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ALMOST SURE STABILIZABILITY OF CONTROLLED DEGENERATE DIFFUSIONS*

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Abstract. We develop a direct Lyapunov method for the almost sure open-loop stabilizability and asymptotic stabilizability of controlled degenerate diffusion processes. The infinitesimal decrease condition for a Lyapunov function is a new form of Hamilton–Jacobi–Bellman partial differential inequality of second order. We give local and global versions of the first and second Lyapunov theorems, assuming the existence of a lower semicontinuous Lyapunov function satisfying such an inequality in the viscosity sense. An explicit formula for a stabilizing feedback is provided for affine systems with smooth Lyapunov function. Several examples illustrate the theory.

Key words. degenerate diffusion, almost sure stability, asymptotic stability, asymptotic controllability, stabilizability, stochastic control, viability, viscosity solutions, Hamilton–Jacobi–Bellman inequalities, nonsmooth analysis

AMS subject classifications. 93E15, 49L25, 93D05, 93D20

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1. Introduction.

For controlled diffusion processes in $\mathbb{R}^N$,

$$(CSDE)\begin{cases} dX_t = f(X_t,\alpha_t)dt + \sigma(X_t,\alpha_t)dB_t, & \alpha_t \in A, \quad t > 0, \\ X_0 = x, \end{cases}$$

there are various possible notions of Lyapunov stability of an equilibrium, say, the origin. The stability in probability has been studied for a long time; we recall here the contributions of Kushner [31, 32], Has'minskii [26], and the recent book of Mao [36] for uncontrolled systems, and the work of Florchinger [21, 22, 23] and Deng, Krstić, and Williams [18] on feedback stabilization for $(CSDE)$; see also the references therein. The almost sure exponential stability was introduced and studied by Kozin [29] (see also [26]), and it implies that, for each fixed sample in a set of probability 1, the (uncontrolled) system is exponentially stable in the usual sense. In this paper we consider a property that we call almost sure stability, or uniform stability with probability 1. For an uncontrolled system it says that for any $\eta > 0$ there exists $\delta > 0$ such that, for any x with $|x| \le \delta$, the process satisfies $|X_t| \le \eta$ for all $t \ge 0$ almost surely (a.s.). Equivalently, for some increasing, continuous function γ null at 0, and for small $|x|$,

$$|X_t| \le \gamma(|x|) \quad \forall t \ge 0 \ \text{ a.s.} \tag{1.1}$$

This property describes a behavior very similar to a stable deterministic system. It is stronger than stability in probability and pathwise stability and, in fact, it is never verified by a nondegenerate process. More precisely, we study the *almost sure*

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(stochastic open-loop) stabilizability of $(CSDE)$, namely, that for each x as above there exists an admissible control function whose trajectory $\overline{X}_.$ verifies a.s. $|\overline{X}_t| \leq \eta$ (and $|\overline{X}_t| \leq \gamma(|x|)$) for all t. If, in addition, $\lim_{t\to+\infty} \overline{X}_t = 0$ a.s., we say the system is *a.s. (stochastic open-loop) asymptotically stabilizable.* For deterministic systems ($\sigma \equiv 0$) the last property reduces to the well-known *asymptotic controllability.*

We follow the Lyapunov direct method and find that the *infinitesimal decrease condition* to be satisfied by a Lyapunov function V for our problem is

$$\max_{\alpha\in A,\, \sigma(x,\alpha)^T DV(x)=0} \left\{-DV(x)\cdot f(x,\alpha) - trace\left[a(x,\alpha)D^2V(x)\right]\right\} \geq l(x), \tag{1.2}$$

with $l \geq 0$ for mere Lyapunov stability and $l > 0$ for $x \neq 0$ for asymptotic stability, where $a := \sigma\sigma^T/2$. This is not a standard Hamilton–Jacobi–Bellman inequality, because the constraint on the control α depends on V. In fact it should be viewed rather as a system of PDEs and inequalities which, in the special case of uncontrolled diffusion, i.e., $\sigma = \sigma(x)$, reads

$$\begin{cases} \max_{\alpha\in A}\{-DV(x)\cdot f(x,\alpha)\} - trace\left[a(x)D^2V(x)\right] \geq l(x), \\ \sigma_i(x)\cdot DV(x) = 0 \quad \forall\, i, \end{cases} \tag{1.3}$$

where σ_i denotes the ith column of the matrix σ. To motivate the infinitesimal decrease condition (1.3), let us give a formal argument in the case V is of class $\mathcal{C}^2$. By applying Ito's formula to the inequality $dV(X_t)/dt \leq l(X_t)$, we get

$$\left[DV(X_t)\cdot f(X_t,\alpha_t) + trace\left(a(X_t)D^2V(X_t)\right)\right]dt + \sigma^T(X_t)DV(X_t)dB_t \leq l(X_t).$$

Now the properties of the Brownian motion lead to the conditions

$$\begin{aligned} DV(X_t)\cdot f(X_t,\alpha_t) + trace\left(a(X_t)D^2V(X_t)\right) &\leq l(X_t), \\ \sigma^T(X_t)DV(X_t) &= 0, \end{aligned}$$

and the existence of a control α_t verifying this is clearly related to (1.3). A more detailed, yet still formal, derivation of (1.3) is the following. The Dynkin formula gives, for any control,

$$\mathbf{E}V(X_t) - V(x) = \mathbf{E}\int_0^t \left[DV(X_s)\cdot f(X_s,\alpha_s) + trace(a(X_s)D^2V(X_s))\right]ds,$$

and from the inequality in (1.3) one argues the existence of a control function such that

$$\mathbf{E}V(X_t) - V(x) \leq -\mathbf{E}\int_0^t l(X_s)ds \leq 0. \tag{1.4}$$

Therefore, the process $V(X_t)$ is a positive supermartingale. Following this argument, it can be proved that a function satisfying merely the Hamilton–Jacobi–Bellman inequality in (1.3) is a Lyapunov function for the stability in probability. The additional equalities $\sigma_i(x)\cdot DV(x) = 0$ in (1.3) say that there is diffusion only in the directions tangential to the level sets of V, and they are necessary conditions for the invariance of the sublevel sets of V for the process ($CSDE$). It turns out that the whole set (1.3) of equalities and inequalities implies the weak invariance, or viability, of the sublevel sets of V, i.e., the existence of a control that maintains forever a.s. the system in such a set if the initial position is in the set. From this property it is possible to infer that, for some control,

$$V(X_t) - V(x) \leq -\int_0^t l(X_s)ds \leq 0 \quad \text{almost sure},$$

a stronger monotonicity-type property than (1.4), which allows us to prove the almost sure stability.

