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A PDE APPROACH TO FINITE TIME INDICATORS IN ERGODIC THEORY

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For dynamical systems defined by vector fields over a compact invariant set, we introduce a new class of approximated first integrals based on finite time averages and satisfying an explicit first order partial differential equation. These approximated first integrals can be used as finite time indicators of the dynamics. On the one hand, they provide the same results on applications than other popular indicators; on the other hand, their PDE based definition — that we show robust under suitable perturbations — allows one to study them using the traditional tools of PDE environment. In particular, we formulate this approximating device in the Lyapunov exponents framework and we compare the operative use of them to the common use of the Fast Lyapunov Indicators to detect the phase space structure of quasi-integrable systems.

Keywords: Ergodic Theory; Lyapunov exponents; Fast Lyapunov Indicators; approximated first integrals; PDE viscous techniques.

1. Introduction

The existence of first integrals and their qualities, e.g. their number and smoothness, constrain the topological properties in the large of the paths of a dynamical system, see for example [9, 12, 19]. Specifically, for the integrable systems, there exists a set of global first integrals which determines completely the dynamics. Otherwise, for the ergodic ones, non-trivial global first integrals do not exist. Both integrable and ergodic systems are very special extreme situations, which usually do not exist in nature, and the most typical case is represented by systems which are neither integrable nor ergodic, but approximating one or the other situation and sometimes both of them. In fact, after some perturbation steps, the systems typically exhibit some variables with integrable or quasi-integrable behavior and other variables with approximated ergodic behavior; examples of this kind arise in Celestial Mechanics, Statistical Physics and Plasma Physics, see e.g. [5, 7, 23, 24]. The dynamics of such (non-integrable and non-ergodic) systems can be characterized by transient behaviors (such as temporary captures into resonances or stickiness phenomena) which are usually difficult to study formally with mathematical tools defined over the complete orbits.

During the last decades, in the numerical investigations of these systems, a very important role has been played by a practical use of approximated notion of first integrals, of time averages, and other popular indicators of the quality of the motion, like the Lyapunov exponents ([6, 25, 26]).

Since the eighties, important results have been achieved by finite time computations of dynamical indicators on discrete sets of points of the phase space, hereafter called grids, see [24] for a review on the subject. In particular, we recall the methods based on the Fourier analysis of the solutions, such as the frequency analysis method ([20, 21]), or on the Lyapunov exponents theory, such as the Fast Lyapunov Indicators (FLI), introduced in [13]. The finite time computation of a dynamical indicator on grids of the phase space provides effective criteria for determining integrability, quasi-integrability or stochasticity of motions (see [15] for the case of FLI).

In this paper, we introduce a new class of approximated global first integrals. Keeping in mind the Birkhoff–Khinchin Theorem — the time average of any L^1 integrable function f is a global first integral — we specifically study an indicator based on finite time averages of functions, with particular attention devoted to the FLI and their applications. We restrict ourselves to a general dynamical system defined by the flow ϕ_X^t of a smooth vector field X over a compact invariant set $\Omega \subset \mathbb{R}^n$. Whenever global differentiable first integrals F of such a system do exist, they are solutions on Ω of the PDE:

$$\nabla F \cdot X(x) = 0. \tag{1.1}$$

Clearly, this is not the case of the finite time average G_T of functions f on $[0, T]$, whose Lie derivative $\nabla G_T \cdot X(x) = \frac{1}{T}[f(\phi_X^T(x)) - f(x)]$ depends both on x and $\phi_X^T(x)$.

Our first contribution is framed in considering unusual "finite time" approximations F_μ, where substantially $\mu > 0$ plays the role of $1/T$:

$$F_\mu(x) := \mu \int_0^{+\infty} e^{-\mu\tau} f(\phi_X^\tau(x)) d\tau. \tag{1.2}$$

The crucial advantage of F_μ with respect to the traditional finite time averages G_T is represented by the fact that it satisfies an explicit PDE, precisely:

$$\nabla F_\mu \cdot X(x) = \mu(F_\mu - f)(x). \tag{1.3}$$

In view of the right-hand side of the previous equation, we regard F_μ as an approximated first integral. We remark that in this paper we are concerned with fixed finite $\mu > 0$, while the classical limit $\mu \to 0^+$ of both F_μ and $G_{1/\mu}$ already appeared in the general context of Tauberian integrals [29] and — more recently — in stochastic applications, see [1, 8].

