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Asymmetric algebraic Riccati equation: A homeomorphic parametrization of the set of solutions☆

Augusto Ferrante [a,*], Michele Pavon [b], Stefano Pinzoni [b]

[a] *Dipartimento di Elettronica e Informazione, Politecnico di Milano, via Ponzio 34/5, 20133 Milano, Italy*
[b] *Dipartimento di Elettronica e Informatica, Università di Padova, via Gradenigo 6/A, and LADSEB-CNR, 35131 Padova, Italy*

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Abstract

In this paper, asymmetric algebraic Riccati equations are analyzed. In particular, we derive a new parametrization of the set of solutions. Generalizing on the symmetric case, the proposed parametrization is obtained in terms of pairs of invariant subspaces of two related "feedback" matrices. Moreover, the connection is clarified between the new parametrization and the classical homeomorphic one based on graph invariant subspaces of the pseudo-Hamiltonian matrix associated with the equation. We finally show that also the newly introduced parametrization is given by a homeomorphic map. © 2001 Published by Elsevier Science Inc. All rights reserved.

Keywords: Algebraic Riccati equation; Invariant subspaces; Feedback matrix; Homeomorphism

1. Introduction

This paper is concerned with the real quadratic matrix equation

$$A_1X + XA_2 + XPX + Q = 0. \tag{1.1}$$

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* Corresponding author.

E-mail addresses: ferrante@elet.polimi.it (A. Ferrante), pavon@dei.unipd.it (M. Pavon), pinzoni@dei.unipd.it (S. Pinzoni).

PII: S 0 0 2 4 - 3 7 9 5 (0 1) 0 0 2 4 1 - 5

In the symmetric case, i.e., when the parameter matrices A_1, A_2, P and Q satisfy the relations $A_1 = A_2^{\mathrm{T}}$, $P = P^{\mathrm{T}}$ and $Q = Q^{\mathrm{T}}$, Eq. (1.1) reduces to the classical algebraic Riccati equation (ARE).

Asymmetric AREs of the form (1.1) have received increasing interest in recent times due to their relevance in many problems of applied mathematics and system theory. In particular, it has been shown that *feedback control* [1], optimal strategy in *differential games* [5,7,32,29], H^2- and H^∞-control problems [4,13], *factorization of polynomials* [10,26,27], *J-spectral factorization* [22], *stability of solutions* of the symmetric ARE under small perturbations of the coefficients [8], and *singular perturbation* of a general boundary value problem [9] may all be reformulated in terms of asymmetric Riccati equations. This is due to the fact that AREs may be viewed as the algebraic counterpart of the problem of *factorization of rational functions* [3,17,18,20,31,33,39,43], which is the essence of a great number of control and applied mathematics problems, see [2,19], and references therein. A description of the manifold applications of asymmetric AREs and many important results in this field may be found in [16,24,25,28,30,34], and references therein.

An extensive study of the ARE has produced a large variety of results. In particular, starting from the classical work [40], various parametrizations of the set of solutions (or of classes of solutions), both in the symmetric and in the general case, have been established, see [11,14,15,18,21,35,38,41,42], to mention but a few.

In particular, it is well known that the set of solutions of (1.1) may be characterized in terms of graph invariant subspaces of the *pseudo-Hamiltonian* matrix

$$H = \begin{bmatrix} A_2 & P \\ -Q & -A_1 \end{bmatrix}. \tag{1.2}$$

More precisely, denoting by $\mathscr{X}$ the set of solutions of (1.1) and by $\mathscr{L}(H)$ the set of graph invariant subspaces of *H*, i.e., invariant subspaces of *H* of the form

$$V = \operatorname{im} \begin{bmatrix} I \\ Y \end{bmatrix},$$

the map

$$\begin{aligned} \varphi : \mathscr{X} &\to \mathscr{L}(H) \\ X &\mapsto \operatorname{im} \begin{bmatrix} I \\ X \end{bmatrix} \end{aligned} \tag{1.3}$$

is a homeomorphic bijection of $\mathscr{X}$ onto $\mathscr{L}(H)$ [21, pp. 545–546].

In the symmetric case, this classification may be compared to the geometric parametrization of J.C. Willems [40], formulated in terms of invariant subspaces of a certain feedback matrix. In this comparison, the classification provided by (1.3) has the disadvantage that the set $\mathscr{L}(H)$ is not easily described. On the other hand, it has been observed in [38] that also Willems' parametrization suffers from some drawbacks, namely:

(i) It does not lead naturally to a concept of solution at infinity.
(ii) It does not admit an obvious generalization to the non-symmetric case.

This paper, together with [34], may also be viewed as an attempt to overcome difficulty (ii). In fact, we extend to the non-symmetric case the parametrization of [40], introducing a *pair* of "feedback" matrices and proving a correspondence between the set of solutions of (1.1) and pairs of invariant subspaces of such feedback matrices. Moreover, extending on [14,42], we show that the proposed parametrization is given by a homeomorphic map.

The paper is organized as follows: in Section 2, we set some notation and recall some preliminary results. In Section 3, we derive a new parametrization for the set of solutions of (1.1) and discuss the connection between this and the classical homeomorphic one given by (1.3). Section 4 is devoted to proving that the newly proposed parametrization is also a homeomorphism. In Section 5, we briefly draw some conclusions. Moreover, Appendix A contains an alternative direct proof of Theorem 4.1.

2. Notation and mathematical background

The vector space $\mathbb{R}^n$ is equipped with the usual Euclidean norm, denoted by $\|\cdot\|$, which assigns to any $x \in \mathbb{R}^n$ the non-negative real number $\|x\| := [x^{\mathrm{T}}x]^{1/2}$. Given a matrix $Y \in \mathbb{R}^{m\times n}$, we denote by $\|Y\| := \max\{\|Yx\| : \ x \in \mathbb{R}^n, \ \|x\| = 1\}$ the spectral norm of Y. It is well known that $\|Y\|$ equals the largest singular value of the matrix Y, i.e., the square root of the largest eigenvalue of $Y^{\mathrm{T}}Y$. Moreover, if $m = n$, $\sigma(Y)$ denotes the spectrum of Y and $s_m(Y)$ the smallest singular value of Y.

We endow the set of linear subspaces of $\mathbb{R}^n$ with the *gap metric*. This is defined as follows. Given two subspaces S_1, S_2, let d be the function

$$d(S_1, S_2) := \|P_{S_1} - P_{S_2}\|, \tag{2.1}$$

where P_{S_i} denotes the orthogonal projection onto the space S_i, $i = 1, 2$. The function d is a distance on the set of subspaces of $\mathbb{R}^n$; we refer to [21] for a discussion of the properties of the gap metric induced by d. Finally, given a square matrix $Y \in \mathbb{R}^{n\times n}$, we denote by $\mathscr{S}_k(Y)$ the set of k-dimensional invariant subspaces of Y.

