized Jacobian  $Jf(x_0)$  is always nonempty and compact. Another standard notion in the literature is the notion of the *Clarke Jacobian* [3, section 2.6], denoted by  $J^\circ f(x_0)$  (respectively, *Clarke subdifferential*  $\partial^\circ f(x_0)$ ), which is defined as the convex hull of  $Jf(x_0)$ .

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In the case of real-valued functions, the Clarke subdifferential is known to contain the so-called *Fréchet* (or *regular* [16]) subdifferential

$$(2) \quad \hat{\partial}f(x_0) := \left\{ p \in R^n : \liminf_{x \rightarrow x_0, x \neq x_0} \frac{f(x) - f(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \geq 0 \right\}.$$

The above subdifferential  $\hat{\partial}f(x_0)$  is sometimes called a *viscosity* subdifferential, because in every separable Asplund space (more generally, in every Banach space admitting a Fréchet differentiable renorming) the elements of  $\hat{\partial}f(x_0)$  can be identified as the *first-order viscosity subjects* in the following sense (see [2], [7], [10], and the references therein).

$p \in \hat{\partial}f(x_0)$  if and only if there exists a  $C^1$  local minorant  $\varphi$  of  $f$  satisfying

$$(3) \quad \begin{cases} \varphi(x_0) = f(x_0), \\ D\varphi(x_0) = p. \end{cases}$$

In the above formula, since  $x_0$  is a local minimizer of the (nonsmooth) function  $f - \varphi$ , the second line of (3) actually interprets the first-order necessary optimality condition  $0 \in \hat{\partial}(f - \varphi)(x_0)$ . At the same time, formula (3) relates subdifferential theory to the notion of first-order viscosity supersolutions of partial differential equations (see, e.g., [4]). During the last two decades the second-order viscosity theory has been developed to cover fully nonlinear partial differential equations (see [5] for details). A natural question is thus arising, namely whether or not the previous correspondence can be extended to a second-order setting. This is one of the main motivations of this work.

In what follows,  $U \subset R^n$  is open and  $f : U \rightarrow R$  denotes a  $C^{1,1}$  function (i.e.,  $f$  is everywhere differentiable and  $Df : U \rightarrow R^n$  is locally Lipschitz continuous). Then formula (1) applied to the derivative function  $Df$  gives the so-called *generalized Hessian* of  $f$  at  $x_0$

$$(4) \quad \mathcal{H}f(x_0) := \left\{ A \in R^{n \times n} : A = \lim_{x_n \rightarrow x_0} D^2f(x_n), x_n \in \text{dom } D^2f \right\},$$

and its convex envelope defines the *Clarke Hessian*  $\mathcal{H}^o f(x_0)$ . In the literature of nonsmooth analysis there are many works on this topic; see, for example, [8], [11], [15], and the references therein. See also [14], [18], [19] for applications to positive semidefinite optimization, as well as [12], [13] for some generalizations. In this setting, a second-order result—analogue to (3)—reads as follows:

(Q) Does there exist a local minorant  $\varphi$  of  $f$  around  $x_0$  of class  $C^2$  satisfying

$$(i) \quad \varphi(x_0) = f(x_0), \quad (ii) \quad D\varphi(x_0) = Df(x_0), \quad (iii) \quad D^2\varphi(x_0) \in \mathcal{H}f(x_0) ?$$

Since  $x_0$  is a local minimizer of the function  $f - \varphi$ , a relaxation of (iii) to the condition

$$(iv) \quad \text{for some } A \in \mathcal{H}f(x_0) \text{ we have } A \succeq D^2\varphi(x_0) \quad (\text{cone relaxation})$$

would correspond to a second-order necessary optimality condition for  $f - \varphi$ . (As usual, the notation  $A \succeq D^2\varphi(x_0)$  means that the (symmetric) matrix  $A - D^2\varphi(x_0)$  has nonnegative eigenvalues.) This weaker version of (Q) is always true (Remark 10).

However,  $(\mathcal{Q})$  is much more demanding, since it asserts equality:  $A = D^2\varphi(x_0)$ .

Besides being natural in nonsmooth analysis,  $(\mathcal{Q})$  has an interesting motivation coming from the theory of partial differential equations. Consider a fully nonlinear second-order (degenerate) elliptic PDE

$$(5) \quad F(x, f(x), Df(x), D^2f(x)) = 0,$$

where  $f : U \rightarrow R$  ( $U \subset R^n$  open) and where elliptic means that  $F(x, r, p, M_1) \leq F(x, r, p, M_2)$  for any  $n \times n$  symmetric matrices  $M_1, M_2$  satisfying  $M_1 \succeq M_2$ . It is natural to say that a  $C^{1,1}$  function  $f$  solves (5) if the equation is satisfied a.e., namely, at all points  $x$  where the second derivative  $D^2f(x)$  exists. However, in general, this concept of weak solution guarantees neither existence nor uniqueness of solutions for the most common boundary value problems encompassed into (5). In order to obtain well-posedness one needs another concept of weak solution, namely the Crandall–Lions notion of *viscosity solution*; see [5]. In the viscosity sense, one defines a solution of (5) by means of two inequalities. If, for instance,  $f \in C^{1,1}$ , then  $f$  is said to be a viscosity supersolution if for any twice differentiable local minorant  $\varphi \leq f$  satisfying  $\varphi(x_0) = f(x_0)$  we have

$$(6) \quad F(x_0, f(x_0), Df(x_0), D^2\varphi(x_0)) \geq 0.$$

It turns out that (5) will also be satisfied by a viscosity solution at all points where the latter is twice differentiable. However, when  $f \in C^{1,1}$ , the inequality (6) has to be checked at every point of the domain since there is always a local minorant  $\varphi$  of  $f$  satisfying  $\varphi(x_0) = f(x_0)$ . Notice that by ellipticity, (6) may be strict for many local minorants. A natural question is then the following: when  $f$  is a viscosity solution of (5), can the inequality (6) be strict at some point  $x_0$  for all local minorants  $\varphi$  satisfying  $\varphi(x_0) = f(x_0)$  (strict supersolution at  $x_0$ )? An indirect way to answer negatively to this question is by solving  $(\mathcal{Q})$ . Indeed, we always have

$$F(x_0, f(x_0), Df(x_0), M) = 0$$

for all  $M \in \mathcal{H}f(x_0)$  ( $F$  here is a continuous map). When  $F$  is the Hamiltonian of a stochastic optimal control problem, the equality in the differential operator for some  $M = D^2\varphi(x_0)$ ,  $\varphi$  local minorant satisfying  $\varphi(x_0) = f(x_0)$ , is also the crucial starting point to build an optimal feedback for the control problem when the value function is nonsmooth. This is an important, long-standing open problem.

The above question  $(\mathcal{Q})$  has a positive answer whenever the generalized Hessian has a minimum element (Corollary 4); that is, whenever there exists  $A_* \in \mathcal{H}f(x_0)$  such that for all  $A \in \mathcal{H}f(x_0)$  we have  $A \succeq A_*$ . In particular, notice that this is always the case in dimension one (Corollary 5). Nevertheless, the element of  $\mathcal{H}f(x_0)$  that satisfies  $(\mathcal{Q})$  might not be unique (Example 6). In  $R^n$  the existence of a minimum for the generalized Hessian is a rather restrictive condition. Still, it can be guaranteed in some particular cases (cf. Theorem 17 or section 4).

Our sufficient condition is then extended to the case where  $\mathcal{H}f(x_0)$  admits a *directional minimum* (Theorem 3), a notion that we make precise in section 2. In what follows we show that directional minima may not exist (Example 7) or may not be unique (Example 8), while under an additional regularity assumption, they characterize the elements of  $\mathcal{H}f(x_0)$  for which  $(\mathcal{Q})$  has a positive answer (Theorem 23).

However,  $(\mathcal{Q})$  has a negative answer in general: we provide an example of a  $C^{1,1}$  function in  $R^2$  for which  $(\mathcal{Q})$  fails (Example 9), even if condition (iii) is relaxed as

follows:

$$(iii)' \quad D^2\varphi(x_0) \in \mathcal{H}^o f(x_0) \quad (\text{Clarke relaxation}).$$

At this stage, let us observe that a stronger statement asserting that all generalized Hessians can be represented as second derivatives of such minorants would not have any chance to hold. To see this, let us consider the following simple one-dimensional example:

$$(7) \quad f(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 0, \\ x^2 & \text{if } x \leq 0. \end{cases}$$

Indeed, one easily sees that  $\mathcal{H}^o f(0) = [1, 2]$  while any  $C^2$  minorant  $\varphi$  of  $f$  with  $\varphi(0) = f(0) = 0$  and  $\varphi'(0) = f'(0) = 0$  should necessarily satisfy  $\varphi''(0) \leq 1$ . This example seems to indicate that in the one-dimensional case—where (Q) has a positive answer—the only element of the Clarke Hessian with this property is its minimum. Even if this comes about to hold for most functions we meet in practice, an easy example reveals that this assertion is not true in general (Example 6). Example 7 reflects the same situation of nonuniqueness in  $n$ -dimensions but the generalized Hessian does not have directional minima. In Example 16, all matrices in the generalized Hessian are diagonal, there is no minimum element and (Q) is satisfied by a unique element.