In Sec. 2 we formulate this approximating device to define finite time approximations of the usual Lyapunov exponents, which can be considered as time averages of suitable functions defined on the tangent space. This requires to rewrite the variational equation on a compact invariant set of the tangent space $\mathbb{R}^{2n}$. We will call exponentially damped Lyapunov indicators these finite time approximations of the Lyapunov exponents. In Sec. 3, we compare the operative use of the exponentially damped Lyapunov indicators for the numerical detection of resonances and invariant tori of quasi-integrable Hamiltonian systems to the common use of the FLI.

Referring to the PDE framework for existence and uniqueness results for (1.3), we recognize the viscosity robustness of the proposed approximated first integral F_μ. More precisely — see Sec. 4 — this property consists in the fact that the solution of Dirichlet's problem given by the following elliptic perturbation of Eq. (1.3):

$$\begin{cases} \dfrac{\nu^2}{2}\Delta F_\mu^\nu(x) + X \cdot \nabla F_\mu^\nu(x) = \mu(F_\mu^\nu - f)(x) \\ F_\mu^\nu(x)|_{\partial\Omega} = \psi(x) \end{cases} \tag{1.4}$$

converges pointwise, for $\nu \to 0$, exactly to the above function F_μ. In particular, the proposed representation (1.2) of the solution for (1.3) is stable under stochastic perturbations. In fact, denoting by (X^ν_τ, P_x) the Markov process solving the stochastic differential equation related to (1.4), we have the representation:

$$F^\nu_\mu(x) = M_x\left[\mu\int_0^{+\infty} e^{-\mu\tau} f(X^\nu_\tau)d\tau\right] \stackrel{\nu\to 0}{\longrightarrow} F_\mu(x), \quad \forall x\in\Omega, \tag{1.5}$$

see [11] for some more detail. We note that as a consequence of the invariance of Ω under ϕ^t_X, any boundary datum in (1.4) does not play any role, since all orbits with initial condition into Ω never reach the boundary $\partial\Omega$.

The discussion above shows that, among any possible perturbation of (1.1), Eq. (1.3) is solved by approximated first integrals coming from a regularizing viscosity technique.

This stable behavior under viscous perturbations (and with arbitrary boundary data) suggests analogies with other important situations in which vanishing artificial viscosity is introduced in order to select a special solution: this occurs for example in the viscosity solutions theory to the Hamilton–Jacobi equation, we refer to [3, 4, 22] for an exhaustive treatment of the matter. However, the approximate first integral F_μ here introduced is better comparable to Fokker–Planck equation — see for example [2, 27]. In fact, if we consider in particular divergence-free systems like the Hamiltonian ones, searching first integrals is equivalent to searching (smooth) invariant measures. We remark that by adding to (1.1) a μ-small relative *friction* (relative, to an assigned function f) and a ν-small *diffusion*, we obtain the following stationary Fokker–Planck equation:

$$\nabla F\cdot X(x) + \mu(f - F)(x) + \frac{\nu^2}{2}\Delta F(x) = 0.$$

In this order of ideas, we infer that the present F_μ is thus robust under the vanishing (i.e. for $\nu\to 0$) diffusion action. A previous use of relative friction and vanishing viscosity has been numerically implemented in [18].

2. PDE Definition of Approximated First Integrals

In this section, starting from the Birkhoff–Khinchin Theorem (see for example [10, Chap. 1]), we propose a natural notion of approximated global first integral and we discuss it from a PDE point of view.

Let X be a smooth (i.e. at least C^1) vector field defined over a compact invariant set $\Omega\subset\mathbb{R}^n$ and ϕ^t_X its flow. By the Birkhoff–Khinchin Theorem, the time average of every real-valued continuous function f:

$$\mathcal{F}(x) := \lim_{t\to+\infty}\frac{1}{t}\int_0^t f(\phi^\tau_X(x))d\tau \tag{2.1}$$

exists a.e. and it is a first integral, that is $\mathcal{F}(\phi^t_X(x)) = \mathcal{F}(x)$ for all $t\in\mathbb{R}$.

On the one hand, the function $\mathcal{F}$ can be highly irregular and its operative use is limited. On the other hand, with reference to some possible applications to perturbation theory and allied topics, the finite time approximation of (2.1):

$$G_T(x) := \frac{1}{T}\int_0^T f(\phi^\tau_X(x))d\tau \tag{2.2}$$

is an approximated first integral, in the sense that its Lie derivative is given by:

$$X\cdot\nabla G_T(x) = \frac{d}{dt}G_T(\phi^t_X(x))|_{t=0} = \frac{1}{T}[f(\phi^T_X(x)) - f(x)]. \tag{2.3}$$

Let us denote $\mu := \frac{1}{T}$. Starting from this format, we consider below a different finite time approximation of (2.1), which offers a better notion of approximated global first integral.

