Arch. Mech., **62**, 1, pp. 49–72, Warszawa 2010

Some theorems of incremental thermoelectroelasticity

A. MONTANARO

Department of Mathematical Methods
and Models for Scientific Aplications
University of Padova
via Trieste 63, 35121 Padova, Italy
e-mail: montanaro@dmsa.unipd.it

We extend to incremental thermoelectroelasticity with biasing fields certain classical theorems, which have been stated and proved in linear thermopiezoelectricity referred to a natural configuration. A uniqueness theorem for the solutions to the initial boundary value problem, the generalized Hamilton principle and the theorem of reciprocity of work are deduced for incremental fields, superposed on finite biasing fields in a thermoelectroelastic body.

Key words: thermoelectroelasticity, uniqueness of solution, incremental thermoelectroelasticity, Hamilton principle, theorem of reciprocity of work.

1. Introduction

In the last decades, with increasing wide use in sensing and actuation, the materials exhibiting couplings between elastic, electric, magnetic and thermal fields have attracted much attention.

In order to give certainty to experimental results and applications, the interest of many researchers turned to mathematical fitting of these topics.

Many applications have their mathematical formulation within a linear framework, and the theoretical study began from this context.

Foundamental is Nowacki's paper [1], where a uniqueness theorem for the solutions of the initial boundary value problems is proved in linear thermopiezoelectricity referred to a natural state, i.e., without biasing (or initial) fields. Hence Nowacki [2] also deduced the generalized Hamilton principle and a theorem of reciprocity of work.

Li [3] generalized the uniqueness and reciprocity theorems for linear thermo-electro-magneto-elasticity referred to a natural state.

Aouadi [4] establishes a reciprocal theorem for a linear theory in which the heat flux is considered as a constitutive independent variable, a rate-type evolution equation for it is added to the system of constitutive equations, and the entropy inequality is stated in the form proposed by Müller [5].

Iesan [6] uses the Green–Naghdi theory of thermomechanics of continua to derive a linear theory of thermoelasticity with internal structure where, in particular, a uniqueness result holds.

Related works on thermoelasticity and thermoelectromagnetism can be found in [7] and [8].

The classical linear theory of thermopiezoelectricity assumes infinitesimal deviations of the field variables from the reference state, where there are no initial mechanical and electric fields. In order to describe the response of thermoelectroelastic materials in presence of the initial fields, one needs the theory for infinitesimal fields superposed on initial fields, and this can only be derived from the fully nonlinear theory of thermoelectroelasticity. The equations of nonlinear thermoelectroelasticity were given in Tiersten [12]. Yang [13] derived then from [12] the equations for infinitesimal incremental fields superposed on finite biasing fields in a thermoelectroelastic body, with no assumption on the biasing fields.

Here we extend the aforementioned three Nowacki's theorems [1, 2] to incremental thermoelectroelasticity with initial fields.

We explicitly refer to the incremental theory [13], hence below we rewrite from this paper, using the same notations, some formulae and results on constitutive equations of incremental thermoelectroelasticity.

Of course, the theorems proved here just reduce to the ones used in Nowacki's paper [2] by neglecting the initial fields.

In the uniqueness theorem of Sec. 4 we assume that in the initial state, entropy does not depend on time and the temperature is uniform. For the theorem of reciprocity of work in Sec. 6 we assume that in the initial state, both the entropy and temperature fields do not depend on time.

2. Equations of nonlinear thermoelectroelasticity

2.1. Balance laws and constitutive equations

Consider a thermoelectroelastic body $\mathcal{B}$ that, in the reference configuration, occupies a region V with boundary surface S. The motion of the body is described by

$$y_i = y_i(X_L, t),$$

where y_i denotes the present coordinates and X_L the reference coordinates of material points with respect to the same Cartesian coordinate system.

Let K_{Lj}, ρ_0, f_j, $\varDelta_L$, ρ_E, θ, η, Q_L and γ respectively denote: the first Piola–Kirchoff stress tensor, the mass density in the reference configuration, the body force per unit mass, the reference electric displacement vector, the free charge density per unit undeformed volume, the absolute temperature, the entropy per unit mass, the reference heat flux vector, and the body heat source per unit mass. Then we have the following equations of motion, electrostatics and heat conduction written in material form with respect to the reference configuration:

$$K_{Li,L} + \rho_0 f_i = \rho_0 \ddot{y}_i, \tag{2.1}$$

$$\Delta_{L,L} = \rho_E, \tag{2.2}$$

$$\rho_0 \theta \dot{\eta} = -Q_{L,L} + \rho_0 \gamma. \tag{2.3}$$

Note that, following [2] and [13], here we directly write the heat equation (2.3) without writing the entropy inequality, the energy equation and a deduction of (2.3) from them; such a deduction can be found e.g. in [14].

The above equations are adjoined by constitutive relations defined by the specification of the free energy ψ and heat flux Q_L:

$$\psi = \psi(E_{MN}, W_M, \theta), \qquad Q_L = Q_L(E_{MN}, W_M, \theta, \Theta_M), \tag{2.4}$$

where

$$E_{MN} = (y_{j,M} y_{j,N} - \delta_{MN})/2, \qquad W_M = -\phi_{,M}, \qquad \Theta_M = \theta_{,M} \tag{2.5}$$

are the finite strain tensor, the reference electric potential gradient, and the reference temperature gradient; of course, δ_{MN} is the Kronecker delta, and ϕ is the electric potential. Hence, by using ψ the constitutive relations (4) of [13] are deduced for K_{Li}, Δ_L, η; here we rewrite them from [13]:

$$\begin{aligned} K_{Li} &= y_{i,A} \rho_0 \frac{\partial \psi}{\partial E_{AL}} + J X_{L,j} \varepsilon_0 \left(E_j E_i - \frac{1}{2} E_i E_i \delta_{ji} \right), \\ \Delta_L &= \varepsilon_0 J X_{L,j} E_j - \rho_0 \frac{\partial \psi}{\partial W_L}, \qquad \eta = -\frac{\partial \psi}{\partial \theta}, \end{aligned} \tag{2.6}$$

with $E_i = -\phi_{,i}$. Recall that the heat-flux constitutive relation $(2.4)_2$ is restricted by

$$Q_L \Theta_L \leq 0. \tag{2.7}$$

For a deduction of the Fourier inequality (2.7) see e.g. [14]. Note that, in particular, $(2.4)_2$ includes the case in which Q_M is linear in Θ_L, that is,

$$Q_M = -\kappa_{ML}(\theta, W_A) \Theta_L. \tag{2.8}$$

2.2. The initial boundary value problem for a thermoelectroelastic body

To describe the corresponding boundary conditions added to the field equations (2.1)–(2.3), three partitions (S_{i1}, S_{i2}), $i = 1, 2, 3$, of the boundary surface $S = \partial\mathcal{B}$ can be assigned. For mechanical boundary conditions, deformation $\widetilde{y}_i$ and traction $\widetilde{t}_i$ per unit undeformed area are prescribed, respectively, on S_{11} and S_{12}; for electric boundary conditions, electric potential $\widetilde{\phi}$ and surface-free charge $\widetilde{\Delta}$ per unit undeformed area are prescribed, respectively, on S_{21} and S_{22}; while for thermal boundary conditions, temperature $\widetilde{\theta}$ and normal heat flux $\widetilde{Q}$ per unit undeformed area are prescribed, respectively, on S_{31} and S_{32}. Hence, we can write

$$y_i = \widetilde{y}_i \quad \text{on } S_{11}, \qquad K_{Li}N_L = \widetilde{K}_i \quad \text{on } S_{12} \qquad (\text{'mechanical'}), \tag{2.9}$$

$$\phi = \widetilde{\phi} \quad \text{on } S_{21}, \qquad \Delta_L N_L = -\widetilde{\Delta} \quad \text{on } S_{22} \qquad (\text{'electric'}), \tag{2.10}$$

$$\theta = \widetilde{\theta} \quad \text{on } S_{31}, \qquad Q_L N_L = \widetilde{Q} \quad \text{on } S_{32} \qquad (\text{'thermal'}), \tag{2.11}$$

where $\mathbf{N} = (N_L)$ is the unit exterior normal to S and

$$S_{i1} \cup S_{i2} = S, \qquad S_{i1} \cap S_{i2} = \emptyset \qquad (i = 1, 2, 3). \tag{2.12}$$

We put

$$\mathcal{A}_{\text{body}} := (f_i, \rho_E, \gamma), \qquad \mathcal{A}_{\text{surf}} := (\widetilde{y}_i, \widetilde{K}_i, \widetilde{\phi}, \widetilde{\Delta}, \widetilde{\theta}, \widetilde{Q}), \tag{2.13}$$

$$\mathcal{A} := (\mathcal{A}_{\text{body}}, \mathcal{A}_{\text{surf}}) = (f_i, \rho_E, \gamma, \widetilde{y}_i, \widetilde{K}_i, \widetilde{\phi}, \widetilde{\Delta}, \widetilde{\theta}, \widetilde{Q}). \tag{2.14}$$

$\mathcal{A}_{\text{body}}$, $\mathcal{A}_{\text{surf}}$ and $\mathcal{A}$ are called the (*external*) *body-action*, *surface-action*, and *action*, respectively. The initial conditions have the form

$$\begin{aligned} &y_i(\mathbf{X}, 0) = f_i(\mathbf{X}), \quad \dot{y}_i(\mathbf{X}, 0) = g_i(\mathbf{X}), \\ &\theta(\mathbf{X}, 0) = h(\mathbf{X}), \quad \phi(\mathbf{X}, 0) = l(\mathbf{X}) \qquad (\mathbf{X} \in B,\ t = 0), \end{aligned} \tag{2.15}$$

where

$$\mathcal{I} = (f_i,\ g_i, h, l)$$

are prescribed smooth functions of domain V. The initial boundary value problem is then stated as: *assigned* $\mathcal{A}_{\text{body}}$*, to find the solution* (ϕ, θ, y_i) *in* V *to the constitutive relations* (2.6) *and field equations* (2.1)–(2.3)*, which satisfies the boundary conditions* (2.9)–(2.11) *and initial conditions* (2.15) *for given* $\mathcal{A}_{\text{surf}}$ *and* $\mathcal{I}$*.*

3. Biasing and incremental fields

In incremental theories, three configurations are distinguished: the reference, initial and present configuration.