We define a Lyapunov function for the almost sure stability as a *lower semicontinuous* proper function V, continuous at 0 and satisfying (1.2) in the *viscosity sense*, and we call it a strict Lyapunov function if $l > 0$ off 0; see Definitions 2.3 and 2.4 below. Our main results are the natural extensions of the first and second Lyapunov theorems to the controlled diffusions:

> *The existence of a local Lyapunov function implies the almost sure (open-loop) stabilizability of (CSDE); a strict Lyapunov function implies the almost sure (open-loop) asymptotic stabilizability.*

The same proof provides their global versions as well: if V satisfies (1.2) in $\mathbb{R}^N \setminus \{0\}$, then $(CSDE)$ is also a.s. (open-loop) *Lagrange stabilizable*, i.e., for all initial points x there is a control such that (1.1) holds; moreover, if V is strict, then the system is *globally* a.s. (open-loop) asymptotically stabilizable. We also give sufficient conditions for the stability of viable (controlled invariant) sets more general than an equilibrium point, and for the a.s. exponential stability.

These facts are much easier to prove when the Lyapunov function is smooth, but this assumption is not necessary and would limit considerably their applicability. The nonexistence of smooth Lyapunov functions is well known in the deterministic case; see [30, 6] for stable uncontrolled systems, and see the surveys [43, 6] for asymptotically stable controlled systems. Here we give an example of an uncontrolled degenerate diffusion process that is a.s. stable but cannot have a continuous Lyapunov function (Example 1 in section 6). Moreover, in a companion paper [12] the second author proves a *converse Lyapunov theorem*, stating that any a.s. stabilizable system $(CSDE)$ has a lower semicontinuous (l.s.c.) local viscosity Lyapunov function.

All the results listed above refer to open-loop almost sure stabilizability. They raise the question of the existence of a stabilizing feedback. Here we give an answer only for affine systems with a smooth strict Lyapunov function. We adapt Sontag's method [41] to the stochastic setting and find an explicit formula for a feedback that renders the system a.s. asymptotically stable. The feedback stabilizability of controlled diffusions in the case of nonsmooth Lyapunov functions seems considerably harder and we are not aware of any paper on the subject.

In the last section we study some simple applications and examples. For instance, we consider a deterministic, asymptotically controllable system $\dot{X}_t = f(X_t, \alpha_t)$ with Lyapunov pair (V, L) and look for conditions on a stochastic perturbation that keep the system a.s. stabilizable with the same Lyapunov function V for some $l \leq L$.

Our proof of the first Lyapunov-type theorem is based on the observation that the infinitesimal decrease condition (1.2) has the rescaling property of the geometric PDEs arising in the level set approach to front propagation (see, e.g., [9, 40] and the references therein), and on a recent result of the first author and Jensen [11] on the viability, or controlled invariance, of general closed sets for controlled diffusions (see [3, 4] and the references therein for earlier work on viability for stochastic processes). For the second Lyapunov-type theorem we use also martingale inequalities and other properties of diffusions.

The first Lyapunov-type theorem on local almost sure stabilizability was announced in [8], where we presented the simpler proof for uncontrolled processes. In the forthcoming paper [13], the second author shows that the existence of an l.s.c. viscosity solution of the Hamilton–Jacobi–Bellman inequality,

$$\max_{\alpha\in A}\left\{-DV(x)\cdot f(x,\alpha)-trace\left[a(x,\alpha)D^2V(x)\right]\right\}\geq l(x),$$

implies the open-loop stabilizability in probability of $(CSDE)$. Converse theorems in this setting appears in the Ph.D. thesis [14] of the second author.

We conclude with some additional references. Nonsmooth Lyapunov functions for uncontrolled diffusion processes were studied by Ladde and Lakshmikantham [33] with Dini-type derivatives along sample paths, and by Aubin and Da Prato [5] by means of a stochastic contingent epiderivative. Recently, Arnold and Schmalfuss [1] gave an extension of Lyapunov's second method to random dynamical systems. Turning to deterministic controlled systems, we recall that Soravia [45] gave direct and inverse Lyapunov theorems for the open-loop stabilizability by means of viscosity solutions (in the more general context of differential games); Sontag and Sussmann [41, 44] did it for the asymptotic controllability (i.e., asymptotic open-loop stabilizability) by using Dini directional derivatives. Viscosity methods for stability problems were also used in [28, 46, 24]. There is a large literature on feedback stabilization: see [2, 42, 16], the surveys [43, 15, 6], and the references therein. We refer to [17, 7] for the basic theory of viscosity solutions, and to [34, 35, 9, 20, 48] for its applications to deterministic and stochastic optimal control.

The paper is organized as follows. In section 2 we give the main definitions and state the first and second Lyapunov-type theorems. Section 3 recalls some viability theory and then gives the proofs of the two main theorems. Section 4 covers feedback stabilization of affine systems with smooth Lyapunov functions. Section 5 contains some extensions to exponential stability, general equilibrium sets, and target problems. Section 6 is devoted to the examples.

2. Lyapunov functions for almost sure stabilizability and asymptotic stabilizability. We consider a controlled Ito stochastic differential equation,

$$(CSDE)\begin{cases} dX_t = f(X_t,\alpha_t)dt+\sigma(X_t,\alpha_t)dB_t, \quad t>0,\\ X_0=x,\end{cases}$$

where B_t is an M-dimensional Brownian motion. Throughout the paper we assume that f,σ are continuous functions defined in $\mathbb{R}^N\times A$, where A is a compact metric space, which take values, respectively, in $\mathbb{R}^N$ and in the space of $N\times M$ matrices, and satisfy

$$(2.1)\quad |f(x,\alpha)-f(y,\alpha)|+\|\sigma(x,\alpha)-\sigma(y,\alpha)\|\leq C|x-y| \quad \forall x,y\in\mathbb{R}^N, \quad \forall\alpha\in A.$$

We adopt the definition of admissible control function, or admissible system, of Haussmann and Lepeltier [27, Def. 2.2, p. 853]. For a given $x\in\mathbb{R}^N$ we denote by $\mathcal{A}_x$ the set of admissible control functions, by $\alpha_.$ its generic element (although it is not a standard function $\mathbb{R}\to A$), and by $X_.$ the corresponding solution of $(CSDE)$.

