Small parameter limit for discrete-time partially observed risk-sensitive control problems

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Abstract

We show that risk-sensitive control problems and deterministic dynamic games can be connected, under rather mild assumptions, by a small noise limit. In order to control this limit, new techniques are developed to study propagation of large deviations through conditional probabilities.

1 Introduction

Properties of risk-sensitive control problems and their connections with dynamic games have been widely investigated in recent years [16, 18, 11, 8, 9, 12, 13, 14, 6, 4], in part inspired by seminal results for linear quadratic models contained in [10, 17, 1]. In particular, it has been shown [8, 9, 12, 13, 4] that under a suitable small parameter limit (*small noise limit*) a family of risk-sensitive stochastic control problems becomes equivalent to a deterministic dynamic game. In other words, this means that optimal risk sensitive control with small noise and suitably rescaled risk parameter is almost equivalent to deterministic robust control (worst-case approach).

Although this result is conceptually natural, its proof usually involves rather sophisticated mathematical techniques, and fairly strong requirements on the model. For continuous-time, totally observable systems a quite satisfactory theory has been developed in [8, 9] by using viscosity solution techniques to analyze the Hamilton-Jacobi equation associated to the optimal control problem. It has been shown, in particular, that the value function of the risk sensitive control problem converges, as the noise parameter goes to zero, to the upper value function of a related two players, zero-sum differential game.

The analysis of the small parameter limit for nonlinear, partially observed, risk sensitive control problems has been initiated by P.Whittle ([18]), whose mostly non rigorous results have inspired most of the further development. A considerable advancement in the understanding of these models is represented by the results in [12, 13], where the *information state* approach is used. This approach consists in reformulating the partially observed control problem as a completely observed one, in a way that, in a suitable sense, is "preserved" in

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the small parameter limit. One consequence of this method is that it provides a natural notion of information state for the limiting dynamic game. In [12] the information state approach is applied to discrete time systems. The result obtained is parallel to the one given in [8], i.e. the connection with partially observed dynamic game is established in terms of the convergence of the value function of the equivalent totally observed model. This value function is the solution of a dynamic programming equation in an infinite dimensional space. The small parameter limit of this equation is obtained by using large deviation techniques.

The corresponding result in continuous time has been obtained, at a non-rigorous level, in [13]. In this context, one has to deal with the small parameter limit of an Hamilton-Jacobi equation (Mortensen's equation) with infinite dimensional state space; the current mathematical understanding of this problem has not allowed a complete proof yet. A different approach to the small parameter limit in continuous time is used in [3] where, rather than the convergence of the value function, it is (*rigorously*) shown the convergence of the *cost functional* for *any* control u in a suitably defined admissible class. This requires considerable work for controlling the small parameter limit of the information state, but avoids the use of Mortensen's equation.

In this paper we consider discrete-time, finite time-horizon partially observed systems, and we develop further some large deviation techniques that were introduced in [4] for totally observed systems. The models for which we can analyze the small parameter limit include those of type

$$\begin{aligned} x_{n+1} &= f_n(x_n, u_n, w_n) \\ y_n &= \phi_n(x_n, v_n) \end{aligned} \tag{1.1}$$

for $n = 0, \ldots, N-1$, with $x_n \in \mathcal{X}, u_n \in \mathcal{U}, w_n \in \mathcal{W}, y_n, v_n \in \mathbb{R}^d$, where $\mathcal{X}, \mathcal{W}$ are metric spaces, and $\mathcal{U}$ is a compact metric space. Moreover, w_n, v_n are independent random variables, $w_n \sim \mu_n^\epsilon$, $v_n \sim \nu_n^\epsilon$, where $(\mu_n^\epsilon)_{\epsilon>0}$, $(\nu_n^\epsilon)_{\epsilon>0}$ are families of probability measures satisfying a Large Deviation Principle (see Section 2). Some further regularity assumptions will be needed for ϕ_n and ν_n^ϵ (see Section 4), while f_n is only supposed to be continuous. We associate to (1.1) a cost functional of the form

$$J^\epsilon(\mathbf{u}) = \epsilon \log E\Big\{ \exp\Big[\epsilon^{-1}\Big(\sum_{n=0}^{N-1} g_n(x_n, u_n) + g_N(x_N)\Big)\Big]\Big\} \tag{1.2}$$

defined for the control sequences $\mathbf{u} = (u_0, \ldots, u_{N-1})$ that are nonanticipative functions of the output sequence $(y_0, \ldots, y_N)$. Note that the parameter ϵ appears both in the noise distribution and in (1.2), where it can be interpreted as a *risk parameter*, a measure of controller's aversion to risk. We show that, as $\epsilon \to 0$, the risk sensitive control problem (1.1) (1.2) converges, in a suitable sense, to the deterministic game with dynamics (1.1) (where w_n, v_n are thought of as deterministic but unknown disturbances) and cost

$$J(\mathbf{u}) = \sup_{\mathbf{v},\mathbf{w}} \Big[\sum_{n=0}^{N-1} \Big(g_n(x_n, u_n) - h_n(w_n) - k_{n+1}(v_{n+1})\Big) + g_N(x_N)\Big] \tag{1.3}$$

where h_n, k_n are *rate functions* (see Section 2) associated to $\mu_n^\epsilon, \nu_n^\epsilon$. This convergence is expressed, similarly to [8] and [12], in terms of the convergence of the value function for an equivalent totally observed problem, so that our result can be seen as a generalization of [12]. In fact, models of type (1.1) (1.2) include the ones studied in [12], but we can deal with fairly more general noise distribution, state-space and dynamical equations. The results of this paper have been, in part, announced in [5] where, however, much stronger conditions were required.

This paper is organized as follows. In Sections 2 and 3 we develop some new large deviation techniques that are suitable for the problem we deal with. Section 4, that contains the main results of this paper, is devoted to the analysis of the small parameter limit for risk-sensitive control problems.

2 Preliminary notions

In this section we recall some notions from large deviation theory that will be used throughout the paper, and introduce some new ones. We let $\mathcal{X}$ be a metric space. All measures on $\mathcal{X}$ are intended to be defined on its Borel σ-field.

Definition 2.1 A family of probability measures $\{P^\epsilon : \epsilon > 0\}$ on $\mathcal{X}$ is said to satisfy a *Large Deviation Principle* (LDP) with rate function $H : \mathcal{X} \to [0, +\infty]$ if

i) H is lower semicontinuous and $\{x : H(x) \leq l\}$ is compact for every $l \geq 0$.

ii) For every $A \subset \mathcal{X}$ measurable

$$
\begin{aligned}
-\inf_{x \in \overset{\circ}{A}} H(x) &\leq \liminf_{\epsilon \to 0} \epsilon \log P^\epsilon(A) \\
&\leq \limsup_{\epsilon \to 0} \epsilon \log P^\epsilon(A) \leq -\inf_{x \in \bar{A}} H(x)
\end{aligned}
$$

where $\overset{\circ}{A}, \bar{A}$ denote respectively the interior and the closure of A.

In [4] a modification of the above definition has been introduced for the case of a family of probability measures depending on a further parameter.

Definition 2.2 Let Θ be a set. A family of probability measures $\{P^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ on $\mathcal{X}$ is said to satisfy a *Uniform Large Deviation Principle* (ULDP) with rate function $H : \mathcal{X} \times \Theta \to [0, +\infty]$ if

i) For every fixed $\theta \in \Theta$, $H(\cdot, \theta)$ is lower semicontinuous and $\{x : H(x, \theta) \leq l\}$ is compact for every $l \geq 0$.

ii) For every $A \subset \mathcal{X}$ measurable and $M > 0$

$$
\limsup_{\epsilon \to 0} \sup_{\theta \in \Theta} \left[\epsilon \log P^\epsilon(A;\theta) + \min\left(M, \inf_{x \in \bar{A}} H(x;\theta) \right) \right] \leq 0
$$

and

$$
\liminf_{\epsilon \to 0} \inf_{\theta \in \Theta} \left[\epsilon \log P^\epsilon(A;\theta) + \inf_{x \in \overset{\circ}{A}} H(x;\theta) \right] \geq 0.
$$

One of the main consequences of a LDP is the well known Varadhan's Lemma ([15]). In [4] the following version of Varadhan's Lemma has been proved.

Lemma 2.3 *Suppose the family* $\{P^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ *satisfies a ULDP. Then for every* $F : \mathcal{X} \to \mathbb{R}$ *bounded and continuous*

$$
\lim_{\epsilon \to 0} \epsilon \log \int e^{\epsilon^{-1} F(x)} P^\epsilon(dx;\theta) = \sup_{x \in X} \left[F(x) - H(x, \theta) \right] \tag{2.1}
$$

uniformly for $\theta \in \Theta$.

Remark 2.4 Identity (2.1) is the crucial large deviation property in applications to risk sensitive control. When Θ is a singleton (and therefore pointwise in θ) it is well known ([7], Bryc's Theorem) that, under rather mild assumptions (namely: exponential tightness, see definition below), identity (2.1) is equivalent to the LDP. It is natural to ask whether, for general Θ, (2.1) implies the ULDP. The answer is no. A simple counterexample is the following: $\mathcal{X} = \mathbb{R}$, $\Theta = [0,1]$, $P^\epsilon(dx;\theta) = \frac{1}{2\epsilon}\chi_{[\theta-\epsilon,\theta+\epsilon]}(x)dx$, where χ denotes the characteristic function of a set. As we will see later in Remark 2.8, the family $P^\epsilon(dx;\theta)$ satisfies (2.1) with $H(x;\theta) = +\infty$ for $x \neq \theta$, and $H(\theta,\theta) = 0$. Now take $A = (-\infty, 0]$. If $P^\epsilon(dx;\theta)$ satisfied a ULDP, then $\lim_{\epsilon\to 0} \epsilon \log P^\epsilon(A;\epsilon/2) = -\infty$, since $\inf_{x\in A} H(x;\epsilon/2) = +\infty$ for all $\epsilon > 0$. However it holds that $P^\epsilon(A;\epsilon/2) = 1/4$.

We introduce now a notion which is weaker than the one of ULDF. For the rest of this section we assume Θ to be a metric space.

Definition 2.5 A family $\{P^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ of positive finite measures on $\mathcal{X}$ is called a *Weakly Uniform Large Deviation Family* (WULDF) with rate function $H : \mathcal{X} \times \Theta \to (-\infty, +\infty]$ if

i) For every fixed $\theta \in \Theta$, $H(\cdot, \theta)$ is lower semicontinuous and $\{x : H(x,\theta) \leq l\}$ is compact for every $l \in \mathbb{R}$.

ii) The map $\theta \to \inf_{x\in X} H(x,\theta)$ is real valued, and is bounded on the compact subsets of Θ.

iii) For every $F : \mathcal{X} \to \mathbb{R}$ bounded and continuous

$$\lim_{\epsilon\to 0} \epsilon \log \int e^{\epsilon^{-1}F(x)} P^\epsilon(dx;\theta) = \sup_{x\in X} \Big[F(x) - H(x,\theta)\Big] \tag{2.2}$$

uniformly for θ in the compact subsets of Θ.

Remark 2.6 For reasons that will become apparent later, we have chosen to allow P^ϵ in Definition 2.5 to be a positive finite measure, not necessarily a probability measure. For technical reasons, we will need condition ii) in Definition 2.5, that roughly says that $P^\epsilon(\mathcal{X};\theta)$ does not either go to zero or grow too fast as $\epsilon \to 0$. Indeed, by using ii) and letting $F \equiv 0$ in iii) the following statement is easy to prove: for each $K \subset \Theta$ compact, there exists $M(K) > 0$ such that, for ϵ sufficiently small,

$$e^{-\epsilon^{-1}M(K)} \leq P^\epsilon(\mathcal{X};\theta) \leq e^{\epsilon^{-1}M(K)} \tag{2.3}$$

for all $\theta \in K$. Note that, if all $P^\epsilon(dx;\theta)$ are probability measures, then ii) is automatically satisfied, since $\inf_{x\in\mathcal{X}} H(x,\theta) \equiv 0$ (see [7]).

We now state a proposition that serves both as a technical lemma for later use and as a preliminary justification of the notion of WULDF. Its proof will be given in Section 3.

Proposition 2.7 *Let $\mathcal{W}$ be a metric space, $f : \Theta \times \mathcal{W} \to \mathcal{X}$ a continuous map, and $\{\mu^\epsilon : \epsilon > 0\}$ a family of probability measures on $\mathcal{W}$ that satisfy a LDP with rate function $h(w)$. Define $P^\epsilon(dx;\theta)$, a probability measure on X, by*

$$P^\epsilon(A;\theta) = \mu^\epsilon\{w : f(\theta,w) \in A\}. \tag{2.4}$$

Then $\{P^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ is a WULDF with rate function

$$H(x;\theta) = \inf\{h(w) : f(\theta,w) = x\}. \tag{2.5}$$

Remark 2.8 The family of probability measures in (2.4) does not necessarily satisfy a ULDP, even if Θ is compact. For example, consider $\Theta = [0,1]$, $\mathcal{X} = \mathcal{W} = \mathbb{R}$, $\mu^\epsilon(dw) = \frac{1}{2\epsilon}\chi_{[-\epsilon,\epsilon]}(w)dw$, $f(\theta, w) = \theta + w$, and we end up with the counterexample in Remark 2.4. Note that this shows that the family $P^\epsilon(dx;\theta)$ is a WULDF, since it is easy to prove that $\{\mu^\epsilon(dw) : \epsilon > 0\}$ satisfies a LDP with rate function $H(w) = +\infty$ if $w \neq 0$, and $H(0) = 0$.

We now introduce a further notion that will be useful later.

Definition 2.9 A family $\{P^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ of positive finite measure on $\mathcal{X}$ is called *exponentially tight* if, for every $L > 0$ and every $K \subset \Theta$ compact, there exists $C \subset \mathcal{X}$ compact such that

$$P^\epsilon(C^c;\theta) \leq e^{-\epsilon^{-1}L} \tag{2.6}$$

for all $\theta \in K$ and ϵ sufficiently small, where C^c is the complement of C.

Note that when the measures are probability measures and Θ is a singleton, the above definition reduces to the usual one of exponential tightness of Large Deviation Theory ([7]).

We conclude this section by stating three easy lemmas, that will be used in Section 3.