Definition 1. Let $\mu > 0$ and $f \in C^0(\Omega; \mathbb{R})$. The function

$$F_\mu(x) := \mu \int_0^{+\infty} e^{-\mu\tau} f(\phi_X^\tau(x)) d\tau \tag{2.4}$$

is an approximated first integral in the sense that its Lie derivative equals:

$$X \cdot \nabla F_\mu(x) = \mu(F_\mu - f)(x). \tag{2.5}$$

The function F_μ, which is defined pointwise in (2.4), can be equivalently introduced as the solution of the PDE (2.5), which has F_μ as its unique solution. This fact shows the plain advantage of F_μ with respect to other popular approximated first integrals like $G_{1/\mu}$ (see (2.2)), whose Lie derivative depends both on x and on the flow $\phi_X^{1/\mu}$. We stress that in this paper we are concerned with fixed $\mu > 0$, while the classical limit $\mu \to 0^+$ of both F_μ and $G_{1/\mu}$ already appeared in the general context of Tauberian integrals [29] and — more recently — in stochastic applications, see [1, 8]. Specifically, from [28], the above limit reads:

$$\lim_{\mu \to 0^+} \mu \int_0^{+\infty} e^{-\mu\tau} f(\phi_X^\tau(x)) d\tau = \lim_{T \to +\infty} \frac{1}{T} \int_0^T f(\phi_X^\tau(x)) d\tau.$$

Remarks. (1) Let us consider the following change of the integral parameter:

$$[0, +\infty] \ni \tau \mapsto t(\tau) = (1 - e^{-\mu\tau})T \in [0, T].$$

We note that the function (2.4) can be obtained from a modification of the equivalent representation:

$$G_{1/\mu}(x) = \mu \int_0^{+\infty} e^{-\mu\tau} f(\phi_X^{t(\tau)}(x)) d\tau.$$

(2) The function F_μ, viewed as a linear operator on $C^0(\Omega; \mathbb{R})$, provides a precise characterization for exact global first integrals, in the sense stated by the following

Proposition 1. *Let $\mu > 0$ be fixed. A function $f \in C^0(\Omega; \mathbb{R})$ is a global first integral for the vector field X if and only if, $\forall x \in \Omega$, $F_\mu(x) = f(x)$, where F_μ is defined in* (2.4).

Proof. Let us first suppose that f is a global first integral for X, that is: $f(\phi_X^t(x)) = f(x)$ $\forall t \in \mathbb{R}$. Therefore we have

$$F_\mu(x) = \mu \int_0^{+\infty} e^{-\mu\tau} f(\phi_X^\tau(x)) d\tau = \mu \int_0^{+\infty} e^{-\mu\tau} f(x) d\tau = f(x).$$

Conversely, let $F_\mu(x) = f(x)$, $\forall x \in \Omega$. Then, according to (2.5), the Lie derivative of f is equal to zero:

$$L_X f(x) = X \cdot \nabla F_\mu(x) = \mu(F_\mu - f)(x) = 0.$$

Equivalently, f is a global first integral for the vector field X. ◻

We finally underline that, in the previous proposition, the choice of the parameter $\mu > 0$ is arbitrary.

In the Introduction, we have motivated finite time approximations of Lyapunov exponents, such as the FLI. Here below, we make use of the previous notion of approximated first integral in the framework of Lyapunov exponents theory. To provide the PDE formulation of the finite time Lyapunov exponents, which we will call exponentially damped Lyapunov indicators, we define a PDE on a suitable domain of the tangent space. We start with the usual definition of Lyapunov exponent.

Definition 2. Given a pair $(x, v) \in \Omega \times (\mathbb{R}^n \backslash \{0\})$, the Lyapunov exponent associated to (x, v) is defined as

$$\chi(x, v) := \lim_{t \to +\infty} \frac{1}{t} \log\left(\frac{\|v_t\|}{\|v\|}\right), \tag{2.6}$$

where $v_t := D\phi_X^t(x)v \in T_{\phi_X^t(x)}\Omega$ is the tangent vector at the time $t > 0$ and $\|\cdot\|$ denotes a norm.