3. Asymmetric algebraic Riccati equations

For $A_1 \in \mathbb{R}^{n_1\times n_1}$, $A_2 \in \mathbb{R}^{n_2\times n_2}$, $P \in \mathbb{R}^{n_2\times n_1}$ and $Q \in \mathbb{R}^{n_1\times n_2}$, let

$$\mathscr{R}(X) := A_1X + XA_2 + XPX + Q \tag{3.1}$$

and consider the asymmetric ARE

$$\mathscr{R}(X) = 0. \tag{3.2}$$

We will establish a parametrization of the solutions $X \in \mathbb{R}^{n_1\times n_2}$ of (3.2) in terms of pairs of subspaces of two generalized feedback matrices, in analogy with the well-known theory for the symmetric ARE. To this aim, we assume that we have a solution X_0 of (3.2), such that the two “feedback” matrices

$$\Gamma_{10} := A_1 + X_0 P, \qquad \Gamma_{20} := -A_2 - P X_0 \tag{3.3}$$

have non-intersecting spectra:

$$\sigma(\Gamma_{10}) \cap \sigma(\Gamma_{20}) = \emptyset. \tag{3.4}$$

In such a way, in the same spirit as [37], we may parametrize the solutions of (3.2) in terms of those of the *homogeneous* algebraic Riccati equation (HARE)

$$\Gamma_{10} D - D\Gamma_{20} + DPD = 0. \tag{3.5}$$

In fact, we may rewrite the equality $\mathscr{R}(X_0) = 0$ as

$$\Gamma_{10} X_0 - X_0 \Gamma_{20} - X_0 P X_0 + Q = 0. \tag{3.6}$$

By subtracting (3.6) from (3.2), it is immediate to check that, given an arbitrary solution X of (3.2), the difference $D = X - X_0$ satisfies the homogeneous asymmetric ARE (3.5). Conversely, summing Eq. (3.5) and (3.6), one can check that to any solution D of (3.5) there corresponds a solution $X = D + X_0$ of (3.2).

Remark 3.1. By applying the characterization (1.3) to the homogeneous ARE (3.5), the set of solutions D is seen to be in a one-to-one correspondence with the set $\mathscr{L}(H_1)$ of graph invariant subspaces of the block triangular matrix

$$H_1 = \begin{bmatrix} -\Gamma_{20} & P \\ 0 & -\Gamma_{10} \end{bmatrix}. \tag{3.7}$$

The latter is related to the original pseudo-Hamiltonian matrix H defined in (1.2) by the similarity transformation

$$H_1 = \begin{bmatrix} I & 0 \\ X_0 & I \end{bmatrix}^{-1} H \begin{bmatrix} I & 0 \\ X_0 & I \end{bmatrix}, \tag{3.8}$$

so that we have

$$\mathscr{L}(H) = \begin{bmatrix} I & 0 \\ X_0 & I \end{bmatrix} \mathscr{L}(H_1). \tag{3.9}$$

Notice that (3.9) exactly reproduces the relation between solutions to the ARE (3.2) and solutions to the associated HARE (3.5), in terms of the parametrization (1.3). In fact,

$$\varphi(D) = \operatorname{im} \begin{bmatrix} I \\ D \end{bmatrix} \in \mathscr{L}(H_1),$$

i.e., D solves (3.5), if and only if

$$\varphi(X) = \operatorname{im} \begin{bmatrix} I \\ X \end{bmatrix} \in \mathscr{L}(H),$$

i.e., X solves (3.2), being $X = X_0 + D$, i.e.,

$$\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I & 0 \\ X_0 & I \end{bmatrix} \begin{bmatrix} I \\ D \end{bmatrix}. \qquad \square$$

3.1. A new parametrization of the set of solutions

To parametrize the solutions of (3.5), and hence of (3.2), let $S_1 \in \mathscr{S}_{k_1}(\Gamma_{10}^{\mathrm{T}})$ and $S_2 \in \mathscr{S}_{k_2}(\Gamma_{20})$, with $n_1 - k_1 = n_2 - k_2 =: l$, where $0 \leqslant l \leqslant \min(n_1, n_2)$. Moreover, let

$$T_1 = \begin{bmatrix} T_{11} | T_{12} \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}, \qquad T_2 = \begin{bmatrix} T_{21} | T_{22} \end{bmatrix} \in \mathbb{R}^{n_2 \times n_2} \tag{3.10}$$

be orthogonal matrices, such that

$$S_1 = \operatorname{im} T_{11}, \qquad S_2 = \operatorname{im} T_{21}. \tag{3.11}$$

Then, we have

$$\begin{aligned} N &:= T_1^{\mathrm{T}} \Gamma_{10} T_1 = \begin{bmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{bmatrix}, \\ M &:= T_2^{\mathrm{T}} \Gamma_{20} T_2 = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}, \end{aligned} \tag{3.12}$$

where N_{22} and M_{22} are square matrices of dimension l. Moreover, define the $n_2 \times n_1$ matrix

$$L := T_2^{\mathrm{T}} P T_1 = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \tag{3.13}$$

partitioned in such a way that also L_{22} is a square matrix of dimension l.

Clearly, from (3.4) it follows that $\sigma(N_{22}) \cap \sigma(M_{22}) = \emptyset$, and hence the Sylvester equation

$$\Delta N_{22} - M_{22} \Delta + L_{22} = 0 \tag{3.14}$$

has a unique solution $\Delta \in \mathbb{R}^{l \times l}$, see e.g. [23, Theorem 4.4.6].

Remark 3.2. It should be obvious that, in the case when $l = 0$, one formally has $T_1 = T_{11}$, $T_2 = T_{21}$, $N = N_{11}$, $M = M_{11}$, $L = L_{11}$, and, in particular, there is no Eq. (3.14) to consider.

On the other hand, for $l > 0$, we observe that N_{22}, M_{22} and L_{22} not only depend on the pair (S_1, S_2), but also on the choice of the matrices T_1 and T_2. However, let

$$T_1' = \begin{bmatrix} T_{11}' | T_{12}' \end{bmatrix} \neq T_1 \quad \text{and} \quad T_2' = \begin{bmatrix} T_{21}' | T_{22}' \end{bmatrix} \neq T_2$$

be orthogonal matrices, such that $S_1 = \operatorname{im} T_{11}'$ and $S_2 = \operatorname{im} T_{21}'$, and denote by N_{22}', M_{22}' and L_{22}' the corresponding matrices obtained as in (3.12) and (3.13). It is easy to check that $T_1^{\mathrm{T}} T_1'$ and $T_2^{\mathrm{T}} T_2'$ are block-diagonal matrices. This fact, taking into account that $T_i^{\mathrm{T}} = T_i^{-1}$, $i = 1, 2$, implies that there exist two non-singular matrices W_1 and W_2, such that

$$N'_{22} = W_1^{-1} N_{22} W_1, \quad M'_{22} = W_2^{-1} M_{22} W_2, \quad L'_{22} = W_2^{-1} L_{22} W_1. \tag{3.15}$$