*Notation.* In what follows, for any set  $K \subset R^m$ , we denote by  $\overline{K}$  (respectively,  $\text{int}K$ ) the topological closure (respectively, interior) of  $K$ . Further, given  $t \in R$  and  $a \in R^m$ , we shall use the abbreviate notation  $a + tK$  for the set  $\{a + tx : x \in K\}$ , and we denote by  $B(x, r)$  the closed ball centered at  $x \in R^n$  with radius  $r > 0$ . We denote by  $\|x\|$  the norm of  $x \in R^n$  and by  $\langle x, y \rangle$  the scalar product of  $x, y \in R^n$ . Let  $W^\top$  stand for the transpose of any  $n \times k$  matrix  $W$ . Under this notation, identifying the vector  $x \in R^n$  with an  $n \times 1$  matrix, we have  $x^\top \cdot x = \langle x, x \rangle = \|x\|^2 \in R$ , while  $x \cdot x^\top$  gives rise to a symmetric  $n \times n$  matrix.

Further, given a real-valued function  $f$  defined on some subset of  $R^m$ , we shall denote by  $\text{dom } f$  its domain and by  $Df(x)$  (respectively,  $D^2f(x)$ ) the first (respectively, second) derivative of the function  $f$  at a point  $x \in R^m$ , whenever they exist. We also denote by  $\text{dom } Df$  and  $\text{dom } D^2f$  the domains of  $Df$  and  $D^2f$ , respectively. In case  $m = 1$ , the simplified notation  $f'(x)$  (respectively,  $f''(x)$ ) will be in use. If the domain of  $f$  is a closed interval  $[a, b]$  of  $R$  we say that  $f$  is (twice) differentiable at  $x = a$  if the (second) right derivative exists, and we keep the same notation  $f'(a)$  (respectively,  $f''(a)$ ) to denote the right derivatives.

Further, let  $S^n$  denote the set of symmetric matrices. This set is partially ordered by the cone  $S_+^n$  of positive semidefinite matrices: for  $A, B \in S^n$  we write  $A \succeq B$  whenever  $A - B \in S_+^n$ . Given  $K \subset S^n$ , if there exists  $A_* \in K$  with the property  $K \subset A_* + S_+^n$ , then  $A_*$  is the minimum of  $K$  with respect to the aforementioned order and will be denoted by  $\min K$  (note that  $\min K$  might not exist).

**2. Main results.** We consider the set of local  $C^2$  minorants of  $f$  at  $x_0$  coinciding with  $f$  at zero (and first) order at  $x_0$ ,

$$(8) \quad \mathcal{M}_f(x_0) = \{ \varphi \in C^2(V) : \varphi \leq f, \varphi(x_0) = f(x_0) \},$$

where  $V$  is some open neighborhood of  $x_0$ . Note that  $\varphi \in \mathcal{M}_f(x_0)$  implies  $D\varphi(x_0) = Df(x_0)$ . According to [5], the set of *second-order subjects* of  $f$  at  $x_0$  is defined by

$$\mathcal{J}_f(x_0) = \{ (D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in \mathcal{M}_f(x_0) \} \subset R^n \times S^n.$$

In our analysis we rather use the projection of the above set onto  $S^n$ , that is,

$$(9) \quad \mathcal{J}_f^2(x_0) = \{ D^2\varphi(x_0) : \varphi \in \mathcal{M}_f(x_0) \}.$$

With this notation, (Q) can be reformulated as follows:

(Q) Is it true that for any  $C^{1,1}$  function  $f$  and any  $x_0 \in \text{dom } f$

$$\mathcal{H}f(x_0) \cap \mathcal{J}_f^2(x_0) \neq \emptyset?$$

Note that (Q) obviously yields that the set  $\mathcal{H}^o f(x_0) \cap \mathcal{J}_f^2(x_0)$  is nonempty. Notice that thanks to the convexity of  $\mathcal{H}^o f(x_0)$  and  $\mathcal{J}_f^2(x_0)$ , this set is always convex.

We first deal with a positive result to the problem (Q) in the case that the generalized Hessian of  $f$  at  $x_0$  has *directional minimum*. Let us define formally this notion, for any  $C^{1,1}$  function  $f : R^n \rightarrow R$ . Given  $X \in \mathcal{H}f(x_0)$  we denote

$$\mathcal{T}_{f,x_0}(X) = \left\{ v \in R^n : \exists \{x_n\}_n \subset \text{dom } D^2f, x_n \rightarrow x_0, \right. \\ \left. D^2f(x_n) \rightarrow X, \frac{x_n - x_0}{\|x_n - x_0\|} \rightarrow v \right\}.$$

DEFINITION 1 (directional minimum element of a generalized Hessian). *We say that  $M \in \mathcal{H}f(x_0)$  is a directional minimum of  $\mathcal{H}f(x_0)$  if for all  $X \in \mathcal{H}f(x_0)$  and all  $v \in \mathcal{T}_{f,x_0}(X)$  we have*

$$\langle Mv, v \rangle \leq \langle Xv, v \rangle.$$

Obviously, a minimum element of  $\mathcal{H}f(x_0)$  (with respect to the cone  $S_+^n$  of the positive semidefinite matrices) is necessarily a directional minimum, since the above inequality should then hold for all  $v \in R^n$ , and not only for those in  $\mathcal{T}_{f,x_0}(X)$ . Thus the notion of directional minimum is weaker. Notice that since  $f \in C^{1,1}$ , for all unit vectors  $v$  we have

$$K_{f,x_0}(v) := \mathcal{T}_{f,x_0}^{-1}(v) = \{X \in \mathcal{H}f(x_0) : v \in \mathcal{T}_{f,x_0}(X)\} \neq \emptyset.$$

In particular,  $M$  is a directional minimum of  $\mathcal{H}f(x_0)$  if and only if for all  $v \in R^n$  and  $X \in K_{f,x_0}(v)$  we have

$$\langle Mv, v \rangle \leq \langle Xv, v \rangle.$$

We better specify how to construct elements in  $K_{f,x_0}(v)$ , using the following notation: for a unit vector  $v \in R^n$  define

$$C_v = \{y \in R^n : \langle y, v \rangle > 0, \langle y, v \rangle^2 > \|y - \langle y, v \rangle v\|^2\}.$$

Notice that  $C_v$  is a nonempty open set satisfying  $0 \in \overline{C_v} \setminus C_v$  and  $tv \in C_v$  for all  $t > 0$ .

LEMMA 2 (convergence within a prescribed tangent). *If  $\{x_n\}_n \subset C_v$  is such that  $x_n \rightarrow 0$ , then  $x_n/\|x_n\| \rightarrow v$ .*

*Proof.* Let  $\{x_n\}_n$  be a sequence as in the statement, then by definition of  $C_v$  if we set

$$r_n = \left\langle \frac{x_n}{\|x_n\|}, v \right\rangle \in [0, 1],$$

we obtain

$$1 \geq r_n^2 > \frac{1}{\|x_n\|} \left\| \frac{x_n}{\|x_n\|} - \left\langle \frac{x_n}{\|x_n\|}, v \right\rangle v \right\| \geq \frac{1}{\|x_n\|} (1 - r_n) \geq 0.$$

As  $n \rightarrow +\infty$  we get the conclusion.  $\square$

In what follows we define for all  $x_0 \in R^n$ ,  $r > 0$ , and  $v \in R^n$  with  $\|v\| = 1$  the set

$$C_{x_0,v}^r = (x_0 + C_v) \cap B(x_0, r).$$

Our main sufficient condition for solving problem (Q) is contained in the following result.