For convenience, in the sequel we use the Euclidean norm. We remark that, by an easy computation, the Lyapunov exponent χ admits also an integral representation as a time average:

$$\chi(x, v) = \lim_{t \to +\infty} \frac{1}{t} \int_0^t f_\chi(\phi_{\mathbf{X}}^\tau(x, v)) d\tau, \tag{2.7}$$

where the function f_χ corresponds to:

$$f_\chi(x, v) = \frac{v \cdot DX(x)v}{\|v\|^2} \tag{2.8}$$

and $\phi_{\mathbf{X}}^\tau$ denotes the tangent flow of ϕ_X^τ. More precisely, denoting by DX the Jacobian matrix related to X, the variational vector field $\mathbf{X}(x, v) := (X(x), DX(x)v)$ is defined on $\Omega \times \mathbb{R}^n$ and the corresponding flow $\phi_{\mathbf{X}}^t$ is given by:

$$\phi_{\mathbf{X}}^t(x, v) = (\phi_X^t(x), D\phi_X^t(x)v).$$

The time average representation formula (2.7) pushes to consider, in the light of the previous considerations, the following

Definition 3. (Exponentially damped Lyapunov indicator) Let $\mu > 0$. Given a pair $(x, v) \in \Omega \times (\mathbb{R}^n \backslash \{0\})$, the exponentially damped Lyapunov indicator associated to (x, v) is defined as

$$\mathcal{K}_\mu(x, v) := \mu \int_0^{+\infty} e^{-\mu\tau} f_\chi(\phi_{\mathbf{X}}^\tau(x, v)) d\tau. \tag{2.9}$$

This definition of $\mathcal{K}_\mu$, though formally correct, is not completely equivalent to the case discussed before, because the flow of $\mathbf{X}$ is not restricted to a compact connected invariant set (the norm of tangent vectors can diverge to infinity). In order to solve this technical problem, we give alternative representations of χ and $\mathcal{K}_\mu$ through a reformulation of the variational dynamics as the flow of a vector field defined on a compact connected invariant domain of the tangent space $\Omega \times \mathbb{R}^n$. This formulation turns out to be crucial also for establishing the robustness of $\mathcal{K}_\mu$ under viscous perturbations of the related PDE (see Sec. 4).

We proceed in two steps. We start by defining the following vector field $\mathbf{Y}$ on $\Omega \times (\mathbb{R}^n \backslash \{0\})$, which is substantially the v-orthogonal projection of $\mathbf{X}$:

$$\mathbf{Y}(x, v) := \left(X(x), DX(x)\frac{v}{\|v\|} - \frac{v}{\|v\|}\left[\frac{v}{\|v\|} \cdot DX(x)\frac{v}{\|v\|}\right]\right). \tag{2.10}$$

We prove now the next technical results.

Lemma 1. *For all* $(x, v) \in \Omega \times (\mathbb{R}^n \backslash \{0\})$, *it holds*:

$$\chi(x, v) = \lim_{t \to +\infty} \frac{1}{t} \int_0^t f_\chi\left(\phi_{\mathbf{Y}}^\tau\left(x, \frac{v}{\|v\|}\right)\right) d\tau,$$

where the function f_χ *is given by formula* (2.8). *Moreover, every subset* $\Omega \times \partial B^n(0, r)$, $r > 0$, *is invariant under the flow* $\phi_{\mathbf{Y}}^t$.

Proof. We start by introducing the retraction Π:

$$\Pi : \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) \to \mathbb{R}^n \times \mathbb{S}^{n-1}$$

$$(x, v) \mapsto \Pi(x, v) = \left(x, \frac{v}{\|v\|} \right).$$

In view of the homogeneity of degree zero of the function f_χ, that is $f_\chi(x, \lambda v) = f_\chi(x, v)\ \forall \lambda \neq 0$, we have that

$$f_\chi(\phi^t_{\mathbf{X}}(x, v)) = f_\chi(\Pi \circ \phi^t_{\mathbf{X}}(x, v)).$$

Therefore, we study the retraction of the dynamics $\phi^t_{\mathbf{X}}(x, v)$, proving that it corresponds to the flow of the vector field $\mathbf{Y}$, that is:

$$\Pi \circ \phi^t_{\mathbf{X}}(x, v) = \phi^t_{\mathbf{Y}} \circ \Pi(x, v). \tag{2.11}$$