3.1. The reference configuration

In the reference state the body is undeformed and free of all fields. A generic point at this state is denoted by $\mathbf{X}$ with rectangular coordinates X_N. The mass density in the reference configuration is denoted by ρ_0.

3.2. The initial configuration

In the initial state, the body is deformed finitely under the action of a prescribed initial action

$$\mathcal{A}^0 := (\mathcal{A}^0_{\text{body}}, \mathcal{A}^0_{\text{surf}}) = (f^0_i, \rho^0_E, \gamma^0, \widetilde{y}^0_i, \widetilde{K}^0_i, \widetilde{\phi}^0, \widetilde{\Delta}^0, \widetilde{\theta}^0, \widetilde{Q}^0), \tag{3.1}$$

$$\mathcal{A}^0_{\text{body}} := (f^0_i, \rho^o_E, \gamma^0), \qquad \mathcal{A}^0_{\text{surf}} := (\widetilde{y}^0_i, \widetilde{K}^0_i, \widetilde{\phi}^0, \widetilde{\Delta}^0, \widetilde{\theta}^0, \widetilde{Q}^0). \tag{3.2}$$

The position of the material point associated with $\mathbf{X}$ is given by

$$y^0_\alpha = y^0_\alpha(\mathbf{X}, t),$$

with the Jacobian of the initial configuration denoted by

$$J_0 = \det(y^0_{\alpha,L}).$$

The initial fields

$$y^0_\alpha = y^0_\alpha(\mathbf{X}, t), \qquad \phi^0 = \phi^0(\mathbf{X}, t), \qquad \theta^0 = \theta^0(\mathbf{X}, t) \tag{3.3}$$

satisfy the equations of nonlinear thermoelectroelasticity (2.1)–(2.12) under the prescribed action $\mathcal{A}^0$. The electric potential, electric field and temperature field are denoted by $\phi^0(\mathbf{X}, t)$, $W^0_\alpha = -\phi^0_{,\alpha}$ and $\theta^0(\mathbf{X}, t)$, respectively.

In studying the incremental fields, the solution to the initial state problem is assumed to be known.

3.3. The present configuration

To the deformed body at the initial configuration, infinitesimal deformations, electric and thermal fields are applied. The present position of the material point associated with $\mathbf{X}$ is given by $y_i(\mathbf{X}, t)$, with electric potential $\phi(\mathbf{X}, t)$ and temperature $\theta(\mathbf{X}, t)$.

The fields $y_i(\mathbf{X}, t)$, $\phi(\mathbf{X}, t)$, $\theta(\mathbf{X}, t)$ satisfy Eqs. (2.1)–(2.3) under the action of the external action (2.14).

3.4. Equations for the incremental fields

Let ε be a small and dimensionless number. The incremental process $\varepsilon(y^1, \phi^1, \theta^1)$ for (y, ϕ, θ) superposed on the initial process (y^0, ϕ^0, θ^0) is assumed to be infinitesimal and, therefore, we write:

$$y_i = \delta_{i\alpha}(y^0_\alpha + \varepsilon y^1_\alpha), \qquad \phi = \phi^0 + \varepsilon\phi^1, \qquad \theta = \theta^0 + \varepsilon\theta^1. \tag{3.4}$$

Corresponding to Eq. (3.4), other quantities of the present state can be written as:

$$\mathcal{A} \cong \mathcal{A}^0 + \varepsilon\mathcal{A}^1, \tag{3.5}$$

where, due to nonlinearity, higher powers of ε may appear. For the incremental action we have

$$\mathcal{A}^1_{\text{body}} := (f^1_i, \rho^1_E, \gamma^1), \qquad \mathcal{A}^1_{\text{surf}} := (\widetilde{y}^1_i, \widetilde{K}^1_i, \widetilde{\phi}^1, \widetilde{\Delta}^1, \widetilde{\theta}^1, \widetilde{Q}^1), \tag{3.6}$$

$$\mathcal{A}^1 := (\mathcal{A}^1_{\text{body}}, \mathcal{A}^1_{\text{surf}}) = (f^1_i, \rho^1_E, \gamma^1, \widetilde{y}^1_i, \widetilde{K}^1_i, \widetilde{\phi}^1, \widetilde{\Delta}^1, \widetilde{\theta}^1, \widetilde{Q}^1). \tag{3.7}$$

We want to derive equations governing the incremental process

$$(\mathbf{u} := \mathbf{y}^1,\ \phi^1,\ \theta^1).$$

From Eqs. (3.4) and (3.5), we can further write:

$$E_{KL} \cong E^0_{KL} + \varepsilon E^1_{KL}, \qquad W_L \cong W^0_L + \varepsilon W^1_L, \qquad \Theta_L \cong \Theta^0_L + \varepsilon\Theta^1_L, \tag{3.8}$$

where

$$\begin{gathered} E^0_{KL} = (y^0_{\alpha,K} y^0_{\alpha,L} - \delta_{KL})/2, \qquad E^1_{KL} = (y^0_{\alpha,K} y^1_{\alpha,L} + y^0_{\alpha,L} y^1_{\alpha,K})/2, \\ W^0_L = -\phi^0_{,L}, \qquad W^1_L = -\phi^1_{,L}, \qquad \Theta^0_L = \theta^0_{,L}, \qquad \Theta^1_L = \theta^1_{,L}. \end{gathered} \tag{3.9}$$

Substituting Eqs. (3.4)–(3.9) into the constitutive relations (2.1)–(2.3), with some very lengthy algebra, the following expressions are obtained [13]:

$$\begin{aligned} K_{Mi} &\cong \delta_{i\alpha}(K^0_{M\alpha} + \varepsilon K^1_{M\alpha}), \qquad & \Delta_M &\cong \Delta^0_M + \varepsilon\Delta^1_M, \\ \eta &\cong \eta^0 + \varepsilon\eta^1, & Q_M &\cong Q^0_M + \varepsilon Q^1_M, \end{aligned} \tag{3.10}$$

where

$$K^1_{M\alpha} = G_{M\alpha L\gamma} u_{\gamma,L} + R_{LM\alpha}\phi^1_{,L} - \rho_0 \Lambda_{M\alpha}\theta^1, \tag{3.11}$$

$$\Delta^1_M = R_{MN\gamma} u_{\gamma,N} - L_{MN}\phi^1_{,N} + \rho_0 P_M \theta^1, \tag{3.12}$$

$$\eta^1 = \Lambda_{M\gamma} u_{\gamma,M} - P_M \phi^1_{,M} + \alpha\theta^1, \tag{3.13}$$

$$Q^1_M = A_{MN\alpha} u_{\alpha,N} - B_{MN}\phi^1_{,N} + C_M\theta^1 + F_{MN}\theta^1_{,N}. \tag{3.14}$$

By putting

$$\kappa_{MN\alpha} = -A_{MN\alpha}, \quad \kappa^E_{MN} = B_{MN}, \quad \kappa_M = -C_M, \quad \kappa_{MN} = -F_{MN},$$

the latter result takes the form:

$$Q^1_M = -\kappa_{MN\alpha} u_{\alpha,N} - \kappa^E_{MN}\phi^1_{,N} - \kappa_M\theta^1 - \kappa_{MN}\theta^1_{,N}. \tag{3.15}$$

In Eqs. (3.11)–(3.14), $G_{M\alpha L\gamma}$ are the effective elastic constants, $R_{LM\alpha}$ are the effective piezoelectric constants, $\Lambda_{M\alpha}$ are the effective thermoelatic constants, L_{MN} are the effective dielectric constants, P_M are the effective pyrolectric constants, α is related to the specific heat. Their expressions are [13]:

$$\begin{aligned}
G_{K\alpha L\gamma} &= y^0_{\alpha,M}\rho_0\frac{\partial^2\psi}{\partial E_{KM}\partial E_{LN}}(\theta^0, E^0_{AB}, W^0_A)y^0_{\alpha,L} \\
&\qquad\qquad + \rho_0\frac{\partial\psi}{\partial E_{KL}}(\theta^0, E^0_{AB}, W^0_A)\delta_{\alpha\gamma} + g_{K\alpha L\gamma}, \\
R_{LM\gamma} &= -\rho_0\frac{\partial^2\psi}{\partial W_K\partial E_{ML}}(\theta^0, E^0_{AB}, W^0_A)y^0_{\gamma,M} + r_{KL\gamma}, \\
\Lambda_{M\gamma} &= -\frac{\partial^2\psi}{\partial E_{LM}\partial\theta}(\theta^0, E^0_{AB}, W^0_A)y^0_{\gamma,L}, \\
L_{MN} &= -\rho_0\frac{\partial^2\psi}{\partial W_M\partial W_N}(\theta^0, E^0_{AB}, W^0_A) + l_{MN}, \\
P_M &= -\frac{\partial^2\psi}{\partial W_M\partial\theta}(\theta^0, E^0_{AB}, W^0_A), \\
\alpha &= -\frac{\partial^2\psi}{\partial\theta^2}(\theta^0, E^0_{AB}, W^0_A), \\
A_{MN\gamma} &= \frac{\partial Q_M}{\partial E_{LN}}(\theta^0, E^0_{AB}, W^0_A)y^0_{\gamma,L} = -\kappa_{MN\gamma}, \\
B_{MN} &= \frac{\partial Q_M}{\partial W_N}(\theta^0, E^0_{AB}, W^0_A) = \kappa^E_{MN}, \\
C_M &= \frac{\partial Q_M}{\partial\theta}(\theta^0, E^0_{AB}, W^0_A) = -\kappa_M, \\
F_{MN} &= \frac{\partial Q_M}{\partial\Theta_N}(\theta^0, E^0_{AB}, W^0_A) = -\kappa_{MN},
\end{aligned} \tag{3.16}$$

where

$$
\begin{aligned}
g_{K\alpha L\gamma} &= \varepsilon_0 J_0 \Big[W^0_\alpha W^0_\beta \big(X_{K,\beta} X_{L,\gamma} - X_{K,\gamma} X_{L,\beta} \big) \\
&\qquad\qquad + W^0_\beta W^0_\gamma \big(X_{K,\alpha} X_{L,\beta} - X_{K,\beta} X_{L,\alpha} \big) \\
&\qquad + W^0_\beta W^0_\beta \big(X_{K,\gamma} X_{L,\alpha} - X_{K,\alpha} X_{L,\gamma} \big)/2 - W^0_\alpha W^0_\gamma X_{K,\beta} X_{L,\beta} \Big], \\
r_{KL\gamma} &= \varepsilon_0 J_0 \big(W^0_\alpha X_{K,\alpha} X_{L,\gamma} - W^0_\alpha X_{K,\gamma} X_{L,\alpha} - W^0_\gamma X_{K,\alpha} X_{L,\alpha} \big), \\
l_{MN} &= \varepsilon_0 J_0 X_{M,\alpha} X_{N,\alpha}.
\end{aligned} \tag{3.17}
$$