In conclusion, a different choice of T_1 and T_2 leads to the equation

$$\begin{aligned} 0 &= \varDelta' N'_{22} - M'_{22}\varDelta' + L'_{22} \\ &= \varDelta' W_1^{-1} N_{22} W_1 - W_2^{-1} M_{22} W_2 \varDelta' + W_2^{-1} L_{22} W_1 \\ &= W_2^{-1}[W_2 \varDelta' W_1^{-1} N_{22} - M_{22} W_2 \varDelta' W_1^{-1} + L_{22}] W_1, \end{aligned} \tag{3.16}$$

whose unique solution $\varDelta'$ is related to the solution $\varDelta$ of (3.14) by

$$\varDelta = W_2 \varDelta' W_1^{-1}. \tag{3.17}$$

Thus, given the ARE (3.2) and the reference solution X_0, the rank of the unique solution $\varDelta$ of Eq. (3.14) only depends on the pair of invariant subspaces (S_1, S_2). □

In view of the previous remark, the set

$$\begin{aligned} \mathscr{I} := \{ (S_1, S_2) : S_1 \in \mathscr{S}_{k_1}(\Gamma_{10}^{\mathrm{T}}),\ S_2 \in \mathscr{S}_{k_2}(\Gamma_{20}),\ n_1 - k_1 = n_2 - k_2 =: l; \\ \text{if } l > 0, \text{ the solution } \varDelta \text{ of (3.14) is non-singular} \} \end{aligned} \tag{3.18}$$

is well defined.

Remark 3.3. It is interesting to notice that, under the assumption of non-intersecting spectra: $\sigma(N_{22}) \cap \sigma(M_{22}) = \emptyset$, observability of the pair (N_{22}, L_{22}) and controllability of the pair (M_{22}, L_{22}) are necessary conditions for invertibility of the (unique) solution $\varDelta$ of (3.14) [12]. Such conditions are also sufficient in the special case when L_{22} has rank 1 [36]. □

The following theorem gives a complete parametrization of the set of solutions of (3.5), and, consequently, of the asymmetric ARE (3.2).

Theorem 3.1. *Let $\mathscr{D}$ be the set of solutions of the homogeneous ARE* (3.5). *Under assumption* (3.4), *there is a bijective correspondence between $\mathscr{D}$ and the set $\mathscr{I}$ defined in* (3.18). *This correspondence is given by the map*

$$\begin{aligned} \Theta : \mathscr{D} &\to \mathscr{I} \\ D &\mapsto (\ker D^{\mathrm{T}}, \ker D). \end{aligned} \tag{3.19}$$

Proof. As a first step, we prove that if $D \in \mathscr{D}$, then $\Theta(D) \in \mathscr{I}$. Post-multiplication of Eq. (3.5) by a vector $v \in \ker D$ shows that $\ker D$ is a Γ_{20}-invariant subspace. In the same way, it can be checked that $\ker D^{\mathrm{T}}$ is a Γ_{10}^{T}-invariant subspace. Moreover, it is clear that if k_1 and k_2 denote the dimensions of $\ker D^{\mathrm{T}}$ and $\ker D$, respectively, then the differences $n_1 - k_1$ and $n_2 - k_2$ coincide, both equaling $l := \operatorname{rank} D = \operatorname{rank} D^{\mathrm{T}}$. It remains to show that to the pair $(S_1, S_2) := (\ker D^{\mathrm{T}}, \ker D)$, with $l > 0$, there corresponds a non-singular solution of Eq. (3.14). To this aim, let the orthogonal matrices T_1 and T_2 be chosen accordingly to (3.10) and (3.11). Then, multiplying Eq. (3.5) on the left by T_1^{T} and on the right by T_2, we get

$$T_1^{\mathrm{T}}\Gamma_{10}T_1T_1^{\mathrm{T}}DT_2 - T_1^{\mathrm{T}}DT_2T_2^{\mathrm{T}}\Gamma_{20}T_2 + T_1^{\mathrm{T}}DT_2T_2^{\mathrm{T}}PT_1T_1^{\mathrm{T}}DT_2 = 0. \tag{3.20}$$

Taking into account (3.10) and (3.11), we see that $T_1^{\mathrm{T}}DT_2$ has the form

$$\begin{bmatrix} 0 & 0 \\ 0 & D_{22} \end{bmatrix},$$

where D_{22} is a square non-singular matrix of dimension l. Then, Eq. (3.20) reduces to

$$N_{22}D_{22} - D_{22}M_{22} + D_{22}L_{22}D_{22} = 0, \tag{3.21}$$

where N_{22}, M_{22} and L_{22} are defined by (3.12) and (3.13). Since D_{22} is non-singular, $\Delta := D_{22}^{-1}$ is the unique solution of (3.14) and it is clearly non-singular.

We now prove that the map Θ is injective. Let $D', D'' \in \mathscr{D}$ and assume that $S_1 := \ker D'^{\mathrm{T}} = \ker D''^{\mathrm{T}}$ and $S_2 := \ker D' = \ker D''$, with $l := \operatorname{rank} D' = \operatorname{rank} D'' \geqslant 0$. Now, if $l = 0$, we get equality, since $D' = D'' = 0$. If $l > 0$, let T_1 and T_2 be orthogonal matrices satisfying (3.10) and (3.11). Then, we have

$$T_1^{\mathrm{T}}D'T_2 = \begin{bmatrix} 0 & 0 \\ 0 & D'_{22} \end{bmatrix}, \qquad T_1^{\mathrm{T}}D''T_2 = \begin{bmatrix} 0 & 0 \\ 0 & D''_{22} \end{bmatrix}, \tag{3.22}$$

where both D'_{22} and D''_{22} are square non-singular matrices of dimension l, solving Eq. (3.21). This clearly implies that both $(D'_{22})^{-1}$ and $(D''_{22})^{-1}$ are solutions of (3.14). Hence, in view of the uniqueness of the solution, $(D'_{22})^{-1} = (D''_{22})^{-1}$. Thus, $D'_{22} = D''_{22}$ and $D' = D''$.

The last step is to prove that the map Θ is surjective. Clearly, the pair $(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}) \in \mathscr{I}$ corresponds to the solution $D = 0$. Moreover, let $(S_1, S_2) \in \mathscr{I}$, with $l := n_1 - \dim S_1 = n_2 - \dim S_2 > 0$, and the orthogonal matrices T_1 and T_2 be as in (3.10) and (3.11). Then, the corresponding Eq. (3.14) admits a unique solution Δ, which is invertible. Now, Δ^{-1} solves (3.21) and

$$D = T_1 \begin{bmatrix} 0 & 0 \\ 0 & \Delta^{-1} \end{bmatrix} T_2^{\mathrm{T}} = T_{12}\Delta^{-1}T_{22}^{\mathrm{T}} \tag{3.23}$$

is a solution of (3.5), such that $(S_1, S_2) = (\ker D^{\mathrm{T}}, \ker D)$. $\square$

Remark 3.4. Condition (3.4) is rather stringent. In fact, there is no guarantee that to any reference solution X_0 of the ARE (3.2), there correspond "feedback" matrices Γ_{10} and Γ_{20} with non-intersecting spectra. Even worse, examples can be found where no X_0 exists such that (3.4) is satisfied.