In order to do this, we denote $(x(t), v(t)) := \phi^t_{\mathbf{X}}(x, v)$ and compute:

$$\begin{aligned}
\frac{d}{dt}\Pi \circ \phi^t_{\mathbf{X}}(x, v) &= \frac{d}{dt}\left(x(t), \frac{v(t)}{\|v(t)\|} \right) \\
&= \left(\dot{x}(t), \frac{\dot{v}(t)}{\|v(t)\|} - \frac{v(t)}{\|v(t)\|^2} \frac{v(t) \cdot \dot{v}(t)}{\|v(t)\|} \right) \\
&= \left(\dot{x}(t), \frac{DX(x(t))v(t)}{\|v(t)\|} - \frac{v(t)}{\|v(t)\|^2} \frac{v(t) \cdot DX(x(t))v(t)}{\|v(t)\|} \right) \\
&= \left(X(x(t)), DX(x(t))\frac{v(t)}{\|v(t)\|} - \frac{v(t)}{\|v(t)\|} \left[\frac{v(t)}{\|v(t)\|} \cdot DX(x(t))\frac{v(t)}{\|v(t)\|} \right] \right) \\
&= \mathbf{Y}(x(t), v(t)) = \mathbf{Y}\left(x(t), \frac{v(t)}{\|v(t)\|} \right).
\end{aligned}$$

The use of the previous relation together with the initial condition $\Pi \circ \phi^0_{\mathbf{X}}(x, v) = \phi^0_{\mathbf{Y}} \circ \Pi(x, v)$ imply the relation (2.11). As a straightforward consequence $f_\chi(\Pi \circ \phi^t_{\mathbf{X}}(x, v)) = f_\chi(\phi^t_{\mathbf{Y}} \circ \Pi(x, v))$, and the statement of the lemma follows. Finally, from the relation:

$$v \cdot \mathbf{Y}^{(v)}(x, v) = v \cdot \left(DX(x)\frac{v}{\|v\|} - \frac{v}{\|v\|}\left[\frac{v}{\|v\|} \cdot DX(x)\frac{v}{\|v\|} \right] \right) = 0,$$

we immediately obtain an invariance of every subset $\Omega \times \partial B^n(0, r)$, $r > 0$, under the flow $\phi^t_{\mathbf{Y}}$. □

By using the same arguments of the previous proof, we gain the analogous result for the exponentially damped Lyapunov indicators.

Lemma 2. *Let $\mu > 0$. For all $(x, v) \in \Omega \times (\mathbb{R}^n \backslash \{0\})$, it holds*:

$$\mathcal{K}_\mu(x, v) = \mu \int_0^{+\infty} e^{-\mu\tau} f_\chi\left(\phi^\tau_{\mathbf{Y}}\left(x, \frac{v}{\|v\|} \right) \right) d\tau,$$

where the function f_χ is given by formula (2.8).

We remark now that, although the flow $\phi^t_{\mathbf{Y}}$ leaves invariant $\Omega \times \partial B^n(0, r)$, $r > 0$, we cannot consider a PDE on the compact domain $\Omega \times B^n(0, 1)$, essentially because $\mathbf{Y}$ cannot be extended by continuity in $v = 0$. To solve this problem, the second step consists in modifying the vector field $\mathbf{Y}$ in a small neighborhood of $0 \in B^n(0,1)$. More precisely, given $h \in C^\infty(B^n(0,1),\mathbb{R})$, such that $h(v) = 1$ for $\|v\| \geq \varepsilon$ and $h(v) = 0$ for $\|v\| < \frac{\varepsilon}{2}$, we introduce the following vector field $\widehat{\mathbf{Y}}$ on $\Omega \times B^n(0,1)$:

$$\widehat{\mathbf{Y}}(x,v) = \begin{cases} \mathbf{Y}(x,v) & \text{if } \|v\| \geq \varepsilon \\ h(v)\mathbf{Y}(x,v) & \text{if } \|v\| < \varepsilon. \end{cases} \tag{2.12}$$

As a consequence, the function $\mathcal{K}_\mu$ can now be obtained as the solution of the following PDE:

$$\widehat{\mathbf{Y}}^{(x)}\nabla_x \mathcal{K}_\mu(x,v) + \widehat{\mathbf{Y}}^{(v)}\nabla_v \mathcal{K}_\mu(x,v) = \mu(\mathcal{K}_\mu - f_\chi)(x,v) \tag{2.13}$$

defined on the domain $\Omega \times B^n(0,1)$.