We define

$$a(x,\alpha):=\frac{1}{2}\sigma(x,\alpha)\sigma(x,\alpha)^T$$

and assume

$$\{(a(x,\alpha), f(x,\alpha)) \, : \, \alpha \in A\} \quad \text{is convex } \forall x \in \mathbb{R}^N. \tag{2.2}$$

DEFINITION 2.1 (almost sure stabilizability). *The system* $(CSDE)$ *is* a.s. (stochastic open-loop Lyapunov) stabilizable *at the origin if for every* $\eta > 0$ *there exists* $\delta > 0$ *such that, for any initial point* x *with* $|x| \leq \delta$*, there exists an admissible control function* $\overline{\alpha}_\cdot \in \mathcal{A}_x$ *whose corresponding trajectory* $\overline{X}_\cdot$ *verifies* $|\overline{X}_t| \leq \eta$ *for all* $t \geq 0$ *a.s.*

The system is a.s. (stochastic open-loop) Lagrange stabilizable*, or it has the property of* uniform boundedness of trajectories*, if for each* $R > 0$ *there is* $S > 0$ *such that for any initial point* x *with* $|x| \leq R$ *there exists an admissible control function* $\overline{\alpha}_\cdot \in \mathcal{A}_x$ *whose corresponding trajectory* $\overline{X}_\cdot$ *verifies* $|\overline{X}_t| \leq S$ *for all* $t \geq 0$ *a.s.*

Remark 1. The almost sure stabilizability implies that the origin is a *controlled equilibrium* of $(CSDE)$, i.e.,

$$\exists\, \overline{\alpha} \in A \, : \, f(0,\overline{\alpha}) = 0, \; \sigma(0,\overline{\alpha}) = 0.$$

In fact, the definition gives for any $\varepsilon > 0$ an admissible control such that the corresponding trajectory starting at the origin satisfies a.s. $|X_t| \leq \varepsilon$ for all t, so $\mathbf{E}_x \int_0^{+\infty} |X_t| e^{-\lambda t} dt \leq \frac{\varepsilon}{\lambda}$ for any $\lambda > 0$. Then $\inf_{\alpha_\cdot \in \mathcal{A}_x} \mathbf{E}_x \int_0^{+\infty} |X_t| e^{-\lambda t} dt = 0$. The convexity assumption (2.2) and an existence theorem for optimal controls [27] imply that the inf is attained, and the minimizing control produces a trajectory satisfying a.s. $|X_t| = 0$ for all $t \geq 0$. The conclusion follows from standard properties of stochastic differential equations.

Remark 2. As is common in the modern deterministic stability theory, the previous definitions can be reformulated in terms of the *comparison functions* introduced by Hahn [25]. We will use the class $\mathcal{K}$ of continuous functions $\gamma : [0,+\infty) \to [0,+\infty)$ strictly increasing and such that $\gamma(0) = 0$ and the class $\mathcal{K}_\infty$ of functions $\gamma \in \mathcal{K}$ such that $\lim_{r \to +\infty} \gamma(r) = +\infty$.

The system $(CSDE)$ is a.s. (open-loop) stabilizable at 0 if there exists $\gamma \in \mathcal{K}$ and $\delta_o > 0$ such that for any starting point x with $|x| \leq \delta_o$

$$\exists\, \overline{\alpha}_\cdot \in \mathcal{A}_x : \quad |\overline{X}_t| \leq \gamma(|x|) \quad \forall t \geq 0 \text{ a.s.}, \tag{2.3}$$

where $\overline{X}_t$ is the trajectory corresponding to $\overline{\alpha}_\cdot$. If (2.3) holds for some $\gamma \in \mathcal{K}_\infty$ and for all $x \in \mathbb{R}^N$, then the system is also a.s. (open-loop) Lagrange stabilizable.

DEFINITION 2.2 (almost sure asymptotic stabilizability). *The system* $(CSDE)$ *is* a.s. (stochastic open-loop) locally asymptotically stabilizable *(or* a.s. locally asymptotically controllable*) at the origin if for every* $\eta > 0$ *there exists* $\delta > 0$ *such that, for all* $|x| \leq \delta$*, there exists an admissible control function* $\overline{\alpha}_\cdot \in \mathcal{A}_x$ *whose corresponding trajectory* $\overline{X}_\cdot$ *verifies a.s.*

$$|\overline{X}_t| \leq \eta \quad \forall t \geq 0, \qquad \lim_{t \to +\infty} |\overline{X}_t| = 0.$$

The system is a.s. (stochastic open-loop) globally asymptotically stabilizable *(or* a.s. asymptotically controllable*) at the origin if there is* $\gamma \in \mathcal{K}_\infty$ *and for all* $x \in \mathbb{R}^N$ *there exists* $\overline{\alpha}_\cdot \in \mathcal{A}_x$ *whose trajectory* $\overline{X}_\cdot$ *satisfies a.s.*

$$|\overline{X}_t| \leq \gamma(|x|) \quad \forall t \geq 0, \qquad \lim_{t \to +\infty} |\overline{X}_t| = 0.$$

Next we give the appropriate definition of a Lyapunov function for the study of almost sure stabilizability. We recall the definition of the second order semijet of an l.s.c. function V at a point x:

$$\mathcal{J}^{2,-}V(x) := \Bigg\{ (p,Y) \in \mathbb{R}^N \times S(N) : \text{ for } y \to x$$
$$V(y) \geq V(x) + p \cdot (y-x) + \frac{1}{2}(y-x)\cdot Y(y-x) + o(|y-x|^2) \Bigg\}.$$