Yet, our standing assumption is not strictly necessary. On one hand, the map $\Theta : D \mapsto (\ker D^{\mathrm{T}}, \ker D)$ can be anyhow defined, regardless of (3.4), mapping the set $\mathscr{D}$ of solutions of the HARE (3.5) surjectively onto a set $\mathscr{I}'$ of pairs of invariant subspaces. Actually, it is not difficult to show that such $\mathscr{I}'$ can be obtained by slightly modifying the definition of the set $\mathscr{I}$, given by (3.18). Namely, in general we have to consider

$$\mathscr{I}' := \{(S_1, S_2) : \; S_1 \in \mathscr{S}_{k_1}(\Gamma_{10}^{\mathrm{T}}), \; S_2 \in \mathscr{S}_{k_2}(\Gamma_{20}), \; n_1 - k_1 = n_2 - k_2 =: l; \\ \text{if } l > 0, \; (3.14) \text{ has a non-singular solution } \Delta\}. \tag{3.24}$$

However, though always surjective, the map $\Theta : \mathscr{D} \to \mathscr{I}'$ is not generally injective. In fact, to turn it into a bona fide parametrization of the set $\mathscr{D}$, we need that the Sylvester equation (3.14), associated to any pair $(S_1, S_2) \in \mathscr{I}'$, with $l > 0$, has just one non-singular solution. Thus, a necessary and sufficient condition to prove the result of Theorem 3.1 should be equivalent to $\sigma(N_{22}) \cap \sigma(M_{22}) = \emptyset$, for all pairs N_{22}, M_{22} of matrices associated to $(S_1, S_2) \in \mathscr{I}'$. To express such a condition directly in terms of the coefficients of the ARE (3.2) or of the HARE (3.5), is still a matter of research. □

Remark 3.5. The parametrization given by Theorem 3.1 has been derived, by suitably choosing coordinates in the spaces $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$. Actually, a completely coordinate-free characterization of the set of solutions D of (3.5) would be possible, by recognizing that $\ker D \subset \mathbb{R}^{n_2}$ is a (right-)invariant subspace for Γ_{20}, an operator from $\mathbb{R}^{n_2}$ to $\mathbb{R}^{n_2}$, while left $\ker D \subset (\mathbb{R}^{n_1})^*$ is a (left-)invariant subspace for Γ_{10}, to be interpreted as a map of the dual space $(\mathbb{R}^{n_1})^*$ into itself. In this set-up, D itself should be either interpreted as a transformation of $\mathbb{R}^{n_2}$ into $\mathbb{R}^{n_1}$, or of $(\mathbb{R}^{n_1})^*$ into $(\mathbb{R}^{n_2})^*$, while P should be taken as defining a bilinear form on $(\mathbb{R}^{n_2})^* \times \mathbb{R}^{n_1}$. We might then define the operators D_{22}, N_{22}, M_{22} and L_{22} in coordinate-free terms, to end up with a parametrization of the solution set of (3.5), completely equivalent to the one, which has been presented.

However, we have preferred the other way, relying on matrices for the sake of computability, especially in view of the discussion of the continuity issue in Section 4 and in Appendix A. □

3.2. *The relation between $\mathscr{L}(H_1)$ and $\mathscr{I}$*

Since the family $\mathscr{D}$ of solutions of the HARE (3.5) may be parametrized both in terms of $\mathscr{I}$ and in terms of the set $\mathscr{L}(H_1)$ of graph invariant subspaces of the pseudo-Hamiltonian matrix H_1 defined in (3.7), it is clear that these two sets are in a one-to-one correspondence. Such a connection is rendered explicit in this section.

To this purpose, under the standing assumption (3.4), let $E \in \mathbb{R}^{n_2 \times n_1}$ be the unique solution to the Sylvester equation

$$E\Gamma_{10} - \Gamma_{20}E + P = 0 \tag{3.25}$$

and define the compound matrix

$$V := \begin{bmatrix} I & E \\ 0 & I \end{bmatrix}. \tag{3.26}$$

Then, the following relation holds.

Theorem 3.2. *Let $D \in \mathscr{D}$ be any solution of* (3.5) *and $\mathscr{L} = \varphi(D)$ be the corresponding graph invariant subspace of H_1. Moreover, let $(S_1, S_2) = \Theta(D)$. Then,*

$$\mathscr{L} = V \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \ x \in S_2, \ y \in S_1^{\perp} \right\}. \tag{3.27}$$

Relation (3.27) explicitly describes the bijective composed map $\Xi = \varphi \circ \Theta^{-1}$, thus providing a complete parametrization of $\mathscr{L}(H_1)$ in terms of $\mathscr{I}$. The proof of the theorem is based on the following general fact from linear algebra.

Lemma 3.1. *Let $D \in \mathbb{R}^{n_1 \times n_2}$. Then,*

$$\operatorname{im} \begin{bmatrix} I \\ D \end{bmatrix} = \begin{bmatrix} I & E \\ 0 & I \end{bmatrix} \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \in \ker D, \ y \in \operatorname{im} D \right\},$$

where $E \in \mathbb{R}^{n_2 \times n_1}$ is any matrix such that $DED = D$.

Proof. We need to show that any vector of the form

$$v = \begin{bmatrix} w \\ Dw \end{bmatrix}$$

with $w \in \mathbb{R}^{n_2}$ can also be written as

$$v = \begin{bmatrix} x + Ey \\ y \end{bmatrix}$$

with $x \in \ker D$ and $y \in \operatorname{im} D$, and vice versa. In fact, given w, we take $y = Dw \in \operatorname{im} D$ and $x = w - Ey \in \ker D$, since $Dx = (D - DED)w = 0$. Conversely, w corresponding to given $x \in \ker D$ and $y \in \operatorname{im} D$ is obviously computed as $w = x + Ey$, yielding $Dw = Dx + DEy = y$. □

Proof of Theorem 3.2. For the trivial solution $D = 0$, the result is obviously true. Also, for any non-zero matrix $D \in \mathscr{D}$, let

$$(S_1, S_2) = \Theta(D) \in \mathscr{I}, \quad T_1 = \left[T_{11} | T_{12} \right], \quad T_2 = \left[T_{21} | T_{22} \right]$$

be orthogonal matrices satisfying (3.11). Then, by multiplying Eq. (3.25) on the left by T_2^{T} and on the right by T_1, we get

$$\begin{bmatrix} T_{21}^{\mathrm{T}} E T_{11} & T_{21}^{\mathrm{T}} E T_{12} \\ T_{22}^{\mathrm{T}} E T_{11} & T_{22}^{\mathrm{T}} E T_{12} \end{bmatrix} \begin{bmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{bmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} \begin{bmatrix} T_{21}^{\mathrm{T}} E T_{11} & T_{21}^{\mathrm{T}} E T_{12} \\ T_{22}^{\mathrm{T}} E T_{11} & T_{22}^{\mathrm{T}} E T_{12} \end{bmatrix} + \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = 0, \tag{3.28}$$

so that the unique non-singular solution $\varDelta$ of (3.14) corresponding to (S_1, S_2) is given by $\varDelta = T_{22}^{\mathrm{T}} E T_{12}$. Recalling, now, that the associated solution D of (3.5) has been computed in (3.23) as $D = T_{12}\varDelta^{-1}T_{22}^{\mathrm{T}}$, we can verify in a direct way that $DED = T_{12}\varDelta^{-1}T_{22}^{\mathrm{T}}ET_{12}\varDelta^{-1}T_{22}^{\mathrm{T}} = D$.