3. Fast and Exponentially Damped Lyapunov Indicators

In the last years, the so called Fast Lyapunov Indicators [13] have been extensively used to numerically detect the phase space structure, i.e. the distribution of KAM tori and resonances, of quasi-integrable systems, see [14, 15]. For the equation $\dot{x} = X(x)$, the simplest definition of Fast Lyapunov Indicator of a point x and of a tangent vector v, at time T, is:

$$\mathrm{FLI}_T(x,v) = \log\left(\frac{\|v_T\|}{\|v\|}\right), \tag{3.1}$$

where $v_T = D\phi_X^T(x)v$. In [15] it is proved that, for Hamiltonian vector fields, if T is suitably long (respect to some inverse power of the perturbing parameter, see [15] for precise statements) and v is generic, the value of $\mathrm{FLI}_T(x,v)$ is different, at order 0 in ε, in the case x belongs to an invariant KAM torus from the case x belongs to a resonant elliptic torus. Therefore, the computation of the FLI on grids of initial conditions in the phase space allows one to detect the distribution of invariant tori and resonances in relatively short CPU times. Let us remark that if at a first glance the function FLI_T seems a crude way of estimating Lyapunov exponents from finite time computations, in [15] it is proved that it provides information on the dynamics of x that cannot be obtained with the largest Lyapunov exponent, which in fact is equal to zero for all KAM tori and resonant elliptic tori. Moreover, resonances and KAM tori are detected by the FLI on times T which are much smaller than the times required to compute finite time approximations of the largest Lyapunov exponent. These computational advantages allows one to use the FLI for extensive dynamical analysis of dynamical systems representing accurate models of real systems, such as the dynamical model for the outer solar system ([16, 17]).

In this section we propose a practical use of the exponentially damped Lyapunov indicators, which we have defined in Sec. 2, as a global definition of Fast Lyapunov Indicators, in the sense specified in the introduction. In fact, on the one hand the definition (3.1) of Fast Lyapunov Indicators is a pointwise definition, on the other hand the definition (2.9) with $\mu = 1/T$ provides substantially the same information as the FLI. In fact, the FLI can be obtained by the integration of a function defined on the tangent space (see (2.7)) on a finite time interval $[0,T]$, while the exponentially damped indicator is obtained by the integration on $[0,+\infty[$ of the same function multiplied by the damping factor $e^{-\mu t}$, which plays the role of limiting the integration to a finite interval of some few multiples of $1/\mu$.

We compare the results provided by the two indicators on the quasi-integrable Hamiltonian system defined in [14]:

$$H = \frac{I_1^2}{2} + \frac{I_1^2}{2} + I_3 + \varepsilon f(\varphi_1,\varphi_2,\varphi_3), \tag{3.2}$$

where $I_1, I_2, I_3 \in \mathbb{R}$, $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{T}^1$, the underlying symplectic structure is $dI \wedge d\varphi$, $\varepsilon > 0$ is the perturbing parameter and the perturbation f is given by:

$$f(\varphi_1, \varphi_2, \varphi_3) = \frac{1}{\cos(\varphi_1) + \cos(\varphi_2) + \cos(\varphi_3) + 4}. \tag{3.3}$$

Hamiltonian system (3.2) is particularly suited for the detection of the KAM tori and web of resonances (see [14]), in fact: for $\varepsilon > 0$ suitable small, the KAM Theorem applies to (3.2); each KAM torus of the system intersects transversely in only one point the section of phase space:

$$S := \{(I_1, I_2, I_3, \varphi_1, \varphi_2, \varphi_3) \quad \text{with} \quad (\varphi_1, \varphi_2, \varphi_3) = (0, 0, 0)\},$$

which we call action space; the perturbation (3.3) is non-generic in the sense of the Poincaré Theorem about the non-integrability of quasi-integrable systems.

We now compute the exponentially damped Lyapunov indicators defined by $\mu = \varepsilon$ for a grid of equally spaced initial conditions on the section S. The practical computation of the integral in (2.9) is done by restricting the integration interval up to a total time T_1 such that the exponential damp $\exp(-\mu T_1)$ is smaller than the numerical precision adopted for the computation. For example, we set T_1 such that: $\exp(-\mu T_1) < 10^{-16}$. The initial vector for any initial condition was chosen as: $(v_{I_1}, v_{I_2}, v_{I_3}, v_{\varphi_1}, v_{\varphi_2}, v_{\varphi_3}) = (1/\sqrt{5}, \sqrt{2/5}, 0, 1/\sqrt{5}, 1/\sqrt{5}, 0)$. The result of the computation is reported in Fig. 1, where we report for any initial actions (I_1, I_2) the value of the computed