Definition 2.3 (control Lyapunov function). *Let $\mathcal{O} \subseteq \mathbb{R}^N$ be an open set containing the origin. A function $V : \mathcal{O} \to [0,+\infty)$ is a* control Lyapunov function *for the almost sure stability of $(CSDE)$ if*

(i) *V is lower semicontinuous;*

(ii) *V is continuous at 0 and* positive definite*, i.e., $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$;*

(iii) *V is* proper*, i.e., $\lim_{|x|\to+\infty} V(x) = +\infty$ or, equivalently, the level sets $\{x|V(x) \leq \mu\}$ are bounded for every $\mu \in [0,\infty)$;*

(iv) *for all $x \in \mathcal{O} \setminus \{0\}$ and $(p,Y) \in \mathcal{J}^{2,-}V(x)$ there exists $\overline{\alpha} \in A$ such that*

$$\sigma(x,\overline{\alpha})^T p = 0 \qquad \text{and} \qquad -p\cdot f(x,\overline{\alpha}) - trace\,[a(x,\overline{\alpha})Y] \geq 0. \tag{2.4}$$

Remark 3. The conditions (ii) and (iii) in the previous definition can be stated as

$$\exists\, \gamma_1, \gamma_2 \in \mathcal{K}_\infty \ : \ \gamma_1(|x|) \leq V(x) \leq \gamma_2(|x|)\ \forall\, x \in \mathbb{R}^N. \tag{2.5}$$

Therefore the level sets $\{V(x) \leq \mu\}$ of the Lyapunov function form a basis of neighborhoods of 0.

Remark 4. If the dispersion matrix σ does not depend on the control, then condition (iv) can be reformulated as follows:

V is a solution in viscosity sense in $\mathcal{O} \setminus \{0\}$ of the system

$$\begin{cases} \sigma(x)^T DV(x) = 0, \\ \max_{\alpha\in A} \left\{ -DV(x)\cdot f(x,\alpha) - trace\left[a(x,\alpha)D^2V(x)\right]\right\} \geq 0. \end{cases}$$

In the general case, we can observe that if condition (iv) holds, then V in particular is a viscosity supersolution of

$$\max_{\alpha\in A} \left\{ -DV(x)\cdot f(x,\alpha) - trace\left[a(x,\alpha)D^2V(x)\right]\right\} = 0. \tag{2.6}$$

Moreover, if the function V is at least differentiable, then condition (iv) can be stated more concisely as follows:

V is a supersolution in viscosity sense in $\mathcal{O} \setminus \{0\}$ of the equation

$$\max_{\{\alpha\in A \ |\ \sigma(x,\alpha)^T DV(x)=0\}} \left\{ -DV(x)\cdot f(x,\alpha) - trace\left[a(x,\alpha)D^2V(x)\right]\right\} = 0.$$

Definition 2.4 (strict control Lyapunov function). *A function $V : \mathcal{O} \to [0,+\infty)$ is a* strict control Lyapunov function *for the almost sure stability of $(CSDE)$ if it satisfies conditions* (i), (ii), (iii) *in Definition* 2.3 *and* (iv)$'$ *for all $x \in \mathcal{O} \setminus \{0\}$ and $(p,Y) \in \mathcal{J}^{2,-}V(x)$ there exists $\overline{\alpha} \in A$ such that*

$$\sigma^T(x,\overline{\alpha})p = 0 \quad \text{and} \quad -p\cdot f(x,\overline{\alpha}) - trace\,[a(x,\overline{\alpha})Y] - l(x) \geq 0 \tag{2.7}$$

for some positive definite and Lipschitz continuous $l : \mathcal{O} \to \mathbb{R}$.

Remark 5. In the inequality in $(\text{iv})'$ we could take

$$p \cdot f(x, \overline{\alpha}) - trace\,[a(x, \overline{\alpha})Y] - l(x, \overline{\alpha}) \geq 0$$

for some continuous $l : \mathcal{O} \times A \to \mathbb{R}$, Lipschitz continuous in x uniformly in α, with $l(x, A)$ convex for all $x \in \mathcal{O}$, and such that $\tilde{l}(x) := \min_{\alpha \in A} l(x, \alpha)$ is positive definite. However, this would not increase the generality of the definition because V would also satisfy condition (2.7) with l replaced by $\tilde{l}$.

Our main results are the following versions for stochastic controlled systems of the first and the second Lyapunov theorems.

THEOREM 2.5 (almost sure stabilizability). *Assume* (2.1), (2.2)*, and the existence of a control Lyapunov function* V*. Then*

(i) *the system* $(CSDE)$ *is a.s. stabilizable at the origin;*

(ii) *if, in addition, the domain* $\mathcal{O}$ *of* V *is all* $\mathbb{R}^N$*, the system is also a.s. Lagrange stabilizable, and for all* $x \in \mathbb{R}^N$ *there exists* $\overline{\alpha}_{\cdot} \in \mathcal{A}_x$ *such that the corresponding trajectory* $\overline{X}_{\cdot}$ *satisfies*

$$|\overline{X}_t| \leq \gamma_1^{-1}(\gamma_2(|x|)) \quad \forall\, t \geq 0 \quad a.s. \tag{2.8}$$

with $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ *verifying* (2.5).

THEOREM 2.6 (almost sure asymptotic stabilizability). *Assume* (2.1), (2.2)*, and the existence of a strict control Lyapunov function* V*. Then*

(i) *the system* $(CSDE)$ *is a.s. locally asymptotically stabilizable at the origin;*

(ii) *if, in addition, the domain* $\mathcal{O}$ *of* V *is all* $\mathbb{R}^N$*, the system is a.s. globally asymptotically stabilizable.*

3. A viability theorem and the proofs of stabilizability. In this section we prove Theorems 2.5 and 2.6. Our main tool is a recent result in [11] about the almost sure viability (called also *controlled invariance* and *weak invariance*) of an arbitrary closed set for a controlled diffusion process. (See [3, 4] and the references therein for earlier related results.)