**THEOREM 3** (Hessian with directional minimum element). *Let  $U$  be an open subset of  $R^n$ ,  $x_0 \in U$ , and  $f : U \rightarrow R$  be a  $C^{1,1}$  function.*

(i) *If a matrix  $M \in S^n$  satisfies  $\langle Mv, v \rangle \leq \langle Xv, v \rangle$  for all  $X \in \mathcal{H}f(x_0)$  and  $v \in \mathcal{T}_{f,x_0}(X)$ , then there exists  $\varphi \in \mathcal{M}_f(x_0)$  such that  $M = D^2\varphi(x_0)$ .*

(ii) *If, in addition,  $M \in \mathcal{H}f(x_0)$  (i.e.,  $M$  is a directional minimum of  $\mathcal{H}f(x_0)$ ), then (Q) holds.*

*Proof.* There is no loss of generality to assume that  $U$  is convex and that  $f(x_0) = 0$  and  $Df(x_0) = 0$ . Consider now some  $r_0 > 0$  such that  $Df$  is Lipschitz continuous on  $B(x_0, r_0) \subset U$ . Let  $M \in S^n$  be as in the statement and fix  $d \in R^n$  with  $\|d\| = 1$  and  $r \in [0, r_0]$ . We define

$$(10) \quad \varphi_d(r) := \begin{cases} \inf_{x \in C_{x_0,d}^r} \inf_{T \in \mathcal{H}f(x) \cup \{M\}} \langle Td, d \rangle & \text{if } r > 0, \\ \langle Md, d \rangle & \text{if } r = 0. \end{cases}$$

Note that  $\varphi_d$  is a real-valued nonincreasing function, so that

$$\varepsilon_d(r) := \varphi_d(0) - \varphi_d(r) \geq 0$$

defines a nondecreasing function which is in fact upper bounded by a constant independent of  $d$ . Using the upper semicontinuity of the multifunction  $x \mapsto \mathcal{H}f(x)$  at  $x_0$ , we deduce that for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\mathcal{H}f(B(x_0, \delta)) \subset \mathcal{H}f(x_0) + B(0, \varepsilon).$$

We now claim, shrinking  $\delta > 0$  if necessary, that the following stronger conclusion holds:

$$x \in C_{x_0,d}^\delta \implies \mathcal{H}f(x) \subset K_{f,x_0}(d) + B(0, \varepsilon).$$

Indeed, if this is not the case, we can find  $\bar{\varepsilon} > 0$  and a sequence  $\{x_n\}_n \subset C_{x_0,v}^{r_0}$  satisfying  $x_n \neq x_0$ ,  $x_n \rightarrow x_0$ , and  $(x_n - x_0)/\|x_n - x_0\| \rightarrow d$  such that there exist  $A_n \in \mathcal{H}f(x_n)$  with  $A_n \notin K_{f,x_0}(d) + B(0, \bar{\varepsilon})$ . By the definition of  $\mathcal{H}f(x_n)$  we can slightly modify the sequence  $\{x_n\}_n$  to a sequence  $\{y_n\}_n \subset C_{x_0,d}^{r_0} \cap \text{dom } D^2f$  so that  $y_n \rightarrow x_0$  and  $A_n = D^2f(y_n)$ . Taking a subsequence if necessary, we may assume  $A_n \rightarrow X \in S^n$ . Using the upper semicontinuity of  $x \mapsto \mathcal{H}f(x)$  we deduce  $X \in \mathcal{H}f(x_0)$ . The fact that  $(y_n - x_0)/\|y_n - x_0\| \rightarrow d$  now yields  $X \in K_{f,x_0}(d)$ , providing a contradiction.

For some  $x \in C_{x_0,d}^\delta$  and  $T \in \mathcal{H}f(x) \cup \{M\}$  (both depending on  $\varepsilon$  and  $d$ ) we have  $\varphi_d(\delta) \geq \langle Td, d \rangle - \varepsilon$ . Let us assume that  $T \in \mathcal{H}f(x)$  (the case  $T = M$  is simpler).

Since  $T$  can be written as  $T_0 + \varepsilon T_1$ , where  $T_0 \in K_{f,x_0}(d)$  and  $T_1 \in S^n \cap B(0,1)$ , we get  $\varphi_d(\delta) \geq \langle T_0 d, d \rangle + \varepsilon \langle T_1 d, d \rangle - \varepsilon \geq \langle M d, d \rangle - 2\varepsilon$ , i.e.,  $\varepsilon_d(\delta) \leq 2\varepsilon$ . We conclude that the function

$$(11) \quad \varepsilon(r) := \sup_{\|d\|=1} \varepsilon_d(r)$$

(which is real-valued and nondecreasing) satisfies

$$(12) \quad \lim_{r \rightarrow 0} \varepsilon(r) = 0.$$

Consider the function

$$f_1(x) := f(x) - \frac{1}{2} \langle M(x - x_0), x - x_0 \rangle \quad \text{for } x \in U.$$

Note that  $f_1$  is  $C^{1,1}$ ,  $f_1(x_0) = 0$ , and  $Df_1(x_0) = 0$ . Further, fix any  $x \in U \setminus \{x_0\}$  (with  $[x_0, x] \subset U$ ) and apply the second-order expansion of  $f_1$  (cf. [8, Theorem 2.3]). Then for some  $z \in (x_0, x)$  and  $S \in \text{co } \mathcal{H}f_1(z)$  we have

$$(13) \quad f_1(x) - f_1(x_0) = \frac{1}{2} \langle S(x - x_0), x - x_0 \rangle = \frac{1}{2} \langle Sd, d \rangle \|x - x_0\|^2,$$

where  $d := (x - x_0)/\|x - x_0\|$ . Then there exist  $\{\lambda_i\}_{i=1}^k \subset [0, 1]$  with  $\sum_{i=1}^k \lambda_i = 1$  and  $\{T_i\}_{i=1}^k \subset \mathcal{H}f(z)$  such that

$$S = \sum_{i=1}^k \lambda_i (T_i - M).$$

Set  $r := \|x - x_0\| \geq \|z - x_0\|$ . From (10), (11), and (13) we deduce

$$\begin{aligned} f_1(x) - f_1(x_0) &= \frac{1}{2} \sum_{i=1}^k \lambda_i (\langle T_i d, d \rangle - \langle M d, d \rangle) \|x - x_0\|^2 \\ &\geq \frac{1}{2} (\varphi_d(r) - \varphi_d(0)) r^2 \geq -\varepsilon(r) r^2, \end{aligned}$$

which in view of Lemma 25 (see the appendix) yields

$$f_1(x) - f_1(x_0) \geq -\psi(\|x - x_0\|),$$

where  $\psi(0) = \psi'(0) = \psi''(0) = 0$  for some  $C^2$  function  $\psi$ , and whenever  $\|x - x_0\|$  is sufficiently small. Defining

$$x \mapsto \varphi(x) := \frac{1}{2} \langle M(x - x_0), x - x_0 \rangle - \psi(\|x - x_0\|)$$

we deduce that  $\varphi \in \mathcal{M}_f(x_0)$  and  $D^2\varphi(x_0) = M \in \mathcal{J}_f^2(x_0)$ .  $\square$

**COROLLARY 4** (Hessian with minimum element). *Let  $U$  be an open convex subset of  $\mathbb{R}^n$ ,  $x_0 \in U$ , and let  $f : U \rightarrow \mathbb{R}$  be a  $C^{1,1}$  function. If  $\mathcal{H}f(x_0)$  has a minimum element, then (Q) holds.*

In the one-dimensional case ( $n = 1$ ), since  $\mathcal{H}^o f(x_0) = \text{co } \mathcal{H}f(x_0)$  is a segment, the subset  $\mathcal{H}f(x_0)$  has a minimum; thus the previous result applies.

**COROLLARY 5** (one-dimensional case). *Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^{1,1}$  function and  $x_0 \in I$ . Then for some  $\varphi \in \mathcal{M}_f(x_0)$  we have*

$$\varphi''(x_0) = \min \mathcal{H}f(x_0).$$

The above corollary might lead to the erroneous conclusion that the only element of the Clarke Hessian that can be represented by a subset is its minimum (see also the example given by (7) in the introduction). In fact this is true for piecewise  $C^2$  functions (in this case for some  $\delta > 0$  the restrictions of  $f$  onto  $[x_0, x_0 + \delta]$  and  $(x_0 - \delta, x_0]$  are  $C^2$  and  $\mathcal{H}f(x_0)$  contains at most two elements) but fails for  $C^{1,1}$  functions with oscillating second derivatives, as the following example shows.

*Example 6* (nonuniqueness of solution of  $(\mathcal{Q})$  in  $R$ ). Let  $f : R \rightarrow R$  be the following  $C^{1,1}$  function:

$$(14) \quad f(x) = \int_0^x g(s)ds \quad \text{with} \quad g(s) = \begin{cases} s^2 \sin(\frac{1}{s}) & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

Note that  $f''(0) = 0$  (but  $f$  is not  $C^2$  around 0) and  $\mathcal{H}f(0) = [-1, 1]$ . Since  $f$  admits a second-order Taylor series around  $x = 0$ , we deduce using Lemma 25 that there exists  $\varphi \in \mathcal{M}_f(0)$  with  $\varphi''(0) = 0$ . It follows easily that

$$\mathcal{H}f(0) \cap \mathcal{J}_f^2(0) = [-1, 0],$$

showing that the element of  $\mathcal{H}f(x_0)$  that satisfies  $(\mathcal{Q})$  need not be unique. □

A slight modification of the above example provides an example in  $R^n$  which proves that the existence of a (directional) minimum element of the generalized Hessian is not a necessary condition for  $(\mathcal{Q})$  to hold.