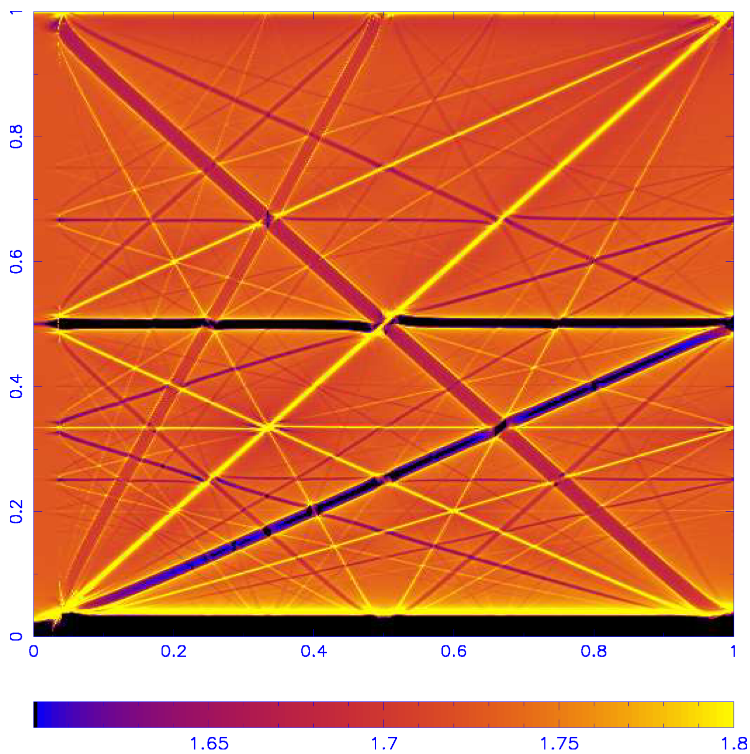

Fig. 1. Computation of the exponentially damped Lyapunov indicators for initial conditions $(I_1, I_2, I_3, \varphi_1, \varphi_2, \varphi_3)$ on the section S, $\varepsilon = 0.004$ and $\mu = \sqrt{\varepsilon}/10$. The x-axis corresponds to the value of I_1, the y-axis corresponds to the value of I_2; for each initial condition the value of the exponentially damped Lyapunov indicator is reported using the color scale reported below the picture. The well known structure of resonances of this system is clearly detected by the highest and lowest values of the exponentially damped Lyapunov indicator, see [14, 15].

exponentially damped Lyapunov indicator using a color scale[a] such that dark gray corresponds to the lowest values of the indicator and light gray corresponds to the highest values of the indicator. Following [15], the KAM tori are characterized by intermediate gray, hyperbolic motions by light gray and resonant elliptic tori by dark gray. It is clear that the distribution of the values of the exponentially damped Lyapunov indicator shown in Fig. 1 corresponds to the distribution of resonances and KAM tori as it is described in [14].

4. Robustness Under Viscous Perturbations

The aim of this section is to show the relevance of the previous notion of approximated first integral (see Definition 1) inside the PDE framework and related viscosity techniques, see [2, 11].

Considering the classical theory for equations of elliptic type, we take into account the following regularization of (2.5), with vanishing viscosity $\nu > 0$ and a sort of friction $\mu > 0$:

$$\frac{\nu^2}{2}\Delta F^\nu(x) + X \cdot \nabla F^\nu(x) = \mu(F^\nu - f)(x). \tag{4.1}$$

The existence and the asymptotic behavior of the solutions for (4.1), namely the convergence of the functions F^ν for $\nu \to 0$, has been largely investigated: for the convenience to the reader, we briefly summarize below the main results (see [11] for some more details).

Referring to the elliptic differential operator:

$$L^\nu := \frac{\nu^2}{2}\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(x)\frac{\partial}{\partial x^i},$$

we are interested on the related Dirichlet's problem:

$$\begin{cases} L^\nu F^\nu(x) + c(x)F^\nu(x) = g(x) \\ F^\nu(x)|_{\partial\Omega} = \psi(x) \end{cases}$$

that is,

$$\begin{cases} \dfrac{\nu^2}{2}\displaystyle\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 F^\nu}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{n} b^i(x)\frac{\partial F^\nu}{\partial x^i}(x) + c(x)F^\nu(x) = g(x) \\ F^\nu(x)|_{\partial\Omega} = \psi(x) \end{cases} \tag{4.2}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth connected boundary $\partial\Omega$ and ψ is supposed to be continuous.

We recall below the existence and uniqueness result.