DEFINITION 3.1 (viable set). *A closed set* $K \subset \mathbb{R}^N$ *is* viable *or* controlled invariant *or* weakly invariant *for the stochastic system* $(CSDE)$ *if for all initial points* $x \in K$ *there exists an admissible control* $\alpha_{\cdot} \in \mathcal{A}_x$ *such that the corresponding trajectory* $X_{\cdot}$ *satisfies* $X_t \in K$ *for all* $t > 0$ *a.s.*

It is easy to see from its definition that the almost sure stabilizability follows from the viability of all the sublevel sets of any function satisfying conditions (i)–(iii) of Definition 2.3. The next result gives a geometric characterization of viable sets. It will allow us to check that the sublevel sets of a control Lyapunov function are viable by means of condition (iv) in Definition 2.3. The Nagumo-type geometric condition in the viability theorem is given in terms of the following *second order normal cone* to a closed set $K \subset \mathbb{R}^N$, first introduced in [10]:

$$\mathcal{N}_K^2(x) := \Big\{(p, Y) \in \mathbb{R}^N \times S(N) : \text{ for } \ y \to x, \ \ y \in K, \\ p \cdot (y - x) + \frac{1}{2}(y - x) \cdot Y(y - x) \geq o(|y - x|^2)\Big\},$$

where $S(N)$ is the set of symmetric $N \times N$ matrices. Note that, if $(p, Y) \in \mathcal{N}_K^2(x)$ and $x \in \partial K$, the vector p is a generalized (proximal or Bony) interior normal to the set K at x. In particular, if ∂K is a smooth surface in a neighborhood of x, $p/|p|$ is the interior normal and Y is related to the second fundamental form of ∂K at x; see [10].

THEOREM 3.2 (viability theorem [11]). *Assume* (2.1) *and* (2.2). *Then a closed set $K \subseteq \mathbb{R}^N$ is viable for $(CSDE)$ if and only if*

$$\forall x \in \partial K,\ \forall (p,Y) \in \mathcal{N}_K^2(x),\ \exists \alpha \in A\ :\ \ f(x,\alpha)\cdot p + trace\,[a(x,\alpha)Y] \geq 0. \tag{3.1}$$

The second tool for the proof of the Lyapunov-type theorem, Theorem 2.5, is the following lemma on the change of unknown for second order PDEs. It says that the Hamilton–Jacobi–Bellman inequality in condition (2.4) in the definition of a control Lyapunov function behaves as a *geometric equation* if the unknown satisfies also the condition in (2.4) of orthogonality between its gradient and the columns of the dispersion matrix σ. We refer the interested reader to the chapters by Evans and Souganidis in the book [9] for an introduction to the geometric PDEs of the theory of front propagation.

LEMMA 3.3. *Let v satisfy condition* (2.4) *for all $(p,Y) \in \mathcal{J}^{2,-}V(x)$, $x \in \mathbb{R}^N\setminus\{0\}$. Let ϕ be a twice continuously differentiable strictly increasing real map. Then $w = \phi\circ v$ is a viscosity supersolution of*

$$\max_{\alpha\in A}\left\{-DV(x)\cdot f(x,\alpha) - trace\left[a(x,\alpha)D^2V(x)\right]\right\} = 0. \tag{3.2}$$

Proof. It is easy to check that, if $(p,Y) \in \mathcal{J}^{2,-}w(x)$, then

$$(\psi'(w(x))p, \psi'(w(x))Y + \psi''(w(x))p\otimes p) \in \mathcal{J}^{2,-}v(x),$$

where ψ is the inverse of ϕ and $p\otimes p$ is the $N\times N$ matrix whose (i,j) entry is p_ip_j. Then, for $(p,Y) \in \mathcal{J}^{2,-}w(x)$ and $x \neq 0$ there exists $\overline{\alpha}$ such that

$$\{-\psi'(w(x))p\cdot f(x,\overline{\alpha}) - trace\,[a(x,\overline{\alpha})\cdot(\psi'(w(x))Y + \psi''(w(x))p\otimes p)]\} \geq 0$$

and

$$trace\,[a(x,\overline{\alpha})\cdot\psi''(w(x))p\otimes p] = \frac{\psi''(w(x))}{(\psi'(w(x)))^2}|\sigma(x,\overline{\alpha})^T\psi'(w(x))p|^2 = 0.$$

Therefore

$$-\psi'(w(x))p\cdot f(x,\overline{\alpha}) - trace\,[a(x,\overline{\alpha})\cdot\psi'(w(x))Y] \geq 0$$

and we can conclude that

$$\sup_{\alpha\in A}\{-p\cdot f(x,\alpha) - trace\,[a(x,\alpha)\cdot Y]\} \geq 0. \qquad \square$$

Proof of Theorem 2.5. We begin with the proof of (ii). We fix an arbitrary $\mu > 0$ and consider the sublevel set of the function V,

$$K := \{x \mid V(x) \leq \mu\}.$$

We claim that K is viable. Then for all initial points $x \in \mathbb{R}^N$ there exists $\overline{\alpha}_\cdot \in \mathcal{A}_x$ such that the associated trajectory $\overline{X}_\cdot$ satisfies

$$\gamma_1(|\overline{X}_t|) \leq V(\overline{X}_t) \leq V(x) \leq \gamma_1(|x|) \quad \forall t \geq 0 \quad \text{a.s.},$$

which gives estimate (2.8). Then the system is a.s. stabilizable and Lagrange stabilizable because $\gamma_1^{-1} \circ \gamma_2 \in \mathcal{K}_\infty$.

To prove that K is viable we will check condition (3.1) of the viability theorem, Theorem 3.2. For a given $\lambda > 0$ we define the nondecreasing continuous real function

$$\psi_\lambda(t) = \begin{cases} 0, & t \leq \mu, \\ \lambda(t-\mu), & \mu \leq t, \leq \mu + \frac{1}{\lambda}, \\ 1, & t \geq \mu + \frac{1}{\lambda}. \end{cases}$$

We claim that the function $\psi_\lambda \circ V$ is a viscosity supersolution of (3.2) for every λ. To prove the claim we choose a sequence ψ_n of strictly increasing, smooth real maps that converge uniformly on compact sets to ψ_λ. Then, for every n, the map $\psi_n \circ V$ is a viscosity supersolution of (3.2) by Lemma 3.3. By the stability of viscosity supersolutions with respect to uniform convergence, we get the claim.