In Eq. (3.14) we have introduced the κ-notation to allow a comparison between the proofs written here and those used in [2]. The following symmetries hold:

$$G_{K\alpha L\gamma} = G_{L\gamma K\alpha}, \qquad L_{MN} = L_{NM}. \tag{3.18}$$

3.5. Restriction on the incremental heat flux

Now we show that the restriction (2.7) on the heat flux $(2.4)_2$, together with the condition

$$Q^0_L = 0 \quad \text{for} \quad \Theta^0_L = 0, \tag{3.19}$$

implies an analogous restriction on the incremental heat flux (3.14), that is

$$Q^1_L \Theta^1_L \le 0. \tag{3.20}$$

Indeed, substituting $Q_L = Q^0_L + \varepsilon Q^1_L$, $\Theta_L = \Theta^0_L + \varepsilon \Theta^1_L$ in (2.7), we obtain

$$(Q^0_L + \varepsilon Q^1_L)(\Theta^0_L + \varepsilon \Theta^1_L) \le 0, \tag{3.21}$$

which for $\Theta^0_L = 0$, by (3.19), yields (3.20). Note that the choice (2.8) for the heat flux response function satisfies the condition (3.19).

3.6. Incremental field equations

By substituting (3.4)–(3.10) into (2.1)–(2.3) and (2.9)–(2.11), we find the governing equations for the incremental fields

$$K^1_{M\alpha,M} + \rho_0 f^1_\alpha = \rho_0 \ddot{u}_\alpha, \tag{3.22}$$

$$\Delta^1_{M,M} = \rho^1_E, \tag{3.23}$$

$$\rho_0(\theta^0 \dot{\eta}^1 + \theta^1 \dot{\eta}^0) = -Q^1_{M,M} + \rho_0 \gamma^1. \tag{3.24}$$

Introducing the constitutive relations (3.11)–(3.14) into the incremental equations of motion (3.22), the equation of the electric field (3.23) and the heat equation (3.24), for $f^1_\alpha = 0$ we have

$$G_{M\alpha L\gamma} u_{\gamma,LM} + R_{LM\alpha}\phi^1_{,LM} - \rho_0 \Lambda_{M\alpha}\theta^1_{,M} = \rho_0 \ddot{u}_\alpha, \tag{3.25}$$

$$R_{MN\gamma} u_{\gamma,NM} - L_{MN}\phi^1_{,NM} + \rho_0 P_M \theta^1_M = \rho^1_E, \tag{3.26}$$

$$\rho_0\theta^0(\Lambda_{M\gamma}\dot{u}_{\gamma,M} - P_M\dot{\phi}^1_{,M} + \alpha\dot{\theta}^1) + \rho_0\theta^1\dot{\eta}^0 = \kappa^E_{MN}\phi^1_{,NM} + \kappa_M\theta^1_{,M} + \kappa_{MN}\theta^1_{,NM} + \kappa_{MN\alpha}u_{\alpha,NM} + \rho_0\gamma^1. \tag{3.27}$$

4. Uniqueness theorem of the solution of the incremental differential equations

In the present section we assume $\dot{\eta}^0 = 0$ and $\Theta^0_L = 0$, i.e. the initial temperature field θ^0 is uniform. This holds true when the initial state is static. We follow step by step the proof of NOWACKI [2] and put in evidence any difference when it will appear.

A modified version of energy balance is needed. It follows the substitution of the virtual increments by the real increments

$$\delta u_\alpha = \frac{\partial u_\alpha}{\partial t} dt = v_\alpha dt, \qquad \delta u_{\alpha,M} = \dot{u}_{\alpha,M} dt, \quad \ldots$$

in the principle of virtual work

$$\int\limits_{V^0} (f^1_\alpha - \rho_0\ddot{u}_\alpha)\delta u_\alpha dV + \int\limits_{S^0} \widetilde{K}_\alpha \delta u_\alpha dS = \int\limits_{V^0} K^1_{M\alpha}\delta u_{\alpha,M} dV. \tag{4.1}$$

Thus the fundamental energy equation

$$\int\limits_{V^0} (f^1_\alpha - \rho_0\dot{v}_\alpha) v_\alpha dV + \int\limits_{S^0} \widetilde{K}_\alpha v_\alpha dS = \int\limits_{V^0} K^1_{M\alpha}\dot{u}_{\alpha,M} dV \tag{4.2}$$

is obtained, where we substitute the constitutive relations (3.11).

Incidentally, let us note that, by (3.22), multiplication by v_α and an obvious identity provide

$$\rho_0(f^1_\alpha - \dot{v}_\alpha)v_\alpha = -(K^1_{M\alpha}v_\alpha)_{,M} + K_{M\alpha}v_{\alpha,M}, \tag{4.3}$$

whence Eq. (4.2) follows.

Hence

$$\int_{V^0} (f^1_\alpha - \rho_0 \dot{v}_\alpha) v_\alpha dV + \int_{S^0} \widetilde{K}_\alpha v_\alpha dS = \int_{V^0} (G_{M\alpha L\gamma} u_{\gamma,L} + R_{LM\alpha} \phi^1_{,L} - \rho_0 \Lambda_{M\alpha} \theta^1) \dot{u}_{\alpha,M} dV, \tag{4.4}$$

thus

$$\frac{d}{dt}(\mathcal{W}+\mathcal{K}) = \int_{V^0} f^1_\alpha v_\alpha dV + \int_{S^0} \widetilde{K}_\alpha v_\alpha dS + \int_{V^0} (\rho_0 \Lambda_{M\alpha} \theta^1 - R_{LM\alpha} \phi^1_{,L}) \dot{u}_{\alpha,M} dV, \tag{4.5}$$

where $\mathcal{W}$ is the work of deformation and $\mathcal{K}$ is the kinetic energy:

$$\mathcal{W} = \frac{1}{2} \int_{V^0} G_{M\alpha L\gamma} u_{\alpha,M} u_{\gamma,L} dV, \qquad \mathcal{K} = \frac{1}{2} \int_{V^0} \rho_0 v_\alpha v_\alpha dV. \tag{4.6}$$

Now, to eliminate the term $\int_{V^0} \rho_0 \Lambda_{M\alpha} \theta^1 \dot{u}_{\alpha,M} dV$, we multiply by θ^1 the heat-conduction equation (3.27), where $\dot{\eta}^0 = 0$, and integrate over V^0; after simple transformations we obtain

$$\begin{aligned} \int_{V^0} \rho_0 \theta^1 \Lambda_{M\alpha} \dot{u}_{\alpha,M} dV &= \frac{\kappa^E_{ML}}{\theta^0} \int_{S^0} \theta^1 \phi^1_{,L} N_M dS \\ &+ \frac{\kappa_L}{\theta^0} \int_{S^0} \theta^1 N_L dS + \frac{\kappa_{ML}}{\theta^0} \int_{S^0} \theta^1 \theta^1_{,L} N_M dS + \frac{\kappa_{ML\alpha}}{\theta^0} \int_{S^0} \theta^1 u_{\alpha,L} N_M dS \\ &+ P_L \int_{V^0} \rho_0 \theta^1 \dot{\phi}^1_{,L} dV + \frac{1}{\theta^0} \int_{V^0} \rho_0 \theta^1 \gamma^1 dV - \frac{d}{dt}\mathcal{P} - (\chi + \chi_\theta + \chi_\phi + \chi_u), \end{aligned} \tag{4.7}$$

where

$$\mathcal{P} = \frac{\alpha}{2\theta^0} \int_{V^0} \rho_0 \theta^1 \theta^1 dV, \tag{4.8}$$

$$\begin{aligned} \chi_\phi &= \frac{\kappa^E_{ML}}{\theta^0} \int_{V^0} \theta^1_{,M} \phi^1_{,L} dV, \qquad & \chi &= \frac{\kappa_M}{\theta^0} \int_{V^0} \theta^1_{,M} \theta^1 dV, \\ \chi_\theta &= \frac{\kappa_{ML}}{\theta^0} \int_{V^0} \theta^1_{,M} \theta^1_{,L} dV, \qquad & \chi_u &= \frac{\kappa_{ML\alpha}}{\theta^0} \int_{V^0} \theta^1_{,M} u_{\alpha,L} dV. \end{aligned} \tag{4.9}$$