Theorem 1 (Existence and uniqueness, [11]). *We assume that the following conditions are satisfied.*

1. *The function c is uniformly continuous, bounded and $c(x) \leq 0$ for all $x \in \mathbb{R}^n$.*
2. *The coefficients of L^ν satisfy a Lipschitz condition.*
3. $$k^{-1}\sum_{i=1}^{n}\lambda_i^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\lambda_i\lambda_j \leq k\sum_{i=1}^{n}\lambda_i^2$$

   for every real $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $x \in \mathbb{R}^n$, where k is a positive constant.

Under these assumptions, for every $\nu > 0$ there exists a unique solution F^ν to the problem (4.2).

[a]The color version of the figure can be found on the electronic version of the paper so that light gray corresponds there to yellow and darker gray corresponds there to darker orange.

In order to investigate the asymptotic behavior of such a solution, we remind the following

Theorem 2 (Pointwise limit, [11]). *Suppose conditions* 1, 2 *and* 3 *are satisfied and* $c(x) < 0$ *for all* $x \in \Omega$. *If, for a given* $x \in \Omega$, *the trajectory* $\phi_b^t(x), t \geq 0$, *does not leave* Ω, *then*

$$\lim_{\nu \to 0} F^\nu(x) = F(x) = -\int_0^{+\infty} g(\phi_b^\tau(x)) \exp\left[\int_0^\tau c(\phi_b^v(x))dv\right] d\tau. \tag{4.3}$$

It is now interesting to underline the following fact: the representation of the function F depends decisively on the behavior of the flow ϕ_b^t. In particular, since the vector field b admits the invariant bounded domain Ω, the pointwise limit $\lim_{\nu\to 0} F^\nu(x) = F(x)$, $\forall x \in \Omega$, does not depend on the boundary datum in (4.2).

Now we are ready to come back to our original elliptic equation (2.5): with respect to the previous general setting, it corresponds to the coefficients $a^{ij}(x) = \delta^{ij}$, $b^i(x) = X^i(x)$, $c(x) = -\mu$ and $g(x) = -\mu f(x)$ and it trivially satisfies the three conditions of Theorems 1 and 2. In such a case, the pointwise limit for $\nu \to 0$ is *just* given by the above introduced function (2.4):

$$F(x) = \mu \int_0^{+\infty} e^{-\mu\tau} f(\phi_X^\tau(x))d\tau, \quad \forall x \in \Omega. \tag{4.4}$$

This fact shows that, among any possible perturbation of $X \cdot \nabla F(x) = 0$, the one proposed in Sec. 2, that is $X \cdot \nabla F(x) = \mu(F - f)(x)$, is solved exactly by the approximated first integral descending from a regularizing viscosity procedure. This argument points out a viscosity motivation of (4.4) and seems to mark a step towards the recognition of a robust notion of approximated global first integral.

We remark that these considerations hold also in the Lyapunov exponents case, in view of the reformulation done in Sec. 2. In such a case, Dirichlet's problem reads:

$$\begin{cases} \dfrac{\nu^2}{2}\Delta\mathcal{K}_\mu^\nu(x,v) + \widehat{\mathbf{Y}}^{(x)}\nabla_x\mathcal{K}_\mu^\nu(x,v) + \widehat{\mathbf{Y}}^{(v)}\nabla_v\mathcal{K}_\mu^\nu(x,v) = \mu(\mathcal{K}_\mu^\nu - f_\chi)(x,v) \\ \mathcal{K}_\mu^\nu(x,v)|_{\partial D} = \psi(x,v) \end{cases} \tag{4.5}$$

where $D := \Omega \times B^n(0,1)$, and the following convergence result holds.

Proposition 2. *Let* $\nu, \mu > 0$. *The solution* K_μ^ν *of Dirichlet's problem* (4.5), *when restricted to* $\Omega \times (B^n(0,1)\backslash \overset{\circ}{B}{}^n(0,\varepsilon))$, *is independent on the regularizing function in* (2.12) *and the following pointwise limit holds*:

$$\lim_{\nu\to 0} \mathcal{K}_\mu^\nu(x,v) = \mathcal{K}_\mu(x,v).$$

5. Conclusions

For a large class of nonlinear dynamical systems, we have introduced a new notion of approximated first integrals, inspired by finite time averages, and we have discussed their properties. These approximated first integrals satisfy an explicit first order partial differential equation and they are stable under viscosity perturbations of such equation. Moreover, their numerical implementation provides results on applications comparable to the ones given by other popular indicators. We have formulated specifically this approximating device to define finite time approximations of Lyapunov exponents and we have considered their use to detect the phase space structure of quasi-integrable systems.

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