Next we observe that the net $\psi_\lambda \circ V$ is increasing and converges as $\lambda \to +\infty$ to the indicator function

$$C(x) = \begin{cases} 0, & x \in K, \\ 1, & x \notin K. \end{cases}$$

Viscosity supersolutions are stable with respect to the pointwise increasing convergence (see, e.g., Prop. V.2.16, p. 306 of [7]). Therefore the indicator function C of K is a viscosity supersolution of (3.2). From the definitions it is easy to check that

$$\mathcal{J}^{2,-}C(x) = -\mathcal{N}_K^2(x) \quad \forall\, x \in \partial K.$$

By plugging this formula into (3.2) we obtain exactly condition (3.1) of the viability theorem and complete the proof of (ii).

To prove (i) we choose $\overline{\mu} > 0$ small enough so that $K := \{x \in \mathcal{O} : V(x) \leq \mu\}$, for $\mu \leq \overline{\mu}$, is closed in $\mathbb{R}^N$ (for instance, $\overline{\mu} < \inf_{y\in\partial\mathcal{O}} \liminf_{x\to y} V(x)$). Then the preceding part of this proof gives the viability of K and the estimate (2.8) for all x such that $V(x) \leq \overline{\mu}$. Therefore, for some $\delta_o > 0$, (2.8) holds for all x with $|x| \leq \delta_o$, and this gives the almost sure stabilizability of the origin. □

Next we give the proof of Theorem 2.6 about asymptotic stability. It is obtained by first applying Theorem 2.5 to a new system with an extra variable and then using martingale inequalities as, e.g., in [18].

Proof of Theorem 2.6. We consider the differential system

$$\begin{cases} dX_t &= f(X_t,\alpha_t)dt + \sigma(X_t,\alpha_t)dB_t, \\ dZ_t &= l(X_t)dt \end{cases}$$

with initial data $X_0 = x$ and $Z_0 = 0$. We rewrite this system in $\mathbb{R}^{N+1}$ as

$$(CSDE2) \begin{cases} d(X_t,Z_t) = \overline{f}(X_t,Z_t,\alpha_t)dt + \overline{\sigma}(X_t,Z_t,\alpha_t)d(B_t,0), \quad t>0, \\ (X_0,Z_0) = (x,0), \end{cases}$$

where $\overline{f}(x,z,\alpha) = (f(x,\alpha), l(x))$ and $\overline{\sigma}(x,z,\alpha) = (\sigma(x,\alpha),0)$. Clearly it satisfies conditions (2.1) and (2.2). Let us consider the function

$$\begin{aligned} W(x,z) :& \mathcal{O} \times \mathbb{R} \to \mathbb{R}, \\ & (x,z) \longmapsto V(x) + |z|. \end{aligned}$$

We claim that it is a Lyapunov function for ($CSDE2$). In fact, W is positive definite (because $W \geq 0$ and $W = 0$ only for $(x,z) = (0,0)$); W is l.s.c., continuous at $(0,0)$, and proper since V is so. We have only to prove that W satisfies condition (2.4). Fix $x \neq 0$ and (x,z) with $z > 0$ and a smooth function ϕ such that $W - \phi$ has a local minimum at (x,z), i.e.,

$$V(x) + z - \phi(x,z) \leq V(y) + w - \phi(y,w),$$

for every (y,w), $w > 0$ in a neighborhood of (x,z). If we choose $w = z$ we get a minimum in x for the function $V(\cdot) - \phi(\cdot, z)$; therefore $(D_x\phi(x,z), D^2_{xx}\phi(x,z)) \in J^{2,-}V(x)$. If we choose $y = x$ we find a minimum in z for the smooth function $w \longmapsto w - \phi(x,w)$, so $D_z\phi(x,z) = 1$. Then there exists $\overline{\alpha} \in A$ such that $(\sigma(x,\overline{\alpha}), 0)^T (D_x\phi(x,z), 1) = 0$ and

$$\left\{-D\phi(x,z) \cdot \overline{f}(x,z,\overline{\alpha}) - trace\left[\overline{a}(x,z,\overline{\alpha}) D^2\phi(x,z)\right]\right\}$$

$$= \left\{-(D_x\phi(x,z), 1)\begin{pmatrix} f(x,\overline{\alpha}) \\ l(x) \end{pmatrix} - trace\left[\begin{pmatrix} a(x,\overline{\alpha}) & 0 \\ 0 & 0 \end{pmatrix} D^2\phi(x,z)\right]\right\}$$

$$= \left\{-D_x\phi(x,z) \cdot f(x,\overline{\alpha}) - trace\left[a(x,\overline{\alpha}) D^2_{xx}\phi(x,z)\right]\right\} - l(x) \geq 0,$$

since V is a strict Lyapunov function. Now fix (x,z) with $z < 0$ and let ϕ be a smooth function such that

$$V(x) - z - \phi(x,z) \leq V(y) - w - \phi(y,w)$$

for every (y,w), $w < 0$ in a neighborhood of (x,z). We argue as before and now get that there exists $\overline{\alpha} \in A$ such that $(\sigma(x,\overline{\alpha}), 0)^T \cdot (D_x\phi(x,z), -1) = 0$ and

$$-D_x\phi(x,z) \cdot f(x,\overline{\alpha}) - trace\left[a(x,\overline{\alpha}) D^2_{xx}\phi(x,z)\right] + l(x)$$

$$> -D_x\phi(x,z) \cdot f(x,\overline{\alpha}) - trace\left[a(x,\overline{\alpha}) D^2_{xx}\phi(x,z)\right] - l(x) \geq 0$$

because l is positive and V is a Lyapunov function. Finally, we consider $(x,0)$ and a smooth function ϕ such that

$$V(x) - \phi(x,0) \leq V(y) - w - \phi(y,w)$$

for every (y,w), $w < 0$ in a neighborhood of $(x,0)$ and

$$V(x) - \phi(x,0) \leq V(y) + w - \phi(y,w)$$

for all (y,w), $w > 0$ in a neighborhood of $(x,0)$. Then $(D_x\phi(x,z), D^2_{xx}\phi(x,z)) \in J^{2,-}V(x)$, $D_z\phi(x,0) \geq -1$, and $D_z\phi(x,0) \leq 1$. Therefore there exists $\overline{\alpha} \in A$ such that $(\sigma(x,\overline{\alpha}), 0)^T \cdot (D_x\phi(x,z), D_z\phi(x,z)) = 0$ and