*Example 7* ( $(\mathcal{Q})$  holds but  $\mathcal{H}f(x_0)$  has no directional minima). Let  $F : R^n \rightarrow R$  be the following function ( $n \geq 2$ ):

$$F(x) = f(\|x\|), \quad x \in R^n,$$

where  $f$  is given by (14). Then it can be easily verified that  $F$  is twice differentiable everywhere, with

$$D^2F(x) = \begin{cases} g'(\|x\|)uu^\top + (g(\|x\|)/\|x\|)(Id_n - uu^\top) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where  $u := x/\|x\|$ ,  $Id_n$  denotes the  $n \times n$  identity matrix, and  $u^\top$  stands for the transposed of the (column) vector  $u$ . It follows easily that  $F$  is  $C^{1,1}$  (but not  $C^2$  around 0) and

$$\mathcal{H}F(0) = \{tvv^\top : t \in [-1, 1], \|v\| = 1\}.$$

Moreover, for  $\|v\| = 1$  we have

$$K_{F,0}(u) = \{tuu^\top : t \in [-1, 1]\}.$$

We first remark that  $\mathcal{H}F(0)$  does not possess a directional minimum element. Indeed, whenever  $w$  is a unit vector normal to  $v$ , each element of  $\mathcal{H}F(0)$  of the form  $A = tvv^\top$  is never a minorant of  $B = -ww^\top \in K_{F,0}(w)$  in the direction of  $w$ .

Consider now some  $\varphi \in \mathcal{M}_f(0)$  with  $\varphi''(0) = 0$  (see Example 6); we define  $\Phi := (\varphi \circ \|\cdot\|)$  and we observe that  $\Phi$  is of class  $C^2$ , with

$$D^2\Phi(x) = \begin{cases} \varphi''(\|x\|)uu^\top + (\varphi'(\|x\|)/\|x\|)(Id_n - uu^\top) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



Thus  $\Phi \in \mathcal{M}_F(0)$  and  $D^2\Phi(0) = 0$ , yielding

$$0 \in \mathcal{H}F(0) \cap \mathcal{J}_F^2(0).$$

We also remark that the null matrix is not the unique solution. We can easily characterize all solutions of  $(\mathcal{Q})$ : a solution  $tvv^\top$  satisfies  $t \in [-1, 0]$ . Conversely, if  $t \in [-1, 0]$ , we set  $\Phi(x) = \varphi(\|x\|) + \frac{1}{2}t \langle v, x \rangle^2$ , and observe that  $D^2\Phi(0) = tvv^\top$ . Thus,

$$\mathcal{H}F(0) \cap \mathcal{J}_F^2(0) = \{tvv^\top : t \in [-1, 0], \|v\| = 1\}. \quad \square$$

We provide a further example to illustrate Theorem 3 and the fact that directional minima of  $\mathcal{H}f(x_0)$ , when they do exist, may not be unique.

*Example 8* (nonuniqueness of directional minima). Consider the following regions in the first quadrant of  $R^2$ :

$$\Omega_1 := \{(x_1, x_2) \in R^2 : x_1 > 0, x_2 \in (0, \frac{1}{2}x_1)\},$$

$$\Omega_2 := \{(x_1, x_2) \in R^2 : x_1 > 0, x_2 \in (\frac{1}{2}x_1, 2x_1)\},$$

$$\Omega_3 := \{(x_1, x_2) \in R^2 : x_1 > 0, x_2 \in (2x_1, +\infty)\}.$$

And let  $f : R^2 \rightarrow R$  be defined as follows:

$$(15) \quad f(x_1, x_2) = \begin{cases} \frac{3x_2^2}{2} & \text{if } (x_1, x_2) \in \text{cl}\Omega_1, \\ 2x_1x_2 - \frac{x_1^2 + x_2^2}{2} & \text{if } (x_1, x_2) \in \text{cl}\Omega_2, \\ \frac{3x_1^2}{2} & \text{if } (x_1, x_2) \in \text{cl}\Omega_3, \end{cases}$$

and  $f(x_1, x_2) = f(|x_1|, |x_2|)$  for  $(x_1, x_2) \in R^2$ . Then we obtain

$$\mathcal{H}f(0, 0) = \left\{ \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \right\}.$$

The first two matrices are bigger than the last two, which are not mutually comparable. Moreover, one easily checks that each of the two matrices

$$(16) \quad \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$$

is a directional minimum. Consequently both matrices are elements of  $\mathcal{H}f(0, 0) \cap \mathcal{J}_f^2(0, 0)$ . Further, notice that  $\varphi \equiv 0 \in \mathcal{M}_f(0, 0)$ , but  $0 = D^2\varphi(0, 0) \in \mathcal{J}_f^2(0, 0)$  is not comparable with either of the two elements in (16); although from the point of view of optimization  $\varphi$  is the most interesting test function since  $f$  has a minimum point at  $(0, 0)$ .  $\square$

We shall now present an example of a piecewise quadratic function in  $R^2$  (whose Hessian at a given point  $x_0$  has no directional minima and) for which  $(\mathcal{Q})$  has a negative answer.

*Example 9* (failure of  $(\mathcal{Q})$  for a  $C^{1,1}$ -function in  $R^2$ ). There exists a (piecewise quadratic)  $C^{1,1}$  function  $f : R^2 \rightarrow R$  such that for some  $\bar{x} \in R^2$ ,

$$(17) \quad \mathcal{H}^o f(\bar{x}) \cap \mathcal{J}_f^2(\bar{x}) = \emptyset.$$

*Construction of the counterexample.* Let us consider the following six open subsets of  $R^2$ :

$$\begin{aligned} \Omega_1 &:= \{(x_1, x_2) \in R^2 : x_1 > 0, x_2 \in (0, \frac{1}{2}x_1)\}, & \Omega_4 &:= (-\infty, 0) \times (0, +\infty), \\ \Omega_2 &:= \{(x_1, x_2) \in R^2 : x_1 > 0, x_2 \in (\frac{1}{2}x_1, 2x_1)\}, & \Omega_5 &:= (-\infty, 0) \times (-\infty, 0), \\ \Omega_3 &:= \{(x_1, x_2) \in R^2 : x_1 > 0, x_2 \in (2x_1, +\infty)\}, & \Omega_6 &:= (0, +\infty) \times (-\infty, 0). \end{aligned}$$

Observe that  $R^2 = \bigcup_{i=1}^6 \text{cl } \Omega_i$ , and let  $f : R^2 \rightarrow R$  be defined as follows:  
(18)

$$f(x_1, x_2) = \begin{cases} \frac{3x_2^2}{2} & \text{if } (x_1, x_2) \in \text{cl } \Omega_1, & | & \frac{x_1^2}{4} & \text{if } (x_1, x_2) \in \text{cl } \Omega_4, \\ 2x_1x_2 - \frac{x_1^2 + x_2^2}{2} & \text{if } (x_1, x_2) \in \text{cl } \Omega_2, & | & \frac{x_1^2 + x_2^2}{4} & \text{if } (x_1, x_2) \in \text{cl } \Omega_5, \\ \frac{3x_1^2}{2} & \text{if } (x_1, x_2) \in \text{cl } \Omega_3, & | & \frac{x_2^2}{4} & \text{if } (x_1, x_2) \in \text{cl } \Omega_6. \end{cases}$$

It is straightforward to see that  $f$  is well defined and continuous. On each of the six open subsets  $\Omega_i$  the function  $f$  coincides with a quadratic function. One can easily verify that  $f$  is differentiable at each  $(x_1, x_2) \in R^2$  and the derivative function  $Df$  is Lipschitz continuous of constant  $K := 3$  in  $R^2$  (the spectral radius of each Hessian matrix  $D^2f(x_1, x_2)$  is less than or equal to 3 for all  $(x_1, x_2) \in \bigcup_{i=1}^6 \Omega_i$ ). Let  $\bar{x} = (0, 0)$  and note that  $f(0, 0) = 0$  and  $Df(0, 0) = (0, 0)$ , while the generalized Hessian  $\mathcal{H}f(0, 0)$  consists of the following six elements (corresponding to each of the six quadratic forms in the definition of  $f$ ):

$$\mathcal{H}f(0, 0) := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix} \right\}.$$

Let us show that none of these matrices can be the second derivative of a minorant  $\varphi \in \mathcal{M}_f(0, 0)$ . Indeed, the five diagonal matrices can be easily excluded: if for some  $\varphi \in \mathcal{M}_f(0, 0)$  one has

$$D^2\varphi(0, 0) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where  $\lambda_1, \lambda_2 \in R_+$  and at least one of them is (strictly) positive, then since  $\varphi(0, 0) = f(0, 0) = 0$  and  $D\varphi(0, 0) = Df(0, 0) = (0, 0)$ , the second-order Taylor series of  $\varphi$  around  $(0, 0)$  would yield that  $f(x_1, x_2) \geq 2^{-1}(\lambda_1x_1^2 + \lambda_2x_2^2) + o(x_1, x_2)$ , where  $\|(x_1, x_2)\|^{-2}o(x_1, x_2) \rightarrow 0$ . A contradiction arises by considering points either of the form  $(t, 0)$  or of the form  $(0, t)$ , for  $|t|$  sufficiently small. The second matrix can also be excluded, since it would correspond to a minorant

$$\varphi(x_1, x_2) = 2x_1x_2 - 2^{-1}(x_1^2 + x_2^2) + o(x_1, x_2),$$

which would fail to be majorated by  $f$  on the line  $\{(t, t) \in R^2 : t < 0\}$ . For essentially the same reason, no convex combination of the matrices in  $\mathcal{H}f(0, 0)$  is representable by a subset. Thus (17) holds true and the construction is complete.