Thus, we may apply Lemma 3.1, with

$$\mathscr{L} = \varphi(D) = \operatorname{im}\begin{bmatrix} I \\ D \end{bmatrix} \quad \text{and} \quad (S_1, S_2) = \varTheta(D) = (\ker D^{\mathrm{T}}, \ker D).$$

The proof is now complete. □

Finally, letting the pair (S_1, S_2) vary into $\mathscr{I} = \varTheta(\mathscr{D})$, we obtain the following representation of the set $\mathscr{L}(H_1) = \varphi(\mathscr{D})$.

Corollary 3.1. *The set $\mathscr{L}(H_1)$ of graph invariant subspaces of the pseudo-Hamiltonian matrix H_1 can be parametrized by the set $\mathscr{I}$ of pairs of subspaces defined in* (3.18), *as*

$$\mathscr{L}(H_1) = \left\{ V\mathscr{M} : \ \mathscr{M} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \ x \in S_2, \ y \in S_1^{\perp} \right\}, \ (S_1, S_2) \in \mathscr{I} \right\}. \tag{3.29}$$

3.3. *Symmetric algebraic Riccati equations*

If $A_1 = A_2^{\mathrm{T}}$, $P = P^{\mathrm{T}}$ and $Q = Q^{\mathrm{T}}$, then (3.2) is a standard symmetric ARE. If, moreover, the reference solution X_0 is also symmetric, then $\varGamma_{20} = -\varGamma_{10}^{\mathrm{T}}$ and (3.5) is a symmetric HARE. In this case, if D is a symmetric solution, then $\ker D = \ker D^{\mathrm{T}} =: S$, so that $\varTheta(D) = (S, S) \in \mathscr{I}$. Notice also that one can take $T_1 = T_2$ in (3.12) and (3.13), yielding $M_{22} = -N_{22}^{\mathrm{T}}$ and $L_{22} = L_{22}^{\mathrm{T}}$ and hence turning (3.14) into a Lyapunov equation. Conversely, it is easy to check that, if (3.5) is symmetric and S is a $\varGamma_{20}$-invariant subspace, such that (3.14) has a non-singular solution, i.e., $(S, S) \in \mathscr{I}$, then the corresponding solution D is a symmetric matrix. In this way, we can recover the classical parametrization of symmetric solutions of symmetric AREs, as a function of a single invariant subspace S. To all other pairs $(S_1, S_2) \in \mathscr{I}$, with $S_1 \neq S_2$, there correspond asymmetric solutions, even if the ARE is a symmetric one.

Let us consider the following example:

$$A^{\mathrm{T}}X + XA + XPX + Q = 0 \tag{3.30}$$

with

$$A = A^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.31}$$

Let $X_0 = 0$, so that $\Gamma_{10} = A$ and $\Gamma_{20} = -A$, satisfying the standing assumption (3.4). There are four different Γ_{20}-invariant subspaces, namely

$$S_0 = \mathbb{R}^2, \quad S_1 = \operatorname{im}\begin{bmatrix}1\\0\end{bmatrix}, \quad S_2 = \operatorname{im}\begin{bmatrix}0\\1\end{bmatrix}, \quad S_3 = \{0\}.$$

However, the symmetric solutions of (3.30) are only two,

$$X_0 = D_0 = 0 = \Theta^{-1}(S_0, S_0) \quad \text{and} \quad X_3 = D_3 = \begin{bmatrix}0 & -3\\-3 & 0\end{bmatrix} = \Theta^{-1}(S_3, S_3).$$

In fact, for $i = 1, 2$, the pair $(S_i, S_i) \notin \mathscr{I}$, since the unique solution of the associated Lyapunov equation (3.14) is singular.

Moreover, the set $\mathscr{I}$ contains the pairs (S_1, S_2) and (S_2, S_1), corresponding to the two asymmetric solutions of the symmetric ARE (3.30). In fact, let

$$T_1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \qquad T_2 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \tag{3.32}$$

and compute

$$N_{22} = 2, \quad M_{22} = -1, \quad L_{22} = 1. \tag{3.33}$$

Now, solving (3.14), one gets $\Delta = -1/3$, yielding

$$X_1 = D_1 = \Theta^{-1}(S_1, S_2) = \begin{bmatrix}0\\1\end{bmatrix}\left(-\frac{1}{3}\right)^{-1}\begin{bmatrix}1 & | & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\-3 & 0\end{bmatrix}. \tag{3.34}$$

The other asymmetric solution of (3.30) is

$$X_2 = D_2 = \Theta^{-1}(S_2, S_1) = \begin{bmatrix}0 & -3\\0 & 0\end{bmatrix} = X_1^{\mathrm{T}}. \tag{3.35}$$

4. Continuity results

We have established in Theorem 3.1 a parametrization of the set $\mathscr{D}$ of solutions of the HARE (3.5), in terms of elements of the set $\mathscr{I}$ defined in (3.18), by means of the map Θ given by (3.19). We now endow the set $\mathscr{I}$ with the metric d_I, which associates to any pair $I = (S_1, S_2)$, $\bar{I} = (\bar{S}_1, \bar{S}_2)$ the distance $d_I(I, \bar{I}) := d(S_1, \bar{S}_1) + d(S_2, \bar{S}_2)$, and with the induced topology. Moreover, the set $\mathscr{D}$ is endowed with the topology induced by the matrix norm $\|\cdot\|$.

Theorem 4.1. *Let the set $\mathscr{D}$ and the set $\mathscr{I}$ be endowed with the topologies defined above. Then, the map Θ is a homeomorphism.*

Proof. As mentioned in Section 1, the map

$$\begin{aligned}\varphi : \mathscr{D} &\to \mathscr{L}(H_1)\\ D &\mapsto \operatorname{im}\begin{bmatrix}I\\D\end{bmatrix}\end{aligned} \tag{4.1}$$

is a homeomorphism, when the set $\mathscr{D}$ is endowed with the topology induced by the matrix norm $\|\cdot\|$ and the set $\mathscr{L}(H_1)$ with the topology induced by the gap metric [21, pp. 545–546].