$$\left\{-D\phi(x,z) \cdot \overline{f}(x,z,\overline{\alpha}) - trace\left[\overline{a}(x,z,\overline{\alpha}) D^2\phi(x,z)\right]\right\}$$

$$= \left\{-D_x\phi(x,z) \cdot f(x,\overline{\alpha}) - trace\left[a(x,\overline{\alpha}) D^2_{xx}\phi(x,z)\right]\right\} - D_z\phi(x,z) l(x)$$

$$= \left\{-D_x\phi(x,z) \cdot f(x,\overline{\alpha}) - trace\left[a(x,\overline{\alpha}) D^2_{xx}\phi(x,z)\right]\right\} - l(x) \geq 0.$$

This completes the proof of the claim, so we can apply Theorem 2.5 to get for every $x \in \mathcal{O}$ an admissible control $\overline{\alpha}_. \in \mathcal{A}_x$ such that the corresponding trajectory $(\overline{X}_., \overline{Z}_.)$ of $(CSDE2)$ with initial data $(x, 0)$ remains a.s. in the level set $K = \{(y, w) \in \mathcal{O} \times \mathbb{R} \,|\, W(y, w) \leq W(x, 0)\}$. Then, for all $t \geq 0$ and a.s., $\overline{X}_t \in \mathcal{O}$,

$$W(\overline{X}_t, \overline{Z}_t) = V(\overline{X}_t) + \overline{Z}_t = V(\overline{X}_t) + \int_0^t l(\overline{X}_s) ds \leq W(x, 0) = V(x)$$

and

$$0 \leq V(\overline{X}_t) \leq V(x) - \int_0^t l(\overline{X}_s) ds. \tag{3.3}$$

In particular, since $l \geq 0$, for some $r > 0$, $|\overline{X}_t| \leq r$ for all t a.s.

Next we claim that $l(\overline{X}_t) \to 0$ a.s. as $t \to +\infty$. Let us assume by contradiction that the claim is not true: then there exist $\varepsilon > 0$, a subset $\Omega_\varepsilon \subseteq \Omega$ with $\mathbf{P}(\Omega_\varepsilon) > 0$, and for every $\omega \in \Omega_\varepsilon$ a sequence $t_n(\omega) \to +\infty$ such that $l(\overline{X}_{t_n}(\omega)) > \varepsilon$. We define

$$F(r) := \max_{|x| \leq r, \alpha \in A} |f(x, \alpha)|, \qquad \Sigma(r) := \max_{|x| \leq r, \alpha \in A} \|\sigma(x, \alpha)\|.$$

We compute

$$\mathbf{E}\left\{ \sup_{t \leq s \leq t+h} |\overline{X}_s - \overline{X}_t|^2 \right\}$$

$$= \mathbf{E}\left\{ \sup_{t \leq s \leq t+h} \left| \int_t^s f(\overline{X}_u, \overline{\alpha}_u) du + \int_t^s \sigma(\overline{X}_u, \overline{\alpha}_u) dB_u \right|^2 \right\}$$

$$\leq 2\mathbf{E}\left\{ \sup_{t \leq s \leq t+h} \left| \int_t^s f(\overline{X}_u, \overline{\alpha}_u) du \right|^2 \right\} + 2\mathbf{E}\left\{ \sup_{t \leq s \leq t+h} \left| \int_t^s \sigma(\overline{X}_u, \overline{\alpha}_u) dB_u \right|^2 \right\}$$

$$\leq 2F^2(r)h^2 + 2\mathbf{E}\left\{ \sup_{t \leq s \leq t+h} \left| \int_t^s \sigma(\overline{X}_u, \overline{\alpha}_u) dB_u \right|^2 \right\} =: K.$$

By Theorem 3.4 in [19] (the process $|\int_t^s \sigma(\overline{X}_u, \overline{\alpha}_u) dB_u|$ is a positive semimartingale) we get

$$K \leq 2F^2(r)h^2 + 8 \sup_{t \leq s \leq t+h} \mathbf{E}\left\{ \left| \int_t^s \sigma(\overline{X}_u, \overline{\alpha}_u) dB_u \right|^2 \right\}$$

and by the Ito isometry,

$$K \leq 2F^2(r)h^2 + 8\mathbf{E}\left\{ \int_t^{t+h} |\sigma(\overline{X}_u, \overline{\alpha}_u)|^2 du \right\} \leq 2F^2(r)h^2 + 8\Sigma^2(r)h.$$

Then, the Chebyshev inequality gives

$$\mathbf{P}\left\{ \sup_{t \leq s \leq t+h} |\overline{X}_s - \overline{X}_t| > k \right\} \leq \frac{\mathbf{E}\left\{ \sup_{t \leq s \leq t+h} |\overline{X}_s - \overline{X}_t|^2 \right\}}{k^2}$$

$$\leq \frac{2F^2(r)h^2 + 8\Sigma^2(r)h}{k^2}.$$

Since l is continuous, we can fix δ such that $|l(x) - l(y)| \leq \frac{\varepsilon}{2}$ if $|x - y| \leq \delta$ and $|x|, |y| \leq r$. We define

$$C := \left\{ \omega \in \Omega \ : \ \sup_{0 \leq s \leq h} |\overline{X}_s - x| \leq \delta \right\}$$

and choose $0 < k < \mathbf{P}(\Omega_\varepsilon)$ and $h > 0$ depending on δ and ε such that

$$\mathbf{P}_x(C) \geq 1 - \frac{2F^2(r)h^2 + 8\Sigma^2(r)h}{\delta^2} \geq 1 + k - \mathbf{P}(\Omega_\varepsilon).$$