Note that this equation differs from the corresponding Eq. (25) in [2] by the terms χ_ϕ, χ and χ_u. Now, substituting (4.7) into (4.5), we are lead to the equation

$$\begin{aligned}(4.10)\qquad \frac{d}{dt}(\mathcal{W}+\mathcal{K}+\mathcal{P})+(\chi+\chi_\theta+\chi_\phi+\chi_u) &= \int\limits_{V^0} f^1_\alpha v_\alpha dV + \int\limits_{S^0} \widetilde{K}_\alpha v_\alpha dS \\ &+ \frac{\kappa^E_{ML}}{\theta^0}\int\limits_{S^0}\theta^1\phi^1_{,L}N_M dS + \frac{\kappa_L}{\theta^0}\int\limits_{S^0}\theta^1 N_L dS + \frac{\kappa_{ML}}{\theta^0}\int\limits_{S^0}\theta^1\theta^1_{,L}N_M dS \\ &+ \frac{1}{\theta^0}\int\limits_{V^0}\rho_0\theta^1\gamma^1 dV - \int\limits_{V^0}(R_{LM\alpha}\phi^1_{,L}\dot{u}_{\alpha,M}-\rho_0 P_M\theta^1\dot{\phi}^1_{,M})dV.\end{aligned}$$

To eliminate the term

$$\int\limits_{V^0}(R_{LM\alpha}\phi^1_{,L}\dot{u}_{\alpha,M}-\rho_0 P_M\theta^1\dot{\phi}^1_{,M})dV$$

in Eq. (4.10), we substitute the constitutive relations (3.12) into the time-derivative of the equation of the electric field (3.23) with $\rho^1_E = 0$. Multiplying the obtained equation by ϕ^1 and integrating over the region of the body, we obtain

$$(4.11)\qquad \int\limits_{S^0}\dot{\Delta}_M\phi^1 N_M dV + \int\limits_{V^0}\dot{\Delta}_M W^1_M dV = 0.$$

Using the relations (3.12) and (4.11), after simple transformations we obtain

$$\begin{aligned}\int\limits_{V^0}\dot{\Delta}_L W^1_L dV &= \int\limits_{V^0}\left(R_{LM\alpha}\dot{u}_{\alpha,M}W^1_L + L_{LM}\dot{W}^1_M W^1_L + \rho_0 P_L\frac{d}{dt}(\theta^1 W^1_L) - \rho_0 P_L\theta^1\dot{W}^1_L\right)dV \\ &= -\int\limits_{S^0}\dot{\Delta}^1_L N_L\phi^1 dS,\end{aligned}$$

from which

$$\begin{aligned}(4.12)\qquad &\int\limits_{V^0}(R_{KM\alpha}\dot{u}_{\alpha,M}W^1_K - \rho_0 P_K\theta^1\dot{W}^1_K)dV \\ &= -\int\limits_{S^0}\dot{\Delta}^1_K N_K\phi^1 dS - \frac{d}{dt}\mathcal{E} - \frac{d}{dt}\left(\rho_0 P_K\int\limits_{V^0}\theta^1 W^1_K dV\right),\end{aligned}$$

where

$$\mathcal{E} = \frac{1}{2} L_{KM} \int\limits_{V^0} W^1_M W^1_K dV. \tag{4.13}$$

In view of Eqs. (4.10) and (4.12), we arrive at the modified energy balance

$$\begin{aligned} \frac{d}{dt}\left(\mathcal{W} + \mathcal{K} + \mathcal{P} + \mathcal{E} + \rho_0 P_K \int\limits_{V^0} \theta^1 W^1_K dV \right) + (\chi + \chi_\theta + \chi_\phi + \chi_U) \\ = \int\limits_{V^0} f^1_\alpha v_\alpha dV + \int\limits_{S^0} \widetilde{K}_\alpha v_\alpha dS + \frac{\kappa^E_{ML}}{\theta^0} \int\limits_{S^0} \theta^1 \phi^1_{,L} N_M dS + \frac{\kappa_L}{\theta^0} \int\limits_{S^0} \theta^1 N_L dS \\ + \frac{\kappa_{ML}}{\theta^0} \int\limits_{S^0} \theta^1 \theta^1_{,L} N_M dS + \frac{1}{\theta^0} \int\limits_{V^0} \rho_0 \theta^1 \gamma^1 dV - \int\limits_{S^0} \dot{\Delta}^1_K N_K \phi^1 dS. \end{aligned} \tag{4.14}$$

The energy balance (4.14) makes possible the proof of the uniqueness of the solution.

We assume that two distinct solutions $(u'_i, \phi^{1\prime}, \theta^{1\prime})$ and $(u''_i, \phi^{1\prime\prime}, \theta^{1\prime\prime})$ satisfy Eqs. (3.22)–(3.24) and the appropriate boundary and initial conditions. Their difference

$$(\hat{u}_i = u'_i - u''_i,\ \hat{\phi} = \phi^{1\prime} - \phi^{1\prime\prime},\ \hat{\theta} = \theta^{1\prime} = \theta^{1\prime\prime})$$

satisfies therefore the homogeneous equations (3.22)–(3.24) and the homogeneous boundary and initial conditions. Equation (4.14) holds for $(\hat{u}_i, \hat{\phi}, \hat{\theta})$.

In view of homogeneity of the equations and the boundary conditions, the right-hand side of Eq. (4.14) vanishes. Hence

$$\frac{d}{dt}\left(\mathcal{W} + \mathcal{K} + \mathcal{P} + \mathcal{E} + \rho_0 P_K \int\limits_{V^0} \theta^1 W^1_K dV \right) = -(\chi + \chi_\theta + \chi_\phi + \chi_u) \leq 0, \tag{4.15}$$

where the last inequality is true since by (3.15), (4.9) and (3.20), we have

$$-(\chi + \chi_\theta + \chi_\phi + \chi_u) = \frac{1}{\theta^0} \int\limits_{V^0} Q^1_M \Theta^1_M dV \leq 0. \tag{4.16}$$

The integral on the left-hand side of Eq. (4.15) vanishes at the initial instant, since the functions $\hat{u}_i, \hat{\phi}, \hat{\theta}$ satisfy the homogeneous initial conditions. On the other hand, by the inequality in (4.15) the left-hand side is either negative or zero.

Now we assume (i)–(iii) below; note that (iii) is the sufficient condition of J. Ignaczak, written in [2] on pages 176–177.

(i) The initial deformation y^0_α realizes that the tensor $G_{M\alpha L\gamma}$ is positive definite, so that $\mathcal{W} \geq 0$ by (4.6).

(ii) The tensor L_{KN} is positive definite so that, by (4.13), $\mathcal{E} \geq 0$.

(iii) L_{IJ} is a known positive definite symmetric tensor, $g_I = \rho_0 P_I$ is a vector, and $c = \rho_0\alpha/2\theta^0 > 0$; consider the function

$$A(\theta^1, W_L) = \ (\theta^1)^2 + 2\theta^1 g_I W^1_I + L_{IJ} W^1_I W^1_J;$$

A is non-negative for every real pair (θ^1, W^1_k), provided

$$|g_I| \leq c\lambda_m$$

where λ_m is the smallest positive eigenvalue of the tensor L_{IJ}.

Under these three assumptions, (4.15) yields

$$\hat{u}_{i,L} = 0, \qquad \hat{\theta} = 0, \qquad \hat{W}_L = 0,$$

which imply the uniqueness of solutions of the incremental thermoelectroelastic equations, i.e.,

$$u'_i = u''_i, \qquad \theta^{1\prime} = \theta^{1\prime\prime}, \qquad W^{1\prime}_I = W^{1\prime\prime}_I.$$

Moreover, from the constitutive relations we obtain

$$K^1_{I\alpha}{}' = K^1_{I\alpha}{}'', \qquad \Delta^1_L{}' = \Delta^1_L{}'', \qquad \eta^{1\prime} = \eta^{1\prime\prime}.$$

5. On the generalized Hamilton's principle

We define the free energy, electric enthalpy, and potential of the heat flow respectively by

$$\psi^1 = \frac{1}{2} G_{M\alpha L\gamma} u_{\alpha,M} u_{\gamma,L} + R_{LM\alpha}\phi^1_{,L} u_{\alpha,M} - \rho_0\theta^1\left[\Lambda_{M\alpha} u_{\alpha,M} - P_M\phi^1_{,M} + \frac{\alpha}{2}\theta^1\right], \tag{5.1}$$

$$H^1 = \psi^1 - \frac{1}{2} L_{AB} W^1_A W^1_B = \psi^1 - \frac{1}{2} L_{AB}\Phi^1_{,A}\Phi^1_{,B}, \qquad \Gamma = Q^1_M \theta^1_{,M}. \tag{5.2}$$

Note that, by (3.15), the latter becomes

$$\Gamma = -\left(\kappa_{MN\alpha} u_{\alpha,N}\theta^1_{,M} + \frac{1}{2}\kappa_{MN}\theta^1_{,M}\theta^1_{,N} + \kappa^E_{MN}\theta^1_{,M}\phi^1_{,N} + \kappa_M\theta^1\theta^1_{,M}\right), \tag{5.3}$$

whence

$$\frac{\partial H^1}{\partial u_{\alpha,M}} = K^1_{M\alpha}, \quad \frac{\partial H^1}{\partial W^1_L} = -\Delta^1_L, \quad \frac{\partial H^1}{\partial \theta} = -\rho_0\eta^1, \tag{5.4}$$

$$Q^1_M = \frac{\partial \Gamma}{\partial \theta^1_{,M}}. \tag{5.5}$$

Finally, we define two functionals

$$\Pi = \int_{V^0} (H^1 + \rho_0\eta^1\theta^1 - f^1_\alpha u_\alpha)dV - \int_{S^0} (\widetilde{K}^1_\alpha u_\alpha - \widetilde{\Delta}^1\phi^1)dS \tag{5.6}$$

and

$$\Psi = \int_{V^0} \big(\Gamma - \rho_0(\eta^1\theta^0\dot{\theta}^1 + \eta^1\dot{\theta}^0\theta^1 + \eta^0\theta^1\dot{\theta}^1 + \gamma^1\theta^1)\big)dV + \int_{S^0} \theta^1\widetilde{Q}dS; \tag{5.7}$$

Eqs. (5.1)–(5.7) generalize Eqs. [2, (36)–(38)].