The result then follows by composition, writing $\Theta = \Xi^{-1} \circ \varphi$, if we show that the map

$$\begin{aligned} &\Xi : \mathscr{I} \to \mathscr{L}(H_1) \\ &\quad (S_1, S_2) \mapsto V \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \ x \in S_2, \ y \in S_1^{\perp} \right\} \end{aligned} \tag{4.2}$$

with V given by (3.26) is also a homeomorphism. This is, in turn, an immediate consequence of the following facts:

(i) V is a constant non-singular matrix;
(ii) $d(S, \bar{S}) \to 0$ if and only if $d(S^{\perp}, \bar{S}^{\perp}) \to 0$. □

The above proof is based on the connection between the two parametrizations of $\mathscr{D}$, provided by Theorem 3.2, and on well-known facts about the continuity of φ. A direct proof, which does not employ the parametrization based on graph invariant subspaces, is derived in Appendix A.

5. Conclusions

In this paper, we have considered the algebraic Riccati equation with no symmetry constraint. A parametrization of the solution set of such an equation has been obtained, in terms of pairs of linear subspaces. In fact, by choosing a particular solution of the original equation, an equivalent homogeneous ARE has been derived. We have then proved that the solution set of such HARE is in a one-to-one correspondence with a subset of pairs of Γ_{10}^{T}- and Γ_{20}-invariant subspaces, respectively, where Γ_{10} and Γ_{20} denote two related “feedback” matrices. When the natural topologies are used for the set of real matrices and the set of (pairs of) linear subspaces, such a parametrization turns out to be a homeomorphic map. Both the parametrization and its homeomorphic characterization appear to be the extension of classical results to the non-symmetric case. As a particular example, the newly introduced parametrization allows one to classify in a natural way asymmetric solutions of symmetric AREs.

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Appendix A. A direct proof of the continuity of Θ and Θ^{-1}

For a direct proof of Theorem 4.1, we need to establish some preliminary results. First, we refer to [21, Theorem 13.5.1] for the proof of the next lemma.

Lemma A.1. *Let $D \in \mathbb{R}^{n\times m}$. There exists a constant $K \geqslant 0$, such that*

$$d(\ker D, \ker \bar{D}) \leqslant K\|D - \bar{D}\| \tag{A.1}$$

for all $\bar{D} \in \mathbb{R}^{n\times m}$, with $\operatorname{rank} \bar{D} = \operatorname{rank} D$.

Now, let $\mathscr{P}_n := \{(N, M, L) : \ N, M, L \in \mathbb{R}^{n\times n}, \ \sigma(N) \cap \sigma(M) = \emptyset\}$. Then, the following lemma holds true.

Lemma A.2. *For any $(N, M, L) \in \mathscr{P}_n$, the Sylvester equation*

$$\varDelta N - M\varDelta + L = 0 \tag{A.2}$$

has a unique solution $\varDelta$, which is a continuous function of the coefficients (N, M, L). If $\varDelta$ is non-singular, the inverse $\varDelta^{-1}$ is a continuous function of (N, M, L), as well.

Proof. Existence and uniqueness of the solution are well-known facts. Continuity of $\varDelta$ is easy to check, by writing the solution of (A.2) as a contour integral [6, p. 206]. Continuity of $\varDelta^{-1}$ is obvious. □

Proposition A.1. *Let $Y \in \mathbb{R}^{n\times p}$ and $m = \operatorname{rank} Y$. Partition Y as*

$$Y = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{21} & Y_2 \end{bmatrix}, \tag{A.3}$$

where $Y_2 \in \mathbb{R}^{m\times m}$, and let $n_1 = n - m$ and $p_1 = p - m$ be the number of rows and columns, respectively, of Y_1. Moreover, let U and V be orthogonal matrices of dimension $p \times p$ and $n \times n$, respectively, such that the first p_1 columns of U are a basis for $\ker Y$ *and the first n_1 columns of V^{T} are a basis for* $\ker Y^{\mathrm{T}}$:

$$U = \begin{bmatrix} U_1 & U_{12} \\ U_{21} & U_2 \end{bmatrix}, \quad U^{\mathrm{T}}U = I, \quad \ker Y = \operatorname{im} \begin{bmatrix} U_1 \\ U_{21} \end{bmatrix}, \tag{A.4a}$$

$$V = \begin{bmatrix} V_1 & V_{12} \\ V_{21} & V_2 \end{bmatrix}, \quad V^{\mathrm{T}}V = I, \quad \ker Y^{\mathrm{T}} = \operatorname{im} \begin{bmatrix} V_1^{\mathrm{T}} \\ V_{12}^{\mathrm{T}} \end{bmatrix}, \tag{A.4b}$$

where the partition of U and V is such that U_2 and V_2 are square matrices of dimension m. Finally, let

$$Z = \begin{bmatrix} 0 & 0 \\ 0 & Z_2 \end{bmatrix} \tag{A.5}$$

be an $n \times p$ matrix, with Z_2 being a non-singular square matrix of dimension m. Then, we have:

$$d(\ker Y, \ker Z) = \|U_{12}\| = \|U_{21}\|, \tag{A.6a}$$

$$d(\ker Y^{\mathrm{T}}, \ker Z^{\mathrm{T}}) = \|V_{12}\| = \|V_{21}\|. \tag{A.6b}$$

If, moreover, $\|U_{12}\| < 1/2$ *and* $\|V_{21}\| < 1/2$, *then* U_2 *and* V_2 *are non-singular and, setting*

$$T_R := (U_{12}U_2^{-1})^{\mathrm{T}}, \qquad T_L := (V_2^{-1}V_{21})^{\mathrm{T}}, \tag{A.7}$$

we have:

$$\|U_{12}\| \leqslant \|T_R\| \leqslant \sqrt{2}\|U_{12}\|, \tag{A.8a}$$

$$\|V_{21}\| \leqslant \|T_L\| \leqslant \sqrt{2}\|V_{21}\|, \tag{A.8b}$$

and

$$Y = \begin{bmatrix} T_L \\ I \end{bmatrix} Y_2 \begin{bmatrix} T_R & I \end{bmatrix}, \tag{A.9}$$

where the matrix Y_2 *is non-singular. Finally, the inequality*

$$\|Y - Z\| \leqslant \Big[\|T_L\| \cdot \|T_R\| + \max\{\|T_L\|,\ \|T_R\|\}\Big] \cdot \|Y_2\| + \|Y_2 - Z_2\| \tag{A.10}$$

holds.

Proof. Equality (A.6a) is proven in [42, Lemma 3.1]. Equality (A.6b) is the transposed version of (A.6a). The fact that, if $\|U_{12}\| < 1/2$, then U_2 is non-singular and inequality (A.8a) holds, is proven in [42, Lemma 4.1] under slightly stronger assumptions. These assumptions, however, are not used there to establish (A.8a), but only to derive other results of that lemma. Inequality (A.8b) is the transposed version of (A.8a).