Lemma 2.10 *Under the assumptions of Proposition 2.7, let $P^\epsilon(dx;\theta)$ be defined by (2.4). If $\{\mu^\epsilon\}$ is exponentially tight then so is $\{P^\epsilon(dx;\theta)\}$.*

Proof. The proof is straightforward, since compactness is preserved by continuous mapping. ∎

Lemma 2.11 *Suppose that $\{P^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ is a WULDF with rate function H, and it is exponentially tight. Then the rate function is* proper, *i.e. for every $L > 0$ and every $K \subset \Theta$ compact, there exists $C \subset \mathcal{X}$ compact such that $H(x;\theta) \geq L$ for all $(x,\theta) \in C^c \times K$.*

Proof. Let $L > 0$ be given, $M > L$ and C be a compact subset of $\mathcal{X}$ such that (2.6) holds for all $\theta \in K$. Also, let C_δ denote the δ-neighborhood of C. Consider the bounded continuous function

$$F(x) = \min\Big\{\frac{M}{\delta}d(x,C) - M, 0\Big\}. \tag{2.7}$$

It is easily seen that $F(x) = -M$ for $x \in C$, $F(x) = 0$ on C_δ^c and $F \leq 0$. By using the definition of WULDF we have, for all $\theta \in K$,

$$\begin{aligned}
\inf_{x \in C_\delta^c} H(x,\theta) &\geq -\sup_{x\in\mathcal{X}}[F(x) - H(x,\theta)] \\
&= -\lim_{\epsilon\to 0} \epsilon \log \int e^{\epsilon^{-1}F(x)} P^\epsilon(dx;\theta) \\
&= -\lim_{\epsilon\to 0} \epsilon \log \Big[\int_C e^{\epsilon^{-1}F(x)} P^\epsilon(dx;\theta) + \int_{C^c} e^{\epsilon^{-1}F(x)} P^\epsilon(dx;\theta)\Big] \\
&\geq -\liminf_{\epsilon\to 0} \epsilon \log \Big[e^{-\epsilon^{-1}M} P^\epsilon(X;\theta) + e^{-\epsilon^{-1}L}\Big] = \min\Big[M - \inf_{x\in\mathcal{X}} H(x,\theta), L\Big].
\end{aligned}$$

Thus, if we choose M large enough, using ii) of Definition 2.5, we have that, for all $\theta \in K$ and $\delta > 0$

$$\inf_{x\in C_\delta^c} H(x,\theta) \geq L \tag{2.8}$$

that clearly concludes the proof. ∎

Lemma 2.12 *Under the assumptions of Lemma 2.11, let* $F^\epsilon : \mathcal{X} \to \mathbb{R}$, $\epsilon \geq 0$ *be such that* $\sup_{\epsilon \geq 0} \|F^\epsilon\|_\infty < \infty$, $F^\epsilon \to F^0$ *as* $\epsilon \to 0$ *uniformly on the compact subsets of* $\mathcal{X}$ *and* F^0 *is continuous. Then*

$$\lim_{\epsilon \to 0} \epsilon \log \int e^{\epsilon^{-1} F^\epsilon(x)} P^\epsilon(dx;\theta) = \sup_{x \in \mathcal{X}} \left[F^0(x) - H(x,\theta) \right] \tag{2.9}$$

uniformly on the compact subsets of Θ.

Proof. If $F^\epsilon \to F^0$ uniformly in all $\mathcal{X}$ then the conclusion follows by (2.2) and by

$$\left| \epsilon \log \int e^{\epsilon^{-1} F^\epsilon(x)} P^\epsilon(dx;\theta) - \epsilon \log \int e^{\epsilon^{-1} F^0(x)} P^\epsilon(dx;\theta) \right| \leq \|F^\epsilon - F^0\|_\infty. \tag{2.10}$$

By using exponential tightness one easily reduces to this case. ∎

3 Propagation of WULDF's

In our analysis of the small parameter limit for risk-sensitive control problems the *filtering probabilities* and the *information states* will play a key role. They are both families of positive measures that satisfy recursive relations. To prove that they form WULDF's we show that the property of being a WULDF is preserved under four basic operations, namely: 1. *state augmentation*; 2. *composition*; 3. *contraction*; 4. *conditioning.*

In the rest of this section $\mathcal{X}, \mathcal{Y}$ and Θ are metric spaces.

Proposition 3.1 *(State augmentation). Let* $\{P^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ *be an exponentially tight WULDF on* $\mathcal{X}$ *with rate function* $H_P(x;\theta)$. *Define the measures on* $\mathcal{X} \times \Theta$

$$Q^\epsilon(dx, d\zeta;\theta) = P^\epsilon(dx;\theta) \otimes \delta_\theta(d\zeta),$$

with δ *denoting the Dirac measure. Then* $\{Q^\epsilon(dx, d\zeta;\theta) : \epsilon > 0, \theta \in \Theta\}$ *is an exponentially tight WULDF with rate function:*

$$H_Q(x,\zeta;\theta) = \begin{cases} H_P(x;\theta) & \text{if } \zeta = \theta \\ +\infty & \text{if } \zeta \neq \theta \end{cases}$$

Proof. First we prove that the measures $Q^\epsilon(dx, d\zeta;\theta)$ form a WULDF. It is easy to see that the function $H_Q(x,\zeta;\theta)$ satisfies properties i), ii) in Definition 2.5, so we only prove that property iii) holds.

Let $K \subseteq \Theta$ be a compact set, and $F : \mathcal{X} \times \Theta \to \mathbb{R}$ be a continuous and bounded function. Let $L > 0$ be such that $|F(x,\zeta)| \leq L$, and $M(K) = \sup_{\theta \in K} |\inf_x H_P(x;\theta)|$, which is finite by ii) of Definition 2.5. Notice that, by the definition of H_Q, we have:

$$\sup_{(x,\zeta) \in \mathcal{X} \times \Theta} \Big[F(x,\zeta) - H_Q(x,\zeta;\theta) \Big] = \sup_{x \in \mathcal{X}} \Big[F(x,\theta) - H_P(x;\theta) \Big].$$

Since H_P is proper (see Lemma 2.11), there exists a compact set $C \subseteq \mathcal{X}$ such that $H_P(x;\theta) \geq 3L + M(K)$ for all $x \in C^c$ and all $\theta \in K$. Moreover, for all $\theta \in K$, we have:

$$\sup_{x \in \mathcal{X}} (F(x,\theta) - H_P(x;\theta)) \geq -L - M(K);$$

$$F(x,\theta) - H_P(x;\theta) \le -2L - M(K), \quad \forall\, x \in C^c.$$

Thus

$$\sup_{x\in\mathcal{X}} \Big[F(x,\theta) - H_P(x;\theta)\Big] = \sup_{x\in C} \Big[F(x,\theta) - H_P(x;\theta)\Big]. \tag{3.1}$$

Let $\beta > 2L + 2M(K)$. Since $\{P^\epsilon(dx;\theta)\}$ is exponentially tight, there exists a compact set $C_M \subseteq \mathcal{X}$ such that

$$P^\epsilon(C_M^c;\theta) \le \exp\{-\epsilon^{-1}\beta\}; \tag{3.2}$$

for all $\theta \in K$. Without loss of generality, we may assume that $C \subseteq C_M$ and that for ϵ small enough $P^\epsilon(C_M;\theta) \ge e^{-2\epsilon^{-1}M(K)}$ for all $\theta \in K$ (see (2.3)). We have:

$$\epsilon \log \int \exp\{\epsilon^{-1}F(x,\theta)\} P^\epsilon(dx;\theta) =$$

$$= \epsilon \log \left(\int_{C_M} \exp\{\epsilon^{-1}F(x,\theta)\} P^\epsilon(dx;\theta) + \int_{C_M^c} \exp\{\epsilon^{-1}F(x,\theta)\} P^\epsilon(dx;\theta) \right) \le$$

$$\le \epsilon \log \int_{C_M} \exp\{\epsilon^{-1}F(x,\theta)\} P^\epsilon(dx;\theta) + \epsilon \log \Big(1 + 2\exp\{\epsilon^{-1}(2L + 2M(K) - \beta)\}\Big).$$

Now choose an arbitrary $\delta > 0$. Since $2L + 2M(K) - \beta < 0$, there exists ϵ_0 such that, for all $\epsilon \le \epsilon_0$ and for all $\theta \in K$, we have:

$$\left| \epsilon \log \int \exp\{\epsilon^{-1}F(x,\theta)\} P^\epsilon(dx;\theta) \, - \, \epsilon \log \int_{C_M} \exp\{\epsilon^{-1}F(x,\theta)\} P^\epsilon(dx;\theta) \right| \le \delta. \tag{3.3}$$

Since $F|_{C_M \times K}$ is uniformly continuous, we have that for each $\theta \in K$ there exists an open neighborhood U_θ of θ such that $|F(x,\theta_1) - F(x,\theta_2)| < \delta$ for all $\theta_1, \theta_2 \in U_\theta$ and all $x \in C_M$. K being compact, there exists $\theta_1, \dots, \theta_n$ such that $K \subseteq \cup_{i=1}^n U_{\theta_i}$.

Let $\theta \in K$; then there exists $\bar{i}$ such that $\theta \in U_{\theta_{\bar{i}}}$. Since $|F(x,\theta) - F(x,\theta_{\bar{i}})| < \delta$ for all $x \in C_M$, we have:

$$\left| \epsilon \log \int_{C_M} \exp\{\epsilon^{-1}F(x,\theta)\} P^\epsilon(dx;\theta) - \epsilon \log \int_{C_M} \exp\{\epsilon^{-1}F(x,\theta_{\bar{i}})\} P^\epsilon(dx;\theta) \right| < \delta, \tag{3.4}$$

and:

$$\left| \sup_{x\in C_M} [F(x,\theta_{\bar{i}}) - H_P(x;\theta)] - \sup_{x\in C_M} [F(x,\theta) - H_P(x;\theta)] \right| < \delta. \tag{3.5}$$

Moreover, by definition of WULDF, we also have:

$$\left| \epsilon \log \int \exp\{\epsilon^{-1}F(x,\theta_{\bar{i}})\} P^\epsilon(dx;\theta) - \sup_{x\in\mathcal{X}} [F(x,\theta_{\bar{i}}) - H_P(x;\theta)] \right| < \lambda_i(\epsilon), \tag{3.6}$$

where $\lim_{\epsilon\to 0} \lambda_i(\epsilon) = 0$.

Now let $\lambda(\epsilon) = \sup_{i=1,\dots,n} \lambda_i(\epsilon)$, and note that $\lim_{\epsilon\to 0} \lambda(\epsilon) = 0$. Using (3.1), (3.3), (3.4), (3.5), and (3.6), for all $\epsilon \le \epsilon_0$, we have:

$$\left| \epsilon \log \int \exp\{\epsilon^{-1}F(x,\zeta)\} Q^\epsilon(dx, d\zeta;\theta) - \sup_{(x,\zeta)\in\mathcal{X}\times\Theta} [F(x,\zeta) - H_Q(x,\zeta;\theta)] \right| =$$

$$= \left| \epsilon \log \int \exp\{\epsilon^{-1}F(x,\theta)\} P^\epsilon(dx;\theta) - \sup_{x\in\mathcal{X}} [F(x,\theta) - H_P(x;\theta)] \right| \le$$

$$\delta + \left| \epsilon \log \int_{C_M} \exp\{\epsilon^{-1} F(x,\theta)\} P^\epsilon(dx;\theta) - \sup_{x \in C_M} [F(x,\theta) - H_P(x;\theta)] \right| \leq$$

$$\delta + \left| \epsilon \log \int_{C_M} \exp\{\epsilon^{-1} F(x,\theta)\} P^\epsilon(dx;\theta) - \epsilon \log \int_{C_M} \exp\{\epsilon^{-1} F(x,\theta_{\bar{i}})\} P^\epsilon(dx;\theta) \right| +$$

$$+ \left| \epsilon \log \int \exp\{\epsilon^{-1} F(x,\theta_{\bar{i}})\} P^\epsilon(dx;\theta) - \epsilon \log \int_{C_M} \exp\{\epsilon^{-1} F(x,\theta_{\bar{i}})\} P^\epsilon(dx;\theta) \right| +$$

$$+ \left| \epsilon \log \int \exp\{\epsilon^{-1} F(x,\theta_{\bar{i}})\} P^\epsilon(dx;\theta) - \sup_{x \in \mathcal{X}} [F(x,\theta_{\bar{i}}) - H_P(x;\theta)] \right| +$$

$$+ \left| \sup_{x \in C_M} [F(x,\theta_{\bar{i}}) - H_P(x;\theta)] - \sup_{x \in C_M} [F(x,\theta) - H_P(x;\theta)] \right| \leq \delta + \delta + \delta + \lambda(\epsilon) + \delta.$$

Note that to get the last inequality we used the fact that equation (3.3) still holds when we replace $F(x,\theta)$ by $F(x,\theta_{\bar{i}})$. Thus

$$\limsup_{\epsilon \to 0} \quad \sup_{\theta \in K} \quad \Big| \epsilon \log \int \exp\{\epsilon^{-1} F(x,\zeta)\} Q^\epsilon(dx, d\zeta;\theta) - \sup_{(x,\zeta) \in \mathcal{X} \times \Theta} [F(x,\zeta) - H_Q(x,\zeta;\theta)] \Big| \leq 4\delta.$$

Since δ is arbitrary the previous limit must be zero. Thus we have proved that the measures $Q^\epsilon(dx, d\zeta;\theta)$ form a WULDF. It remains to show that this family is also exponentially tight. Let $K \subseteq \Theta$ be a compact set. Since P^ϵ is exponentially tight, for every $L > 0$ there exists $C \subseteq \mathcal{X}$ compact such that

$$P^\epsilon(C^c;\theta) \leq e^{-\epsilon^{-1} L}$$

for every $\theta \in K$. Let $\tilde{C} = C \times K$. Clearly $\tilde{C} \subseteq \mathcal{X} \times \Theta$ is compact, and

$$Q^\epsilon(\tilde{C}^c;\theta) = P^\epsilon(C^c;\theta).$$

The exponential tightness is therefore easily proved. ∎

Next corollary restates in a different but equivalent way the result of the previous proposition. We give it explicitly, for further references.

Corollary 3.2 *Let $\{P^\epsilon(dx;y,\theta) : \epsilon > 0, (y,\theta) \in \mathcal{Y} \times \Theta\}$ be an exponentially tight WULDF on $\mathcal{X}$ with rate function $H_P(x;y,\theta)$. Define the measures on $\mathcal{X} \times \mathcal{Y}$, $Q^\epsilon(dx, dz; y,\theta) = P^\epsilon(dx;y,\theta) \otimes \delta_y(dz)$, with δ denoting the Dirac measure. Then $\{Q^\epsilon(dx, dz; y,\theta) : \epsilon > 0, (y,\theta) \in \mathcal{Y} \times \Theta\}$ is an exponentially tight WULDF with rate function:*

$$H_Q(x,z;y,\theta) = \begin{cases} H_P(x;y,\theta) & \text{if } z = y \\ +\infty & \text{if } z \neq y \end{cases}$$

As a simple application of Proposition 3.1, we give the proof of Proposition 2.7.

Proof of Proposition 2.7. By Proposition 3.1 the following identities are easily obtained:

$$\lim_{\epsilon \to 0} \epsilon \log \int e^{\epsilon^{-1} F(x)} P^\epsilon(dx;\theta) = \lim_{\epsilon \to 0} \epsilon \log \int e^{\epsilon^{-1} F(f(\theta,w))} \mu^\epsilon(dw)$$

$$= \lim_{\epsilon \to 0} \epsilon \log \int e^{\epsilon^{-1} F(f(\gamma,w))} \mu^\epsilon \otimes \delta_\theta(dw, d\gamma) = \sup_{w \in \mathcal{W}} \Big[F(f(\theta,w)) - h(w) \Big] = \sup_{x \in \mathcal{X}} \Big[F(x) - H(x;\theta) \Big]$$

where the limit is uniform in the compact subsets of Θ. Moreover, property i) of Definition 2.5 is easily shown for $H(x;\theta)$, while property ii) comes automatically from the fact that the P^ϵ are probability measures. ∎

Next lemma presents an easy technical fact that we will need in the proof of Proposition 3.4.

