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FAITHFUL REPRESENTATIONS FOR CONVEX HAMILTON–JACOBI EQUATIONS*

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Abstract. When a Hamiltonian $H = H(t, x, p)$ is convex in the adjoint variable p, the corresponding Hamilton–Jacobi equation

$$u_t + H(t, x, u_x) = 0 \tag{0.1}$$

is known to be the Bellman equation of a suitable optimal control problem. Of course, the latter is not unique, so it is interesting to select a *good* optimal control problem among those representing (0.1). We call such a representation *faithful* if (i) it involves a dynamics which is locally Lipschitz continuous in the state variable—so that a unique trajectory corresponds to any given control and initial point—and (ii) *the Lagrangian displays the same regularity as H in the x variable.* The main result of the present paper establishes the existence of faithful representations for a large class of Hamiltonians, including those for which the standard comparison theorems (of viscosity solution theory) are valid. Moreover, our investigation includes t-measurable Hamiltonians as well.

If a faithful control-theoretical representation does exist (and (0.1) enjoys uniqueness properties), one can infer sharp regularity results for the solution of (0.1) just by studying the regularity of the value function of the associated optimal control problem. A further application consists of a simple interpretation of the front propagation phenomenon in terms of optimal trajectories of the underlying minimum problem.

Key words. HJ equations, representation of Hamiltonians, parameterization of set-valued maps

AMS subject classifications. 70H20, 49J24, 35E10

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1. Introduction.

1.1. Some notation and conventions.

We shall call *modulus* any increasing, continuous function $\omega : [0, +\infty[\to [0, +\infty[$ such that $\omega[0] = 0$. A *local modulus* will be a continuous map $\omega : [0, +\infty[\times[0, +\infty[\to [0, +\infty[$ that is increasing in the first variable and is a modulus in the second variable. The closed ball of $\mathbb{R}^n$ of radius $R \geq 0$ will be denoted by $\mathbf{B}_R$, and $\mathbf{B}$ will stand in place of $\mathbf{B}_1$. For each map $\varphi : \mathbb{R}^n \to \mathbb{R} \cup +\infty$, the domain of φ, i.e., the subset of those $v \in \mathbb{R}^n$ such that $\varphi(v) < +\infty$, will be denoted by $\mathrm{dom}(\varphi(\cdot))$. For any map $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, H^* will denote the conjugate map with respect to the third variable; that is, we shall set

$$H^*(t, x, v) \doteq \sup_{p \in \mathbb{R}^n} \{p \cdot v - H(t, x, p)\}$$

for all $(t, x, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$.

If $w = w(y_1, \ldots, y_q)$ is a map of many (possibly vector-valued) variables, for any $i = 1, \ldots, q$ we shall use w_{y_i} to denote the gradient with respect to the y_i variable. It will be clear by the context whether this has to be intended in the sense of viscosity solution theory.

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1.2. Statement of the problem. For every $(t,x) \in [0,T] \times \mathbb{R}^n$ let us consider the Bolza optimal control problem

$$(\mathcal{P}_{t,x}) \qquad \begin{array}{ll} \text{minimize} & \int_t^T l(s,y(s),a(s))ds + g(y(T)), \\ & \dot{y}(s) = f(s,y(s),a(s)), \\ & y(t) = x, \end{array}$$

where the controls $a(\cdot)$ (are measurable maps on $[t,T]$ and) take values in some subset A of a Euclidean vector space and the *Lagrangian-dynamics* pair (l,f) verify suitable hypotheses which will be made precise later. The function g will be assumed continuous, even though weaker assumptions could be considered (see Remark 2.1 below).

The Bellman–Cauchy problem corresponding to the family of optimal control problems $\{\mathcal{P}_{t,x},\ (t,x) \in [0,T] \times \mathbb{R}^n\}$ is defined as the Hamilton–Jacobi equation

$$u_t + H(t,x,u_x) = 0 \qquad \text{in }]0,T[\, \times \mathbb{R}^n \tag{1.1}$$

with the initial condition

$$u(0,x) = g(x) \qquad \forall x \in \mathbb{R}^n, \tag{1.2}$$

where

$$H(t,x,p) \doteq \sup_{a \in A} \{p \cdot f(t,x,a) - l(t,x,a)\}. \tag{1.3}$$

As is well known, the connection between the Bolza problems $(\mathcal{P}_{t,x})$ and the initial value problem (1.1)–(1.2) relies on the fact that if $V(t,x)$ is the value function of $(\mathcal{P}_{t,x})$, that is,

$$V(t,x) = \inf_{a(\cdot)} \int_t^T l(s,y(s),a(s))ds + g(y(T)), \tag{1.4}$$

then the map $u(t,x) = V(T-t,x)$ is a solution (e.g., viscosity [BCD] or minmax [Su] solution) of (1.1)–(1.2). Notice, in particular, that the Hamiltonian is convex in the gradient variable.

Let us consider the converse question. Suppose the Cauchy problem (1.1)–(1.2) is given, with only the information that H is convex in the gradient variable (plus other technical conditions which guarantee existence and uniqueness of *solutions* to (1.1)–(1.2)). Then it is natural to wonder whether (1.1)–(1.2) is the Bellman–Cauchy problem of a family $\{\mathcal{P}_{t,x},\ (t,x) \in [0,T] \times \mathbb{R}^n\}$ of optimal control problems. This means that one looks for a triple (A,f,l) such that (1.3) is verified. Such a triple will be called a (*control theoretical*) *representation of* H.

It is easily seen that if a representation of H exists, then infinitely many others exist.[1] So, we may consider a further question, namely, that of choosing a representation verifying some given properties.

[1] For instance, the map $H(p) = |p|$ is the Hamiltonian corresponding to the trivial optimal control problem

$$\begin{array}{ll} \text{minimize} & \int_t^T (1-|a(s)|)ds, \\ & \dot{y}(s) = a(s), \qquad a(s) \in [-1,1], \\ & y(t) = x. \end{array}$$

On the other hand, for every pair $(h(\cdot),k(\cdot))$ of positive maps, $H(p) = |p|$ is also the Hamiltonian of

Indeed, this is our aim, which, loosely speaking, consists of finding representations that allow both *uniqueness of trajectories of* f (for any given control) and *a Lagrangian with the same* x*-regularity* as the given Hamiltonian.

In order to define the problem let us begin by stating properties (A1)–(A3) below, which are the properties we wish to be satisfied by a family of optimal control problems. They will imply certain conditions on H, which have to be considered as sort of *minimal assumptions* for our problem.

Given a family $\{\mathcal{P}_{t,x},\ (t,x)\in[0,T]\times\mathbb{R}^n\}$ of Bolza optimal control problems, we shall consider the following hypotheses on the triple (A,f,l):

(A1) *There exists a constant* Q *such that*

$$|f(t,0,a,)|,|l(t,0,a)|\le Q$$

for all $t\in[0,T]$ *and* $a\in A$.

(A2) *The maps* f *and* l *are continuous from* $[0,T]\times\mathbb{R}^n\times A$ *into* $\mathbb{R}^n$ *and* $\mathbb{R}$, *respectively, and for every* $R>0$ *there exists a nonnegative number* E_R *such that*

$$|f(t,x,a)-f(t,y,a)|\le E_R|x-y|, \tag{1.5}$$

$$|l(t,x,a)-l(t,y,a)|\le \nu[R,|x-y|] \tag{1.6}$$

for all $(t,x,a),(t,y,a)\in[0,T]\times\mathbf{B}_R\times A$, *where* ν *is a suitable local modulus.*

(A3) *There is* $C>0$ *such that*

$$|f(t,x,a)|\le C(1+|x|)$$

for all $(t,x,a)\in[0,T]\times\mathbb{R}^n\times A$.