Notice that  $f \geq 0$  and it has a minimum at the origin; in particular,  $\varphi \equiv 0 \in \mathcal{M}_f(0, 0)$  so that the null matrix is in  $\mathcal{J}_f^2(0, 0)$  and it is a maximal element of this

set, but it is not an element of  $\mathcal{H}^o f(0,0)$ . Nonetheless,  $\mathcal{H}f(0,0)$  contains an element with strictly negative eigenvalue. Indeed the set  $\mathcal{H}f(0,0)$  seems to contain too much information from the point of view of nonsmooth analysis, as it contradicts standard second-order necessary conditions for local minima.  $\square$

Notice that the function  $f$  in the previous two examples is *semialgebraic*, that is, its graph is a semialgebraic set. We recall that a set  $A \subseteq R^l$  is called semialgebraic if it is defined via a finite Boolean combination of real polynomial equations and inequalities. The broad applicability of semialgebraic sets follows largely from the Tarski–Seidenberg principle, which asserts that the projection

$$(x_1, x_2, \dots, x_n) \mapsto (x_2, x_3, \dots, x_n)$$

preserves the semialgebraic property. This fact is the origin of the important stability properties of the semialgebraic sets and yields a variety of qualitative properties for semialgebraic functions, particularly useful and largely explored in variational analysis; see [1] for an example. Still, Example 9 reveals that this property is not helpful by its own to guarantee  $(\mathcal{Q})$ . As we shall see in the next section, the latter demands other restrictive properties of different nature. On the other hand, as already stated in the introduction, the cone relaxation (iv) of condition (iii) of  $(\mathcal{Q})$  always yields a positive answer. This is the aim of the following remark.

*Remark 10* (cone relaxation). Let  $f : U \subset R^n \rightarrow R$  be a  $C^{1,1}$  function and  $x_0 \in U$ . Then for any  $M \in \mathcal{J}_f^2(x_0)$  there exists  $A \in \mathcal{H}f(x_0)$  such that  $A \succeq M$ .

*Proof.* Let us recall that every  $C^{1,1}$ -function, being lower- $C^2$  (e.g., [6, Remark 3.3]), is semiconvex. Thus the above statement is a simple consequence of Jensen’s Lemma (see [5, Appendix]). Let us recall this argument for completeness. Fix any  $\varphi \in \mathcal{M}_f(x_0)$  and set  $M := D^2\varphi(x_0)$ . Then the function  $f - \varphi$  attains a local minimum at  $x_0$ . Moreover, modifying  $\varphi$  if necessary (but keeping invariant its second derivative at  $x_0$ ), we may assume that the above minimum is strict and we can construct two sequences  $\{x_n\}_n$  and  $\{p_n\}_n$  in  $R^n$  with  $\{x_n\} \rightarrow x_0$  such that  $f$  is twice differentiable at  $x_n$ , and such that the function

$$x \mapsto f_n(x) := f(x) - \varphi(x) - \langle p_n, x \rangle$$

has a local minimum at  $x_n$  for all  $n \in N$ . Applying the second-order optimality condition for the function  $f_n$  at  $x_n$  we deduce that  $D^2f(x_n) \succeq D^2\varphi(x_n)$  and passing to the limit (for a subsequence) we obtain

$$A := \lim_{x_n \rightarrow x_0} D^2f(x_n) \succeq D^2\varphi(x_0) = M.$$

The proof is complete.  $\square$

**3. The case of piecewise  $C^2$  functions.** Comparing with Theorem 3, Example 9 reveals an important difference in case the dimension of the space is more than one, namely the lack of a total ordering in the elements of  $\mathcal{H}f(x_0)$ . In particular, neither the minimum element  $\min \mathcal{H}f(x_0)$  nor the directional minima of  $\mathcal{H}f(x_0)$  necessarily exist, so there is no *prior* natural candidate to fulfill with  $(\mathcal{Q})$ . In what follows, we seek for conditions guaranteeing the existence of a minimum (thus of a second-order subset). The term hypersurface refers to a  $C^2$  submanifold of  $R^n$  of codimension 1.

**DEFINITION 11** ( $C^{1,1}$  functions that are piecewise  $C^2$ ). *Let  $U$  be an open subset of  $R^n$ . A  $C^{1,1}$  function  $f : U \rightarrow R$  is called piecewise  $C^2$  near  $x_0 \in U$  if for some  $r > 0$  with  $B(x_0, r) \subset U$  there exist finitely many hypersurfaces  $\{\Gamma_j\}_j$  such that*

- (i)  $x_0 \in \overline{\Gamma}_j$  for all  $j$ ;
- (ii)  $B(x_0, r) \setminus \cup_j \overline{\Gamma}_j$  has a finite number of connected components  $\Omega_1, \dots, \Omega_k$  with  $x_0 \in \overline{\Omega}_i$ ;
- (iii)  $f \in C^2(\Omega_i)$  and  $x \mapsto D^2f(x)$  is uniformly continuous on  $\Omega_i$  for all  $i \in \{1, \dots, k\}$ .

*Remark 12.* (a) Condition (ii) of Definition 11 is automatically fulfilled whenever the hypersurfaces  $\Gamma_j$  are analytic (or semialgebraic) manifolds.

(b) As a consequence of Definition 11, each function  $D^2f|_{\Omega_i}$  can be continuously extended to  $\overline{\Omega}_i$  in a unique manner, yielding that the generalized Hessian  $\mathcal{H}f(x_0)$  consists of the (not necessarily distinct) matrices  $\{A_1, \dots, A_k\}$ , each of which is obtained as a (unique) limit coming from each region  $\Omega_i$ , that is,

$$A_i := \lim_{\substack{x \rightarrow x_0 \\ x \in \Omega_i}} D^2f(x) \quad \text{for } i \in \{1, \dots, k\}.$$

*Example 13* (particular case:  $k = 2$ ). Let  $f \in C^{1,1}(U)$  and assume that for some  $r > 0$  with  $B(x_0, r) \subset U \subset R^n$  there exist two open, nonempty, connected sets  $\Omega^+, \Omega^-$  and a hypersurface  $\Gamma \ni x_0$  such that  $\Omega^+ \cup \Gamma$  and  $\Omega^- \cup \Gamma$  are manifolds with boundary,  $f \in C^2(\Omega^+ \cup \Gamma) \cup C^2(\Omega^- \cup \Gamma)$ , and for all  $x \in B(x_0, r)$ , one and only one of the following occurs:

$$x \in \Omega^+, \quad x \in \Omega^-, \quad x \in \Gamma.$$

In this case

$$\mathcal{H}f(x_0) = \{D^2f(x_0^+), D^2f(x_0^-)\},$$

where  $D^2f(x_0^+), D^2f(x_0^-)$  are the limits at  $x_0$  of Hessians of  $f$  obtained from each side of the hypersurface  $\Gamma$ .

In the situation of the previous example the generalized Hessian at  $x_0$  has a minimum element, thus providing (in view of Theorem 3) a positive answer for (Q).