The generalized Hamilton's principle has the form

$$\delta\int_{t_1}^{t_2} (\mathcal{K} - \Pi)dt = 0, \qquad \delta\int_{t_1}^{t_2} \Psi dt = 0. \tag{5.8}$$

The virtual processes

$$(\delta u_\alpha, \delta\theta^1, \delta\phi^1)$$

of the body must be compatible with the conditions restricting the process of the body. Moreover, the virtual processes must satisfy the conditions

$$\delta u_\alpha(\mathbf{x}, t_1) = \delta u_\alpha(\mathbf{x}, t_2) = 0,$$
$$\delta\theta^1(\mathbf{x}, t_1) = \delta\theta^1(\mathbf{x}, t_2) = 0,$$
$$\delta\phi^1(\mathbf{x}, t_1) = \delta\phi^1(\mathbf{x}, t_2) = 0.$$

Hence, performing the variations in the first of Eqs. (5.8) and observing that

$$\delta H^1 = K^1_{M\alpha}\delta u_{\alpha,M} - \rho_0\eta^1\delta\theta^1 + \Delta^1_L\delta\Phi^1_{,L}, \tag{5.9}$$

and

$$\int_{t_1}^{t_2} (\mathcal{K} - \Pi)dt \tag{5.10}$$
$$= \int_{t_1}^{t_2} dt\bigg[\int_{V^0} \bigg(\frac{\rho_0}{2}\dot{u}_\alpha\dot{u}_\alpha - H^1 - \rho_0\eta^1\theta^1 + f^1_\alpha u_\alpha\bigg)dV + \int_{S^0} (\widetilde{K}^1_\alpha u_\alpha - \widetilde{\Delta}^1\phi^1)dS\bigg],$$

we have

$$
(5.11)\qquad \delta\int_{t_1}^{t_2}(\mathcal{K}-\Pi)dt
= \int_{t_1}^{t_2} dt\Bigg[\int_{V^0}(-\rho_0\ddot{u}_\alpha\delta u_\alpha - K^1_{M\alpha}\delta u_{\alpha,M} - \varDelta^1_L\delta\varPhi^1_{,L} + f^1_\alpha\delta u_\alpha)dV
+ \int_{S^0}(\widetilde{K}^1_\alpha\delta u_\alpha - \widetilde{\varDelta}^1\delta\phi^1)dS\Bigg].
$$

Hence, by the identities

$$
(5.12)\qquad \begin{aligned}
-K^1_{L\alpha}(\delta u_\alpha)_{,L} &= -(K^1_{L\alpha}\delta u_\alpha)_{,L} + (K^1_{L\alpha,L})\delta u_\alpha,\\
\varDelta^1_L(\delta\phi^1)_{,L} &= (\varDelta^1_L\delta\phi^1)_{,L} - (\varDelta^1_{L,L})\delta\phi^1,
\end{aligned}
$$

we have

$$
(5.13)\qquad \delta\int_{t_1}^{t_2}(\mathcal{K}-\Pi)dt
= \int_{t_1}^{t_2} dt\Bigg[\int_{V^0}\big[(-\rho_0\ddot{u}_\alpha\delta u_\alpha + K^1_{M\alpha,M} + f^1_\alpha)\delta u_\alpha + \varDelta^1_{M,M}\delta\phi^1\big]dV
+ \int_{S^0}(-K^1_{M\alpha}\delta u_\alpha N_M dS - \varDelta^1_M\delta\phi^1 N_M)dS + \int_{S^0}(\widetilde{K}^1_\alpha\delta u_\alpha - \widetilde{\varDelta}^1\delta\phi^1)dS\Bigg].
$$

Thus we have

$$
(5.14)\qquad \int_{t_1}^{t_2} dt\Bigg[\int_{V^0}(-\rho_0\ddot{u}_\alpha + K^1_{M\alpha,M} + f^1_\alpha)\delta u_\alpha dV + \int_{V^0}\varDelta^1_{M,M}\delta\phi^1 dV
+ \int_{S^0}(\widetilde{K}^1_\alpha - K^1_{M\alpha}N_M)\delta u_\alpha dS - \int_{S^0}(\widetilde{\varDelta}^1 + \varDelta^1_M N_M)\delta\phi^1 dS\Bigg] = 0.
$$

Since the variations δu_α and $\delta\phi^1$ are arbitrary, Eq. (5.14) is equivalent to the equations governing the incremental motion and electric field, completed by the appropriate boundary conditions. These equations and boundary conditions coincide with those written above.

Next we perform the required variation in the second of Eqs. (5.8) by observing that

$$\delta\Gamma = \frac{\partial\Gamma}{\partial u_{\alpha,N}}\delta u_{\alpha,N} + \frac{\partial\Gamma}{\partial\theta^1_{,L}}\delta\theta^1_{,L} + \frac{\partial\Gamma}{\partial\phi^1_{,L}}\delta\phi^1_{,L} + \frac{\partial\Gamma}{\partial\theta^1}\delta\theta^1 = -\kappa_{MN\alpha}\theta^1_{,M}\delta u_{\alpha,N} + Q^1_L\delta\theta^1_{,L} - \kappa^E_{MN}\theta^1_{,M}\delta\phi^1_{,L} - \kappa_M\theta^1_{,M}\delta\theta^1. \tag{5.15}$$

By (5.7) we have

$$\delta\int_{t_1}^{t_2}\Psi dt = \int_{t_1}^{t_2} dt\bigg[\int_{V^0}\big(\delta\Gamma - \rho_0\eta^1(\theta^0\delta\dot\theta^1 + \dot\theta^0\delta\theta^1) - \rho_0\eta_0(\theta^1\delta\dot\theta^1 + \dot\theta^1\delta\theta^1) - \rho_0\gamma^1\delta\theta^1\big)dV + \int_{S^0}\delta\theta^1\widetilde{Q}dS\bigg]$$
$$= \int_{t_1}^{t_2} dt\bigg[\int_{V^0}\big(-\kappa_{ML\alpha}\theta^1_{,M}\delta u_{\alpha,L} + Q^1_L\delta\theta^1_{,L} - \kappa^E_{ML}\theta^1_{,M}\delta\phi^1_{,L} - \kappa_M\theta^1_{,M}\delta\theta^1 + \rho_0\big[\dot\eta^1\theta^0\delta\theta^1 - \dot{\overline{(\eta^1\theta^0\delta\theta^1)}}\big] + \rho_0\big[\dot\eta^0\theta^1\delta\theta^1 - \dot{\overline{(\eta^0\theta^1\delta\theta^1)}}\big] - \rho_0\gamma^1\delta\theta^1\big)dV + \int_{S^0}\delta\theta^1\widetilde{Q}dS\bigg]. \tag{5.16}$$

Note that

$$\int_{t_1}^{t_2}\dot{\overline{(\eta^\nu\theta^\tau\delta\theta^1)}}dt = \big[\eta^\nu\theta^\tau\delta\theta^1\big]^{t_2}_{t_1} = 0, \qquad (\nu,\tau = 0,1), \tag{5.17}$$

since $\delta\theta^1 = 0$ at t_1 and t_2. Also, by using the identity

$$(a_L b)_{,L} = a_{L,L}b + a_L b_{,L} \tag{5.18}$$

we obtain

$$\delta\int_{t_1}^{t_2}\Psi dt = \int_{t_1}^{t_2} dt\bigg[\int_{V^0}\big(Q^1_{L,L} + \rho_0[\dot\eta^1\theta^0 + \dot\eta^0\theta^1 - \gamma^1]\big)\delta\theta^1 dV - \int_{S^0}(Q^1_L N_L - \widetilde{Q})\delta\theta^1 dS\bigg] - \int_{t_1}^{t_2} dt\bigg[\int_{V^0}(\kappa_{ML\alpha}\theta^1_{,M}\delta u_{\alpha,L} + \kappa^E_{IJ}\theta^1_{,I}\delta\phi^1_{,J} + \kappa_L\theta^1_{,L}\delta\theta^1)dV\bigg] \tag{5.19}$$

with

$$\int_{V^0} \kappa_{ML\alpha}\theta^1_{,M}\delta u_{\alpha,L}dV = \kappa_{ML\alpha}\left[-\int_{V^0}\theta^1_{,ML}\delta u_\alpha dV + \int_{S^0}\theta^1_{,M}N_L\delta u_\alpha dS\right], \tag{5.20}$$

$$\int_{V^0} \kappa^E_{ML}\theta^1_{,M}\delta\phi^1_{,L}dV = \kappa^E_{ML}\left[-\int_{V^0}\theta^1_{,ML}\delta\phi^1 dV + \int_{S^0}\theta^1_{,M}N_L\delta\phi^1 dS\right]. \tag{5.21}$$

Hence, by performing the variation (5.19) with the variations δu_α, $\delta\phi^1$ which vanish, and with $\delta\theta^1$ being arbitrary, we obtain that (5.19) reduces to

$$\delta\int_{t_1}^{t_2}\Psi dt = \int_{t_1}^{t_2} dt\left[\int_{V^0}\left(Q^1_{L,L} - \kappa_L\theta^1_{,L} + \rho_0[\dot{\eta}^1\theta^0 + \dot{\eta}^0\theta^1 - \gamma^1]\right)\delta\theta^1 dV - \int_{S^0}(Q^1_L N_L - \widetilde{Q})\delta\theta^1 dS\right]. \tag{5.22}$$

Thus,

(i) *the variational equation* $(5.8)_2$ *performed with*

$$\delta u_\alpha = 0 = \delta\phi^1 \tag{5.23}$$

is equivalent to the entropy balance

$$Q^1_{L,L} + \rho_0(\dot{\eta}^1\theta^0 + \dot{\eta}^0\theta^1 - \gamma^1) = 0 \tag{5.24}$$

and the boundary condition for the heat flow

$$Q^1_L N_L = \widetilde{Q}, \qquad (\mathbf{x}\in S), \tag{5.25}$$

if and only if the expression (3.15) *of the heat flux holds for*

$$\kappa_L = 0. \tag{5.26}$$

Alternatively, by performing the variation $(5.8)_2$ with all the variations δu_α, $\delta\phi^1$, $\delta\theta^1$ being arbitrary, we deduce that:

(ii) *the variational equation* $(5.8)_2$ *is equivalent to the entropy balance* (5.24) *and the boundary condition for the heat flow* (5.25) *if and only if the expression* (3.15) *of the heat flux holds for*

$$\kappa_L = 0, \qquad \kappa^E_{ML} = 0, \qquad \kappa_{ML\alpha} = 0. \tag{5.27}$$

6. Theorem of reciprocity of work

Next we extend the theorem of reciprocity of work following some steps in [2] on pages 179–182, where it is referred to linear thermoelectroelasticity in a natural configuration. Here there are some essential changes imposed by the presence of the initial fields. We assume that the body is homogeneous and moreover, that the initial state is static, so that in particular $\dot{\theta}^0 = 0$, $\dot{\eta}^0 = 0$. Here we do not assume that θ^0 is uniform.