From a control theoretical viewpoint these are rather standard hypotheses for the triple (A,f,l). In turn, it is straightforward to verify that they imply the following properties for the Hamiltonian H defined in (1.3):

(H1) *For any* $(t,x)\in[0,T]\times\mathbb{R}^n$, *the map* $q\mapsto H(t,x,q)$ *is convex from* $\mathbb{R}^n$ *into* $\mathbb{R}$.

(H2) *There exist local moduli* ω_1, ω_2, *and* ω_3 *such that for any* $R>0$, *one has*

$$|H(t,x,p)-H(t,y,p)|\le\omega_1[R,|x-y|(1+|p|)] \tag{1.7}$$

and

$$|H(t,x,p)-H(s,x,p)|\le|p|\omega_2[R,|t-s|]+\omega_3[R,|t-s|] \tag{1.8}$$

for all $x,y\in\mathbf{B}_R$, $p\in\mathbb{R}^n$, *and* $t,s\in[0,T]$.

(H3) *There exists a constant* C *such that*

$$|H(t,x,p)-H(t,x,q)|\le C(1+|x|)|p-q| \tag{1.9}$$

for all $(t,x)\in[0,T]\times\mathbb{R}^n$ *and* $p,q\in\mathbb{R}^n$.

(H4) *For every* $R>0$, *there exists a nonnegative number* N_R *such that*

$$|H^*(t,x,v)|\le N_R$$

for all $(t,x)\in[0,T]\times\mathbf{B}_R$ *and* $v\in\mathrm{dom}(H^*(t,x,\cdot))$.

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the (much more involved) optimal control problem

$$\begin{aligned}\text{minimize}\quad &\int_t^T \frac{h(y(s))}{a^2(s)}ds,\\ &\dot y(s)=\frac{a(s)k(y(s))}{1+|a(s)|k(y(s))}\cdot\frac{1-a^2(s)}{1+a^2(s)},\qquad a(s)\in\mathbb{R},\\ &y(t)=x.\end{aligned}$$

1.3. Aim. Assumptions (H1)–(H4), beyond being *necessary* for (A1)–(A3) (see Remark 2.2), are in fact verified with

$$\omega_1[R,s] = E_R(s + \nu[R,s]).$$

That is, the local modulus of continuity (in x) of H turns out to coincide—up to a sum with a linear mapping and a multiplication by a positive number, both depending on the radius R—with the local modulus of l. We say that H *inherits the same continuity* (*in* x) *from* l.

For a given Hamiltonian H verifying (H1)–(H4), we wish to find a representation (A, f, l) such that f *is locally Lipschitz continuous in* x—so that uniqueness of trajectories is guaranteed—and l *has the same kind of continuity* (*in* x) *as* H.

To be more precise, this means that we are looking for a triple (A, f, l) verifying (A1)–(A3) *with*

$$\nu[R,s] = P_R(s + \omega_1[R,s])$$

for suitable coefficients $P_R(\geq 0)$.

Remark 1.1. Up to now, the major contribution to the representation's issue for convex Hamiltonians could be found in Ishii [Is2]. As a matter of fact, in [Is2] representations were provided such that both f and l turn out to have a modulus of continuity equal to $(\omega_1)^{\frac{1}{2}}$ (while the control set turns out to be infinite-dimensional). This implies, for instance, that even in the quite regular case when $\omega_1[R,s] = L \cdot s$, f and l turn out to be just $\frac{1}{2}$-Holder continuous in x. In particular, the Lagrangian is *less regular* than the Hamiltonian, and the Cauchy problems for the control vector field f in general admit multiple solutions (for each control). On the contrary, in such a situation our result implies that both the dynamics f and the Lagrangian l are locally Lipschitz continuous in the state variable.

Remark 1.2. Problems with no convexity were investigated, e.g., by Ishii in [Is3] and by Evans and Souganidis in [ES]. Both papers aimed toward a representation of the solution in terms of an (Elliot–Kalton) upper or lower value of a suitable differential game. The dynamics of Ishii's representation involves infinite-dimensional control sets for the opponents in the game and displays a sort of Lipschitz continuity on compact sets. Instead the Lagrangian is just continuous. On the other hand, in [ES] Hamiltonians as well as initial data are restricted to Lipschitz continuous, bounded, functions. When referred to the case with convexity these results are weaker than ours. However, a comparison actually does not make sense because of the greater generality of the problems treated in [Is3] and [ES]. As a matter of fact, the lack of convexity could well be a serious drawback in the attempt to give a representation with a Lagrangian *as regular (in x)* as the Hamiltonians—apart from the Lipschitz bounded case treated in [ES].

1.4. Main results and an outline of the paper. The main contribution of this paper—see Theorems 2.1 and 2.2 below—consists of accomplishing *the twofold program of finding a locally Lipschitz continuous dynamics f and a Lagrangian l that preserves the same kind of continuity (in x) of the Hamiltonian.* Moreover, the control set A in our representation turns out to be particularly simple, namely, the unit ball of $\mathbb{R}^n$. Lastly (see section 6) we can prove extensions of these results to Hamiltonians measurably dependent on t.

As a first consequence of such results, many statements in the literature that have been proved for a Hamiltonian displaying an explicit control-theoretical form as in (1.3) can now be updated by considering Hamiltonians H that merely verify (H1)–(H4) (and, for some specific purposes, a further technical hypothesis (H5)). Furthermore, some results concerning the solution of the Cauchy problem (1.1) can be sharpened by means of the control-theoretical representation we are providing. For instance, this is the case of the regularity of the solution to (1.1), which is addressed in section 5. Finally, already known results may be interpreted as facts concerning the optimal trajectories of the underlying optimal control problem, as it happens for the phenomenon of front propagation (see section 5).

As for the proof of the main result, let us remark that it is based essentially on the following arguments. First, in Theorem 3.2 below we establish (by means of an argument based on Kakutani's fixed point theorem) that under hypotheses (H1)–(H4) the multifunction that maps (t,x) into $F(t,x) \doteq \mathrm{dom}(H^*(t,x,\cdot))$ is *locally Lipschitz continuous* in x (and an analogous fact holds in the case of Hamiltonians measurable in t). Theorem 3.3 yields a global version of this result. Observe that the presence of the local modulus ω_1 in (H2) would suggest an (at most) ω_1-regularity for this multifunction rather than the local Lipschitz continuity actually obtained by means of our results. Secondly, we exploit a parameterization theorem for convex multifunctions proved in [O] (see also [Lo]). According to this theorem, if $F(t,x)$ is a convex multifunction satisfying suitable regularity assumptions, then there exists a map $f : [0,T] \times \mathbb{R}^n \times \mathbf{B} \to \mathbb{R}^n$ displaying an akin regularity and verifying $F(t,x) = f(t,x,\mathbf{B})$ for all (t,x). Finally, by (H4) one proves that l displays the same kind of continuity (in x) as H.

The outline of the paper is as follows. In section 2 we state the main result (Theorem 2.1) and a version of it involving global regularity. In section 3 we establish that the multivalued map $(t,x) \mapsto \mathrm{dom}(H^*(t,x,\cdot))$ is (continuous and) locally Lipschitz continuous in x. Subsequently, a global version of this result is proven as well. In section 4 we conclude the proof of the main result by exploiting the parameterization theorem for multifunctions mentioned above. Section 5 is devoted to applications to regularity questions and to a control theoretical interpretation of the front propagation phenomenon. Finally, in section 6, we extend the results of the previous sections to the case when H is just measurable in the variable t.