Lemma 3.3 *Let $F : \mathcal{X} \times \Theta \to \mathbb{R}$ be a continuous and bounded map, and $H : \mathcal{X} \times \Theta \to \mathbb{R}^+$ be the rate function of an exponentially tight WULDF whose elements are probability measures. Moreover, suppose H satisfies the following properties:*

(i) Let $A = \{(x,\theta) \ : \ H(x;\theta) < +\infty\}$. Then for every $(x,\theta) \in A$ and every sequence $\theta_n \to \theta$ there exists a sequence $x_n \to x$ such that $H(x_n;\theta_n) \to H(x;\theta)$,

(ii) H is lower semicontinuous as a function of (x,θ).

Then

$$G(\theta) \; = \; \sup_{x \in \mathcal{X}} [F(x,\theta) - H(x;\theta)] \tag{3.7}$$

is a bounded and continuous function.

Proof. Assume that $|F(x,\theta)| \leq L$ for all (x,θ). Since H is the rate function for a family of probability measures we have (see Remark 2.6) $\inf_x H(x,\theta) = 0$ for all $\theta \in T$. This easily implies $|G(\theta)| \leq L$ for all $t \in \Theta$.

We now show that G is upper semicontinuous. First of all we note that, for any $\theta \in \Theta$, there exists $x \in \mathcal{X}$ such that $G(\theta) = F(x,\theta) - H(x,\theta)$, i.e. the supremum in (3.7) is attained. In fact, that supremum can be equivalently taken for $x \in C$, where $C = \{x : H(x,\theta) \leq 3L\}$. Since C is compact and $F - H$ is upper semicontinuous, then it follows that $F(\cdot,\theta) - H(\cdot,\theta)$ has maximum in C. Now let $\theta_n \to \theta$, and x_n be such that $G(\theta_n) = F(x_n,\theta_n) - H(x_n,\theta_n)$. We have to prove that

$$\limsup G(\theta_n) \leq G(\theta). \tag{3.8}$$

Since the limsup is the limit along a subsequence, we can assume, without loss of generality, that the sequence $G(\theta_n)$ has limit. Due to the fact that $\{\theta_n : n \geq 0\} \cup \{\theta\}$ is compact and the properness of H, it follows that the sequence x_n is relatively compact, so it has a convergent subsequence $x_{n_k} \to x$. Thus

$$\lim G(\theta_n) = \lim G(\theta_{n_k}) = \lim \Big[F(x_{n_k},\theta_{n_k}) - H(x_{n_k},\theta_{n_k}) \Big]$$

$$\leq F(x,\theta) - H(x,\theta) \leq G(\theta)$$

where we have used the (joint) upper semicontinuity of $F - H$.

Now we prove lower semicontinuity, i.e. that $\liminf G(\theta_n) \geq G(\theta)$. Let x be such that $G(\theta) = F(x,\theta) - H(x;\theta)$. By property (i), there exists a sequence $x_n \to x$, with $H(x_n;\theta_n) \to H(x,\theta)$. So we get:

$$\liminf G(\theta_n) \geq \liminf [F(x_n,\theta_n) - H(x_n;\theta_n)] = G(\theta).$$

∎

Proposition 3.4 *(Composition) Let $\{P^\epsilon(dx;y,\theta) : \epsilon > 0, (y,\theta) \in \mathcal{Y} \times \Theta\}$ and $\{Q^\epsilon(dy;\theta) : \epsilon > 0, \theta \in \Theta\}$ be two exponentially tight WULDF in $\mathcal{X}$ and $\mathcal{Y}$ respectively with rate functions $H_P(x;y,\theta)$ and $H_Q(y;\theta)$. Assume that the measures P^ϵ are all probability measures. Moreover, assume that the rate function $H_P(x;y,\theta)$ satisfies assumptions (i)-(ii) of Lemma 3.3 (with $\mathcal{Y} \times \Theta$ in place of Θ). Then $\{R^\epsilon(dx,dy;\theta) : \epsilon > 0, \theta \in \Theta\}$ defined by:*

$$\int f(x,y) R^\epsilon(dx,dy;\theta) = \int \left[\int f(x,y) P^\epsilon(dx;y,\theta)\right] Q^\epsilon(dy;\theta),$$

is an exponentially tight WULDF with rate function:

$$H_R(x,y;\theta) = H_P(x;y,\theta) + H_Q(y;\theta).$$

Proof. First we prove that R^ϵ is a WULDF. Since $H_P(\cdot;y,\theta)$ is positive and has minimum zero, property ii) of Definition 2.5 for H_R is easily derived from the corresponding property for H_Q. We now show that i) of Definition 2.5 holds. Since lower semicontinuity is obvious, we only need to prove that for all $L \in \mathbb{R}$, and for each θ, the set $Z = \{(x,y) \,|\, H_R(x,y;\theta) \leq L\}$ is compact. Notice that if $(x,y) \in Z$ then $y \in W = \{y \,|\, H_Q(y;\theta) \leq L\}$, and W is compact. Since H_P is proper there exists a compact set $V \subseteq \mathcal{X}$ such that $H_P(x;y,\theta) \geq L+1$ for all $x \in V^c$ and all $y \in W$ (note that $W \times \{\theta\}$ is compact). Thus we have that

$$Z \subseteq V \times W,$$

and so Z is compact, as desired.

Now we must show that also iii) of Definition 2.5 holds for R^ϵ. Let $F(x,y)$ be a continuous and bounded function, and let $K \subseteq \Theta$ be compact. We need to prove that:

$$\begin{aligned}\lim_{\epsilon\to 0} \quad \sup_{\theta\in K} \quad & \Big[\epsilon \log \textstyle\int \exp\{\epsilon^{-1}F(x,y)\} R^\epsilon(dx,dy;\theta) - \\ & \sup_{(x,y)\in\mathcal{X}\times\mathcal{Y}} [F(x,y) - H_R(x,y;\theta)]\Big] = 0.\end{aligned} \tag{3.9}$$

Notice that:

$$\int \exp\{\epsilon^{-1}F(x,y)\} R^\epsilon(dx,dy;\theta) \;=\; \int \exp\{\epsilon^{-1}G^\epsilon(y,\theta)\} Q^\epsilon(dy,\theta),$$

where:

$$G^\epsilon(y,\theta) \;=\; \epsilon \log \int \exp\{\epsilon^{-1}F(x,y)\} P^\epsilon(dx;y,\theta).$$

Clearly, the functions G^ϵ are uniformly bounded. By Corollary 3.2

$$G^\epsilon(y,\theta) \to G(y,\theta) \equiv \sup_{x\in\mathcal{X}} [F(x,y) - H_P(x;y,\theta)]$$

uniformly on the compact subsets of $Y \times \Theta$. Moreover, by Lemma 3.3, G is a bounded continuous function. Thus (3.9) follows as an application of Lemma 2.12 and Proposition 3.1.

It remains to show that the family R^ϵ is exponentially tight. Let $K \subseteq \Theta$ be a compact set, and $M > 0$. Since Q^ϵ is exponentially tight, there exists a compact set $C_1 \subseteq \mathcal{Y}$ such that for all $\theta \in K$ and for all ϵ small enough:

$$Q^\epsilon(C_1^c;\theta) \leq \frac{e^{-\epsilon^{-1}M}}{2}$$

Moreover, since $P^\epsilon(dx; y, \theta)$ is also exponentially tight, there exists a compact set $C_2 \subseteq \mathcal{X}$ such that

$$P^\epsilon(C_2^c; y, \theta) \leq \frac{e^{-\epsilon^{-1}M}}{2},$$

for all $(y, \theta) \in C_1 \times K$ and for all ϵ small enough. Let $C = C_1 \times C_2$. Then:

$$R^\epsilon(C^c; \theta) \leq R^\epsilon(C_2^c \times C_1; \theta) + R^\epsilon(C_2 \times C_1^c; \theta) \leq \sup_{y \in C_1} P^\epsilon(C_2^c; y, \theta) + Q^\epsilon(C_1^c; \theta).$$

Thus, for all ϵ small enough, and for all $\theta \in K$, we get

$$R^\epsilon(C^c; \theta) \leq e^{-\epsilon^{-1}M},$$

that completes the proof. ∎

Lemma 3.5 *Let $\{P^\epsilon(dx; \theta) : \epsilon > 0, \theta \in \Theta\}$ be a WULDF, with rate function $H(x; \theta)$, and $f : \mathcal{X} \to \mathcal{Y}$ be a continuous function. Then $\{P_f^\epsilon(dy; \theta) : \epsilon > 0, \theta \in \Theta\}$ defined by*

$$P_f^\epsilon(B; \theta) = P^\epsilon(f^{-1}(B); \theta), \quad B \subseteq \mathcal{Y},$$

is again a WULDF with rate function:

$$H_f(y; \theta) \;=\; \inf\{H(x; \theta) : \; f(x) = y\}$$

Moreover if the family $P^\epsilon(dx; \theta)$ is exponentially tight then also the family $P_f^\epsilon(dy; \theta)$ is exponentially tight.

The proof of Lemma 3.5 is easy, and is omitted. From Lemma 3.5 the following Proposition follows.

Proposition 3.6 *(Contraction). Let $\{R^\epsilon(dx, dy; \theta) : \epsilon > 0, \theta \in \Theta\}$ be a WULDF on $\mathcal{X} \times \mathcal{Y}$ with rate function $H_R(x, y; \theta)$. Then $\{P^\epsilon(dx; \theta) : \epsilon > 0, \theta \in \Theta\}$, defined by:*

$$P^\epsilon(A; \theta) \;:=\; R^\epsilon(A \times \mathcal{Y}; \theta),$$

is a WULDF with rate function

$$H_P(x; \theta) \;=\; \inf_{y \in \mathcal{Y}} H_R(x, y; \theta).$$

Moreover if the family $R^\epsilon(dx, dy; \theta)$ is exponentially tight then also the family $P^\epsilon(dx; \theta)$ is exponentially tight.

Proposition 3.7 *(Conditioning) Let $\{P^\epsilon(dx; \theta) : \epsilon > 0, \theta \in \Theta\}$ and $\{Q^\epsilon(dy; x) : \epsilon > 0, x \in \mathcal{X}\}$ be two exponentially tight WULDF's on $\mathcal{X}$ and $\mathcal{Y}$ respectively, with rate functions $H_P(x; \theta)$ and $H_Q(y; x)$. Assume that the measures $Q^\epsilon(dy; x)$ are all probability measures, and that both families of kernels are exponentially tight. Moreover assume that the rate function $H_Q(y; x)$ is always finite and continuous, and that the following properties hold.*

1. *the measure* $Q^\epsilon(dy;x)$ *is of the form:*

$$Q^\epsilon(dy;x) \;=\; q^\epsilon(y;x)\alpha(dy).$$

where $q^\epsilon(y;x) > 0$ *and the measure* $\alpha(dy)$ *satisfies*

$$\inf_{y\in K} \alpha\left(B(y,\gamma)\right) > 0;$$

for every $K \subset \mathcal{Y}$ *compact and* $\gamma > 0$*, where* $B(y,\gamma)$ *is the ball centered at* y *with radius* γ*.*

2. *for any compact sets* $K \subseteq \mathcal{Y}$*,* $C \subseteq \mathcal{X}$*, and any* $\delta > 0$ *there exists* $\delta_1 > 0$ *and* $\epsilon(\delta)$ *such that:*

$$|\epsilon \log q^\epsilon(y_1;x) \;-\; \epsilon \log q^\epsilon(y_2;x)| \;<\; \delta,$$

for all y_1*,* $y_2 \in K$ *such that* $d(y_1,y_2) < \delta_1$*, for all* $\epsilon \le \epsilon(\delta)$*, and for all* $x \in C$*;*

3. *for any compact sets* $K \subseteq \mathcal{Y}$*,* $C \subseteq \mathcal{X}$ *there exists* $n_{K,C} > 0$ *such that:*

$$\epsilon \log q^\epsilon(y;x) \ge -n_{K,C};$$

for all $y \in K$*,* $x \in C$*, and* $\epsilon > 0$

4. *for any compact set* $K \subseteq \mathcal{Y}$*, there exists* $N_K > 0$ *such that:*

$$\epsilon \log q^\epsilon(y;x) \le N_K;$$

for all $y \in K$*, for all* $x \in \mathcal{X}$*, and for all* $\epsilon > 0$*.*

Then the measures on $\mathcal{X}$

$$R^\epsilon(dx;y,\theta) \;=\; q^\epsilon(y;x)P^\epsilon(dx;\theta)$$

form an exponentially tight WULDF with rate function

$$H_R(x;y,\theta) \;=\; H_Q(y;x) + H_P(x;\theta).$$

Proof. First we prove that the family R^ϵ is exponentially tight. Let $\tilde{K} \subseteq \mathcal{Y}\times\Theta$ be a compact set, and denote by K_1 and K_2 its projection on Y and Θ respectively. By property 4 there exists a constant N_{K_1} such that $q^\epsilon(y;x) \le e^{\epsilon^{-1}N_{K_1}}$, for all $y \in K_1$, all $x \in \mathcal{X}$, and all $\epsilon > 0$. Given any $M > 0$, since P^ϵ is exponentially tight, there exists a compact set $C \subseteq \mathcal{X}$ such that:

$$\sup_{\theta\in K_2} P^\epsilon(C^c;\theta) \;\le\; e^{-\epsilon^{-1}(M+N_{K_1})}.$$

We have:

$$\sup_{(y,\theta)\in\tilde{K}} R^\epsilon(C^c;y,\theta) = \sup_{(y,\theta)\in\tilde{K}} \int_{C^c} q^\epsilon(y;x)P^\epsilon(dx;\theta) \le e^{\epsilon^{-1}N_{K_1}} \sup_{\theta\in K_2} P^\epsilon(C^c;\theta) \le e^{-\epsilon^{-1}M},$$

which proves exponential tightness. To show that the family $\{R^\epsilon(dx;y,\theta)\}$ is a WULDF with rate function H_R, we define

$$M_\theta^P \;=\; \inf_{x\in X} H_P(x;\theta).$$

Notice that, since the measures $Q^\epsilon(dy;x)$ are all probability measure, it holds that $H_Q(y;x) \geq 0$. Then it is clear that $H_R(x;y,\theta) \geq M_\theta^P$, so the map H_R satisfies property ii) of Definition 2.5. Moreover, since:

$$\{\, x \mid H_R(x;y,\theta) \leq L \,\} \subseteq \{\, x \mid H_P(x,\theta) \leq L \,\},$$

also property i) of Definition 2.5 holds.

Now we need to establish property iii) of Definition 2.5. First we prove an intermediate step, that consists in approximating the density $q^\epsilon(y;x)$ with an average on the form

$$\frac{1}{\alpha(B(y,\delta_1))}\int_Y e^{\epsilon^{-1}h(\eta)}q^\epsilon(\eta;x)\alpha(d\eta),$$

$h(\eta)$ being a function suitably concentrate about $\eta = y$.