The Laplace transform of functions $\nu = \nu(\mathbf{x}, t)$,

$$\overline{\nu}(\mathbf{x}, p) = \int_0^\infty e^{-pt}\nu(\mathbf{x}, t)dt, \tag{6.1}$$

will be used below.

Consider two sets of causes $\mathcal{A}^1$, $\mathcal{A}^{1\prime}$ for incremental processes and the respective effects $(u_\alpha, \phi^1, \theta^1)$, $(u'_\alpha, \phi^{1\prime}, \theta^{1\prime})$. Starting from the equations of motion

$$K^1_{L\alpha,L} + \rho_0 f_\alpha = \rho_0 \ddot{u}_\alpha, \qquad K^{1\prime}_{L\alpha,L} + \rho_0 f'_\alpha = \rho_0 \ddot{u}'_\alpha, \tag{6.2}$$

taking their Laplace transform, multiplying each by $\overline{\theta^0}$, then multiplying the first one by $\overline{u}'_\alpha$ and the second one by $\overline{u}_\alpha$, and making the difference of their integrals over the instantaneous region V, assuming that the initial conditions for the displacements are homogeneous, we obtain the integral equation:

$$\int_{V^0} \overline{\theta^0}\big(\overline{F}_\alpha \overline{u}'_\alpha - \overline{F}'_\alpha \overline{u}_\alpha\big)dV + \int_{V^0} \overline{\theta^0}\big(\overline{K^1_{L\alpha,L}}\,\overline{u'_\alpha} - \overline{K^{1\prime}_{L\alpha,L}}\,\overline{u_\alpha}\big)dV = 0, \tag{6.3}$$

where $F_\alpha = \rho_0 f_\alpha$, $F'_\alpha = \rho_0 f'_\alpha$. Now, by the identity (5.18) and the divergence theorem, we have

$$\begin{aligned}\int_{V^0} \overline{\theta^0}\big(\overline{K^1_{L\alpha,L}}\,\overline{u'_\alpha} - \overline{K^{1\prime}_{L\alpha,L}}\,\overline{u_\alpha}\big)dV &= \int_{S^0} \overline{\theta^0}\big(\overline{K^1_{L\alpha}}\,\overline{u'_\alpha} - \overline{K^{1\prime}_{L\alpha}}\,\overline{u_\alpha}\big)N_L dS \\ &\quad - \int_{V^0} \big(\overline{K^1_{L\alpha}}(\overline{\theta^0}\,\overline{u'_\alpha})_{,L} - \overline{K^{1\prime}_{L\alpha}}(\overline{\theta^0}\,\overline{u_\alpha})_{,L}\big)dV,\end{aligned}$$

hence

$$\begin{aligned}\int_{V^0} \overline{\theta^0}\Big(\overline{K^1_{L\alpha,L}}\,\overline{u'_\alpha} - \overline{K^{1\prime}_{L\alpha,L}}\,\overline{u_\alpha}\Big)dV &= \int_{S^0} \overline{\theta^0}\big(\overline{K^1_{L\alpha}}\,\overline{u'_\alpha} - \overline{K^{1\prime}_{L\alpha}}\,\overline{u_\alpha}\big)N_L dS \\ - \int_{V^0} (\overline{\theta^0})_{,L}\big(\overline{K^1_{L\alpha}}\,\overline{u'_\alpha} - \overline{K^{1\prime}_{L\alpha}}\,\overline{u_\alpha}\big)dV &- \int_{V^0} \overline{\theta^0}\big(\overline{K^1_{L\alpha}}(\overline{u'_\alpha})_{,L} - \overline{K^{1\prime}_{L\alpha}}(\overline{u_\alpha})_{,L}\big)dV.\end{aligned} \tag{6.4}$$

Hence by the latter equation and the constitutive relations (3.11), Eq. (6.3) becomes

$$\int\limits_{V^0}\overline{\theta^0}\big(\overline{F}_\alpha\overline{u}'_\alpha-\overline{F}'_\alpha\overline{u}_\alpha\big)dV+\int\limits_{S^0}\overline{\theta^0}\big(\overline{K}^1_{L\alpha}\overline{u}'_\alpha-\overline{K}^{1\prime}_{L\alpha}\overline{u}_\alpha\big)N_LdS$$
$$+\int\limits_{V^0}\overline{\theta^0}\Big[\rho_0\Lambda_{L\alpha}\big(\overline{\theta}^{1\prime}\overline{u}_{\alpha,L}-\overline{\theta}^1\overline{u}'_{\alpha,L}\big)+R_{LN\gamma}\big(\overline{u_{\gamma,N}}\,\overline{W^{1\prime}_L}-\overline{u'_{\gamma,N}}\,\overline{W^1_L}\big)\Big]dV$$
$$-\int\limits_{V^0}(\overline{\theta^0})_{,L}\big(\overline{K^1_{L\alpha}}\,\overline{u'}_\alpha-\overline{K^{1\prime}_{L\alpha}}\,\overline{u}_\alpha\big)dV=0, \tag{6.5}$$

which is an analogue of Eq. (54) in [2].

Next we shall make use of the heat-conduction equation (5.24) for both the systems of loadings, rewritten in the form

$$-\overline{\left(\frac{1}{\theta^0}Q^1_{M,M}\right)}-\rho_0\overline{\dot{\eta}^1}=-\rho_0\overline{\left(\frac{\gamma^1}{\theta^0}\right)}, \tag{6.6}$$

since we have

$$\overline{\dot{\eta}^0}=0. \tag{6.7}$$

Hence by Eqs. (3.15) and (3.13) we obtain

$$\left(\kappa_{LN\alpha}\frac{\overline{u_{\alpha,NL}}}{\theta^0}+\kappa^E_{MN}\frac{\overline{\phi^1_{,NM}}}{\theta^0}+\kappa_L\frac{\overline{\theta^1_{,L}}}{\theta^0}+\kappa_{MN}\frac{\overline{\theta^1_{,NM}}}{\theta^0}\right)$$
$$-p\rho_0\big(\Lambda_{M\gamma}\overline{u}_{\gamma,M}-P_M\overline{\phi}^1_{,M}+\alpha\overline{\theta}^1\big)=-\rho_0\overline{\left(\frac{\gamma^1}{\theta^0}\right)}. \tag{6.8}$$

Multiplying the latter by $\overline{\theta^0}$ we have

$$\overline{\theta^0}\left(\kappa_{LN\alpha}\frac{\overline{u_{\alpha,NL}}}{\theta^0}+\kappa^E_{MN}\frac{\overline{\phi^1_{,NM}}}{\theta^0}+\kappa_L\frac{\overline{\theta^1_{,L}}}{\theta^0}+\kappa_{MN}\frac{\overline{\theta^1_{,NM}}}{\theta^0}\right)$$
$$-p\rho_0\overline{\theta^0}\big(\Lambda_{M\gamma}\overline{u}_{\gamma,M}-P_M\overline{\phi}^1_{,NM}+\alpha\overline{\theta}^1\big)=-\overline{\theta^0}\rho_0\overline{\left(\frac{\gamma^1}{\theta^0}\right)}. \tag{6.9}$$

Write the latter equality for both the states, multiply the first equation by $\overline{\theta^1}'$ and the second by $\overline{\theta^1}$; we obtain

$$\overline{\theta^1}'\overline{\theta^0}\left(\kappa_{LN\alpha}\frac{\overline{u_{\alpha,NL}}}{\theta^0}+\kappa^E_{MN}\frac{\overline{\phi^1_{,NM}}}{\theta^0}+\kappa_L\frac{\overline{\theta^1_{,L}}}{\theta^0}+\kappa_{MN}\frac{\overline{\theta^1_{,NM}}}{\theta^0}\right)$$
$$-p\rho_0\overline{\theta^1}'\overline{\theta^0}\left(\Lambda_{M\gamma}\overline{u}_{\gamma,M}-P_M\overline{\phi}^1_{,NM}+\alpha\overline{\theta}^1\right)=-\overline{\theta^1{}'\theta^0}\rho_0\overline{\left(\frac{\gamma^1}{\theta^0}\right)}, \tag{6.10}$$

and

$$\overline{\theta^1\theta^0}\Bigg(\kappa_{LN\alpha}\frac{\overline{u_{\alpha,NL}{}'}}{\theta^0}+\kappa^E_{MN}\frac{\overline{\phi^1_{,MN}{}'}}{\theta^0}+\kappa_L\frac{\overline{\theta^1_{,L}{}'}}{\theta^0}+\kappa_{MN}\frac{\overline{\theta^1_{,MN}{}'}}{\theta^0}\Bigg)$$
$$-\,p\rho_0\overline{\theta^1\theta^0}\big(\varLambda_{M\gamma}\overline{u}'{}_{\gamma,M}-P_M\overline{\phi}^{1\prime}_{,M}+\alpha\overline{\theta}^{1\prime}\big)=-\overline{\theta^1\theta^0}\rho_0\overline{\left(\frac{\gamma^{1\prime}}{\theta^0}\right)}. \tag{6.11}$$