2. The main result. In the next theorem we shall also consider the following hypothesis on the Hamiltonian H.

(H5) *For every $R > 0$ there exists $K_R > 0$ such that for every $(t,x) \in [0,T] \times \mathbf{B}_R$ and every $v \in \mathrm{dom}(H^*(t,x,\cdot))$, one has*

$$\mathrm{argmax}_p\{p \cdot v - H(t,x,p)\} \cap B_{K_R} \neq \emptyset.$$

Here $\mathrm{argmax}_p\{p \cdot v - H(t,x,p)\}$ *denotes the set of values of p where the map $p \mapsto p \cdot v - H(t,x,p)$ attains its maximum.*

THEOREM 2.1. *Let us consider a Hamiltonian H verifying hypotheses* (H1)–(H4). *Then there exist a dynamics $f = f(t,x,a)$ satisfying* (A1)–(A3) *and a continuous Lagrangian $l = l(t,x,a)$, with the control set A coinciding with the unit ball $\mathbf{B}$, such that*

$$H(t,x,p) = \sup_{a \in \mathbf{B}} \{p \cdot f(t,x,a) - l(t,x,a)\} \qquad \forall (t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n. \tag{2.1}$$

Furthermore, if hypothesis (H5) *is in force as well, then* (A2) *turns out to be satisfied with $\nu[R,s] \doteq \omega_1[R,(1+K_R)s] + D_R s$ for suitable coefficients D_R.*

Remark 2.1. As we have already pointed out in the introduction, the main point of Theorem 2.1 consists of the fact that, on one hand, f *turns out to be locally Lipschitz continuous* (in x), even in the case when H is *not* locally Lipschitz continuous (in x), and, on the other hand, l turns out to *inherit* the regularity (in x) of H. Finally, the control set A turns out to be quite simple, namely, it coincides with the unit ball of $\mathbb{R}^n$.

The following theorem is a global version of the previous one.

THEOREM 2.2. *Let H verify* (H1)–(H5)*, where we assume that ω_1 is a modulus (i.e., it is independent of R) and there exists a constant K such that $K_R = K$ for all $R \geq 0$. Then there exist a dynamics f and a Lagrangian l such that*

$$H(t,x,p) = \sup_{a \in \mathbf{B}} \{p \cdot f(t,x,a) - l(t,x,a)\} \qquad \forall (t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \tag{2.2}$$

holds true, and (A1)–(A3) *are satisfied, with the control set A coinciding with the unit ball $\mathbf{B}$, E_R independent of R, and, for all $R \geq 0$, $\nu[R,s] = \nu(s) = \omega_1[(1+K)s + Ds]$ for a suitable $D \geq 0$.*

Remark 2.2 (*on hypotheses* (H1)–(H4)). Assumptions (H1)–(H4) are *necessary* if we look for a representation (A, f, l) verifying (A1)–(A3). Moreover, they guarantee the (existence and) uniqueness of a viscosity solution to the Cauchy problem (1.1)–(1.2) (see, e.g., [CL]).

Let us observe that by assuming (H3) we are confining our investigation to Hamiltonians which are (convex and) *Lipschitz continuous in the adjoint variable* (not necessarily uniformly with respect to x). Moreover, (H4) prescribes the boundedness of the conjugate map H^*, locally with respect to (t,x). This is motivated by the fact that, on one hand, we are looking for control-theoretical representations (f,l,A) of H such that both the sets $f(t,x,A)$ and $l(t,x,A)$ are bounded, not necessarily uniformly with respect to t and x. And, on the other hand, the sets $f(t,x,A)$ and $l(t,x,A)$ finally will coincide with $\mathrm{dom}(H^*(t,x,\cdot))$ and $(H^*(t,x,\mathbb{R}^n))\backslash\{\infty\}$, respectively. As a matter of fact, we regard this paper as a first step of a wider program which shall allow for more general conditions on H. These should include superlinearity in the adjoint variable, which in turn would force one to look for representations with noncompact (possibly unbounded) control sets.

Finally, let us notice that assumption (H2) is quite standard for comparison (and hence uniqueness) results for a viscosity solution of (1.1); see, e.g., [BCD] and [Ba]. (However, let us remark that there are boundary value problems for which (H2) is no longer sufficient to guarantee uniqueness of the solution. In this case, a faithful representation of the Hamiltonian could be still exploited in order to provide a representation of *all* solutions of the boundary value problem, as, e.g., in [So], where the Hamiltonian is a control-theoretical one.)

Remark 2.3 (*on hypothesis* (H5)). Let us point out that, unlike hypotheses (H1)–(H4), hypothesis (H5) is not necessary for the existence of a representation (A, f, l) verifying (A1)–(A3), i.e., for the theses of Theorems 2.1 and 2.2 to hold true. For instance, let us consider the Hamiltonian

$$\tilde{H}(x,p) = (1+p^2)^{\frac{1}{2}} - 1 - \psi(x),$$

where ψ is just a continuous function. It is straightforward to verify that this Hamiltonian satisfies hypotheses (H1)–(H4), but *it does not satisfy hypothesis* (H5). On the other hand, it is easy to check that *the triple*

$$(\tilde{A}, \tilde{f}, \tilde{l}) = ([-1,1], a, 1 - (1-a^2)^{\frac{1}{2}} + \psi(x))$$

is a representation of H *verifying* (A1)–(A3), that is,

$$\tilde{H}(x,p) = \sup_{a\in[-1,1]} \{p \cdot a - \tilde{l}(x,a)\}.$$

At present we are unable to foresee how (H5) could be weakened, so we leave this question as an open problem.

Remark 2.4. Let us just mention that the representation question can also be addressed by considering only calculus of variations problems (see, e.g., [L] and also [G]) at the cost of allowing the *extended* Lagrangian H^*. Of course there is an intimate relation between the two approaches: roughly speaking, in the control-theoretical approach one is looking for a dynamics-Lagrangian pair so that, in particular, the *forbidden* velocities, that is, those mapped to $+\infty$ by H^*, are not contained in the dynamics.

3. The map $(\mathbf{t},\mathbf{x}) \mapsto \mathbf{dom}(\mathbf{H}^*(\mathbf{t},\mathbf{x},\cdot))$. In this section we prove that the (convex) multifunction $F(t,x) \doteq \text{dom}(H^*(t,x,\cdot))$ is Lipschitz continuous in x. As a matter of fact, the proofs of Theorems 2.1 and 2.2, given in the next section, will be based essentially on the Lipschitz continuity of F and on the application of a parameterization theorem for convex-valued multifunctions; see Theorem 4.1 below.

Let us consider the set-valued map

$$(t,x) \mapsto F(t,x) \doteq \text{dom}(H^*(t,x,\cdot)),$$

which is defined on $[0,T] \times \mathbb{R}^n$.

LEMMA 3.1. *The set-valued map* F *has nonempty, convex, compact values.*

Proof. Since for every $(t,x) \in [0,T] \times \mathbb{R}^n$ the map $v \mapsto H^*(t,x,v)$ is convex, proper (i.e., not everywhere equal to $+\infty$), lower semicontinuous, and bounded on its domain, $F(t,x)$ is a nonempty, convex, closed subset of $\mathbb{R}^n$(see, e.g., [RW]). Moreover, hypothesis (H3) implies that $F(t,x) \subset B_{C(1+|x|)}$ for every $(t,x) \in [0,T] \times \mathbb{R}^n$. Hence, for every $(t,x) \in [0,T] \times \mathbb{R}^n$, $F(t,x)$ is a compact convex subset of $\mathbb{R}^n$.