**PROPOSITION 14** (cobordered regions give rise to comparable Hessians). *Let  $f \in C^{1,1}(U)$  be as in Example 13. Then the matrices  $D^2f(x_0^+)$  and  $D^2f(x_0^-)$  are comparable with respect to the order in the space of symmetric matrices. In particular  $\mathcal{H}f(x_0) = \{D^2f(x_0^+), D^2f(x_0^-)\}$  has a minimum element.*

*If the matrices  $D^2f(x_0^+), D^2f(x_0^-)$  have a common orthonormal base of eigenvectors  $\{v_1, \dots, v_n\}$  with eigenvalues  $\{\lambda_1^+, \dots, \lambda_n^+\}$ , and, respectively,  $\{\lambda_1^-, \dots, \lambda_n^-\}$  and a unit normal vector  $n(x_0)$  to  $T_{x_0}\Gamma$  is not one of the eigenvectors, then  $f$  is twice differentiable at  $x_0$ .*

*Proof.* Let  $v \in T_{x_0}\Gamma$  be in the tangent space, and let  $\gamma : (-1, 1) \rightarrow \Gamma$  be a smooth curve with  $\gamma(0) = x_0, \dot{\gamma}(0) = v$ . Then

$$\left. \frac{d}{dt} Df(\gamma(t)) \right|_{t=0} = D^2f(x_0^+)v = D^2f(x_0^-)v.$$

Since the order of symmetric matrices is an intrinsic fact, if we choose as a base for  $R^N$  the set  $\{v_1, \dots, v_{n-1}, n(x_0)\}$  where the first  $n - 1$  vectors are a base of  $T_{x_0}\Gamma$  and  $n(x_0)$  is a normal unit vector of  $\Gamma$  at  $x_0$ , then the only possible unequal element of the two matrices  $D^2f(x_0^+), D^2f(x_0^-)$  is the element of place  $n \times n$ , i.e.,  $n(x_0) \cdot D^2f(x_0^\pm)n(x_0)$ , or  $n(x_0) \cdot D^2f(x_0^-)n(x_0)$ , respectively. Thus the two matrices are comparable.

In the assumptions of the second part of the statement, we can find a unit tangent vector  $\bar{\tau} \in T_{x_0}\Gamma$  such that  $\bar{\tau} = \sum_{i=1}^n \alpha_i v_i$  and  $\alpha_i \neq 0$  for all  $i$ . Then from the first part of the proof we get that

$$\lambda_i^- \alpha_i = D^2 f(x_0^-) \bar{\tau} \cdot v_i = D^2 f(x_0^+) \bar{\tau} \cdot v_i = \lambda_i^+ \alpha_i$$

for all indices  $i$ . Thus  $\lambda_i^- = \lambda_i^+$  for all  $i$  and the matrices  $D^2 f(x_0^-), D^2 f(x_0^+)$  are identical.  $\square$

Using the above proposition we obtain the following result.

**COROLLARY 15.** *Let  $f \in C^{1,1}(U)$  be a piecewise  $C^2$  function,  $x_0 \in U$ . Under the notation of Definition 11, we set  $\mathcal{H}f(x_0) = \{A_1, \dots, A_k\}$ . Then for every  $i, j \in \{1, \dots, k\}$  such that  $x_0 \in \bar{\Gamma}_\ell \subset \bar{\Omega}_i \cap \bar{\Omega}_j$ , the matrices  $A_i$  and  $A_j$  are comparable.*

*Proof.* Let  $\{x_n\}_n \subset \Gamma_\ell$  be such that  $x_n \rightarrow x_0$ . Denoting by  $D^2 f(x_n^i)$  (respectively,  $D^2 f(x_n^j)$ ) the limit of the Hessians of  $f$  at  $x_n$  obtained from the region  $\Omega_i$  (respectively,  $\Omega_j$ ) we have by Proposition 14 that either  $D^2 f(x_n^i) \succeq D^2 f(x_n^j)$  or  $D^2 f(x_n^i) \preceq D^2 f(x_n^j)$ . Taking a subsequence if necessary, we may assume that the order is fixed throughout the sequence (say,  $D^2 f(x_n^i) \succeq D^2 f(x_n^j)$  for all  $n \geq 1$ ), which then passes to the limit as  $n \rightarrow \infty$  yielding  $A_i \succeq A_j$ .  $\square$

The following example shows that the generalized Hessian  $\mathcal{H}f(\bar{x})$  of a piecewise  $C^2$  function  $f \in C^{1,1}(U)$  might not have a minimum element, even if all generalized Hessians are diagonal matrices.

*Example 16* (diagonal Hessians with no minimum). There exists a  $C^{1,1}$  function  $f : R^2 \rightarrow R$  which is piecewise  $C^2$  around  $\bar{x} = (0, 0)$  and for which all matrices of  $\mathcal{H}f(\bar{x})$  are diagonal, but there is no minimum element.

*Construction of the counterexample.* Let us consider the following five open subsets of  $R^2$  :

$$\begin{aligned} \Omega_1 &:= \{(x_1, x_2) : x_1 > 0, x_2 \in (-x_1^{1/3}, x_1^3)\}, & \Omega_4 &:= (-\infty, 0) \times (-\infty, 0), \\ \Omega_2 &:= \{(x_1, x_2) : x_1 > 0, x_2 > x_1^3\}, & \Omega_5 &:= \{(x_1, x_2) : x_1 > 0, x_2 < -x_1^{1/3}\}. \\ \Omega_3 &:= (-\infty, 0) \times (0, +\infty), \end{aligned}$$

Observe that  $R^2 = \bigcup_{i=1}^5 \text{cl } \Omega_i$  and define the function  $f : R^2 \rightarrow R$  as follows:

$$f(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) \in \text{cl } \Omega_1, & | & -\frac{1}{2}x_2^6 - x_1x_2^3 & \text{if } (x_1, x_2) \in \text{cl } \Omega_4, \\ -\frac{1}{2}(-x_1^3 + x_2)^2 & \text{if } (x_1, x_2) \in \text{cl } \Omega_2, & | & -\frac{1}{2}(x_1 + x_2^3)^2 & \text{if } (x_1, x_2) \in \text{cl } \Omega_5. \\ -\frac{1}{2}x_2^2 & \text{if } (x_1, x_2) \in \text{cl } \Omega_3, & | & \end{cases}$$

It is easily verified that  $f$  is  $C^{1,1}$ , piecewise  $C^2$  around  $\bar{x} := (0, 0)$ , and its generalized Hessian  $\mathcal{H}f(\bar{x})$  consists of the following three diagonal matrices:

$$\mathcal{H}f(\bar{x}) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Then  $\mathcal{H}f(\bar{x})$  does not have a minimum element. Nevertheless, one can check that for an appropriate choice of  $\varepsilon(r)$  (which tends to 0 as  $r \rightarrow 0$ ) the function  $\varphi(x) = -\frac{1}{2}x_2^2 + \varepsilon(\|x\|)\|x\|^2$  belongs to  $\mathcal{M}_f(\bar{x})$  and

$$(19) \quad D^2\varphi(\bar{x}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{H}f(\bar{x}).$$

Thus (Q) is here satisfied. Notice that the above matrix is the unique element of  $\mathcal{H}f(\bar{x})$  that satisfies (Q) and that it is a directional minimum of  $\mathcal{H}f(\bar{x})$ .  $\square$

In the above example we remark that there exist two hypersurfaces (one-dimensional manifolds) separating the regions  $\Omega_1, \Omega_5$  and  $\Omega_4$  having the same tangent  $(0, -1)$ . This is somehow the cause of the nonexistence of a minimum element. In fact, in the two-dimensional setting of the previous example, a qualification condition that avoids the above situation, ensures the existence of a minimum element. In what follows, we denote by  $\tau = (\tau_1, \tau_2)$  the unit vector that generates the tangent space of the hypersurface (smooth curve)  $\Gamma$  at  $\bar{x}$ . (In case  $\bar{x} \in \bar{\Gamma} \setminus \Gamma$ , we assume that the limit of the tangents of  $\Gamma$  at  $x$  as  $x \rightarrow \bar{x}$  exists, which is the case if  $\bar{\Gamma}$  is a manifold with boundary, and we take  $\tau$  equal to this limit.) We also denote by  $E := \{(1, 0), (0, 1), (-1, 0), (0, -1)\}$  the four tangents corresponding to the Cartesian semi-axes.