To prove that the factorization (A.9) holds, we observe that *VYU* has the block-diagonal form diag$\{0, \Lambda\}$, where Λ is non-singular. Hence,

$$\begin{aligned} Y &= V^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix} U^{\mathrm{T}} \\ &= \begin{bmatrix} V_{21}^{\mathrm{T}} \\ V_2^{\mathrm{T}} \end{bmatrix} \Lambda \begin{bmatrix} U_{12}^{\mathrm{T}} & U_2^{\mathrm{T}} \end{bmatrix} \\ &= \begin{bmatrix} T_L \\ I \end{bmatrix} V_2^{\mathrm{T}} \Lambda U_2^{\mathrm{T}} \begin{bmatrix} T_R & I \end{bmatrix}, \end{aligned} \tag{A.11}$$

which clearly implies $Y_2 = V_2^{\mathrm{T}}\Lambda U_2^{\mathrm{T}}$, and hence (A.9). Moreover, since V_2, Λ and U_2 are non-singular matrices, so is Y_2. Finally, (A.10) readily follows from the following decomposition:

$$Y - Z = \begin{bmatrix} T_L Y_2 T_R & T_L Y_2 \\ Y_2 T_R & Y_2 - Z_2 \end{bmatrix}$$

$$= \begin{bmatrix} T_L Y_2 T_R & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & T_L Y_2 \\ Y_2 T_R & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Y_2 - Z_2 \end{bmatrix}. \qquad \square \quad \text{(A.12)}$$

We now come to a direct proof of Theorem 4.1, alternative to the one given in Section 4.

Proof of Theorem 4.1. To show that Θ is continuous, we first prove that if D and $\bar{D}$ solve (3.5), with rank $\bar{D} <$ rank D, then there exists $\delta > 0$, only depending on the coefficients Γ_{10}, Γ_{20} and P, and on the rank l of D, such that

$$\|\bar{D} - D\| \geqslant \delta. \tag{A.13}$$

To this aim, set $S_1 = \ker D^{\mathrm{T}}$ and $S_2 = \ker D$. Let T_1 and T_2 be defined as in (3.10) and (3.11), and N, M and L be defined and partitioned as in (3.12) and (3.13). Then, as given by (3.23),

$$T_1^{\mathrm{T}} D T_2 = \begin{bmatrix} 0 & 0 \\ 0 & \Delta^{-1} \end{bmatrix},$$

where $\Delta \in \mathbb{R}^{l \times l}$ is the unique solution of (3.14). Now, let

$$T_1^{\mathrm{T}} \bar{D} T_2 = \begin{bmatrix} \bar{D}_1 & \bar{D}_{12} \\ \bar{D}_{21} & \bar{D}_2 \end{bmatrix}$$

be partitioned conformably with $T_1^{\mathrm{T}} D T_2$. Note that rank $\bar{D} <$ rank D implies that $\bar{D}_2$ is a singular matrix. Then, we have

$$\|D - \bar{D}\| = \|T_1^{\mathrm{T}}(D - \bar{D})T_2\| \geqslant \|\Delta^{-1} - \bar{D}_2\| \geqslant s_m(\Delta^{-1}) = \frac{1}{\|\Delta\|}, \tag{A.14}$$

where the last inequality derives from the fact that $\bar{D}_2$ is singular.

Now, let γ be a smooth closed contour in the complex plane, leaving $\sigma(\Gamma_{10})$ and a fortiori $\sigma(N_{22})$ in its interior, while $\sigma(\Gamma_{20})$ and hence $\sigma(M_{22})$ are outside of γ. Recalling that Δ solves (3.14) and employing Rosemblum's formula [6, p. 206], we have

$$\Delta = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma} (M_{22} - zI)^{-1} L_{22} (N_{22} - zI)^{-1}\, \mathrm{d}z. \tag{A.15}$$

Then, setting $g_1 = \max\{\|(zI - \Gamma_{10})^{-1}\| : z \in \gamma\} \geqslant \max\{\|(zI - N_{22})^{-1}\| : z \in \gamma\}$ and $g_2 = \max\{\|(zI - \Gamma_{20})^{-1}\| : z \in \gamma\} \geqslant \max\{\|(zI - M_{22})^{-1}\| : z \in \gamma\}$, we have

$$\|\Delta\| \leqslant \frac{|\gamma|}{2\pi} \|P\| g_1 g_2 \tag{A.16}$$

with $|\gamma|$ being the length of γ. Therefore, taking into account (A.14), inequality (A.13) holds with

$$\delta = \left(\frac{|\gamma|}{2\pi} \|P\| g_1 g_2 \right)^{-1}. \tag{A.17}$$

Clearly, δ is positive and it depends only on the coefficients Γ_{10}, Γ_{20} and P.

In particular, inequality (A.13) implies that if two solutions D and $\bar{D}$ of (3.5) are sufficiently close, then they have the same rank. Therefore, in view of Lemma A.1, there exists $K > 0$, such that $d(\ker D, \ker \bar{D}) \leqslant K\|D - \bar{D}\|$ and $d(\ker D^{\mathrm{T}}, \ker \bar{D}^{\mathrm{T}}) \leqslant K\|D - \bar{D}\|$, yielding

$$d_I(\Theta(D), \Theta(\bar{D})) \leqslant 2K\|D - \bar{D}\| \tag{A.18}$$

and, hence, the continuity of the map Θ.

We now prove that the inverse map Θ^{-1} is continuous, as well. Let $(\bar{S}_1, \bar{S}_2) \in \mathscr{I}$ and $\bar{T}_1, \bar{T}_2$ be orthogonal matrices, such that

$$\bar{T}_1 = \left[\bar{T}_{11}|\bar{T}_{12}\right], \quad \operatorname{im} \bar{T}_{11} = \bar{S}_1, \quad \bar{T}_2 = \left[\bar{T}_{21}|\bar{T}_{22}\right], \quad \operatorname{im} \bar{T}_{21} = \bar{S}_2. \tag{A.19}$$

The matrices $\bar{N} := \bar{T}_1^{\mathrm{T}} \Gamma_{10} \bar{T}_1$ and $\bar{M} := \bar{T}_2^{\mathrm{T}} \Gamma_{20} \bar{T}_2$ have the same block-triangular structure as N and M in (3.12). Thus, we can write

$$\begin{aligned} \bar{N} &= \begin{bmatrix} \bar{N}_{11} & 0 \\ \bar{N}_{21} & \bar{N}_{22} \end{bmatrix}, \quad \bar{M} = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ 0 & \bar{M}_{22} \end{bmatrix}, \\ \bar{L} &:= \bar{T}_2^{\mathrm{T}} P \bar{T}_1 = \begin{bmatrix} \bar{L}_{11} & \bar{L}_{12} \\ \bar{L}_{21} & \bar{L}_{22} \end{bmatrix}, \end{aligned} \tag{A.20}$$

where $\bar{N}_{22}$, $\bar{M}_{22}$ and $\bar{L}_{22}$ are square matrices with the same dimensions.