Throughout this paper the Hausdorff distance between two nonempty, compact subsets $A, B \subset \mathbb{R}^n$ will be denoted by $\delta(A,B)$; that is, we set

$$\delta(A,B) \doteq \max\left\{\max_{a\in A} d(a,B),\ \max_{b\in B} d(b,A)\right\}.$$

Let us recall that δ is a metric on the class $\mathcal{K}$ of nonempty, compact subsets of $\mathbb{R}^n$. In what follows, a multivalued map F with compact values from $[0,T] \times \mathbb{R}^n$ into $\mathbb{R}^n$ is said to be Lipschitz continuous (resp., continuous) if it is Lipschitz continuous (resp., continuous) when considered as a (univalued) map from $\mathbb{R}^n$ into the set $\mathcal{K}$ endowed with the metric δ. Actually, for maps with compact values, these definitions are equivalent to the usual ones (see [AC]).

F is said to be locally Lipschitz continuous if it is Lipschitz continuous on compact subsets of $[0,T] \times \mathbb{R}^n$.

THEOREM 3.2. *Let us assume hypotheses* (H1)–(H4). *Then, the set-valued map* $x \mapsto F(t,x)$ *is locally Lipschitz continuous in* x*, uniformly in* t*. That is, for every* $R > 0$*, there exists a number* $M_R \geq 0$ *such that*

$$\delta(F(t,x), F(t,y)) \leq M_R|x-y| \tag{3.1}$$

for all $x, y \in \mathbf{B}_R$ *and* $t \in [0,T]$.

Moreover, for every $R > 0$, there exists a number $\tilde{M}_R \geq 0$ such that

$$\delta(F(t,x), F(s,x)) \leq \tilde{M}_R \omega_2[R, |t-s|] \tag{3.2}$$

for every $x \in \mathbf{B}_R$ and $t, s \in [0,T]$. In particular, for each $x \in \mathbb{R}^n$, $t \mapsto F(t,x)$ is continuous.

In order to prove Theorem 2.2 we also need the following version of the previous result, which involves the global Lipschitz continuity of the map $x \mapsto F(t,x)$.

THEOREM 3.3. *Let us assume that hypotheses* (H1)–(H5) *are verified with both the local modulus ω_1 and the parameter K_R being in fact independent of R (that is, ω_1 is a modulus, and there exists a constant K such that $K_R = K$ for all $R \geq 0$.) Then, the set-valued map $x \mapsto F(t,x)$ is Lipschitz continuous in x, uniformly in t, that is,* (3.1) *holds true, and there exists a constant M such that $M_R = M$ for all R.*

Finally, let us state a simple property of the map H^* that will be used to prove both the regularity of the Lagrangian l and the global issue stated in Theorem 3.3.

PROPOSITION 3.4. *Assume hypotheses* (H1)–(H5)*, and fix $R > 0$. Then, for all $t \in [0,T]$, $x, y \in \mathbf{B}_R$, and $v \in F(t,x)$, $w \in F(t,y)$, one has*

$$|H^*(t,x,v) - H^*(t,y,w)| \leq \omega_1[R, (1+K_R)|x-y|] + K_R|v-w|.$$

Moreover, for all $t, s \in [0,T]$, $x \in \mathbf{B}_R$, and $v \in F(t,x)$, $w \in F(s,x)$, one has

$$|H^*(t,x,v) - H^*(s,x,w)| \leq \omega_2[R, K_R|t-s|] + \omega_3[R, |t-s|] + K_R|v-w|.$$

Proof of Theorem 3.2. Let us prove (3.1). Assume by contradiction that there exist sequences (x_n), (y_n) in $\mathbf{B}_R$ such that $x_n \neq y_n$ for every n and

$$\lim_{n\to\infty} \frac{\delta(F(t,x_n), F(t,y_n))}{|x_n - y_n|} = +\infty. \tag{3.3}$$

Up to the identification of the sequence (x_n, y_n) with a suitable subsequence, condition (3.3) yields either the existence of a selection $v_n \in F(t,x_n) \backslash F(t,y_n)$ verifying

$$\lim_{n\to\infty} \frac{d(v_n, F(t,y_n))}{|x_n - y_n|} = +\infty \tag{3.4}$$

or the existence of a selection $v'_n \in F(t,y_n) \backslash F(t,x_n)$ verifying

$$\lim_{n\to\infty} \frac{d(v'_n, F(t,x_n))}{|x_n - y_n|} = +\infty. \tag{3.5}$$

Suppose that (3.4) is actually verified ((3.5) implying perfectly symmetric considerations). Then, for any selection $w_n \in F(t,y_n)$, one has

$$\lim_{n\to\infty} \frac{|v_n - w_n|}{|x_n - y_n|} = +\infty. \tag{3.6}$$

Setting

$$p_n \doteq \frac{v_n - w_n}{|v_n - w_n||x_n - y_n|},$$

one obtains

$$
\begin{aligned}
\omega_1[R, |x_n - y_n|(1+|p_n|)] &= \omega_1[R, |x_n - y_n| + 1] \\
&\geq H(t, x_n, p_n) - H(t, y_n, p_n) \\
&\geq p_n \cdot v_n - H^*(t, x_n, v_n) \\
&\quad - \max_{w \in \operatorname{dom}(H^*(t, y_n, \cdot))} \{p_n \cdot w - H^*(t, y_n, w)\}.
\end{aligned} \tag{3.7}
$$

In order to achieve a contradiction, let us choose w_n to be a fixed point of the map

$$
\eta_n(w) \doteq \operatorname{argmax} \left\{ \frac{(v_n - w) \cdot \xi}{|x_n - y_n||v_n - w|} - H^*(t, y_n, \xi), \quad \xi \in \operatorname{dom}(H^*(t, y_n, \cdot)) \right\}.
$$

In view of Lemma 3.5 (where one sets $\varphi = H^*(t, y_n, \cdot)$ and $r = |x_n - y_n|$), such a point does exist. Hence one has

$$
\frac{(v_n - w_n) \cdot w_n}{|x_n - y_n||v_n - w_n|} - H^*(t, y_n, w_n) \geq \frac{(v_n - w_n) \cdot \xi}{|x_n - y_n||v_n - w_n|} - H^*(t, y_n, \xi) \tag{3.8}
$$

for all $\xi \in \operatorname{dom}(H^*(t, y_n, \cdot))$. By (3.7)–(3.8) and hypothesis (H4) one obtains

$$
\begin{aligned}
&\omega_1[R, 2R+1] \geq \omega_1[R, |x_n - y_n| + 1] \\
&\geq p_n \cdot (v_n - w_n) - H^*(t, x_n, v_n) + H^*(t, y_n, w_n) \geq p_n \cdot (v_n - w_n) - 2N_R,
\end{aligned} \tag{3.9}
$$

which is a contradiction, for the right-hand side tends to $+\infty$ while the left-hand side is bounded.

In order to prove (3.2) one has to exploit the same arguments with suitable adjustments: more precisely, one has to replace the sequences x_n, y_n with sequences $t_n, s_n \in [0, T]$, and the v_n and w_n must belong to $F(t_n, x)$ and $F(s_n, x)$, respectively. Moreover, the quantities $|x_n - y_n|$ have to be replaced with $\omega_2[R, |t_n - s_n|]$. In particular, one has

$$
\lim_{n \to \infty} \frac{v_n - w_n}{\omega_2[R, |t_n - s_n|]} = +\infty
$$

instead of (3.6). Setting

$$
p_n \doteq \frac{v_n - w_n}{|v_n - w_n| \omega_2[R, |t_n - s_n|]}
$$

one can conclude by arguing as in the first part.