For any $y \in \mathcal{Y}$, $\delta > 0$, $0 < \tilde{\delta} < \delta$, and $M > 0$, let $g_{y,\delta,\tilde{\delta},M}(\eta)$ be a continuous and bounded function such that:

1. $g_{y,\delta,\tilde{\delta},M}(\eta) \leq 0$ for all $\eta \in \mathcal{Y}$, and $g_{y,\delta,\tilde{\delta},M}(\eta) = 0$ for all $\eta \in B(y,\tilde{\delta})$;
2. $g_{y,\delta,\tilde{\delta},M}(\eta) = -M$ if $\eta \notin B(y,\delta)$.

For each given $y \in \mathcal{Y}$, $0 < \tilde{\delta} < \delta$, and $M > 0$, the existence of a function $g_{y,\delta,\tilde{\delta},M}(\cdot)$ satisfying the previous requirements is easily proved. For example one may take

$$g_{y,\delta,\tilde{\delta},M}(\eta) = -\frac{M}{\delta-\tilde{\delta}}\left(\min\{\text{ dist }(\eta, B(y,\tilde{\delta})), \delta-\tilde{\delta}\}\right).$$

Claim 1 For any $K \subseteq \mathcal{Y}$, and $C \subseteq \mathcal{X}$ compact sets, and any $\delta > 0$, $M > 0$ let $\delta_1 > 0$, and $\epsilon(\delta) > 0$ be such that:

$$q^\epsilon(y;x)e^{-\epsilon^{-1}\delta} \leq q^\epsilon(\eta;x) \leq q^\epsilon(y;x)e^{\epsilon^{-1}\delta}, \tag{3.10}$$

for all $x \in C$, $y,\eta \in K$ such that $\eta \in B(y,\delta_1)$, and for all $\epsilon \leq \epsilon(\delta)$ (use property 2). Fix any $\tilde{\delta} < \delta_1$, and let $h(\eta) = g_{y,\delta_1,\tilde{\delta},M}(\eta)$. Then there exists a constant $C(\delta_1) > 0$ such that:

$$\begin{aligned} e^{-\epsilon^{-1}\delta}\left(1 + C(\delta_1)^{-1}e^{-\epsilon^{-1}(M-N_K+\delta)}\right)^{-1} \tfrac{1}{\alpha(B(y,\delta_1))}\textstyle\int_{\mathcal{Y}} e^{\epsilon^{-1}h(\eta)}q^\epsilon(\eta;x)\alpha(d\eta) \leq q^\epsilon(y;x) \leq \\ \leq e^{\epsilon^{-1}\delta}\tfrac{1}{\alpha(B(y,\tilde{\delta}))}\textstyle\int_{\mathcal{Y}} e^{\epsilon^{-1}h(\eta)}q^\epsilon(\eta;x)\alpha(d\eta) \end{aligned} \tag{3.11}$$

for all $y \in K$, for all $x \in C$, $\forall\, \tilde{\delta} < \delta_1$, and $\forall\, \epsilon \leq \epsilon(\delta)$, where N_K is defined in Property 4 in the assumptions.

Proof of the Claim
Let $C(\delta_1) = \inf_{y\in K}\alpha\left(B(y,\delta_1)\right)$. It is easy to see that the following inequalities hold:

$$\int_{\mathcal{Y}} e^{\epsilon^{-1}h(\eta)}q^\epsilon(\eta;x)\alpha(d\eta) \geq \int_{B(y,\tilde{\delta})} e^{-\epsilon^{-1}\delta}q^\epsilon(y;x)\alpha(d\eta) = e^{-\epsilon^{-1}\delta}q^\epsilon(y;x)\alpha\left(B(y,\tilde{\delta})\right),$$

$$\begin{aligned} &\textstyle\int_{\mathcal{Y}} e^{\epsilon^{-1}h(\eta)}q^\epsilon(\eta;x)\alpha(d\eta) \leq \int_{B(y,\delta_1)} e^{\epsilon^{-1}\delta}q^\epsilon(y;x)\alpha(d\eta) + \int_{B(y,\delta_1)^c} e^{-\epsilon^{-1}M}q^\epsilon(\eta;x)\alpha(d\eta) \\ &\leq e^{\epsilon^{-1}\delta}q^\epsilon(y;x)\alpha\left(B(y,\delta_1)\right)\left[1 + C(\delta_1)^{-1}e^{-\epsilon^{-1}(M-N_K+\delta)}\right] \end{aligned}$$

from which (3.11) follows easily. So Claim 1 is proved.

Fix any $K \subseteq \mathcal{Y}$, and $C \subseteq \mathcal{X}$ compact sets, and any $\delta > 0$, $M > 0$. Combining equations (3.11) and (3.10) we get that for all $x \in C$, and all $\tilde{y} \in K$ such that $|\tilde{y} - y| < \delta_1$:

$$\begin{aligned} e^{-\epsilon^{-1}2\delta} \tfrac{\gamma(\delta_1)^{-1}}{\alpha(B(y,\delta_1))} \textstyle\int_{\mathcal{Y}} e^{\epsilon^{-1}h(\eta)} q^\epsilon(\eta; x)\alpha(d\eta) \leq q^\epsilon(\tilde{y}; x) \leq \\ \leq e^{\epsilon^{-1}2\delta} \tfrac{1}{\alpha(B(y,\tilde{\delta}))} \textstyle\int_{\mathcal{Y}} e^{\epsilon^{-1}h(\eta)} q^\epsilon(\eta; x)\alpha(d\eta) \end{aligned} \tag{3.12}$$

where $h(\eta) = g_{y,\delta_1,\tilde{\delta},M}(\eta)$ and $\gamma(\delta_1) = 1 + \frac{1}{C(\delta_1)} e^{-\epsilon^{-1}(M - N_K + \delta)}$, Since $K \subset \cup_{y \in K} B(y, \delta_1)$, and K is compact, there exist $h_1(\eta), \ldots, h_l(\eta)$ all of the type $g_{y_i,\delta_1,\tilde{\delta},M}(\eta)$ for some $y_i \in K$, such that for all $y \in K$ and for all $x \in C$ there exists an index $i \in \{1, \ldots, l\}$ such that

$$\begin{aligned} e^{-\epsilon^{-1}2\delta} \tfrac{\gamma(\delta_1)^{-1}}{\alpha(B(y_i,\delta_1))} \textstyle\int_{\mathcal{Y}} e^{\epsilon^{-1}h_i(\eta)} q^\epsilon(\eta; x)\alpha(d\eta) \leq q^\epsilon(\tilde{y}; x) \\ \leq e^{\epsilon^{-1}2\delta} \tfrac{1}{\alpha(B(y_i,\tilde{\delta}))} \textstyle\int_{\mathcal{Y}} e^{\epsilon^{-1}h_i(\eta)} q^\epsilon(\eta; x)\alpha(d\eta) \end{aligned} \tag{3.13}$$

where (3.13) holds for all $\tilde{\delta} < \delta_1$, $\epsilon \leq \epsilon(\delta)$.

Now we prove that also iii) of Definition 2.5 holds. Fix a compact set $\tilde{K} \subseteq \mathcal{Y} \times \Theta$. Let $K_1 \subseteq \mathcal{Y}$ be its first projection (i.e. $K_1 = \Pi_1(\tilde{K})$), and $K_2 \subseteq \Theta$ be its second projection. Moreover, let N_{K_1} be the positive constant given by property 4. For any continuous and bounded function $F(x)$, we need to show that:

$$\lim_{\epsilon \to 0} \sup_{(y,\theta) \in \tilde{K}} \left[\epsilon \log \int_{\mathcal{X}} e^{\epsilon^{-1}F(x)} R^\epsilon(dx; y, \theta) - \sup_{x \in \mathcal{X}} (F(x) - H_R(x; y, \theta)) \right] = 0.$$

We let:

- $|F(x)| \leq L_1$,
- $x_\theta \in \mathcal{X}$ be such that $H^P(x_\theta; \theta) = M_\theta^P = \inf_x H_P(x; \theta)$,
- $M^P(K_2)$ be such that $|M_\theta^P| \leq M^P(K_2)$ for all $\theta \in K_2$,
- $C_1 \subseteq \mathcal{X}$ be a compact set such that $x_\theta \in C_1$ for all $\theta \in K_2$ (such compact set exists by properness of the rate function),
- L_2 be such that $|H_Q(y; x)| \leq L_2$ for all $y \in K_1$ and all $x \in C_1$ (notice that this constant exists since H_Q is continuous),

Now consider a compact set $C_2 \subset \mathcal{X}$ and a constant $\Lambda > 0$ such that, for ϵ sufficiently small,

$$P^\epsilon(C_2; \theta) \geq e^{-\epsilon^{-1}\Lambda} \tag{3.14}$$

for every $\theta \in K_2$. Note that this can be done by (2.3) and exponential tightness of P^ϵ. Moreover, by property 3, there is a constant n such that

$$-n \leq \epsilon \log q^\epsilon(y; x) \tag{3.15}$$

for all $x \in C_2, y \in K_1$.

Fix any positive constants M, T and $\tilde{M}$ such that:

$$M > 2L_1 + L_2 + 2M^P(K_2) + N_{K_1} \quad T > 2L_1 + \Lambda + n, \quad \tilde{M} > 2L_1 + L_2 + M^P(K_2) + M + \Lambda \tag{3.16}$$

Since H_P is proper, and P^ϵ, R^ϵ are exponentially tight, we have that there is a compact sets $C_3 \subseteq \mathcal{X}$, which satisfies the following inequality for ϵ small enough:

$$H_P(x;\theta) \;>\; \tilde{M} \qquad \text{for all} \;\; x \in C_3^c \;\; \text{and all} \;\; \theta \in K_2, \tag{3.17}$$

$$P^\epsilon(C_3^c;\theta) \leq e^{-\epsilon^{-1}\tilde{M}} \qquad \text{for all} \;\; \theta \in K_2, \tag{3.18}$$

$$R^\epsilon(C_3^c;y,\theta) \;\leq\; e^{-\epsilon^{-1}T} \qquad \text{for all} \;\; (y,\theta) \in \tilde{K}. \tag{3.19}$$

Notice that without loss of generality, we may assume that $C_1, C_2 \subseteq C_3$ Fix any $\delta > 0$. Let $\delta_2 > 0$ be such that

$$|H_Q(y;x) - H_Q(y';x')| \;\leq\; \delta, \tag{3.20}$$

for all $x,\, x' \in C_3$, and $y,\, y' \in K_1$, such that $\text{dist}\,(x,x') < \delta_2$, and $\text{dist}\,(y,y') < \delta_2$.

Now, fix $(y,\theta) \in \tilde{K}$, $x \in C_3$. We have seen that there exist $\delta_1 \leq \delta$, $\epsilon(\delta) > 0$, $y_1, \ldots, y_l \in K_1$ and $i \in \{1,\ldots,l\}$ such that (3.13) holds for all $x \in C_3$, $y \in K_1$, $\tilde{\delta} \leq \delta_1$ and $\epsilon \leq \epsilon(\delta)$.

Upper bound.

$$\epsilon \log \int e^{\epsilon^{-1}F(x)} R^\epsilon(dx;y,\theta)$$

$$= \epsilon \log \int_{C_3} e^{\epsilon^{-1}F(x)} R^\epsilon(dx;y,\theta) + \epsilon \log \left[1 + \frac{\int_{C_3^c} e^{\epsilon^{-1}F(x)} R^\epsilon(dx;y,\theta)}{\int_{C_3} e^{\epsilon^{-1}F(x)} R^\epsilon(dx;y,\theta)} \right]$$

$$\leq \epsilon \log \int_{C_3} e^{\epsilon^{-1}F(x)} R^\epsilon(dx;y,\theta) + \epsilon \log \left(1 + e^{\epsilon^{-1}(2L_1+\Lambda+n-T)} \right)$$

where we have used the inequalities

$$\int_{C_3^c} e^{\epsilon^{-1}F(x)} R^\epsilon(dx;y,\theta) \leq e^{\epsilon^{-1}(L_1-T)}$$

$$\int_{C_3} e^{\epsilon^{-1}F(x)} R^\epsilon(dx;y,\theta) \geq \int_{C_2} e^{\epsilon^{-1}F(x)} R^\epsilon(dx;y,\theta) \geq e^{-\epsilon^{-1}(L_1+\Lambda+n)}$$

for every $(y,\theta) \in \tilde{K}$. Therefore, using (3.13)

$$\epsilon \log \int e^{\epsilon^{-1}F(x)} R^\epsilon(dx;y,\theta) \leq \epsilon \log \left(1 + e^{\epsilon^{-1}(2L_1+\Lambda+n-T)} \right) + 2\delta$$

$$-\epsilon \log \alpha(B(y_i,\tilde{\delta})) + \epsilon \log \int_{C_3 \times Y} e^{\epsilon^{-1}(F(x)+h_i(\eta))} Q^\epsilon(d\eta;x) P^\epsilon(dx;\theta)$$

$$\leq \epsilon \log \left(1 + e^{\epsilon^{-1}(2L_1+\Lambda+n-T)} \right) + 2\delta - \epsilon \log C(\tilde{\delta}) + \epsilon \log \int_{X \times Y} e^{\epsilon^{-1}(F(x)+h_i(\eta))} Q^\epsilon(d\eta;x) P^\epsilon(dx;\theta)$$

$$\leq \epsilon \log \left(1 + e^{\epsilon^{-1}(2L_1+\Lambda+n-T)} \right) + 2\delta - \epsilon \log C(\tilde{\delta}) +$$
$$\sup_{x\in\mathcal{X},\eta\in\mathcal{Y}} [F(x) + h_i(\eta) - H_Q(\eta;x) - H_P(x;\theta)] + \lambda_i(\epsilon)$$

with $\lambda_i(\epsilon) \to 0$ as $\epsilon \to 0$. Note that for the last inequality Proposition 3.4 has been used. Now observe that

$$\sup_{x\in\mathcal{X},\eta\in\mathcal{Y}} \Big[F(x) + h_i(\eta) - H_Q(\eta;x) - H_P(x;\theta) \Big] \geq$$

$$F(x_\theta) + h_i(y_i) - H_Q(y_i; x_\theta) - H_P(x_\theta; \theta) \geq -L_1 - L_2 - M^P(K_2). \tag{3.21}$$

On the other hand, for $x \notin C_3$

$$F(x) + h_i(\eta) - H_Q(\eta; x) - H_P(x; \theta) \leq L_1 - \tilde{M}, \tag{3.22}$$

and, for dist $(\eta, y_i) > \delta_1$,

$$F(x) + h_i(\eta) - H_Q(\eta; x) - H_P(x; \theta) \leq L_1 - M + M^P(K_2). \tag{3.23}$$