By taking the integral over V of the difference between the last two equations, we obtain the analogue of Eq. (57) in [2], that is,

$$\kappa_{LN\alpha}\int\limits_{V^0}\overline{\theta^0}\Bigg(\overline{\theta^1}'\frac{\overline{u_{\alpha,NL}}}{\theta^0}-\overline{\theta^1}\frac{\overline{u'_{\alpha,NL}}}{\theta^0}\Bigg)dV+\kappa^E_{MN}\int\limits_{V^0}\overline{\theta^0}\Bigg(\overline{\theta^{1\prime}}\frac{\overline{\phi^1_{,MN}}}{\theta^0}-\overline{\theta^1}\frac{\overline{\phi^{1\prime}_{,M}}}{\theta^0}\Bigg)dV$$
$$+\kappa_L\int\limits_{V^0}\overline{\theta^0}\Bigg(\overline{\theta^{1\prime}}\frac{\overline{\theta^1_{,L}}}{\theta^0}-\overline{\theta^1}\frac{\overline{\theta^{1\prime}_{,L}}}{\theta^0}\Bigg)dV+\kappa_{MN}\int\limits_{V^0}\overline{\theta^0}\Bigg(\overline{\theta^{1\prime}}\frac{\overline{\theta^1_{,NM}}}{\theta^0}-\overline{\theta^1}\frac{\overline{\theta^{1\prime}_{,M}}}{\theta^0}\Bigg)dV$$
$$+p\int\limits_{V^0}\overline{\rho}_0\overline{\theta^0}\Big[\overline{\theta^{1\prime}}\big(-\varLambda_{M\gamma}\overline{u}_{\gamma,M}-P_M\overline{W^1_M}\big)+\overline{\theta^1}\big(\varLambda_{M\gamma}\overline{u}'{}_{\gamma,M}+P_M\overline{W^{1\prime}_M}\big)\Big]dV$$
$$+\int\limits_{V^0}\rho_0\overline{\theta^0}\Bigg(\overline{\theta^1}\frac{\overline{\gamma^{1\prime}}}{\theta^0}-\overline{\theta^{1\prime}}\frac{\overline{\gamma^1}}{\theta^0}\Bigg)dV=0. \tag{6.12}$$

Finally, we make use of the equation for the electric field

$$\overline{\varDelta^1}_{L,L}=0,\qquad \overline{\varDelta^{1}}'{}_{L,L}=0. \tag{6.13}$$

Multiplying both expressions by $\overline{\theta^0}$, the first one by $\overline{\phi}'$, the second one by $\overline{\phi}$, subtracting the results and integrating over the region of the body, we obtain

$$\int\limits_{V^0}\big(\overline{\varDelta^1{}_{L,L}}(\overline{\theta^0\phi^{1\prime}})-\overline{\varDelta^{1\prime}_{L,L}}(\overline{\theta^0\phi^1})\big)dV=0. \tag{6.14}$$

By the identity (5.18) we have

$$\int\limits_{S^0}\overline{\theta^0}\big(\overline{\varDelta^1}_L\overline{\phi^{1\prime}}-\overline{\varDelta^{1\prime}_L}\overline{\phi^1}\big)N_L dS-\int\limits_{V^0}\big[\overline{\varDelta^1}_L(\overline{\theta^0\phi^{1\prime}})_{,L}-\overline{\varDelta^{1\prime}}_L(\overline{\theta^0\phi^1})_{,L}\big]dV=0, \tag{6.15}$$

and thus

$$\int\limits_{S^0}\overline{\theta^0}\big(\overline{\varDelta^1{}_L\phi^{1\prime}}-\overline{\varDelta^1{}_L{}'\phi^1}\big)N_L dS-\int\limits_{V^0}(\overline{\theta^0})_{,L}\big(\overline{\varDelta^1}_L\overline{\phi^{1\prime}}-\overline{\varDelta^1{}_L{}'\phi^1}\big)dV$$
$$-\int\limits_{V^0}\overline{\theta^0}\big[\overline{\varDelta^1}_L(\overline{\phi^{1\prime}})_L-\overline{\varDelta^{1\prime}}_L(\overline{\phi^1})_{,L}\big]dV=0, \tag{6.16}$$

$$\int_{S^0} \overline{\theta^0}\big(\overline{\Delta^1{}_L}\,\overline{\phi^{1\prime}} - \overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\big)N_L dS - \int_{V^0} (\overline{\theta^0})_{,L}\big(\overline{\Delta^1{}_L}\,\overline{\phi^{1\prime}} - \overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\big)dV + \int_{V^0} \overline{\theta^0}\big(\overline{\Delta^1{}_L}\,\overline{W^{1\prime}_L} - \overline{\Delta^{1\prime}_L}\,\overline{W^1_L}\big)dV = 0. \tag{6.17}$$

Now we substitute the constitutive relation

$$\overline{\Delta^1{}_L} = R_{LN\gamma}\overline{u_{\gamma,N}} - L_{LN}\overline{\phi^1_{,N}} + \rho_0 P_L\overline{\theta^1}$$

in the third integral of the last equation. We obtain

$$\int_{S^0} \overline{\theta^0}\big(\overline{\Delta^1{}_L}\,\overline{\phi^{1\prime}} - \overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\big)N_L dS - \int_{V^0} (\overline{\theta^0})_{,L}\big(\overline{\Delta^1{}_L}\,\overline{\phi^{1\prime}} - \overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\big)dV + \int_{V^0} \overline{\theta^0}\Big[\big(R_{LN\gamma}\overline{u_{\gamma,N}} - L_{LN}\overline{\phi^1_{,N}} + \rho_0 P_L\overline{\theta^1}\big)\overline{W^1_{L\prime}} - \big(R_{LN\gamma}\overline{u'_{\gamma,N}} - L_{LN}\overline{\phi^{1\prime}_{,N}} + \rho_0 P_L\overline{\theta^{1\prime}}\big)\overline{W^1_L}\Big]dV = 0. \tag{6.18}$$

Thus

$$\int_{S^0} \overline{\theta^0}\big(\overline{\Delta^1{}_L}\,\overline{\phi^{1\prime}} - \overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\big)N_L dS - \int_{V^0} (\overline{\theta^0})_{,L}\big(\overline{\Delta^1{}_L}\,\overline{\phi^{1\prime}} - \overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\big)dV + \int_{V^0} \overline{\theta^0}\Big[R_{LN\gamma}\big(\overline{u_{\gamma,N}}\,\overline{W^{1\prime}_L} - \overline{u'_{\gamma,N}}\,\overline{W^1_L}\big) + \rho_0 P_L\big(\overline{\theta^1}\overline{W^{1\prime}_L} - \overline{\theta^{1\prime}}\,\overline{W^1_L}\big)\Big]dV = 0. \tag{6.19}$$

This equation is the analogue of Eq. [2, (61)].

Taking the expression for

$$\int_{V^0} \overline{\theta^0} R_{LN\gamma}\big(\overline{u_{\gamma,N}}\,\overline{W^{1\prime}_L} - R_{LN\gamma}\overline{u'_{\gamma,N}}\,\overline{W^1_L}\big)dV \tag{6.20}$$

deduced from (6.19) and inserting this into (6.5), we obtain

$$\begin{aligned} -\int_{V^0} \overline{\theta^0}\Big[\rho_0 \Lambda_{L\alpha}\big(\overline{\theta}^1\overline{u}'_{\alpha,L} - \overline{\theta}^{1\prime}\overline{u}_{\alpha,L}\big)\Big]dV &= \int_{V^0} \overline{\theta^0}\big(\overline{F}_\alpha\overline{u}'_\alpha - \overline{F}'_\alpha\overline{u}_\alpha\big)dV + \int_{S^0} \overline{\theta^0}\big(\overline{K}^1_{L\alpha}\,\overline{u}'_\alpha - \overline{K}^{1\prime}_{L\alpha}\,\overline{u}_\alpha\big)N_L dS \\ &+ \int_{S^0} \overline{\theta^0}\big(\overline{\Delta^1{}_L}\,\overline{\phi^{1\prime}} + \overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\big)N_L dS - \int_{V^0} (\overline{\theta^0})_{,L}\big(\overline{\Delta^1{}_L}\,\overline{\phi^{1\prime}} - \overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\big)dV \\ &- \int_{V^0} \overline{\theta^0}\rho_0 P_L\big(\overline{\theta^1}\,\overline{W^{1\prime}_L} - \overline{\theta^{1\prime}}\,\overline{W^1_L}\big)dV - \int_{V^0} (\overline{\theta^0})_{,L}\big(\overline{K^1_{L\alpha}}\,\overline{u'_\alpha} - \overline{K^{1\prime}_{L\alpha}}\,\overline{u_\alpha}\big)dV. \end{aligned} \tag{6.21}$$