Proof of Theorem 3.3. If ω_1 and K do not depend on R, in order to prove global Lipschitz continuity let us argue as in the previous proof until estimate (3.9), except for the fact that now (x_n) and (y_n) lie in $\mathbb{R}^n$. In particular, the last inequality of (3.9) is no longer valid. Yet, it is not restrictive to assume that $|x_n - y_n| \leq \frac{1}{2K}$. Hence, in view of Proposition 3.4—where we take ω_1 and K independent of R—one has

$$
|H^*(t, y_n, w_n) - H^*(t, x_n, v_n)| \leq \omega_1 \left[\frac{1+K}{2K} \right] + K|v_n - w_n|.
$$

Hence

$$
\omega_1 \left[\frac{1}{2K} + 1 \right] \geq \frac{|v_n - w_n|}{|x_n - y_n|} - \omega_1 \left[\frac{1+K}{2K} \right] - K|v_n - w_n|. \tag{3.10}
$$

Now, if $|v_n - w_n|$ is bounded, we contradict (3.6). If, on the contrary, (a subsequence of) $|v_n - w_n|$ tends to infinity, by the previous estimate we have

$$\omega_1\left[\frac{1}{2K}+1\right] \geq -\omega_1\left[\frac{1+K}{2K}\right] + K|v_n - w_n|, \tag{3.11}$$

which is a contradiction, in that the left-hand side is bounded while the right-hand side diverges. □

Proof of Proposition 3.4. Let p_v be an element of $\operatorname{argmax}_p\{p\cdot v - H(t,x,p)\}$. In view of hypothesis (H5) we can choose p_v such that the inequality $|p_v| \leq K_R$ is verified. Hence,

$$\begin{aligned} H^*(t,x,v) - H^*(t,y,w) &\leq p_v v - H(t,x,p_v) - p_v w + H(t,y,p_v) \\ &\leq K_R|v-w| + \omega_1[R,(1+K_R)|x-y|]. \end{aligned}$$

In an analogous way one obtains the same estimate for $H^*(t,y,w) - H^*(t,x,v)$, so the first inequality is proved. The proof of the estimate in the t-variable is akin, so we omit it. □

LEMMA 3.5. *Let $r > 0$ and let $\varphi : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a convex, lower semicontinuous, proper map such that* $\operatorname{dom}(\varphi)$ *is compact. Let $v \in \mathbb{R}^n\backslash\operatorname{dom}(\varphi)$ and let us consider the set-valued map $\eta : \operatorname{dom}(\varphi) \to \mathcal{P}(\operatorname{dom}(\varphi))$ defined by*

$$\eta(w) \doteq \operatorname{argmax}\left\{\frac{(v-w)\cdot\xi}{r|v-w|} - \varphi(\xi),\quad \xi \in \operatorname{dom}(\varphi)\right\}.$$

Then η has a fixed point, that is, there exists $\bar{w} \in \operatorname{dom}(\varphi)$ such that $\bar{w} \in \eta(\bar{w})$.

Proof. The map η has compact convex values. Moreover, since φ is continuous on its domain, η is upper semicontinuous (and is defined on a compact convex subset of $\mathbb{R}^n$). Then the lemma follows from Kakutani's fixed point theorem (see, e.g., [AC]). □

4. Proofs of Theorems 2.1 and 2.2. To prove Theorems 2.1 and 2.2 we are going to exploit the parameterization result for convex multifunctions proved in [O] (see also [Lo]). This result involves measurability in t, which will be useful in section 6 in order to address the case with t-measurable Hamiltonians. In particular *t-measurable moduli* will be utilized. We call *t-measurable modulus* every map $w : [0,T]\times[0,*\infty[\to[0,*\infty[$ such that for every $r \in [0,*\infty[$ the map $t \mapsto w[t,r]$ is measurable and for every $t\in[0,1]$ the map $r\mapsto w[t,r]$ is a modulus. Similarly, a *local t-measurable modulus* will be a map $w : [0,*\infty[\times[0,T]\times[0,*\infty[\to[0,*\infty[$, increasing in the first variable and such that for every $R \in [0,*\infty[$ the map $(t,r)\mapsto w[R,t,r]$ is a t-measurable modulus.

Let us recall that a multifunction $\mathcal{M} : [0,T]\to\mathbb{R}^n$ is called *measurable* if for every open subset $V \subset \mathbb{R}^n$ the preimage

$$\mathcal{M}^{-1}(V) \doteq \{t\in[0,T] \,:\, \mathcal{M}(t)\cap V \neq \emptyset\}$$

is a measurable subset of $[0,T]$).

Let us consider a multivalued map $F : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ verifying the following hypotheses.

Hypotheses (H_F):

(a) for every $(t,x) \in [0,T]\times\mathbb{R}^n$, $F(t,x)$ is a nonempty, compact, convex subset of $\mathbb{R}^n$;

(b) for every $x \in \mathbb{R}^n$ the multifunction $t \mapsto F(\cdot, x)$ is measurable;

(c) there exists a t-measurable local modulus w such that for every $R > 0$ and for almost every $t \in [0, T]$ one has

$$\delta(F(t,x), F(t,y)) \leq w[R, t, |x-y|] \tag{4.1}$$

for all $x, y \in \mathbf{B}_R$.

THEOREM 4.1 (see [O], Thm. 1). *Let F verify hypotheses* (H_F)*, and let us set*

$$M(t,x) \doteq \max\{1, |v| \ : \ v \in F(t,x)\}.$$

Then there exists a function $f : [0,T] \times \mathbb{R}^n \times \mathbf{B}$ such that

(i) *$F(t,x) = f(t,x,\mathbf{B})$ for all $x \in \mathbb{R}^n$ and for a.e. $t \in [0,T]$;*

(ii) *$f(\cdot, x, u)$ is measurable for every $(x,u) \in \mathbb{R}^n \times \mathbf{B}$;*

(iii) *there exists $N \geq 0$ such that for all $R > 0$ one has*

$$|f(t,x,u) - f(t,y,v)| \leq N(w[R,t,|x-y|] + M(t,x)|u-v|)$$

for all $x, y \in \mathbf{B}_R$ and for a.e. $t \in [0,T]$.

Moreover, if F and w are continuous, then f is continuous as well.

Remark 4.1. Actually, this theorem was proved (in [O]) under a hypothesis of uniform continuity, which means that in fact the map w is a t-measurable modulus. However, it is easy to verify (by direct inspection of the original proof) that the local statement of the present version can be proved by just replacing moduli with local moduli.

Proofs of Theorems 2.1 *and* 2.2. By Theorem 3.2 the multifunction $F(t,x) = \mathrm{dom}(H^*(t,x,\cdot))$ is continuous and agrees with the hypotheses of Theorem 4.1, with $w[R,t,r] \doteq M_R \cdot r$. Hence there exists a vector field f which verifies (A2) with $A = \mathbf{B}$ and such that $F(t,x) = f(t,x,\mathbf{B})$ for all x and a.e. $t \in [0,T]$.

Setting

$$l(t,x,a) \doteq H^*(t,x,f(t,x,a)) \qquad \forall (t,x,a) \in [0,T] \times \mathbb{R}^n \times \mathbf{B},$$

we get (2.1). Moreover, if hypothesis (H5) is in force, Proposition 3.4 and Theorem 4.1 imply the last part of the thesis. Notice that, since $A = \mathbf{B}$ is compact, (A1) is verified as well. Finally, let us prove that f satisfies the linear growth condition (A3). Indeed (H3) implies

$$|H(t,x,p)| \leq C(1+|x|)|p| + |H(t,x,0)| \tag{4.2}$$

for every $(t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$. For any $(t,x,a) \in [0,T] \times \mathbb{R}^n \times \mathbf{B}$ and $\lambda > 0$ let us take $p = \lambda f(t,x,a)$, thus obtaining

$$\begin{aligned} \lambda f^2(t,x,a) - l(t,x,a) &\leq |H(t,x,\lambda f(t,x,a))| \\ &\leq \lambda C(1+|x|)|f(t,x,a)| + |H(t,x,0)|. \end{aligned} \tag{4.3}$$

If $f(t,x,a) = 0$, we are done. Otherwise, by dividing both members in (4.3) by $\lambda |f(t,x,a)|$ and letting λ go to $+\infty$ one obtains

$$|f(t,x,a)| \leq C(1+|x|).$$

In view of Theorem 3.3, Theorem 2.2 can be proved in a similar way. ∎

5. Some applications. Let us present two simple instances on how the representation results proved in the previous sections can be exploited both to sharpen and to interpret some facts concerning (1.1).