By (3.16), it follows that the r.h.s. of (3.22) and (3.23) are smaller that the r.h.s. of (3.21). Therefore

$$\sup_{x \in \mathcal{X}, \eta \in \mathcal{Y}} \Big[F(x) + h_i(\eta) - H_Q(\eta; x) - H_P(x; \theta)\Big] =$$

$$= \sup_{x \in C_3, \eta \in B(y_i, \delta_1)} \Big[F(x) + h_i(\eta) - H_Q(\eta; x) - H_P(x; \theta)\Big]$$

$$\leq \sup_{x \in C_3} \Big[F(x) - H_Q(y; x) - H_P(x; \theta)\Big] + \delta$$

$$= \sup_{x \in \mathcal{X}} \Big[F(x) - H_Q(y; x) - H_P(x; \theta)\Big] + \delta$$

where the last equality comes from an argument analogous to (3.21)-(3.23). Summing up

$$\epsilon \log \int e^{\epsilon^{-1} F(x)} R^\epsilon(dx; y, \theta) \leq$$

$$\leq \epsilon \log \left(1 + e^{\epsilon^{-1}(2L_1 + \Lambda + n - T)}\right) + 2\delta - \epsilon \log C(\tilde{\delta}) + \sup_{x \in \mathcal{X}} \Big[F(x) - H_Q(y; x) - H_P(x; \theta)\Big] + \lambda(\epsilon)$$

with $\lambda(\epsilon) = \max_{i=1,\ldots,l} \lambda_i(\epsilon)$. By using again (3.16), the last inequality implies

$$\limsup_{\epsilon \to 0} \sup_{(y,\theta) \in \tilde{K}} \epsilon \log \int e^{\epsilon^{-1} F(x)} R^\epsilon(dx; y, \theta) - \sup_{x \in \mathcal{X}} \Big[F(x) - H_Q(y; x) - H_P(x; \theta)\Big] \leq 0. \tag{3.24}$$

Lower bound. By (3.13):

$$\epsilon \log \int e^{\epsilon^{-1} F(x)} R^\epsilon(dx; y, \theta) \geq \epsilon \log \int_{C_3} e^{\epsilon^{-1} F(x)} R^\epsilon(dx; y, \theta)$$

$$\geq -2\delta - \epsilon \log \gamma(\delta_1) - \epsilon \log \alpha(B(y_i, \delta_1)) + \epsilon \log \int_{C_3 \times \mathcal{Y}} e^{\epsilon^{-1}(F(x) + h_i(\eta))} Q^\epsilon(d\eta; x) P^\epsilon(dx; \theta). \tag{3.25}$$

Note that

$$\int_{C_3^c \times \mathcal{Y}} e^{\epsilon^{-1}(F(x) + h_i(\eta))} Q^\epsilon(d\eta; x) P^\epsilon(dx; \theta) \leq e^{\epsilon^{-1}(L_1 - \tilde{M})}$$

and

$$\int_{\mathcal{X} \times \mathcal{Y}} e^{\epsilon^{-1}(F(x) + h_i(\eta))} Q^\epsilon(d\eta; x) P^\epsilon(dx; \theta) \geq e^{-\epsilon^{-1}(L_1 + M + \Lambda)},$$

which implies

$$\epsilon \log \int_{C_3 \times \mathcal{Y}} e^{\epsilon^{-1}(F(x) + h_i(\eta))} Q^\epsilon(d\eta; x) P^\epsilon(dx; \theta) =$$

$$= \epsilon \log \int_{\mathcal{X}\times\mathcal{Y}} e^{\epsilon^{-1}(F(x)+h_i(\eta))} Q^\epsilon(d\eta; x) P^\epsilon(dx;\theta) +$$

$$+ \epsilon \log \left[1 - \frac{\int_{C_3^c \times \mathcal{Y}} e^{\epsilon^{-1}(F(x)+h_i(\eta))} Q^\epsilon(d\eta; x) P^\epsilon(dx;\theta)}{\int_{\mathcal{X}\times\mathcal{Y}} e^{\epsilon^{-1}(F(x)+h_i(\eta))} Q^\epsilon(d\eta; x) P^\epsilon(dx;\theta)} \right]$$

$$\geq \epsilon \log \int_{\mathcal{X}\times\mathcal{Y}} e^{\epsilon^{-1}(F(x)+h_i(\eta))} Q^\epsilon(d\eta; x) P^\epsilon(dx;\theta) + \epsilon \log \left[1 - e^{\epsilon^{-1}(2L_1+M+\Lambda-\tilde{M})} \right].$$

Thus

$$\epsilon \log \int e^{\epsilon^{-1}F(x)} R^\epsilon(dx; y, \theta) \geq -2\delta - \epsilon \log \gamma(\delta_1) - \epsilon \log \alpha(B(y_i, \delta_1))$$

$$+ \epsilon \log \int_{\mathcal{X}\times\mathcal{Y}} e^{\epsilon^{-1}(F(x)+h_i(\eta))} Q^\epsilon(d\eta; x) P^\epsilon(dx;\theta) + \epsilon \log \left[1 - e^{\epsilon^{-1}(2L_1+M+\Lambda-\tilde{M})} \right].$$

After having noticed that, by (3.16), $2L_1 + M + \Lambda - \tilde{M} < 0$, the proof of the lower bound proceeds by repeating the arguments in the proof of the upper bound, yielding

$$\liminf_{\epsilon\to 0} \inf_{(y,\theta)\in\tilde{K}} \epsilon \log \int e^{\epsilon^{-1}F(x)} R^\epsilon(dx; y, \theta) - \sup_{x\in\mathcal{X}} \left[F(x) - H_Q(y; x) - H_P(x;\theta) \right] \geq 0 \quad (3.26)$$

that, together with (3.24), completes the proof. ∎

Note that, if in the statement of Proposition 3.7 we interpret $q^\epsilon(y; x)\alpha(dy)P^\epsilon(dx;\theta)$ as a measure in $\mathcal{X} \times \mathcal{Y}$, the measure R^ϵ has the meaning of *unnormalized* conditional measure. An analogous statement for the normalized version, whose proof follows easily from Proposition 3.7, is given below.

Corollary 3.8 *Under the assumptions of Proposition 3.7, define*

$$R^\epsilon(dx; y, \theta) = \frac{q^\epsilon(y; x)P^\epsilon(dx;\theta)}{\int_{\mathcal{X}} q^\epsilon(y; x)P^\epsilon(dx;\theta)}.$$

Then $\{R^\epsilon(dx; y, \theta) : \epsilon > 0, (y, \theta) \in \mathcal{Y}\times\Theta\}$ *is an exponentially tight WULDF with rate function*

$$H_R(x; y, \theta) = H_Q(y; x) + H_P(x;\theta) - \inf_{x\in\mathcal{X}} \left[H_Q(y; x) + H_P(x;\theta) \right].$$

4 The small parameter limit for partially observed, risk sensitive control problems

4.1 The model

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{X}, \mathcal{Y}$ metric spaces, and $\mathcal{U}$ a compact metric space. Moreover, let $(\mathcal{F}_n)_{n=0}^N$, $(\mathcal{G}_n)_{n=0}^N$ be given filtrations on $(\Omega, \mathcal{F}, P)$. We construct a controlled, partially observed stochastic system with state space $\mathcal{X}$, observation space $\mathcal{Y}$ and control space $\mathcal{U}$. The (discrete) time will vary in $\{0, 1, \ldots, N\}$.

Now, let $\mathcal{M}_1(\mathcal{X})$ ($\mathcal{M}_1(\mathcal{Y})$) denote the set of probability measures on $\mathcal{X}$ ($\mathcal{Y}$), provided with the weak topology and the corresponding Borel σ-field. Suppose that, for $n = 0, \ldots, N-1$ and $\epsilon > 0$, we are given measurable functions (*probability kernels*)

$$\begin{array}{ccc} \mathcal{X}\times\mathcal{U} & \to & \mathcal{M}_1(\mathcal{X}) \\ (x, u) & \to & P_n^\epsilon(\cdot; x, u) \end{array} \quad (4.1)$$

$$\begin{array}{ccc} \mathcal{X} & \rightarrow & \mathcal{M}_1(\mathcal{Y}) \\ x & \rightarrow & Q^\epsilon_{n+1}(\cdot;x). \end{array} \tag{4.2}$$

We assume there exists a σ-finite measure α on $\mathcal{Y}$ such that for all $x \in \mathcal{X}$, $Q^\epsilon_{n+1}(\cdot;x)$ has a density w.r.t. α:

$$Q^\epsilon_{n+1}(dy;x) = q^\epsilon_{n+1}(y;x)\alpha(dy), \tag{4.3}$$

and we assume $q^\epsilon_{n+1}(y;x)$ to be strictly positive everywhere.

For $n = 0, \ldots, N-1$ we let $u_n : \mathcal{Y}^{n+1} \rightarrow \mathcal{U}$ be a measurable function. The sequence $(u_0, \ldots, u_{N-1})$ will be denoted by $\mathbf{u}$, and the set of such sequences (*admissible controls*) will be denoted by $ad(\mathcal{U})$.

For every given $\mathbf{u} \in ad(\mathcal{U})$ we now define $(X^{\epsilon,\mathbf{u}}_n)^N_{n=0}$, $(Y^{\epsilon,\mathbf{u}}_n)^N_{n=0}$ to be respectively $\mathcal{X}$ and $\mathcal{Y}$-valued stochastic process, defined on $(\Omega, \mathcal{F}, P)$, having the following properties:
i) $X^{\epsilon,\mathbf{u}}_n$ is $\mathcal{F}_n$-measurable; $Y^{\epsilon,\mathbf{u}}_n$ is $\mathcal{G}_n$-measurable;
ii) for $n = 0, \ldots, N-1$

$$P\{X^{\epsilon,\mathbf{u}}_{n+1} \in \cdot \,|\mathcal{F}_n \vee \mathcal{G}_n\} = P^\epsilon_n(\cdot\,; X^{\epsilon,\mathbf{u}}_n, u_n(Y^{\epsilon,\mathbf{u}}_0, \ldots, Y^{\epsilon,\mathbf{u}}_n)); \tag{4.4}$$

iii) for $n = 1, \ldots, N$

$$P\{Y^{\epsilon,\mathbf{u}}_n \in \cdot \,|\mathcal{F}_n \vee \mathcal{G}_{n-1}\} = Q^\epsilon_n(\cdot\,; X^{\epsilon,\mathbf{u}}_n). \tag{4.5}$$

Note that i), ii) and iii) completely determine the law of the processes $X^{\epsilon,\mathbf{u}}_n$ and $Y^{\epsilon,\mathbf{u}}_n$, up to the initial condition $X^{\epsilon,\mathbf{u}}_0, Y^{\epsilon,\mathbf{u}}_0$. For simplicity, we assume $X^{\epsilon,\mathbf{u}}_0 = \xi$, $Y^{\epsilon,\mathbf{u}}_0 = \eta$, deterministic and $(\epsilon, \mathbf{u})$-independent. It is clear that for given probability kernels as in (4.1)(4.2), one can construct on a suitable probability space a stochastic process satisfying i), ii) and iii). The dependence on ϵ of the probability kernels in (4.1)(4.2) will be specified later. From now on, the index $(\epsilon, \mathbf{u})$ in $X^{(\epsilon,\mathbf{u})}_n$ and $Y^{(\epsilon,\mathbf{u})}_n$ will be omitted, and we shortly write u_n for $u_n(Y^{\epsilon,\mathbf{u}}_0, \ldots, Y^{\epsilon,\mathbf{u}}_n)$.

Now we define the cost functional for the optimal control problem. Suppose we are given bounded measurable functions

$$g_n : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}, \quad n = 0, \ldots, N-1 \quad g_N : \mathcal{X} \rightarrow \mathbb{R}. \tag{4.6}$$

For $\mathbf{u} \in ad(\mathcal{U})$ define

$$J^\epsilon(\mathbf{u}) = \epsilon \log E\Big\{ \exp\Big[\epsilon^{-1}\Big(\sum_{n=0}^{N-1} g_n(X_n, u_n) + g_N(X_N)\Big)\Big]\Big\}. \tag{4.7}$$

The optimal control problem associated to J^ϵ consists in computing $J^\epsilon_* = \inf\{J^\epsilon(\mathbf{u}) : \mathbf{u} \in ad(\mathcal{U})\}$ and determining a $\mathbf{u}^* \in ad(\mathcal{U})$ such that $J^\epsilon(\mathbf{u}^*) = J^\epsilon_*$.

4.2 Information vector, information measure and dynamic programming

In this section the dependence on ϵ of the objects defined in Section 2.1 is not relevant. So the index ϵ will be dropped.

It is a standard procedure in stochastic control to analyze optimal control problems with partial observation through a redefinition of the model as a completely observed one. Let $n = 0, \ldots, N$. The *information vector* at time n is defined by

$$Z_n = (Y_0, \ldots, Y_n, u_0, \ldots, u_{n-1}) \in \mathcal{Y}^{n+1} \times \mathcal{U}^n \equiv \mathcal{Z}_n. \tag{4.8}$$

In the sequel we often identify Z_{n+1} with the triple (Z_n, u_n, Y_{n+1}).

Note that an admissible control at time n can be thought of as a function of Z_n. The stochastic dynamics of (X_n, Y_n) described in i)-iii) induces the following stochastic dynamics for the information vector

$$P\{Z_{n+1} \in \cdot \,|\mathcal{G}_n\} = \delta_{Y_0} \otimes \cdots \otimes \delta_{Y_n} \otimes P_n^O(dy_{n+1}; Z_n, u_n) \otimes \delta_{u_0} \otimes \cdots \otimes \delta_{u_n}. \tag{4.9}$$

The probability kernels

$$P_n^O : \mathcal{Z}_n \times \mathcal{U} \to \mathcal{M}_1(\mathcal{Y}) \tag{4.10}$$

can be recursively constructed following the procedure below. We also construct an auxiliary sequence of kernels

$$P_n^f : \mathcal{Z}_n \to \mathcal{M}_1(\mathcal{X}) \quad (\textit{filtering probabilities}). \tag{4.11}$$

a) Initialize $P_0^f = \delta_\xi$.
b) Define P_n^O by

$$P_n^O(A; z_n, u_n) = \int \Big(\int Q_{n+1}(A; x_{n+1}) P_n(dx_{n+1}; x_n, u_n) \Big) P_n^f(dx_n; z_n) \tag{4.12}$$

for $A \subset \mathcal{Y}$ measurable.
c) Define P_{n+1}^f by

$$P_{n+1}^f(B; z_{n+1}) = \frac{\int_{\mathcal{X}} \Big(\int_B q_{n+1}(y_{n+1}, x_{n+1}) P_n(dx_{n+1}; x_n, u_n) \Big) P_n^f(dx_n; z_n)}{\int_{\mathcal{X}} \Big(\int_{\mathcal{X}} q_{n+1}(y_{n+1}, x_{n+1}) P_n(dx_{n+1}; x_n, u_n) \Big) P_n^f(dx_n; z_n)}. \tag{4.13}$$

for $B \subset \mathcal{X}$ measurable.

Remark 4.1 Equation (4.13) is the well known *discrete Zakai equation* for the filtering probability. Indeed, P_n^f is a version of the conditional probability of X_n given Z_n, and P_n^O is a version of the conditional probability of Y_{n+1} given Z_n. We assume implicitly that all integrals in (4.13) are finite; this will be guaranteed by later assumptions on the model (Assumption A, Section 4.3), where the function $q_n(y_n; \cdot)$ is assumed to be bounded.