**THEOREM 17** (diagonal matrices in  $R^2$  with a qualification condition). *Let  $f : R^2 \rightarrow R$  be a  $C^{1,1}$  function which is piecewise  $C^2$  around the origin  $\bar{x} = (0, 0)$ . Assume that all elements of the generalized Hessian  $\mathcal{H}f(\bar{x}) = \{A_1, \dots, A_k\}$  are diagonal matrices, and that for any  $\tau \in E$  there is at most one hypersurface (smooth curve)  $\Gamma$  of Definition 11 whose tangent vector at  $\bar{x}$  is  $\tau$ . Then  $\mathcal{H}f(\bar{x})$  has a minimum element (thus (Q) holds).*

*Proof.* Let  $\Omega_1, \dots, \Omega_k$  be the finite regions (open connected sets) of  $\bar{x}$  that give rise to the generalized Hessians  $\mathcal{H}f(\bar{x}) = \{A_1, \dots, A_k\}$  (cf. Definition 11), where

$$A_i := \lim_{\substack{x \rightarrow \bar{x} \\ x \in \Omega_i}} D^2f(x) = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}.$$

Using Corollary 15 and reordering the set  $\{1, \dots, k\}$  if necessary, we may assume that as the index  $i$  moves progressively in the set  $\{1, \dots, k\}$  the matrices  $A_i$  and  $A_{i+1}$  are comparable ( $i$  is taken modulo  $k$ ). By Proposition 14 the matrices  $A_i$  and  $A_{i+1}$  are equal, whenever the tangent of the common boundary  $\Gamma$  of  $\Omega_i$  and  $\Omega_{i+1}$  is not parallel to one the axes. Thus, a change between  $A_i$  and  $A_{i+1}$  may occur whenever the (normalized) tangent of the common boundary of  $\Omega_i$  and  $\Omega_{i+1}$  belongs to  $E$ , i.e.,  $\tau_1\tau_2 = 0$ . In particular, if  $\tau_1 = 0$ , then  $a_i \neq a_{i+1}$  and  $b_i = b_{i+1}$ , while if  $\tau_2 = 0$ , then  $a_i = a_{i+1}$  and  $b_i \neq b_{i+1}$ . Now evoking the qualification condition we deduce that each of the aforementioned changes either appears twice or does not appear at all. The conclusion follows easily.  $\square$

As an easy consequence we obtain the following result.

**COROLLARY 18.** *Let  $f : R^2 \rightarrow R$  be a  $C^{1,1}$  function that is piecewise  $C^2$  around the origin  $\bar{x} = (0, 0)$ . Assume that all matrices in  $\mathcal{H}f(\bar{x})$  are simultaneously diagonalizable and that distinct hypersurfaces  $\Gamma^\ell$  give rise to distinct tangents  $\tau^\ell$  at  $\bar{x}$ . Then  $\mathcal{H}f(\bar{x})$  possesses a minimum element (and (Q) holds).*

*Remark 19.* Note that if all separating curves of Definition 11 are line segments (locally around  $x_0$ ), then the above qualification condition is automatically satisfied.

Let us finally note, for completeness, that the existence of a minimum element of the generalized Hessian  $\mathcal{H}f(\bar{x})$  does not imply the simultaneous diagonalizability of its elements.

*Example 20* (Hessian with minimum element and noncommutative elements). There exists a  $C^{1,1}$  function  $f : R^2 \rightarrow R$  which is piecewise  $C^2$  around  $\bar{x} = (0, 0)$ , such that the generalized Hessian  $\mathcal{H}f(\bar{x})$  has a minimum element but its elements are not simultaneously diagonalizable matrices.

*Construction of the counterexample.* Let us consider the following six open subsets of  $R^2$ :

$$\begin{aligned} \Omega_1 &:= \{(x_1, x_2) \in R^2 : x_1 > 0, x_2 \in (0, \frac{1}{2}x_1)\}, & \Omega_4 &:= (-\infty, 0) \times (0, +\infty), \\ \Omega_2 &:= \{(x_1, x_2) \in R^2 : x_1 > 0, x_2 \in (\frac{1}{2}x_1, 2x_1)\}, & \Omega_5 &:= (-\infty, 0) \times (-\infty, 0), \\ \Omega_3 &:= \{(x_1, x_2) \in R^2 : x_1 > 0, x_2 > 2x_1\}, & \Omega_6 &:= (0, +\infty) \times (-\infty, 0). \end{aligned}$$

Observe that  $R^2 = \bigcup_{i=1}^6 \text{cl } \Omega_i$  and define the function  $f : R^2 \rightarrow R$  as follows:

$$f(x_1, x_2) = \begin{cases} \frac{3}{2}x_2^2 & \text{if } (x_1, x_2) \in \text{cl } \Omega_1, & | & -3x_1^2 & \text{if } (x_1, x_2) \in \text{cl } \Omega_4, \\ 2x_1x_2 - \frac{1}{2}(x_1^2 + x_2^2) & \text{if } (x_1, x_2) \in \text{cl } \Omega_2, & | & -3x_1^2 - 3x_2^2 & \text{if } (x_1, x_2) \in \text{cl } \Omega_5, \\ \frac{3}{2}x_1^2 & \text{if } (x_1, x_2) \in \text{cl } \Omega_3, & | & -3x_2^2 & \text{if } (x_1, x_2) \in \text{cl } \Omega_6. \end{cases}$$

It is easily verified that  $f$  is  $C^{1,1}$ , piecewise  $C^2$  around  $\bar{x}$ , and that

$$\mathcal{H}f(0, 0) := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -6 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix} \right\}.$$

Then  $\mathcal{H}f(\bar{x})$  has a minimum element (coming out from the region  $\Omega_5$ ) but its elements are not simultaneously diagonalizable.  $\square$

**4. The case of the metric projector over the cone  $S_+^n$ .** Apart from the situation of Theorem 17 and Corollary 18 concerning piecewise  $C^2$  functions, or the one-dimensional case of Corollary 5, let us give below an example of a  $SC^1$  function in which Corollary 4 applies. (A  $C^{1,1}$  function  $f$  is called  $SC^1$  if its gradient  $\nabla f$  is semismooth.) We refer to [19, Definition 3.6] for the definition of semismoothness.

Let us denote by  $\Pi_{S_+^n}(X)$  the projection of a symmetric matrix  $X \in S^n$  to the cone of positive semidefinite matrices  $S_+^n$  under the Frobenius norm, and let us consider the function

$$\begin{cases} f : S^n \rightarrow R, \\ f(X) = \frac{1}{2} \|\Pi_{S_+^n}(X)\|^2. \end{cases}$$

Then  $f$  is continuously differentiable with derivative  $\nabla f(X) = \Pi_{S_+^n}(X)$  (see, e.g., [14, section 2.2]). Since  $\nabla f$  is Lipschitz continuous and semialgebraic, we deduce from [1, Theorem 1] that  $f$  is  $SC^1$ . (In fact, according to [19, Corollary 4.15],  $\nabla f$  is also strongly semismooth.) Notice though that  $f$  is not piecewise- $C^2$ , unless  $n = 1$ . Assuming that  $X$  has the following spectral decomposition  $X = P\Lambda P^T$ , where  $\Lambda$  is the diagonal matrix of eigenvalues  $\lambda_1 \geq \dots \geq \lambda_n$  of  $X$  and  $P$  is the corresponding orthogonal matrix of eigenvectors, it follows that  $\Pi_{S_+^n}(X) = P\Lambda_+P^T$ , where  $\Lambda_+ = \text{Diag}(\lambda_1^+, \dots, \lambda_n^+)$  and  $\lambda_i^+ = \max\{\lambda_i, 0\}$ . Following [18], let us define the three index sets of positive, zero, and negative eigenvalues of  $X$  by  $\alpha := \{i : \lambda_i > 0\}$ ,  $\beta := \{i : \lambda_i = 0\}$ , and  $\gamma := \{i : \lambda_i < 0\}$ , respectively, and write

$$\Lambda = \begin{bmatrix} \Lambda_\alpha & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Lambda_\gamma \end{bmatrix} \quad \text{and} \quad P = [P_\alpha \ P_\beta \ P_\gamma]$$

with  $P_\alpha$  (respectively,  $P_\beta$  and  $P_\gamma$ ) being an  $n \times |\alpha|$  matrix (respectively, an  $n \times |\beta|$  and an  $n \times |\gamma|$  matrix). Then according to [18, Proposition 2.2],  $V \in \mathcal{H}f(X)$  if and only if there exists  $W_{|\beta|} \in \mathcal{J}\Pi_{S_+^{|\beta|}}(0)$  such that

$$V(H) = P \begin{bmatrix} \tilde{H}_{\alpha\alpha} & \tilde{H}_{\alpha\beta} & U_{\alpha\gamma} \circ \tilde{H}_{\alpha\gamma} \\ \tilde{H}_{\alpha\beta}^T & W_{|\beta|}(\tilde{H}_{\beta\beta}) & \mathbf{0} \\ \tilde{H}_{\alpha\gamma} \circ U_{\alpha\gamma}^T & \mathbf{0} & \mathbf{0} \end{bmatrix} P^T$$

for every  $n \times n$  symmetric matrix  $H$ , where  $\tilde{H} := P^T H P$  and where  $U$  is an  $n \times n$  symmetric matrix with entries

$$U_{ij} := \frac{\lambda_i^+ + \lambda_j^+}{|\lambda_i| + |\lambda_j|} \quad \text{for } i, j \in \{1, \dots, n\} \quad (\text{we use the convention } 0/0 = 1).$$

Notice that formally  $\mathcal{H}f(X)$  is made up by 2-tensors acting on the space of  $n \times n$  matrices. The natural semidefinite order there is defined as follows:  $S \succeq R$  if and only if  $\langle H, SH \rangle \geq \langle H, RH \rangle$  for all  $H \in S^n$ , where  $\langle \cdot, \cdot \rangle$  is the Frobenius scalar product on  $S^n$ . We refer to [14] for more details.