5.1. Regularity of the solutions of (1.1)–(1.2). The first issue concerns the regularity of the solutions to (1.1)–(1.2), which, in view of the representation results in section 2 (and of the uniqueness of the solution), is nothing but the regularity of the corresponding value function.

Let us begin by briefly recalling some well-known facts concerning the value function of an optimal control problem. Besides (A1)–(A3), let us assume the following hypothesis on the final cost g:

(A4) The map g is continuous, that is, it verifies

$$|g(x) - g(y)| \leq \nu_g[R, |x-y|]$$

for all $x, y \in \mathbb{R}^n$ and a suitable local modulus ν_g.

Let us consider the value function $V = V(t,x)$ defined in (1.4) and the connected Hamiltonian

$$H(t,x,p) \doteq \sup_{a \in A} \{p \cdot f(t,x,a) - l(t,x,a)\}. \tag{5.1}$$

THEOREM 5.1. *Let us assume hypotheses* (A1)–(A4)*. Then the map* $u(t,x) \doteq V(T-t,x)$ *is continuous on* $[0,T] \times \mathbb{R}^n$*, and, for any* $R > 0$*, there exists a coefficient* $L_R \geq 0$ *such that*

$$|u(t,x) - u(t,y)| \leq L_R\left(|x-y| + \nu[R, |x-y|] + \nu_g[L_R R, L_R|x-y|]\right),$$
$$|u(t,x) - u(s,x)| \leq L_R\left(|t-s| + \nu[R, L_R|s-t|] + \nu_g[L_R R, L_R|t-s|]\right)$$

for all $(t,x), (t,y), (s,x) \in [0,T] \times \mathbf{B}_R$*. Moreover,* u *is the unique viscosity solution of the Cauchy problem* (1.1)–(1.2)*.*

We omit the proof of the regularity of V (and hence of u), which is standard and based essentially on Gronwall's lemma. For the uniqueness result see, e.g., [CL, Thm. VI.I]

As a corollary of Theorems 2.1 and 5.1 we obtain the following regularity result.

THEOREM 5.2. *Assume hypotheses* (H1)–(H5) *and let the initial datum* g *satisfy* (A4)*. Then, for any* $R > 0$ *there exists a coefficient* $C_R \geq 0$ *such that the solution* $u(t,x)$ *of* (1.1)–(1.2) *verifies*

$$|u(t,x) - u(t,y)| \leq C_R\big(|x-y| + \omega_1[R, |x-y|] + \nu_g[C_R R, C_R|x-y|]\big),$$
$$|u(t,x) - u(s,x)| \leq C_R\left(|t-s| + \omega_1[R, C_R|s-t|] + \nu_g[C_R R, C_R|t-s|]\right)$$

for all $(t,x), (t,y) \in [0,T] \times \mathbf{B}_R$*.*

Example. Roughly speaking, this theorem shows that the solution *preserves* the (x-)continuity of both the Hamiltonian H and the datum g. For instance, if $\omega_1(\eta) \doteq |\eta|^\alpha$, $\alpha \leq 1$, and g is β-Holder continuous, then the solution u turns out to be γ-Holder continuous, with $\gamma = \min\{\alpha, \beta\}$. As an example, consider the Cauchy problem in $[0,T] \times \mathbb{R}$:

$$u_t + \tilde{H}(x, u_x) = 0, \qquad u(0,x) = 0, \tag{5.2}$$

where

$$\tilde{H}(x,p) = |x \cdot p| - |x|^{\frac{1}{2}}.$$

It is straightforward to check that the map

$$v(t,x) = 2|x|^{\frac{1}{2}}(1 - e^{-\frac{t}{2}}) \tag{5.3}$$

is a viscosity solution of (5.2), and well-known uniqueness results imply that no other solutions do exist. Since

$$|H(x,p) - H(y,p)| \leq |x-y|(1+|p|) + [|x-y|(1+|p|)]^{\frac{1}{2}}$$

and $g = 0$, Theorem 5.2 establishes that for any $R > 0$ and for all $(t,x),(t,y) \in [0,T] \times \mathbf{B}_R$ the solution of 5.2 satisfies

$$|v(t,x) - v(y,t)| \leq C_R(|x-y| + |x-y|^{\frac{1}{2}})$$

for a suitable positive number C_R. On the other hand, by the explicit expression (5.3) we know that this indeed is the case, with $C_R = 2$, for every R.

Let us note that neither the available results based on direct PDE methods nor the application of the representation provided in [Is2] would yield such sharpness in the regularity estimates. Indeed, on one hand, PDE arguments are mainly concerned with local Lipschitz continuity (see, e.g., [Ba], [CL], [Le]). On the other hand, the results in [Is2], when applied to the present example, give at most[2] $\frac{1}{4}$-Holder regularity for the solution (see Remark 1.1).

Example. An even more elementary but significative example is provided by the transport equation

$$u_t + u_x \cdot f(x) - l(x) = 0, \qquad u(0,x) = g(x),$$

where we assume that $f(x)$ and $l(x)$ verify (A2)–(A3) and g is continuous. Denoting the solution at time s of the Cauchy problem

$$\dot{y} = f(y), \qquad y(0) = x,$$

by $y(x,s)$ one can straightforwardly check that

$$u(t,x) = g(y(x,-t)) + \int_0^t l(y(x,-s))ds$$

is the unique viscosity solution of this problem. Moreover, in view of Remark 2.2, the involved Hamiltonian verifies hypotheses (H1)–(H5), with $\omega_1[R,s] = E_R s + \nu[R,s]$. So, comparing the actual regularity of u with the one which can be deduced by Theorem 5.2, we see that the latter is as sharp as possible.

Remark 5.1. As observed in the introduction, since Theorems 2.1 and 2.2 concern just the Hamiltonian H, results for different boundary value problems could be obtained as well. Similarly, the case where the datum g is no longer continuous, possibly equal to $+\infty$—which includes optimal control problems with endpoint constraints—also could be treated (by exploiting the notion of semicontinuous solution; see, e.g., [BJ91] and [Fr]).

[2]We use the expression "at most" because the fact remains that in general no uniqueness of trajectories—for a given control—would be guaranteed.

5.2. Front propagation. A second issue where a representation result can be applied concerns the phenomenon of front propagation. Let us begin with a definition. Let $\mathcal{G}$ be a class of real continuous functions on $\mathbb{R}^n$.