Note that P_{n+1}^O has a density with respect to α given by

$$\rho_n(y_{n+1}; z_n, u_n) = \rho_n(z_{n+1}) = \int \Big(\int q_{n+1}(y_{n+1}, x_{n+1}) P_n(dx_{n+1}; x_n, u_n) \Big) P_n^f(dx_n; z_n). \tag{4.14}$$

Next step consists in writing the cost function $J(\mathbf{u})$ in terms of the information vector Z_n. The main tool is provided by what we define to be the *information measure*. The information measure at time n is a map

$$P_n^I : \mathcal{Z}_n \to \mathcal{M}(\mathcal{X}) \tag{4.15}$$

where $\mathcal{M}(\mathcal{X})$ is the space of positive finite measures on $\mathcal{X}$. The maps P_n^I are recursively defined as follows.

$$\begin{aligned} P_0^I(dx_0) &= \delta_\xi \\ P_{n+1}^I(A; z_{n+1}) &= \frac{\int_{\mathcal{X}} \Big(\int_A e^{g_n(x_n, u_n)} q_{n+1}(y_{n+1}, x_{n+1}) P_n(dx_{n+1}; x_n, u_n) \Big) P_n^I(dx_n; z_n)}{\rho_n(z_{n+1})}. \end{aligned} \tag{4.16}$$

An important property of the information measure is given in next lemma.

Lemma 4.2 *The following identity holds for $n = 0, \dots, N$, for any $f : \mathcal{X} \times \mathcal{Z}_n \to \mathbb{R}$ bounded and measurable and every $\mathbf{u} \in \mathcal{U}$*

$$E\Big\{ \int f(x_n, Z_n) P_n^I(dx_n; Z_n) \Big\} = E\Big\{ f(X_n, Z_n) \exp\Big[\sum_{k=0}^{n-1} g_k(X_k, u_k) \Big] \Big\}. \tag{4.17}$$

Proof. For $n = 0$ there is nothing to prove. The inductive step is proved as follows;

$$\begin{aligned}
& E\Big\{ \textstyle\int f(x_{n+1}, Z_{n+1}) P_{n+1}^I(dx_{n+1}; Z_{n+1}) \Big\} = \\
&= E\Big\{ E\Big\{ \textstyle\int f(x_{n+1}, Z_{n+1}) P_{n+1}^I(dx_{n+1}; Z_{n+1}) \Big| Z_n, u_n \Big\} \Big\} \\
&= E\Big\{ E\Big\{ \textstyle\int f(x_{n+1}, Z_n, u_n, Y_{n+1}) P_{n+1}^I(dx_{n+1}; Z_n, u_n, Y_{n+1}) \Big| Z_n, u_n \Big\} \Big\} \\
&= E\Big\{ \textstyle\int \Big[\int f(x_{n+1}, Z_n, u_n, y_{n+1}) P_{n+1}^I(dx_{n+1}; Z_n, u_n, y_{n+1}) \Big] \\
&\qquad\qquad \rho_n(y_{n+1}, Z_n, u_n) \alpha(dy_{n+1}) \Big\} \\
&\text{(by (4.16))} \\
&= E\Big\{ \textstyle\int \Big[\int \Big(\int f(x_{n+1}, Z_n, u_n, y_{n+1}) q_{n+1}(y_{n+1}; x_{n+1}) P_n(dx_{n+1}; x_n, u_n) \Big) \\
&\qquad\qquad \alpha(dy_{n+1}) \Big] e^{g_n(x_n, u_n)} P_n^I(dx_n; Z_n) \Big\} \\
&\text{(by inductive assumption)} \\
&= E\Big\{ \textstyle\int \Big[\int f(x_{n+1}, Z_n, u_n, y_{n+1}) q_{n+1}(y_{n+1}; x_{n+1}) P_n(dx_{n+1}; X_n, u_n) \Big] \alpha(dy_{n+1}) \\
&\qquad\qquad \exp[\textstyle\sum_{k=0}^{n} g_k(X_k, u_k)] \Big\} \\
&= E\Big\{ E\Big\{ f(X_{n+1}, Z_{n+1}) \Big| \mathcal{F}_n \vee \mathcal{G}_n \Big\} \exp[\textstyle\sum_{k=0}^{n} g_k(X_k, u_k)] \Big\} \\
&= E\Big\{ f(X_{n+1}, Z_{n+1}) \exp[\textstyle\sum_{k=0}^{n} g_k(X_k, u_k)] \Big\}
\end{aligned} \tag{4.18}$$

where we have used elementary properties of conditional expectation. ∎

By using the recursive definition (4.16) it is easily checked that the information measures are indeed finite measures. A bound on $P_n^I(X; z_n)$ which is uniform in z_n will be useful later, and is given in the following Lemma.

Lemma 4.3 *For every $n = 0, \dots, N$ and every $z_n \in \mathbb{Z}_n$ we have*

$$|\log P_n^I(X; z_n)| \leq \sum_{k=0}^{n-1} \|g_k\|_\infty. \tag{4.19}$$

Proof. Since the filtering measures are probability measure, it is enough to show that for any $n = 0, \dots, N$ and every positive measurable function f

$$\int f(x_n) P_n^f(dx_n; z_n) e^{-\sum_{k=0}^{n-1} \|g_k\|_\infty} \leq \int f(x_n) P_n^I(dx_n; z_n) \leq \int f(x_n) P_n^f(dx_n; z_n) e^{\sum_{k=0}^{n-1} \|g_k\|_\infty}. \tag{4.20}$$

The proof of (4.20) comes from an easy induction, and is omitted. ∎

By using Lemma 4.2, we can rewrite the cost functional $J(\mathbf{u})$ as follows:

$$J(\mathbf{u}) = \log E\Big\{ \exp\Big[G_N(Z_N)\Big]\Big\} \tag{4.21}$$

where

$$G_N(Z_N) = \log \int e^{g_N(x_N)} P_N^I(dx_N; Z_N). \tag{4.22}$$

The partially observed stochastic control problem described in Section 4.1 has now been transformed into a totally observed one, with state variable Z_n and cost functional (4.21). For this system the variables Y_n should be thought of as noise variables.

The *value function* associated to (4.22) is defined by

$$V_n(z_n) = \inf_{\mathbf{u}\in ad(\mathcal{U})} \log E\Big\{ \exp\Big[G_N(Z_N)\Big]\Big| Z_n = z_n\Big\}. \tag{4.23}$$

It can be shown (see e.g. [2]) that V_n satisfies the following recursion

$$\begin{aligned} V_N(z_N) &= G_N(z_N) \\ V_n(z_n) &= \inf_{u\in\mathcal{U}} \log \textstyle\int \exp[V_{n+1}(y_{n+1}, z_n, u)] P_n^O(dy_{n+1}; z_n, u). \end{aligned} \tag{4.24}$$

Remark 4.4 By using (4.24) and Lemma 4.3 it is easily seen that the functions V_n are bounded.

Remark 4.5 The stochastic control problem (4.21)-(4.22) is somewhat implicitly stated, since the cost function is given in terms of the solution of the recursion (4.16). Indeed, the cost functional $J(\mathbf{u})$ can be written only in terms of the measures (P_n^f, P_n^I). To see this, consider the stochastic dynamics on $\mathcal{M}_1(X) \times \mathcal{M}(\mathcal{X})$ given by

$$P_{n+1}^f(B) = \frac{\int_{\mathcal{X}} \Big(\int_B q_{n+1}(Y_{n+1}, x_{n+1}) P_n(dx_{n+1}; x_n, u_n)\Big) P_n^f(dx_n)}{\int_{\mathcal{X}} \Big(\int_{\mathcal{X}} q_{n+1}(Y_{n+1}, x_{n+1}) P_n(dx_{n+1}; x_n, u_n)\Big) P_n^f(dx_n)} \tag{4.25}$$

$$P_{n+1}^I(A) = \frac{\int_{\mathcal{X}} \Big(\int_A e^{g_n(x_n,u_n)} q_{n+1}(Y_{n+1}, x_{n+1}) P_n(dx_{n+1}; x_n, u_n)\Big) P_n^I(dx_n)}{\int_{\mathcal{X}} \Big(\int_{\mathcal{X}} q_{n+1}(Y_{n+1}, x_{n+1}) P_n(dx_{n+1}; x_n, u_n)\Big) P_n^f(dx_n)} \tag{4.26}$$

or, in short,

$$(P_{n+1}^f, P_{n+1}^I) = F(P_n^f, P_n^I, u_n, Y_{n+1}). \tag{4.27}$$

In (4.27) the Y_n's play the role of disturbances, whose distribution is determined by

$$P(Y_{n+1} \in A | u_0, \ldots, u_n, Y_0, \ldots, Y_n) = \int \Big(\int Q_{n+1}(A; x_{n+1}) P_n(dx_{n+1}; x_n, u_n)\Big) P_n^f(dx_n). \tag{4.28}$$

If we define the cost functional

$$K(\mathbf{u}) = \log E\left\{ e^{g_N(x)} P_N^I(dx)\right\} \tag{4.29}$$

then we have that $K(\mathbf{u}) = J(\mathbf{u})$ for every $\mathbf{u} \in ad(\mathcal{U})$. This shows that, in a very precise sense, the pair (P_n^f, P_n^I) is a sufficient statistics for the risk sensitive control problem or, in terms more commonly used in control theory, it is an *information state*.

As a consequence, it follows that the value function V_n can be thought of as a function of the information state. In [12], Theorem 3.2, the $\epsilon \to 0$ limit of the value function is studied by looking at the value function as a function of the information state (which is not the same as here, see Remark 4.6 below). In the generality of our model a statement of the type of Theorem 3.2. in [12] does not seem to make sense, and so we prefer to analyze the value function as a function of the information vector. In our construction the information state is only an auxiliary object that allows to express the cost functional $J(\mathbf{u})$ in terms of the information vector.

Remark 4.6 The information state for risk-sensitive control problems, which is rather recent achievement in stochastic control theory, it has been first introduced in [1] and it is usually defined (see [12]) through a measure transformation that decouples the observation from the state. The notion of information state in [12] has, among other advantages, the one that in the $\epsilon \to 0$ limit it induces a quite natural notion of information state for the limit dynamic game. However, when the partially observed control problem is transformed into a totally observed one by means of the information state in [12] one gets a value function which is, in general, unbounded. In order to use large deviation techniques to control the $\epsilon \to 0$ limit of the value function, some growth bound are needed, and these bounds come from assumptions on the dynamics of the model. The assumptions that will be given in Section 4.3 would not imply any growth bound. Our construction guarantees boundedness of the value function, and appears to be more robust in terms of assumptions on the model.

4.3 Small parameter limit

In this section we investigate the limit of the value function in (4.23) as $\epsilon \to 0$. We first introduce the basic assumptions on the model that are needed to study the small parameter limit.

ASSUMPTION A

1. For $n = 0, \dots, N-1$ the families of probability measures $\{P_n^\epsilon(dx_{n+1}; x_n, u_n) : \epsilon > 0, (x_n, u_n) \in \mathcal{X} \times \mathcal{U}\}$ are WULDF's with rate functions $H_n^P(x_{n+1}; x_n, u_n)$, and they are exponentially tight. Moreover, the map $(x_n, u_n) \to P_n^\epsilon(dx_{n+1}; x_n, u_n)$ is weakly continuous.

2. Let $A_n = \{(x, \xi, u) \in \mathcal{X} \times \mathcal{X} \times \mathcal{U} : H_n^P(x, \xi, u) < +\infty\}$. Then for every sequence $(\xi_n, u_n) \to (\xi, u)$ there exists a corresponding sequence $x_n \to x$ such that $H_n^P(x_n, \xi_n, u_n) \to H_n^P(x, \xi, u)$.

3. H_n^P is jointly l.s.c. in (x_{n+1}, x_n, u_n).

4. For $n = 1, \dots, N$ the families of probability measures $\{Q_n^\epsilon(dy_n; x_n) : \epsilon > 0, x_n \in \mathcal{X}\}$ are WULDF's with finite and continuous rate functions $H_n^Q(y_n; x_n)$, and they are exponentially tight.

5. The reference measure α on $\mathcal{Y}$ such that $Q_n^\epsilon(dy_n; x_n) = q_n^\epsilon(y_n; x_n)\alpha(dy_n)$ satisfies
$$\inf_{y \in K} \alpha\left(B(y, \gamma)\right) > 0;$$
for every $K \subset \mathcal{Y}$ compact and $\gamma > 0$, where $B(y, \gamma)$ is the ball centered at y with radius γ. Moreover the density $q_n^\epsilon(y_n; x_n)$ is jointly continuous in $(y_n; x_n)$.

6. For every $K \subset \mathcal{Y}$ compact, $C \subset \mathcal{X}$ compact and every $\delta > 0$ there exists $\delta' > 0$ and $\epsilon' > 0$ such that if $y, y' \in K$ and $d(y, y') < \delta'$ then $|\epsilon \log q_n^\epsilon(y, x) - \epsilon \log q_n^\epsilon(y', x)| < \delta$ for all $x \in C$ and $\epsilon < \epsilon'$.

7. For every $K \subset \mathcal{Y}$, $C \subset \mathcal{X}$ compact the functions $\epsilon \log q_n^\epsilon(y, x)$ are uniformly bounded from above on $K \times \mathcal{X}$ and uniformly bounded from below on $K \times C$ (uniformly means that the bound is independent of ϵ).

8. The functions g_n appearing in the cost functional $J(\mathbf{u})$ are continuous and bounded.

Note that conditions 2-3, 5-7 correspond to the assumptions of Propositions 3.4 and 3.7 respectively.

A sufficient condition for Assumption A to hold is provided by the following.

ASSUMPTION B

1. Let $\mathcal{W}$ be a metric space. For $n = 0, \dots, N-1$ let $f_n : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \to \mathcal{X}$ be continuous functions, and $\{\mu_n^\epsilon : \epsilon > 0\}$ be an exponentially tight family of probability measures on $\mathcal{W}$ satisfying a LDP with rate function $h_n(w)$. The probability measures $P_n^\epsilon(dx_{n+1}; x_n, u_n)$ are defined by

$$P_n^\epsilon(A; x_n, u_n) = \mu_n^\epsilon\{w : f_n(x_n, u_n, w_n) \in A\}.$$

2. Let $\mathcal{Y} = \mathbb{R}^d$, and, for $n = 1, \dots, N$, let $\phi_n : \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous functions. Moreover, let $\{\nu_n^\epsilon : \epsilon > 0\}$, $n = 1, \dots, N$, be exponentially tight families of probability measures satisfying a LDP with rate function $k_n(v)$, that is finite and continuous. Suppose the following conditions are satisfied:

   a) for every fixed $x \in \mathcal{X}$ the map $v \to \phi_n(x, v)$ is a diffeomorphism in $\mathbb{R}^d$. Moreover the inverse map $\phi_n^{-1}(x, y)$ and $D_y \phi_n^{-1}(x, y)$ are continuous on $\mathcal{X} \times \mathbb{R}^d$, where D_y denotes differentiation w.r.t. y.

   b) For every $K \subset \mathbb{R}^d$ compact, the map det $(D_y \phi_n^{-1})$ is bounded on $\mathcal{X} \times K$.

   c) $\nu_n^\epsilon \ll dv$, and $\{\epsilon \log \frac{d\nu_n^\epsilon}{dv} : \epsilon > 0\}$ is a family of functions that, when restricted to any compact subset of $\mathbb{R}^d$, are equicontinuous and uniformly bounded from below, and are uniformly bounded from above on all $\mathbb{R}^d$

   The probability measures $Q_n^\epsilon(dy_n; x_n)$ are defined by

$$Q_n^\epsilon(B; x_n) = \nu_n^\epsilon\{v : \phi_n(x_n, v) \in B\}.$$

3. The functions g_n appearing in the cost functional $J(\mathbf{u})$ are continuous and bounded.

Note that, under Assumption B, the dynamics for X_n, Y_n have the form

$$\begin{aligned} X_{n+1} &= f_n(X_n, u_n, W_n) \\ Y_n &= \phi_n(X_n, V_n) \end{aligned}$$

where $\{W_0, \dots, W_{N-1}, V_1, \dots, V_N\}$ are independent random variable with $W_n \sim \mu_n^\epsilon$ and $V_n \sim \nu_n^\epsilon$.