As pointed out by one of the referees,  $W_{|\beta|} := \mathbf{0} \in \mathcal{J}\Pi_{S_+^{|\beta|}}(0)$  is the minimum element of  $\mathcal{J}\Pi_{S_+^{|\beta|}}(0)$  (as the mapping  $\Pi_{S_+^{|\beta|}}(\cdot)$  is firmly monotone). Thus the element  $V^0 \in \mathcal{H}f(X)$  corresponding to the choice  $W_{|\beta|} := \mathbf{0} \in \mathcal{J}\Pi_{S_+^{|\beta|}}(0)$  is the minimum element of  $\mathcal{H}f(X) = \mathcal{J}\Pi_{S_+^n}(X)$  (with respect to the aforementioned order). Thus Corollary 4 applies yielding  $(\mathcal{Q})$ .

**5. Characterization of directional minima.** Let us now come back to the situation of Theorem 3 and the sufficiency part of its condition, that is, the existence of directional minima of the generalized Hessian  $\mathcal{H}f(x_0)$ . In view of Example 7 the existence of directional minima is clearly not a necessary condition. Thus, in order to obtain a characterization an additional assumption is required. Such an assumption is provided by the following definition.

**DEFINITION 21** (semi- $C^2$  regularity). *Let  $U$  be an open subset of  $R^n$ . A  $C^{1,1}$  function  $f : U \subset R^n \rightarrow R$  is called upper semi- $C^2$  at  $x_0 \in U$  if for every  $X \in \mathcal{H}f(x_0)$  and  $v \in \mathcal{T}_{f,x_0}(X)$ , there exists a sequence  $\{x_n\}_n \subset U \setminus \{x_0\}$  with  $x_n \rightarrow x_0$  and  $\frac{x_n - x_0}{\|x_n - x_0\|} \rightarrow v$  such that*

$$(20) \quad f(x_n) \leq f(x_0) + Df(x_0)(x_n - x_0) + \frac{1}{2} \langle X(x_n - x_0), x_n - x_0 \rangle + o(\|x_n - x_0\|^2).$$

*Remark 22.* It can be easily deduced from the forthcoming result (Theorem 23) that the function  $f : R \rightarrow R$  of Example 6 and the function  $F : R^n \rightarrow R$  of Example 7 do not satisfy the above regularity condition. On the other hand, all other functions considered in the examples of this work are upper semi- $C^2$ . This is a consequence of the fact that piecewise  $C^2$  functions (Definition 11) are upper semi- $C^2$  (in fact semi- $C^2$ , a function  $f$  being called semi- $C^2$ , if (20) holds with equality). To see this, note that for any  $A \in \mathcal{H}f(x_0)$ , the cone  $R_+ \mathcal{T}_{f,x_0}(A)$  can be described as a finite union of tangent cones at  $x_0$  of some  $\overline{\Omega}_i$  (those giving  $A = \lim D^2 f(x)$  with  $x \rightarrow x_0$  in  $\Omega_i$ ).

We are now ready to state and prove the following result.

**THEOREM 23** (characterization of directional minima of  $\mathcal{H}f(x_0)$ ). *Let  $U \subset R^n$*



be a nonempty open set, and let  $f : U \rightarrow R$  be a  $C^{1,1}$  function which is upper semi- $C^2$  at  $x_0$ . Then  $M \in \mathcal{H}f(x_0)$  is a directional minimum of  $\mathcal{H}f(x_0)$  if and only if there exists  $\varphi \in \mathcal{M}_f(x_0)$  such that  $D^2\varphi(x_0) = M$ .

*Proof.* In view of Theorem 3 we need only prove the sufficiency part. To this end, let  $M \in \mathcal{H}f(x_0) \cap \mathcal{J}_f^2(x_0)$ , and let  $\varphi$  be a corresponding minorant of  $f$  at  $x_0$  so that

$$f(x) \geq \varphi(x) = f(x_0) + Df(x_0)(x - x_0) + \frac{1}{2}\langle M(x - x_0), x - x_0 \rangle + o(\|x - x_0\|^2).$$

Let  $X \in \mathcal{H}f(x_0)$  and let  $v \in \mathcal{T}_{f,x_0}(X)$ . Then for some sequence  $\{x_n\}_n \subset U$  with  $x_n \rightarrow x_0$  and  $\frac{x_n - x_0}{\|x_n - x_0\|} \rightarrow v$ , relation (20) holds. Combining with the above we deduce

$$\langle (X - M)(x_n - x_0), x_n - x_0 \rangle \geq o(\|x_n - x_0\|^2).$$

Dividing by  $\|x_n - x_0\|^2$  and taking the limit as  $x_n \rightarrow x_0$ , we deduce  $\langle (X - M)v, v \rangle \geq 0$ , which proves the assertion.  $\square$

We complement the previous result with the following remark.

*Remark 24* (Jensen's lemma revisited). Let  $\varphi \in \mathcal{M}_f(x_0)$  and let  $A = D^2\varphi(x_0)$ . Then

$$f(x) \geq \varphi(x) = f(x_0) + Df(x_0)(x - x_0) + \frac{1}{2}\langle A(x - x_0), x - x_0 \rangle + o(\|x - x_0\|^2).$$

Set  $x = x_0 + tv$ , where  $\|v\| = 1$  and  $t > 0$ , and apply the nonsmooth second-order Taylor expansion [8, Theorem 3.2] to deduce

$$\frac{1}{2}\langle Av, v \rangle + \frac{o(t^2)}{t^2} \leq \frac{1}{t^2} (f(x) - f(x_0) - Df(x_0)(x - x_0)) \in \frac{1}{2}\langle \mathcal{H}^o f(x_0, x)v, v \rangle.$$

The above holds for any sequence  $\{t_n\}_n$  with  $t_n \rightarrow 0$ . Taking a subsequence if necessary, and passing to the limit as  $n \rightarrow \infty$  we deduce using the upper semicontinuity of  $\mathcal{H}f$  that for some  $B \in K_{f,x_0}(v)$  we have

$$\langle Av, v \rangle \leq \langle Bv, v \rangle.$$

Notice that Example 2 shows that the above inequality might not be true for all  $B \in K_{f,x_0}(v)$ . However, whenever the set

$$\{\langle Bv, v \rangle : B \in K_{f,x_0}(v)\}$$

is a singleton, we directly conclude that  $A$  is a directional minimum and  $(\mathcal{Q})$  holds.  $\square$

**6. Appendix: A side lemma.** In this section we state and prove the following standard lemma, which has been used in the text.

**LEMMA 25** (second-order regularization from above). *Let  $\varepsilon : (0, \infty) \rightarrow R$  be a function satisfying  $\lim_{t \rightarrow 0} \varepsilon(t) = 0$ . Then there exists  $\delta > 0$  and a  $C^2$  function  $\psi : [0, \delta) \rightarrow [0, \infty)$  such that*

$$\psi(0) = \psi'(0) = \psi''(0) = 0$$

and

$$\psi(t) \geq \varepsilon(t) t^2 \quad \text{for all } t \in [0, \delta).$$

*Proof.* By [17, Lemma 3.7] (see also [5]) there exists  $\delta_1 > 0$  and a  $C^1$  function  $\alpha : [0, \delta_1) \rightarrow [0, \infty)$  such that  $\alpha(0) = \alpha'(0) = 0$  and  $\alpha(t) \geq t\varepsilon(t)$  for  $t \in (0, \delta_1)$ . Since  $\lim_{t \rightarrow 0} \alpha'(t) = \alpha'(0) = 0$  applying again [17, Lemma 3.7] to the function  $\alpha'$ , we obtain  $0 < \delta < \delta_1$  and a  $C^1$  function  $\beta : [0, \delta) \rightarrow [0, \infty)$  such that  $\beta(0) = \beta'(0) = 0$  and  $\beta(t) \geq t\alpha'(t)$  for  $t \in [0, \delta)$ . Let us set  $\gamma(t) = \alpha(t) + \beta(t)$  for all  $t \in [0, \delta)$ . Obviously  $\gamma \in C^1([0, \delta))$ , thus the function

$$t \longmapsto \psi(t) := \int_0^t \gamma(s) ds$$

is of class  $C^2$  on  $[0, \delta)$ . Since

$$\gamma(s) = \alpha(s) + \beta(s) \geq \alpha(s) + s\alpha'(s) \geq (s\alpha(s))' \quad \text{for all } s \in [0, \delta),$$

we deduce that  $\psi(t) \geq \int_0^t (s\alpha(s))' ds = t\alpha(t)$  for  $t \in [0, \delta)$ . Thus  $\psi(t) = t^2\varepsilon(t)$  for  $t \in [0, \delta)$  and the conclusion follows.  $\square$

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