Now inserting (6.21) in (6.12), we obtain

$$\begin{aligned}
(6.22)\quad &\kappa_{LN\alpha}\int\limits_{V^0}\overline{\theta^0}\Big(\overline{\theta^1}'\frac{\overline{u_{\alpha,NL}}}{\theta^0}-\overline{\theta^1}\frac{\overline{u'_{\alpha,NL}}}{\theta^0}\Big)dV+\kappa^E_{MN}\int\limits_{V^0}\overline{\theta^0}\Big(\overline{\theta^{1\prime}}\frac{\overline{\phi^1_{,MN}}}{\theta^0}-\overline{\theta^1}\frac{\overline{\phi^{1\prime}_{,MN}}}{\theta^0}\Big)dV\\
&+\kappa_L\int\limits_{V^0}\overline{\theta^0}\Big(\overline{\theta^{1\prime}}\frac{\overline{\theta^1_{,L}}}{\theta^0}-\overline{\theta^1}\frac{\overline{\theta^1_{,L}{}'}}{\theta^0}\Big)dV+\kappa_{MN}\int\limits_{V^0}\overline{\theta^0}\Big(\overline{\theta^{1\prime}}\frac{\overline{\theta^1_{,NM}}}{\theta^0}-\overline{\theta^1}\frac{\overline{\theta^{1\prime}_{,NM}}}{\theta^0}\Big)dV\\
&+p\int\limits_{V^0}\overline{\rho}_0\overline{\theta^0}\big(-\overline{\theta^{1\prime}}P_M\overline{W^1_M}+\overline{\theta^1}P_M\overline{W^{1\prime}_M}\big)dV\\
&+p\Bigg[\int\limits_{V^0}\overline{\theta^0}\big(\overline{F}_\alpha\overline{u}'_\alpha-\overline{F}'_\alpha\overline{u}_\alpha\big)dV+\int\limits_{S^0}\overline{\theta^0}\big(\overline{K}^1_{L\alpha}\overline{u}'_\alpha-\overline{K}^{1\prime}_{L\alpha}\overline{u}_\alpha\big)N_LdS\\
&+\int\limits_{S^0}\overline{\theta^0}\big(\overline{\Delta^1}_L\overline{\phi^{1\prime}}-\overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\big)N_LdS-\int\limits_{V^0}(\overline{\theta^0})_{,L}\big(\overline{\Delta^1}_L\overline{\phi^{1\prime}}-\overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\big)dV\\
&-\int\limits_{V^0}\overline{\theta^0}\rho_0P_L\big(\overline{\theta^1}\,\overline{W^{1\prime}_L}-\overline{\theta^{1\prime}}\,\overline{W^1_L}\big)dV-\int\limits_{V^0}\overline{(\theta^0)_{,L}}\Big(\overline{K^1_{L\alpha}}\,\overline{u'_\alpha}-\overline{K^{1\prime}_{L\alpha}}\,\overline{u_\alpha}\Big)dV\Bigg]\\
&+\int\limits_{V^0}\rho_0\overline{\theta^0}\Big(\overline{\theta^1}\frac{\overline{\gamma^{1\prime}}}{\theta^0}-\overline{\theta^{1\prime}}\frac{\overline{\gamma^1}}{\theta^0}\Big)dV=0.
\end{aligned}$$

Next in the latter equality we transform the sum of the first four integrals. Firstly note that by (6.1), we have

$$\begin{aligned}
&\overline{1}=\int\limits_0^\infty e^{-pt}dt=1/p,\\
&\theta^0=\theta^0(\mathbf{x})\ \Rightarrow\ \overline{\theta^0}=\theta^0/p,\\
(6.23)\quad &\overline{\Big(\frac{h(\mathbf{x},t)}{f(\mathbf{x})}\Big)}=\frac{1}{f(\mathbf{x})}\int\limits_0^\infty e^{-pt}h(\mathbf{x},t)dt=\frac{1}{f(\mathbf{x})}\overline{h(\mathbf{x},t)},\\
&\kappa_{\dots}\int\limits_{V^0}\overline{\theta^0}\Big(\overline{\theta^{1\prime}}\frac{\overline{f_{\dots}}}{\theta^0}-\overline{\theta^1}\frac{\overline{f'_{\dots}}}{\theta^0}\Big)dV=\frac{\kappa_{\dots}}{p}\int\limits_{V^0}\big(\overline{\theta^{1\prime}}\,\overline{f_{\dots}}-\overline{\theta^1}\,\overline{f'_{\dots}}\big)dV.
\end{aligned}$$

Hence by these equalities and the constitutive relation for the incremental heat flux (3.15), the aforementioned sum of the four integrals equals

$$(6.24)\qquad \frac{1}{p}\int\limits_{V^0}\big(\overline{\theta^{1\prime}}\,\overline{Q^1_{L,L}}-\overline{\theta^1}\,\overline{Q^{1\prime}_{L,L}}\big)dV.$$

Again, by the identity

$$ab_{,ML} = (ab_{,M})_{,L} - a_{,L}b_{,M}$$

and the divergence theorem, the sum (6.24) equals

$$\frac{1}{p}\left[\int_{S^0} \left(\overline{\theta^{1\prime}}\,\overline{Q^1_L} - \overline{\theta^1}\,\overline{Q^{1\prime}{}_L}\right) N_L dS - \int_{V^0} \left(\overline{\theta^{1\prime}_{,L}}\,\overline{Q^1_L} - \overline{\theta^1_{,L}}\,\overline{Q^{1\prime}_{,L}}\right) dV\right]. \tag{6.25}$$

By substituting the sum of the first four integrals in Eq. (6.22) by (6.25), we obtain

$$\begin{aligned}
&\frac{1}{p}\left[\int_{S^0} \left(\overline{\theta^{1\prime}}\,\overline{Q^1_L} - \overline{\theta^1}\,\overline{Q^{1\prime}{}_L}\right) N_L dS - \int_{V^0} \left(\overline{\theta^{1\prime}_{,L}}\,\overline{Q^1_L} - \overline{\theta^1_{,L}}\,\overline{Q^{1\prime}_{,L}}\right) dV\right] \\
&\qquad + pP_M \int_{V^0} \overline{\rho}_0\, \overline{\theta^0}\left(-\overline{\theta^{1\prime}}\,\overline{W^1_M} + \overline{\theta^1}\,\overline{W^{1\prime}_M}\right) dV \\
&+ p\left[\int_{V^0} \overline{\theta^0}\left(\overline{F}_\alpha \overline{u}'_\alpha - \overline{F}'_\alpha\, \overline{u}_\alpha\right) dV + \int_{S^0} \overline{\theta^0}\left(\overline{K}^1_{L\alpha}\overline{u}'_\alpha - \overline{K}^{1'}_{L\alpha}\,\overline{u}_\alpha\right) N_L dS\right. \\
&+ \int_{S^0} \overline{\theta^0}\left(\overline{\Delta^1{}_L}\,\overline{\phi^{1\prime}} - \overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\right) N_L dS - \int_{V^0} (\overline{\theta^0})_{,L}\left(\overline{\Delta^1{}_L}\,\overline{\phi^{1\prime}} - \overline{\Delta^{1\prime}_L}\,\overline{\phi^1}\right) dV \\
&\left. - \int_{V^0} \overline{\theta^0}\rho_0 P_L\left(\overline{\theta^1}\,\overline{W^{1\prime}_L} - \overline{\theta^{1\prime}}\,\overline{W^1_L}\right) dV - \int_{V^0} \overline{(\theta^0)_{,L}}\left(\overline{K^1_{L\alpha}}\,\overline{u'_\alpha} - \overline{K^{1\prime}_{L\alpha}}\,\overline{u_\alpha}\right) dV\right] \\
&\qquad + \int_{V^0} \rho_0\left(\overline{\theta^1}\,\overline{\frac{\gamma^{1\prime}}{\theta^0}} - \overline{\theta^{1\prime}}\,\overline{\frac{\gamma^1}{\theta^0}}\right) dV = 0.
\end{aligned} \tag{6.26}$$

The latter is the final form of the *theorem of reciprocity of work*, containing all causes and effects. It generalizes Eq. [2, (62)], and reduces exactly to the latter in case of vanishing of the initial fields, that is, when the initial configuration is natural.

References

1. W. NOWACKI, *A Reciprocity Theorem for Coupled Mechanical and Thermoelectric Fields in Piezoelectric Crystals*, Proc. Vibrations Probl., **6**, 1, 3–11, 1965.
2. W. NOWACKI, *Some general Theorems of Thermopiezoelectricity*, J. of Thermal Stresses, **1**, 171–182, 1978.

3. J.Y. LI, *Uniqueness and reciprocity theorems for linear thermo-electro-magnetoelasticity*, Q. J. Mech. Appl. Math., **56**, 35–43, 2003.

4. M. AOUADI, *The Generalized Theory of Thermo-Magnetoelectroelasticity*, Technische Mechanik, **27**, 2, 133–146, 2007.

5. I.M. MÜLLER, *The coldness a universal function in thermoelastic bodies*, Arch. Rational Mech. Anal., **41**, 319–332, 1971.

6. D. IESAN, *Thermopiezoelectricity without energy dissipation*, Proc. R. Soc. A, **631**, 133–656, 2007.

7. V.D. KUPRADZE, T.G. GEGELIA, M.O. BASHELEISHVILI, T.V. BURCHULADZE, *Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, Amsterdam 1979.

8. A. MORRO, M. FABRIZIO, *Electromagnetism of Continuous Media*, Oxford University Press, Oxford 2003.

9. V.D. COLEMAN, E.H. DILL, *Thermodynamic restrictions on the constitutive equations of electromagnetic theory*, Z. Angew. Math. Phys., **22**, 691–702, 1971.

10. G. AMENDOLA, *On thermodynamic conditions for the stability of a thermoelectromagnetic system*, Math. Meth. Appl., **23**, 17–39, 2000.

11. G. AMENDOLA, *Linear stability for a thermoelectromagnetic material with memory*, Math. Mech. Appl., **59**, 67–84, 2001.

12. H.F. TIERSTEN, *On the Nonlinear Equations of Thermoelectroelasticity*, Int. J. Engng. Sci., **9**, 587–604, Pergamon Press, 1971.

13. J.S. YANG, *Equations for Small Fields Superposed on Finite Biasing Fields in a Thermoelectroelastic Body*, IEEE Transactions on Ultrasonics, Ferroelectricts, and Frequency Control, **50**, 2, 187–192, 2003.

14. A. MONTANARO, *On the Constitutive Relations in Thermo-Electroelasticity*, Technical Report No. 102 October 2009, DMMMSA (Dep. Math. Meth. Models Appl. Sci.), Univ. Padua; see the web-pages arXiv:0910.1344 or http://paduaresearch.cab.unipd.it/2184/.

Received 30 May 2009; revised version 9 October 2009.