The following fact will be proved in the Appendix.

Proposition 4.7 *Assumption B implies Assumption A.*

Example 4.8

1. We give first some examples where Assumption B holds. Assume $\mathcal{W} = \mathbb{R}^m$. Suppose also that, for $n = 0, 1, \dots, N-1$ we are given Borel measurable functions $\tilde{h}_n : \mathbb{R}^m \to \mathbb{R}$, $\tilde{k}_n : \mathbb{R}^d \to \mathbb{R}$ such that

   i) $\tilde{h}_n(w), \tilde{k}_n(v) \to +\infty$ as $\|w\| \to +\infty, \|v\| \to +\infty$;

   ii) $e^{-\tilde{h}_n}$, $e^{-\tilde{k}_n}$ are integrable w.r.t. the Lebesgue measure;

   iii) $\tilde{h}_n$ is a.e. bounded from below, $\tilde{k}_n$ is continuous.

   Then we can define

   $$\mu_n^\epsilon(dw) = \frac{e^{-\epsilon^{-1}\tilde{h}_n(w)}dw}{\int e^{-\epsilon^{-1}\tilde{h}_n(w)}dw}, \quad \nu_n^\epsilon(dw) = \frac{e^{-\epsilon^{-1}\tilde{k}_n(v)}dv}{\int e^{-\epsilon^{-1}\tilde{k}_n(v)}dv}.$$

   Then $\mu_n^\epsilon, \nu_n^\epsilon$ satisfy the requirements given in Assumption B, with corresponding rate functions $h_n(w) = \tilde{h}_n(w) - \inf \tilde{h}_n$, $k_n(v) = \tilde{k}_n(v) - \inf \tilde{k}_n$. Note that this example includes the Gaussian noise considered in [12].

   To complete the description of the model we can assign any continuous functions $f_n : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \to \mathcal{X}$ for the state dynamic equations, while examples of output functions ϕ_n satisfying Assumption B are provided by functions of type

   $$\phi_n(x, v) = \beta_n(x) + \gamma_n(x)v$$

   where $\beta_n : \mathcal{X} \to \mathbb{R}^d$, $\gamma_n : \mathcal{X} \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ are continuous functions and, for all $v \in \mathbb{R}^d$, the inequality $\|\gamma_n(x)v\|^2 \geq \delta\|v\|^2$ holds for a constant δ independent of $x \in \mathcal{X}$. Note that no boundedness or growth assumptions on β_n are required.

2. Assumption A has the advantage of being more general than Assumption B, and somewhat more directly usable in the proofs. Besides technical convenience, the description of the model in terms of transition probabilities, rather than difference equations with noise, may be more natural in some contexts, e.g. when $\mathcal{X}$ and/or $\mathcal{Y}$ are finite sets. For instance, in the case of $\mathcal{X}$ finite, one may consider transition probabilities of the form

   $$P_n^\epsilon(x_{n+1}; x_n, u_n) = \frac{e^{-\epsilon^{-1}H_n^P(x_{n+1};x_n,u_n)}}{\sum_{z\in\mathcal{X}} e^{-\epsilon^{-1}H_n^P(z;x_n,u_n)}}. \tag{4.30}$$

   If H_n^P is finite and continuous in u_n, then (4.30) automatically satisfies 1-3 of Assumption A. A similar transition mechanism can be defined for the output, when $\mathcal{Y}$ is finite; if $\mathcal{X}$ and $\mathcal{Y}$ are both finite, conditions 4-7 of Assumption A are trivially satisfied.

   Dynamics of type (4.30) appear naturally in statistical mechanical models of particle systems; in that context ϵ is a temperature parameter, while the control u_n may be seen as an external field perturbing some "free" evolution.

In what follows we will use objects defined in Section 4.2. We find convenient to give a list of all identities we are going to use, showing explicitly the dependence on ϵ.

$$P_{n+1}^{f,\epsilon}(B; z_{n+1}) = \frac{\int_{\mathcal{X}} \left(\int_B q_{n+1}^\epsilon(y_{n+1}, x_{n+1}) P_n^\epsilon(dx_{n+1}; x_n, u_n) \right) P_n^{f,\epsilon}(dx_n; z_n)}{\int_{\mathcal{X}} \left(\int_{\mathcal{X}} q_{n+1}^\epsilon(y_{n+1}, x_{n+1}) P_n^\epsilon(dx_{n+1}; x_n, u_n) \right) P_n^{f,\epsilon}(dx_n; z_n)} \tag{4.31}$$

$$\rho_n^\epsilon(z_{n+1}) = \int \Big(\int q_{n+1}^\epsilon(y_{n+1}, x_{n+1}) P_n^\epsilon(dx_{n+1}; x_n, u_n) \Big) P_n^{f,\epsilon}(dx_n; z_n) \tag{4.32}$$

$$P_{n+1}^{I,\epsilon}(A; z_{n+1}) = \frac{\int_{\mathcal{X}} \Big(\int_A e^{\epsilon^{-1} g_n(x_n,u_n)} q_{n+1}^\epsilon(y_{n+1}, x_{n+1}) P_n^\epsilon(dx_{n+1}; x_n, u_n) \Big) P_n^{I,\epsilon}(dx_n; z_n)}{\rho_n^\epsilon(z_{n+1})} \tag{4.33}$$

$$J^\epsilon(\mathbf{u}) = \epsilon \log E\Big\{ \exp \epsilon^{-1} \Big[G_N^\epsilon(z_N) \Big] \Big\} \tag{4.34}$$

$$G_N^\epsilon(z_N) = \epsilon \log \int e^{\epsilon^{-1} g_N(x_N)} P_N^{I,\epsilon}(dx_N; z_N) \tag{4.35}$$

$$V_n^\epsilon(z_n) = \inf_{u \in \mathcal{U}} \epsilon \log \int \exp \epsilon^{-1} [V_{n+1}^\epsilon(z_n, u, y_{n+1})] P_{n+1}^{O,\epsilon}(y_{n+1}; z_n, u). \tag{4.36}$$

Remark 4.9 By using Assumption B one checks by rather standard arguments that the value functions $V_n^\epsilon(z_n)$ are continuous. By Remark 4.4 we know that they are also bounded, and it is clear that the bound does not depend on ϵ.

We now give the main result of this section.

Theorem 1 *There are functions $V_n : \mathcal{Z}_n \to \mathbb{R}$, for $n = 0, \ldots, N$, such that $V_n^\epsilon \to V_n$ uniformly on the compact subsets of $\mathcal{Z}_n$. Moreover the functions V_n satisfy the following recursion*

$$\begin{aligned} V_N(z_N) &= \sup_x \Big[g_N(x) - H_N^I(x; z_N) \Big] \\ V_n(z_n) &= \inf_{u_n} \sup_{y_{n+1}} \Big[V_{n+1}(z_n, u_n, y_{n+1}) - H_n^O(y_{n+1}; z_n, u_n) \Big] \end{aligned} \tag{4.37}$$

where H_N^I, H_n^O are determined by the following recursions

$$H_0^I(x) = H_0^f(x) \begin{cases} 0 & \text{if } x = \xi \\ +\infty & \text{otherwise} \end{cases}$$

$$H_n^O(y_{n+1}; z_n, u_n) = \inf_{x_{n+1}, x_n} \Big[H_{n+1}^Q(y_{n+1}; x_{n+1}) + H_n^P(x_{n+1}; x_n, u_n) + H_n^f(x_n; z_n) \Big] \tag{4.38}$$

$$\begin{aligned} H_{n+1}^f(x_{n+1}; z_{n+1}) = H_{n+1}^Q(y_{n+1}; x_{n+1}) &+ \inf_{x_n} \Big[H_n^P(x_{n+1}; x_n, u_n) + H_n^f(x_n; z_n) \Big] \\ &- H_n^O(y_{n+1}; z_n, u_n). \end{aligned} \tag{4.39}$$

$$\begin{aligned} H_{n+1}^I(x; z_{n+1}) = H_{n+1}^Q(y_{n+1}; x) &+ \inf_{\eta \in \mathcal{X}} \Big[H_n^P(x, \eta, u_n) + H_n^I(\eta, z_n) - g_n(\eta, u_n) \Big] \\ &- H_n^O(y_{n+1}; z_n, u_n). \end{aligned} \tag{4.40}$$

Remark 4.10 We have observed above that the functions V_n^ϵ are continuous and bounded. It follows by Theorem 1 that also the function V_n are continuous and bounded. Indeed, in any metric space, uniform convergence on compact subsets preserves continuity.

The proof of Theorem 1 is based on the following result, whose proof is given at the end of this section.

Proposition 4.11 *The families* $\{P_n^{O,\epsilon}(dy_{n+1}; z_n, u_n)\}$, $n = 0, \dots, N-1$, $\{P_n^{f,\epsilon}(dx_n; z_n)\}$ *and* $\{P_n^{I,\epsilon}(dx_n, z_n)\}$, $n = 0, \dots, N$, *are WULDF with rate functions* H_n^O, H_n^f, H_n^I *respectively.*

In the proof of Theorem 1 we also use the following technical result, that we prove in the Appendix.

Lemma 4.12 *Let* E *be a metric space,* F *a compact metric space, and* $f^\epsilon : E \times F \to \mathbb{R}$, $\epsilon \geq 0$, *be a family of continuous functions such that* $f^\epsilon \to f^0$ *uniformly on the compact subsets of* $E \times F$. *Define* $g^\epsilon : E \to \mathbb{R}$ *by*

$$g^\epsilon(x) = \inf_{y \in F} f^\epsilon(x, y).$$

Then $g^\epsilon \to g^0$ *uniformly on the compact subsets of* E.

Proof of Theorem 1. We prove the convergence $V_n^\epsilon \to V_n$ by backward induction on n. For $n = N$ the claim is an immediate consequence of (4.35) and Proposition 4.11. We now prove the inductive step. Define

$$T_n^\epsilon(z_n, u) = \epsilon \log \int \exp \epsilon^{-1}[V_{n+1}^\epsilon(z_n, u, y_{n+1})] P_{n+1}^{O,\epsilon}(y_{n+1}; z_n, u), \tag{4.41}$$

so that

$$V_n^\epsilon(z_n) = \inf_{u \in \mathcal{U}} T_n^\epsilon(z_n, u). \tag{4.42}$$

By inductive assumption $V_{n+1}^\epsilon(z_n, u, y_{n+1}) \to V_{n+1}(z_{n+1})$ uniformly on the compact subsets of $\mathcal{Z}_{n+1}$. Thus, by using Lemma 2.12 and Proposition 4.11

$$T_n^\epsilon(z_n, u) \to \sup_{y_{n+1} \in \mathcal{Y}} \Big[V_{n+1}(z_n, u, y_{n+1}) - H_n^O(y_{n+1}; z_n, u_n) \Big]. \tag{4.43}$$

By (4.42) and Lemma 4.12 the conclusion follows. ∎

Proof of Proposition 4.11. We prove by induction that H_n^I and H_n^f are the rate functions for $P_n^{I,\epsilon}$ and $P_n^{f,\epsilon}$ respectively. The $n = 0$ case is clear, since the singleton $\{\delta_\xi\}$ is a WULDF with rate function $H_0^I = H_0^f$. The inductive step, in both cases, is a simple application of (4.31), (4.33), Propositions 3.4 and 3.7. The fact that H_n^O is the rate function for $P_n^{O,\epsilon}$ also follows for (4.12) and Proposition 3.4.

∎

4.4 Interpretation of the limit value function

In this section we show that the limit value function V_n can be interpreted as the value function for a partially observed dynamic game. Although this would not be necessary, for conceptual simplicity Assumption B will be assumed throughout this section. Thus, the stochastic dynamics for the risk-sensitive control problem are given, in short, by

$$\begin{cases} x_{n+1} &= f_n(x_n, u_n, w_n) \\ y_n &= \phi_n(x_n, v_n) \end{cases} \tag{4.44}$$

with $w_n \sim \mu_n^\epsilon$ and $v_n \sim \nu_n^\epsilon$. Now, consider the deterministic, zero sum dynamic game with dynamics given by (4.44) and cost functional

$$J(\mathbf{u}) = \sup_{\mathbf{v},\mathbf{w}} \Big[\sum_{n=0}^{N-1} \Big(g_n(x_n, u_n) - h_n(w_n) - k_{n+1}(v_{n+1}) \Big) + g_N(x_N) \Big] \tag{4.45}$$

defined for $\mathbf{u} \in ad(\mathcal{U})$. The supremum in (4.45) is over all sequences $\mathbf{w} = (w_0, \ldots, w_{N-1}) \in \mathcal{W}^N$, $\mathbf{v} = (v_1, \ldots, v_N) \in (\mathbb{R}^d)^N$. Note that, for $\mathbf{u} \in ad(\mathcal{U})$ fixed, the expression in (4.45) within square brackets is a function of $\mathbf{w}, \mathbf{v}$.