DEFINITION 5.3. *We say that the pair* $(H, \mathcal{G})$ *verifies the* front propagation property *if*

(i) *for every* g *belonging to* $\mathcal{G}$ *the Cauchy problem*

$$\begin{aligned} u_t + H(t, x, u_x) = 0 \qquad & in\ ]0, T[\times \mathbb{R}^n, \\ u(0, x) = g(x) \qquad & \forall x \in \mathbb{R}^n \end{aligned}$$

has a unique (*viscosity*) *solution, say* u_g*;*

(ii) *if* $k \in \mathbb{R}$ *and* $g, \tilde{g} \in \mathcal{G}$ *are such that*

$$\begin{aligned} \Lambda_g^k(0) \doteq \{x \in \mathbb{R}^n \,:\, g(x) < k\} = \{x \in \mathbb{R}^n \,:\, \tilde{g}(x) < k\} \doteq \Lambda_{\tilde{g}}^k(0), \\ \Gamma_g^k(0) \doteq \{x \in \mathbb{R}^n \,:\, g(x) = k\} = \{x \in \mathbb{R}^n \,:\, \tilde{g}(x) = k\} \doteq \Gamma_{\tilde{g}}^k(0), \end{aligned}$$

then

$$\begin{aligned} \Lambda_g^k(t) \doteq \{x \in \mathbb{R}^n \,:\, u_g(t, x) < k\} = \{x \in \mathbb{R}^n \,:\, u_{\tilde{g}}(t, x) < k\} \doteq \Lambda_{\tilde{g}}^k(t), \\ \Gamma_g^k(t) \doteq \{x \in \mathbb{R}^n \,:\, u_g(t, x) = k\} = \{x \in \mathbb{R}^n \,:\, u_{\tilde{g}}(t, x) = k\} \doteq \Gamma_{\tilde{g}}^k(t) \end{aligned}$$

for every $t \in [0, T]$.

In other words, this condition states that the propagations of the k-level and the k-sublevel sets depend only on the k-sublevel set and the k-level set of the initial data. It is straightforward to check that property (ii) holds true for all k as soon as it is valid for one particular value of k. As is well known, a crucial role is played by the following homogeneity assumption:

(H-hom) For each $\lambda \geq 0$ one has

$$H(t, x, \lambda p) = \lambda H(t, x, p) \tag{5.4}$$

for all $(t, x, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$.

In fact, if the Hamiltonian H verifies hypotheses (H1)–(H3) *and* (H-hom) and $\mathcal{G}$ is the set of uniformly continuous functions, then the pair $(H, \mathcal{G})$ has the front propagation property (see, e.g., [BSS]). Thanks to the representation results of the previous sections—which can be applied here, for (H-hom) implies (H4) and (H5)—we can now give a simple control-theoretical explanation to this phenomenon, with $\mathcal{G}$ equal to the set of (not necessarily uniformly) continuous maps.

Remark 5.2. A control-theoretical interpretation of the front propagation phenomenon is nothing new: indeed it was originally proposed in [ES]. However, though the Hamiltonian is allowed to be nonconvex, the regularity assumptions therein assumed are much stronger than those considered here. In particular, they include the *global Lipschitz continuity of* H *in* (x, p), which in a representation like (1.3) means that f has to be *bounded*; see, for instance, assumption (1.1) in [So] in the context of front propagation along normal directions.

In section 6 we shall show that the front propagation property is still valid for Hamiltonians measurable in the variable t.

THEOREM 5.4. *Let us assume* (H1)–(H3)*, and let* $\mathcal{G} \doteq C(\mathbb{R}^n)$*. Then the following are equivalent:*

(i) H *verifies* (H-hom)*;*

(ii) *there exists a representation* (A, f, l) *of* H *satisfying* (A1)–(A3)*, with* l *equal to zero;*

(iii) *for every* $(t, x) \in [0, T] \times \mathbb{R}^n$*, the conjugate map* $v \mapsto H^*(t, x, v)$ *is constant equal to zero on its domain.*

Moreover, they imply the following:

(iv) *for all* $R > 0$ *the local modulus* $\omega_1[R, \cdot]$ *is in fact a linear mapping;*

(v) *the pair* $(H, \mathcal{G})$ *verifies the front propagation property.*

Proof. Since H is convex, the equivalence of (i) and (iii) is straightforward. Moreover, let us observe that (iii) trivially implies hypotheses (H4) and (H5), so Theorem 2.1 applies. Hence (ii) follows from (iii), since l was defined by $l(t, x, a) = H^*(t, x, f(t, x, a))$. The fact that (ii) implies (iii) and (iv) is straightforward as well.

Let us prove that (ii) implies (v). Assume by contradiction that there exist initial data g and $\tilde{g}$, both belonging to $\mathcal{G}$, and a point $(t, x) \in]0, T] \times \mathbb{R}^n$ such that $\Lambda^0_g(0) = \Lambda^0_{\tilde{g}}(0)$, $\Gamma^0_g(0) = \Gamma^0_{\tilde{g}}(0)$, while the corresponding solutions of (5.4) verify $u_g(t, x) = 0$, $u_{\tilde{g}}(t, x) \neq 0$. Let us recall that $u_g(t, x) = V_g(T - t, x)$ and $u_{\tilde{g}}(t, x) = V_{\tilde{g}}(T - t, x)$, where the value functions V_g and $V_{\tilde{g}}$ are defined as follows:

$$V_g(T - t, x) \doteq \inf g(y(T)), \quad \dot{y}(s) = f(s, y(s), a(s)), \quad y(T - t) = x,$$
$$V_{\tilde{g}}(T - t, x) \doteq \inf \tilde{g}(y(T)), \quad \dot{y}(s) = f(s, y(s), a(s)), \quad y(T - t) = x.$$

Let $\hat{a}$ be an optimal control for the datum g, which means

$$\begin{aligned} V_g(T - t, x) &= g(\hat{y}(T)), \\ \dot{\hat{y}}(s) &= f(s, \hat{y}(s), a(s)), \qquad y(T - t) = x. \end{aligned}$$

(This control exists, for $f(s, y, \mathbf{B}) = \mathrm{dom} H^*(t, x, \cdot)$ is convex for every (s, y). However this is not crucial, for one could as well consider an ϵ-optimal control.) Now $0 = V_g(T - t, x) = g(\hat{y}(T))$, which implies $\tilde{g}(\hat{y}(T)) = 0$. Hence it cannot happen that $V_{\tilde{g}}(T - t, x) = u_{\tilde{g}}(t, x) > 0$, for one would get $\tilde{g}(\hat{y}(T)) = 0 < V_{\tilde{g}}(T - t, x)$. In a similar way, the case when $u_{\tilde{g}}(t, x) < 0$ produces a contradiction. Finally, with the same arguments one proves that it cannot happen that $u_g(t, x) > 0$ while $u_{\tilde{g}}(t, x) < 0$. ∎

6. *t*-measurable Hamiltonians. The results presented in the previous sections may be extended, substantially in their full strength, to the case where the Hamiltonian H is measurable in the variable t. The aim of this section is to present the corresponding statements and to point out some needed changes in the assumptions and in the proofs.

6.1. The value function and the Bellman equation. Aiming toward representations of t-measurable Hamiltonians, we have to consider optimal control problems where the data f and l are measurable in t. Accordingly, let us replace assumptions (A1)–(A3) with the following ones:

(A1′) *There exists a constant* Q *such that*

$$|f(t, 0, a)|, |l(t, 0, a)| \leq Q$$

for almost all $t \in [0, T]$ *and* $a \in A$.

(A2′) *The maps* f *and* l *are continuous in* (x, a) *from* $[0, T] \times \mathbb{R}^n \times A$ *into* $\mathbb{R}^n$ *and* $\mathbb{R}$*, respectively, and verify conditions*

$$|f(t, x, a) - f(t, y, a)| \leq E_R |x - y|, \tag{6.1}$$

$$|l(t, x, a) - l(t, y, a)| \leq \nu[R, |x - y|] \tag{6.2}$$

for all $(t,x,a),(t,y,a)\in[0,T]\times\mathbf{B}_R\times A$*, where* ν *is a suitable local modulus.*

(A3′) *There is* $C>0$ *such that*

$$f(t,x,a)\le C(1+|x|)$$

for all $(x,a)\in[0,T]\times\mathbb{R}^n\times A$ *and almost every* $t\in[0,T]$.