Now, let $\bar{D} = \Theta^{-1}(\bar{S}_1, \bar{S}_2)$ be the corresponding solution of (3.5). Then, the matrix

$$Z := \bar{T}_1^{\mathrm{T}} \bar{D} \bar{T}_2 \in \mathbb{R}^{n_1 \times n_2} \tag{A.21}$$

solves the equation

$$\bar{N}Z - Z\bar{M} + Z\bar{L}Z = 0. \tag{A.22}$$

Since Z may be partitioned as

$$Z = \begin{bmatrix} 0 & 0 \\ 0 & Z_2 \end{bmatrix},$$

Z_2 is the inverse of the unique solution to the Sylvester equation

$$\Delta \bar{N}_{22} - \bar{M}_{22} \Delta + \bar{L}_{22} = 0. \tag{A.23}$$

We observe that

$$\bar{S}_1 = \ker \bar{D}^{\mathrm{T}} = \bar{T}_1 \ker Z^{\mathrm{T}}, \quad \bar{S}_2 = \ker \bar{D} = \bar{T}_2 \ker Z. \tag{A.24}$$

Consider now a second pair $(S_1, S_2) \in \mathscr{I}$ and let $D = \Theta^{-1}(S_1, S_2)$ be the corresponding solution of (3.5). Moreover, define the matrix

$$Y = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{21} & Y_2 \end{bmatrix} := \bar{T}_1^{\mathrm{T}} D \bar{T}_2, \tag{A.25}$$

partitioned conformably with Z. Clearly, also Y is a solution of (A.22) and

$$S_1 = \ker D^{\mathrm{T}} = \bar{T}_1 \ker Y^{\mathrm{T}}, \quad S_2 = \ker D = \bar{T}_2 \ker Y. \tag{A.26}$$

Thus, by orthogonality of $\bar{T}_1$ and $\bar{T}_2$, we have

$$d(\bar{S}_1, S_1) = d(\ker Z^{\mathrm{T}}, \ker Y^{\mathrm{T}}), \quad d(\bar{S}_2, S_2) = d(\ker Z, \ker Y). \tag{A.27}$$

Therefore,

$$d_I((S_1, S_2), (\bar{S}_1, \bar{S}_2)) \to 0 \;\Rightarrow\; \begin{cases} d(\ker Z^{\mathrm{T}}, \ker Y^{\mathrm{T}}) \to 0, \\ d(\ker Z, \ker Y) \to 0. \end{cases} \tag{A.28}$$

Now, with reference to Y given by (A.25) and Z given by (A.21), define V and U as in (A.4a) and (A.4b), respectively. If $d_I((S_1, S_2), (\bar{S}_1, \bar{S}_2)) \leqslant 1/2$, then both $d(\ker Z, \ker Y) \leqslant 1/2$ and $d(\ker Z^{\mathrm{T}}, \ker Y^{\mathrm{T}}) \leqslant 1/2$. We can then apply Proposition A.1 and obtain

$$Y = \begin{bmatrix} T_L \\ I \end{bmatrix} Y_2 \begin{bmatrix} T_R & I \end{bmatrix}, \tag{A.29}$$

where T_R and T_L are defined as in (A.7) and Y_2 is non-singular.

Since Y solves (A.22), we have

$$\bar{N}Y - Y\bar{M} + Y\bar{L}Y = 0. \tag{A.30}$$

By multiplying this equation by $\begin{bmatrix} 0 \\ I \end{bmatrix}$ on the right-hand side and by $\begin{bmatrix} 0 & I \end{bmatrix}$ on the left-hand side, and taking into account the factorization (A.29) of Y, we get

$$\begin{aligned} &\begin{bmatrix} \bar{N}_{21} & \bar{N}_{22} \end{bmatrix} \begin{bmatrix} T_L \\ I \end{bmatrix} Y_2 - Y_2 \begin{bmatrix} T_R & I \end{bmatrix} \begin{bmatrix} \bar{M}_{12} \\ \bar{M}_{22} \end{bmatrix} \\ &\quad + Y_2 \begin{bmatrix} T_R & I \end{bmatrix} \bar{L} \begin{bmatrix} T_L \\ I \end{bmatrix} Y_2 = 0. \end{aligned} \tag{A.31}$$

Finally, we have

$$\tilde{N}_{22}Y_2 - Y_2\tilde{M}_{22} + Y_2\tilde{L}_{22}Y_2 = 0, \tag{A.32}$$

where we have defined

$$\tilde{N}_{22} := \begin{bmatrix} \bar{N}_{21} & \bar{N}_{22} \end{bmatrix} \begin{bmatrix} T_L \\ I \end{bmatrix} = \bar{N}_{21}T_L + \bar{N}_{22}, \tag{A.33a}$$

$$\tilde{M}_{22} := \begin{bmatrix} T_R & I \end{bmatrix} \begin{bmatrix} \bar{M}_{12} \\ \bar{M}_{22} \end{bmatrix} = T_R\bar{M}_{12} + \bar{M}_{22}, \tag{A.33b}$$

$$\tilde{L}_{22} := \begin{bmatrix} T_R & I \end{bmatrix} \bar{L} \begin{bmatrix} T_L \\ I \end{bmatrix} = T_R\bar{L}_{11}T_L + \bar{L}_{21}T_L + T_R\bar{L}_{12} + \bar{L}_{22}. \tag{A.33c}$$

We observe that, in view of (A.6a), (A.8a) and (A.28), if $d_I((S_1, S_2), (\bar{S}_1, \bar{S}_2)) \to 0$, then both $\|T_R\| \to 0$ and $\|T_L\| \to 0$, so that

$$\|\tilde{N}_{22} - \bar{N}_{22}\| \to 0, \qquad \|\tilde{M}_{22} - \bar{M}_{22}\| \to 0, \qquad \|\tilde{L}_{22} - \bar{L}_{22}\| \to 0. \tag{A.34}$$

This, in particular, implies that each of the eigenvalues of $\tilde{N}_{22}$ tends to one of the eigenvalues of $\bar{N}_{22}$ and each of the eigenvalues of $\tilde{M}_{22}$ tends to one of the eigenvalues of $\bar{M}_{22}$. Thus, since, by assumption, $\sigma(\bar{N}_{22}) \cap \sigma(\bar{M}_{22}) = \emptyset$, if $d_I((S_1, S_2), (\bar{S}_1, \bar{S}_2))$ is sufficiently small, we also have $\sigma(\tilde{N}_{22}) \cap \sigma(\tilde{M}_{22}) = \emptyset$, so that the Sylvester equation

$$\Delta \tilde{N}_{22} - \tilde{M}_{22}\Delta + \tilde{L}_{22} = 0 \tag{A.35}$$

has a unique solution $\tilde{\Delta}$. By continuity with respect to the coefficients of (A.35) (see Lemma A.2), $\tilde{\Delta} \to \Delta = Z_2^{-1}$. On the other hand, Eq. (A.32) has a unique non-singular solution Y_2. Hence, Y_2^{-1} solves (A.35) and therefore $Y_2 = \tilde{\Delta}^{-1}$. Thus, $Y_2 \to Z_2$, where Z_2 is the inverse of the unique solution of Eq. (A.23).

Therefore, employing (A.10), as $d_I((S_1, S_2), (\bar{S}_1, \bar{S}_2)) \to 0$, $\|Y - Z\| \to 0$. This implies, by the definition of Y and Z, that $\|D - \bar{D}\| \to 0$, which completes the proof. □

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