Proposition 4.13 *The following identity holds for $n = 0, \ldots, N-1$:*

$$\sup \Big[\sum_{l=0}^{n} \Big(g_l(x_l, u_l) - h_l(w_l) - k_{l+1}(v_{l+1}) \Big) : \ \text{(4.44) holds, and}$$

$$w_0, \ldots, w_n, v_1, \ldots, v_{n+1} \ \text{are such that} \ (y_0, \ldots, y_{n+1}, u_0, \ldots, u_n) = z_{n+1}, x_{n+1} = x \Big]$$

$$= -H_{n+1}^I(x; z_{n+1}) - \sum_{k=0}^{n} H_k^O(y_{k+1}; z_k, u_k). \tag{4.46}$$

Proof. Under Assumption B, we can rewrite (4.40) as

$$H_{n+1}^I(x; z_{n+1}) = \inf_{\eta \in \mathcal{X}} \inf_{w \in \mathcal{W}} \inf_{v \in \mathbb{R}^d} \Big[k_{n+1}(v) + h_n(w) + H_n^I(\eta, z_n) - g_n(\eta, u_n)$$

$$: f_n(\eta, u_n, w) = x, \phi_{n+1}(x, v) = y_{n+1} \Big] - H_n^O(y_{n+1}; z_n, u_n). \tag{4.47}$$

We prove (4.46) by induction on n. For $n = 0$ the claim follows using (4.47). Otherwise, by using the inductive assumption and (4.47), we get

$$\sup \Big[\sum_{l=0}^{n} \Big(g_l(x_l, u_l) - h_l(v_l) - k_{l+1}(v_{l+1}) \Big) : \ \text{(4.44) holds, and}$$

$$w_0, \ldots, w_n, v_1, \ldots, v_{n+1} \ \text{are such that} \ (y_0, \ldots, y_{n+1}, u_0, \ldots, u_n) = z_{n+1}, x_{n+1} = x \Big]$$

$$= \sup_{\eta \in \mathcal{X}} \sup_{w \in \mathcal{W}} \sup_{v \in \mathbb{R}^d} \Big[g_n(\eta, u_n) - k_{n+1}(v) - h_n(w) - H_n^I(\eta, z_n) : f_n(\eta, u_n, w) = x, \phi_{n+1}(x, v) = y_{n+1} \Big]$$

$$- \sum_{k=0}^{n-1} H_k^O(y_{k+1}; z_k, u_k)$$

$$= -H_{n+1}^I(x; z_{n+1}) - \sum_{k=0}^{n} H_k^O(y_{k+1}; z_k, u_k). \tag{4.48}$$

■

Proposition 4.13 allows to transform the dynamic game (4.44)(4.45) into a totally observed one, in terms of the information vector

$$z_{n+1} = (z_n, u_n, y_{n+1}) \tag{4.49}$$

$$J(\mathbf{u}) = \sup_{\mathbf{y}} \Big[\sum_{n=0}^{N-1} G_n(z_n, u_n, y_{n+1}) + G_N(z_N) \Big] \tag{4.50}$$

with

$$G_N(z_N) = \sup_{x \in \mathcal{X}} \Big[g_N(x) - H_N^I(x; z_N) \Big], \tag{4.51}$$

$$G_n(z_n, u_n, y_{n+1}) = -H_n^O(y_{n+1}; z_n, u_n), \tag{4.52}$$

and where the supremum in (4.50) is over all sequences $\mathbf{y} = (y_1, \ldots, y_N) \in \mathcal{Y}^N$. We recall that the standard definition of the upper value function $V_n(z_n)$ for the dynamic game (4.49)(4.50) is the infimum of

$$\sup_{y_{n+1}, \ldots, y_N} \sum_{k=n}^{N-1} G_k(z_k, u_k, y_{k+1}) + G_N(z_N)$$

over $\mathbf{u} \in ad(\mathcal{U})$ where the dynamics (4.49) start at time n from z_n. A simple dynamic programming argument yields the following.

Proposition 4.14 *The upper value function V_n for the zero sum, two players dynamic game (4.49) (4.50) is given by (4.37).*

Remark 4.15 We have seen that the pair $(P_n^{f,\epsilon}, P_n^{I,\epsilon})$ is an information state for the risk sensitive control problem. The corresponding pair (H_n^f, H_n^I) can be interpreted as an information state for the limit dynamic game. In fact, the following totally observed dynamic game with state variables (H_n^f, H_n^I) is equivalent to (4.49)(4.50):

$$\begin{aligned} H_{n+1}^f(x) = H_{n+1}^Q(y_{n+1}; x) + \inf_{\eta} \Big[H_n^P(x_{n+1}; \eta, u_n) + H_n^f(\eta) \Big] \\ - \inf_x \Big\{ H_{n+1}^Q(y_{n+1}; x) + \inf_{\xi} \Big[H_n^P(x_{n+1}; \xi, u_n) + H_n^f(\xi) \Big] \Big\} \end{aligned} \tag{4.53}$$

$$\begin{aligned} H_{n+1}^I(x) = H_{n+1}^Q(y_{n+1}; x) + \inf_{\eta \in \mathcal{X}} \Big[H_n^P(x, \eta, u_n) + H_n^I(\eta) - g_n(\eta, u_n) \Big] \\ - \inf_x \Big\{ H_{n+1}^Q(y_{n+1}; x) + \inf_{\xi} \Big[H_n^P(x_{n+1}; \xi, u_n) + H_n^f(\xi) \Big] \Big\}, \end{aligned} \tag{4.54}$$

$$\begin{aligned} J(\mathbf{u}) = \sup_{\mathbf{y}} \Big\{ - \sum_{n=0}^{N-1} \inf_x \Big\{ H_{n+1}^Q(y_{n+1}; x) + \inf_{\xi} \Big[H_n^P(x_{n+1}; \xi, u_n) + H_n^f(\xi) \Big] \Big\} \\ + \sup_{x \in \mathcal{X}} \Big[g_N(x) - H_N^I(x) \Big]. \end{aligned} \tag{4.55}$$

It should be noticed that there is a simpler notion of information state for the dynamic game (4.44)-(4.45), given by the real valued function $K_n : \mathcal{X} \to \mathbb{R}$, evolving according to the equation

$$K_{n+1}(x) = H^Q_{n+1}(y_{n+1}; x) + \inf_{\eta \in \mathcal{X}} \Big[H^P_n(x, \eta, u_n) + K_n(\eta) - g_n(\eta, u_n) \Big]. \tag{4.56}$$

It can be shown that

$$J(\mathbf{u}) = \sup_{\mathbf{y}} \sup_{x \in \mathcal{X}} \Big[g_N(x) - K_N(x) \Big]. \tag{4.57}$$

It can be proved that the function $K_n(x)$ is the rate function of a WULDF that is recursively defined as in (4.33) where the denominator ρ^ϵ_n is dropped. The measure obtained in this way is quite related to the information state in [12]; the use of this measure in place of P^I_n gives rise to an unbounded value function, posing serious difficulty to the small parameter analysis.

Example 4.16 In the case $\mathcal{X} = \mathbb{R}^p$, and

$$\mu^\epsilon_n(dw) = \frac{1}{(2\pi\epsilon)^{p/2}} e^{-\frac{1}{2\epsilon}||w||^2} dw \qquad \text{and} \qquad \nu^\epsilon_n(dv) = \frac{1}{(2\pi\epsilon)^{d/2}} e^{-\frac{1}{2\epsilon}||v||^2} dv,$$

we have $h_n(w) = \frac{1}{2}||w||^2$, and $k_n(w) = \frac{1}{2}||v||^2$, and we recover the model in [12], but with much more general equation for the dynamics.

5 The Completly Observed Case

The risk sensitive control problems satisfying Assumptions A do not include the completely observed case ($Y_n = X_n$). It is clear, however, that the method used in this paper can be easily directly applied to the dynamic programming equation of a completely observed problem.

Consider a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_n)_{n=0}^N$. Define $ad(\mathcal{U})$, the set of admissible controls, to be the set of the $\mathcal{F}_n$-adapted $\mathcal{U}$-valued processes. For $\mathbf{u} \in ad(\mathcal{U})$, we let $X^{\epsilon,\mathbf{u}}_n$ to be the $\mathcal{F}_n$-adapted $\mathcal{X}$-valued process such that:

$$P\{X^{\epsilon,\mathbf{u}}_{n+1} \in \cdot \,|\mathcal{F}_n\} = P^\epsilon_n(\cdot\,; X^{\epsilon,\mathbf{u}}_n, u_n).$$

The cost functional is as in (4.7), but defined in this new set of admissible controls. Consider the value function:

$$V^\epsilon_n(x) = \inf_{\mathbf{u} \in ad(\mathcal{U})} \epsilon \log E\Big\{ \exp\Big[\epsilon^{-1} \Big(\sum_{k=n}^{N-1} g_k(X_n, u_n) + g_N(X_N) \Big) \Big] \Big\}.$$

The following result is proved by induction as in Theorem 1. Note that the assumptions needed are much weaker than those of a similar result given in [4].

Theorem 2 *Assume that part 1 of either Assumptions A or B holds. Then there are functions V_n such that $V^\epsilon_n \to V_n$ as $\epsilon \to 0$, uniformly on compact subsets of $\mathcal{X}$.*

Moreover, if part 1 of Assumptions B holds, then V_n is the upper value function for the deterministic dynamic game with dynamic given by:

$$x_{n+1} = f_n(x_n, u_n, w_n),$$

and with cost functional given by:

$$J(\mathbf{u}) = \sup_{\mathbf{w}} \Big[\sum_{n=0}^{N-1} \Big(g_n(x_n, u_n) - h_n(w_n) \Big) + g_N(x_N) \Big].$$

6 Appendix

6.1 Proof of Proposition 4.7

Properties 1. and 4. of Assumption A follow from Proposition 2.7 and Lemma 2.10.

In the rest of the proof we drop the index n everywhere.

Proof of property 2. Suppose $(x, \xi, u) \in A$, i.e. $H^P(x; \xi, u) < \infty$. First note that, since the set $\{w : f(\xi, u, w) = x\}$ is closed, then there is $w \in \mathcal{W}$ such that $f(\xi, u, w) = x$ and $h(w) = H^P(x; \xi, u)$.

Suppose now we have a sequence $(\xi_k, u_k) \to (\xi, u)$. We construct a sequence $x_k \to x$ such that $H^P(x_k; \xi_k, u_k) \to H^P(x; \xi, u)$. Define $x_k = f(\xi_k, u_k, w)$. By continuity of f, we have that $x_k \to x$. Then let w_k be such that $x_k = f(\xi_k, u_k, w_k)$ and $h(w_k) = H^P(x_k; \xi_k, u_k)$. Clearly $h(w_k) \le h(w)$, and therefore the sequence w_k has has a convergent subsequence $w_{n_k} \to w'$. By lower semicontinuity of h we have

$$h(w') \le \liminf h(w_{n_k}) \le h(w). \tag{6.1}$$

But, again by continuity of f, we also have $x = f(\xi, x, w')$, and therefore

$$h(w') \ge h(w). \tag{6.2}$$

By (6.1) and (6.2) we have

$$\lim_k h(w_k) = h(w)$$

as desired.

Proof of property 3. Consider a sequence $(x_k, \xi_k, u_k) \to (x, \xi, u)$. We must show that

$$\liminf H^P(x_k; \xi_k, u_k) \ge H^P(x; \xi, u). \tag{6.3}$$

It is enough to prove the following statement: suppose there is a subsequence $(x_{n_k}, \xi_{n_k}, u_{n_k})$ such that

$$\lim_k H^P(x_{n_k}; \xi_{n_k}, u_{n_k}) = l < \infty; \tag{6.4}$$

then $l \ge H^P(x; \xi, u)$. To prove this, let w_{n_k} be such that $x_{n_k} = f(\xi_{n_k}, u_{n_k}, w_{n_k})$ and $h(w_{n_k}) = H^P(x_{n_k}; \xi_{n_k}, u_{n_k})$. By (6.4), the sequence w_{n_k} has a limit point w. By continuity of f, $f(\xi, u, w) = x$, and therefore $h(w) \ge H^P(x; \xi, u)$. Finally, by lower semicontinuity of h

$$l = \lim_k h(w_{n_k}) \ge h(w) \ge H^P(x; \xi, u)$$

which completes the proof of property 3.

Proof of properties 5. and 6. Letting

$$\rho^\epsilon = \frac{d\nu^\epsilon}{dv}$$

we easily get

$$\epsilon \log q^\epsilon(y; x) = \epsilon \log \rho^\epsilon(\phi^{-1}(x, y)) + \epsilon \log \Big| \det \Big(D_y \phi^{-1}(x, y)\Big)\Big| \tag{6.5}$$

Property 5. follows from (6.5), equicontinuity of $\epsilon \log \rho^\epsilon$, continuity of ϕ^{-1} and the fact that $\log |\det (D_y \phi^{-1})|$, being continuous, is bounded on the compact subsets of $\mathcal{X} \times \mathbb{R}^d$. Property 6. follows from (6.5) and boundedness of $\epsilon \log \rho^\epsilon$ and $\det (D_y \phi^{-1})$.

The only thing left to prove is the finiteness and continuity of the rate function H^Q. This follows from finiteness and continuity of k and the identity

$$H^Q(y; x) = k(\phi^{-1}(x, y)).$$

6.2 Proof of Lemma 4.11

We first show that, for $\epsilon \geq 0$, g^ϵ is continuous. We omit the index ϵ in this part of the proof. Upper semicontinuity is obvious. To prove lower semicontinuity, observe that, due to the compactness of F, for every $x \in E$ there is a "minimizer" $y \in F$ such that $g(x) = f(x, y)$. So let $x_n \to x$, and y_n be the corresponding sequence of minimizers. We have to show that

$$\liminf g(x_n) \geq g(x).$$

To do so, it is not restrictive to assume that the sequence $g(x_n)$ has limit. Moreover, let y be a limit point of $\{y_n\}$. We have:

$$\lim g(x_n) = \lim f(x_n, y_n) = f(x, y) \geq g(x).$$

Thus the functions g^ϵ, $\epsilon \geq 0$, are continuous.

Now, let $K \subset E$ be compact. Let $\delta > 0$ be arbitrary, and let $\epsilon' > 0$ be such that for every $\epsilon < \epsilon'$

$$\left| f^\epsilon(x, y) - f^0(x, y) \right| < \delta \tag{6.6}$$

for any $x \in K, y \in F$. Given $x \in K$, let y^ϵ be such that $g^\epsilon(x) = f^\epsilon(x, y^\epsilon)$. By (6.6), for $\epsilon < \epsilon'$ and $x \in K$,

$$g^\epsilon(x) = f^\epsilon(x, y^\epsilon) \geq f^0(x, y^\epsilon) - \delta \geq g^0(x) - \delta.$$

To complete the proof we have to show that there exists ϵ'' such that for any $\epsilon < \epsilon''$

$$g^\epsilon(x) \leq g^0(x) + \delta \tag{6.7}$$

for all $x \in K$. Suppose, by contradiction, that there is no such ϵ''. Then there is a sequence $\epsilon_n \to 0$ and a corresponding sequence x_n in K such that

$$g^{\epsilon_n}(x_n) > g^0(x_n) + \delta \tag{6.8}$$

for all n. Denote by x a limit point of $\{x_n\}$ and by y a limit point of the sequence of minimizers $y_n^{\epsilon_n}$. By possibly passing to subsequences, we may assume that $\{x_n\} \to x$ and $y_n^{\epsilon_n} \to y$. Thus

$$\lim_n g^{\epsilon_n}(x_n) = \lim_n f^{\epsilon_n}(x_n, y_n^{\epsilon_n}) = f^0(x, y). \tag{6.9}$$

On the other hand, by continuity of g^0,

$$\lim_n g^0(x_n) = g^0(x). \tag{6.10}$$

Thus, by (6.8), (6.9) and (6.10) we have $f^0(x, y) \geq g^0(x) + \delta$. Therefore, there is a y' with $f^0(x, y) \geq f^0(x, y') + \delta$. This implies

$$\lim_n \left[f^{\epsilon_n}(x_n, y_n^{\epsilon_n}) - f^{\epsilon_n}(x_n, y') \right] \geq \delta$$

which is impossible since $f^{\epsilon_n}(x_n, y_n^{\epsilon_n}) = \inf_z f^{\epsilon_n}(x_n, z)$.

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