PROPOSITION 6.1. *The regularity results stated in Theorem* 5.1 *are still valid under the weaker hypotheses* (A1′), (A2′), (A3′)*, and* (A4).

The proof of this proposition does not present substantial new difficulties with respect to the case where the data are continuous.

A *uniqueness result* analogous to the one stated in Theorem 5.1 holds true for t-measurable Hamiltonians as well, but some care is needed. To begin with, we cannot exploit the classical notion of viscosity solution, for the Hamiltonian H in (5.1) is now merely measurable in the t-variable. A suitable notion of solution for this case was introduced by Ishii in [Is2]. Successively, Lions and Perthame [LP87] provided three equivalent versions of this notion (see also [BJ87]). Recently (see [BR]) density results have been proved for this concept of solution. For the sake of self-consistency, let us recall the notion of subsolution, in one of the versions provided in [LP87].

DEFINITION 6.2. *A continuous map* $u:[0,T]\times\mathbb{R}^n$ *is a* viscosity subsolution *of* (1.1) *at* $(t_0,x_0)\in[0,T]\times\mathbb{R}^n$ *if for every* C^1 *map* ϕ *defined in a neighborhood of* (t_0,x_0) *and* $b\in L^1(0,T)$ *such that* (t_0,x_0) *is a local maximum for*

$$u(t,x)+\int_0^t b(s)ds-\phi(x)$$

one has

$$\lim_{\delta\downarrow 0^+}\operatorname*{ess\,inf}_{|t-t_0|<\delta}\inf\{H(t,x,s,p)-b(t):|x-x_0|\le\delta,|p-\nabla\phi(x_0)|\le\delta,|s-u(t_0,x_0)|\le\delta\}\le 0.$$

The definition of *viscosity supersolution* is perfectly symmetric, and a map is a *viscosity solution* if it is both a subsolution and a supersolution.

Again, it is not difficult to prove that the map $u(t,x)\doteq V(T-t,x)$ is a viscosity solution of the Cauchy problem (1.1)–(1.2).

6.2. A representation theorem for *t*-measurable Hamiltonians. In order to state a representation result for t-measurable Hamiltonians we shall assume suitable hypotheses. It turns out that we have to make only the obvious change due to the lack of continuity in t. Precisely we shall consider those hypotheses, which we label (H1′)–(H5′), respectively, that are obtained from (H1)–(H5) by replacing $[0,T]$ with any full-measure subset. (Of course, condition (1.8), which would imply continuity in t, is no longer assumed.)

In the new framework, the representation Theorems 2.1 and 2.2 assume the following forms, respectively.

THEOREM 6.3. *Let us consider a Hamiltonian* H *verifying hypotheses* (H1′)–(H4′)*. Then there exist a dynamics* $f=f(t,x,a)$ *satisfying* (A1′)–(A3′) *and a Lagrangian* $l=l(t,x,a)$ *(continuous in* (x,a) *for almost every* $t\in[0,T]$*), with the control set* A *coinciding with the unit ball* $\mathbf{B}$*, such that*

$$H(t,x,p)=\sup_{a\in\mathbf{B}}\{p\cdot f(t,x,a)-l(t,x,a)\}\tag{6.3}$$

for almost all $t \in [0,T]$ *and for all* $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$*. Furthermore, if hypothesis* (H5′) *is in force as well, then* l *verifies* (A2′)*, with* $\nu[R,s] \doteq \omega_1[R,(1+K_R)s] + D_R s$*, for suitable coefficients* D_R*.*

THEOREM 6.4. *Let* H *verify* (H1′)–(H5′)*, with* ω_1 *being a modulus (i.e., independent of* R*) and the numbers* K_R *being equal to a constant* K *for all* R*. Then there exist a dynamics* f *and a Lagrangian* l *such that*

$$H(t,x,p) = \sup_{a\in\mathbf{B}} \{p \cdot f(t,x,a) - l(t,x,a)\} \tag{6.4}$$

holds true for almost all $t \in [0,T]$ *and for all* $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$*, conditions* (A1′)–(A3′) *are satisfied, and, moreover, the control set* A *coincides with the unit ball* $\mathbf{B}$*. Furthermore,* E_R *turns out to be independent of* R*, and* $\nu(s) = \omega_1[(1+K)s + Ds]$ *for a suitable* $D \geq 0$*.*

Proofs of Theorems 6.3 *and* 6.4. In view of Theorem 4.1, once we have proved that for every x the map $t \mapsto F(t,x) = \mathrm{dom} H^*(t,x,\cdot)$ is measurable (see definition in section 5) we are done. Indeed the parts of Theorems 3.2 and 3.3 concerning the variable x remain unchanged.

To prove that the map $t \mapsto F(t,x)$ is measurable we need some sharper result from set-valued analysis. Let us fix $x \in \mathbb{R}^n$. Then (see, e.g., [RW]) by the measurability of $t \mapsto H(t,x,p)$, the measurability of $t \mapsto H^*(t,x,v)$ follows, for each $v \in \mathbb{R}^n$.

Moreover, the multivalued map

$$t \mapsto \mathrm{epi}[H^*(t,x,\cdot)] \doteq \{(u,r) \in \mathbb{R}^n \times \mathbb{R} \,:\, r \geq H^*(t,x,u)\}$$

turns out to have a Castaing representation (u_n, r_n) (see, e.g., [RW]).

Hence (u_n) is a Castaing representation of the map $t \mapsto F(t,x)$, which therefore turns out to be measurable (see, e.g., [RW]). □

6.3. Regularity of solutions for *t*-measurable Hamiltonians. By the previous considerations it turns out that Theorem 5.2 on the regularity of solutions is still valid for t-measurable Hamiltonians verifying hypotheses (H1′)–(H5′) (plus some extra condition such that the uniqueness of the solution is guaranteed). Let us point out that the latter can be achieved either according to [Is1] (e.g., by imposing hypothesis (A6) therein) or by following the approach in [BR], which relies on the approximability of H by continuous Hamiltonians.

6.4. Front propagation for *t*-measurable Hamiltonians. Thanks to the representation provided by Theorem 6.3, the front propagation phenomenon can be studied for t-measurable Hamiltonians as well, as soon as the latter verify (H1′)–(H3′). For this purpose let us consider the following weakened version of assumption (H-hom):

(H′-hom) For each $\lambda \geq 0$ one has

$$H(t,x,\lambda p) = \lambda H(t,x,p) \tag{6.5}$$

for all $(t,x,p) \in [0,T]\backslash\mathcal{N} \times \mathbb{R}^n \times \mathbb{R}^n$ where $\mathcal{N}$ has measure zero.

With an unchanged proof with respect to Theorem 5.4, one obtains the following result.

THEOREM 6.5. *Let us assume* (H1′)–(H3′)*, and let* $\mathcal{G} \doteq C(\mathbb{R}^n)$*. Then the following are equivalent:*

(i) H *verifies* (H′-hom)*;*

(ii) *there exists a representation* (f, l) *of* H *satisfying* (A1′)–(A3′)*, with* l *equal to zero in* $[0, T]\backslash\mathcal{N} \times \mathbb{R}^n \times \mathbf{B}$*, for a suitable subset* $\mathcal{N}$ *of measure zero;*

(iii) *there is a zero measure subset* $\mathcal{N}$ *such that, for every* $(t, x) \in [0, T]\backslash\mathcal{N} \times \mathbb{R}^n$*, the conjugate map* $v \mapsto H^*(t, x, v)$ *is constant equal to zero on its domain.*

Moreover, each of them implies the following two conditions:

(iv) *for all* $R > 0$ *the modulus* $\omega_1[R, \cdot]$ *is in fact a linear mapping;*

(v) *the pair* $(H, \mathcal{G})$ *verifies the front propagation property.*

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