By the uniform continuity of l, the set

$$B := \left\{ \omega \in \Omega \ : \ \sup_{0 \leq s \leq h} |l(\overline{X}_s) - l(x)| \leq \varepsilon/2 \right\}$$

contains C and then

$$\mathbf{P}_x(B) \geq 1 + k - \mathbf{P}(\Omega_\varepsilon). \tag{3.4}$$

From inequality (3.3), letting $t \to \infty$, we get

$$V(x) \geq \mathbf{E}_x \int_0^{+\infty} l(\overline{X}_s) ds \geq \int_{\Omega_\varepsilon} \int_0^{+\infty} l(\overline{X}_s)\, ds\, d\mathbf{P} \geq \int_{\Omega_\varepsilon} \sum_n \int_{t_n(\omega)}^{t_n(\omega)+h} l(\overline{X}_s) ds\, d\mathbf{P}$$

$$\geq \int_{\Omega_\varepsilon} \sum_n h \inf_{[t_n(\omega), t_n(\omega)+h]} l(\overline{X}_t) \geq h \sum_n \int_{\Omega_\varepsilon} \inf_{[t_n(\omega), t_n(\omega)+h]} l(\overline{X}_t) d\mathbf{P}$$

$$\geq h \sum_n \frac{\varepsilon}{2} \mathbf{P} \left[\left(\sup_{0 \leq s \leq h} |l(\overline{X}_s) - l(x)| \leq \varepsilon/2 \mid x = \overline{X}_{t_n} \right) \cap \Omega_\varepsilon \right].$$

By the properties of the solutions of $(CSDE)$ estimate (3.4) gives $\mathbf{P}\left(\sup_{0 \leq s \leq h} |l(\overline{X}_s) - l(x)| \leq \varepsilon/2 \| \ x = \overline{X}_{t_n}\right) \geq 1 + k - \mathbf{P}_x(\Omega_\varepsilon)$ for every n. Therefore $\mathbf{P}\left[\left(\sup_{0 \leq s \leq h} |l(\overline{X}_s) - l(x)| \leq \varepsilon/2 \| \ x = \overline{X}_{t_n}\right) \cap \Omega_\varepsilon\right] \geq k$ for every n. Then by the previous inequality, we get

$$V(x) \geq h \sum_n \frac{\varepsilon}{2} k = +\infty.$$

This gives a contradiction; thus $\mathbf{P}(\Omega_\varepsilon) = 0$ for every $\varepsilon > 0$. We have proved that $l(\overline{X}_t) \to 0$ a.s. as $t \to +\infty$, now the positive definiteness of l implies that $|\overline{X}_t| \to 0$ a.s. as $t \to +\infty$. ◻

Remark 6. If the function l is only nonnegative semidefinite, the proof of the last theorem gives, for any x, a control $\overline{\alpha}_.$ whose trajectory $\overline{X}_t$ satisfies a.s. $V(\overline{X}_t) \leq V(x)$ and $l(\overline{X}_t) \to 0$ as $t \to +\infty$. Then the set $\mathcal{L} := \{y \mid l(y) = 0\}$ is an attractor, for a suitable choice of the control, in the sense that $dist(\overline{X}_t, \mathcal{L}) \to 0$ a.s. as $t \to +\infty$. For uncontrolled diffusion processes, results of this kind can be found in [37] and [18] and are considered stochastic versions of a theorem by La Salle. The earlier paper of Kushner [32] also studies a stochastic version of the La Salle invariance principle, namely, that the omega limit set of the process is an invariant subset of $\mathcal{L}$ in a suitable sense.

4. Almost sure feedback stabilization of affine systems. In this section we give a result on the *feedback stabilizability* of systems affine in the control in the case where there exists a smooth strict control Lyapunov function. It is an analogue for the almost sure stability of a celebrated theorem of Artstein [2] and Sontag [42] for deterministic systems, extended by Florchinger [21] to the stability of controlled diffusions in probability.

We begin with the simple case of a single-input affine system with uncontrolled diffusion, that is,

$$dX_t = (f(X_t) + \alpha_t g(X_t))\, dt + \sigma(X_t) dB_t, \tag{4.1}$$

where f, g, σ are vector fields in $\mathbb{R}^N$ with $f(0) = 0$ and $\sigma(0) = 0$, B_t is a one-dimensional Brownian motion, and the control α_t takes values in $\mathbb{R}$. We seek a function $k : \mathbb{R}^N \to \mathbb{R}$, at least continuous in $\mathbb{R}^N \setminus \{0\}$, such that the origin is a.s. asymptotically stable for the stochastic differential equation

$$dX_t = (f(X_t) + k(X_t)g(X_t))\, dt + \sigma(X_t) dB_t. \tag{4.2}$$

Then k is called an *a.s. asymptotically stabilizing feedback* for the control system (4.1).

If there are no constraints on the control, a smooth strict control Lyapunov function V satisfies, in $\mathbb{R}^N \setminus \{0\}$,

$$f \cdot DV + trace\left[\frac{1}{2}\sigma\sigma^T D^2 V\right] + \inf_{\alpha\in\mathbb{R}} \{\alpha g \cdot DV\} \le -l, \qquad \sigma \cdot DV = 0.$$

Set $\gamma(x) := f \cdot DV + trace\left[\sigma\sigma^T D^2 V\right]/2 + l/2$ and observe that the inequality for V means

$$g(x) \cdot DV(x) = 0 \quad \Rightarrow \quad \gamma(x) \le -l(x)/2 < 0.$$

It is clear that $k(x) := -\gamma(x)/g(x) \cdot DV(x)$, $k(x) := 0$ if $g(x) \cdot DV(x) = 0$ could be a stabilizing feedback, but it is discontinuous where $g(x) \cdot DV(x)$ vanishes. If this case occurs, we build a continuous feedback by means of Sontag's universal formula [42], i.e.,

$$k(x) := -\frac{\gamma(x) + \sqrt{\gamma^2(x) + (g(x)\cdot DV(x))^4}}{g(x) \cdot DV(x)} \quad \text{if } g(x) \cdot DV(x) \ne 0, \tag{4.3}$$

and $k(x) = 0$ if $g(x) \cdot DV(x) = 0$. By the argument in [42], $k \in C(\mathbb{R}^N \setminus \{0\})$ if $V \in C^2(\mathbb{R}^N \setminus \{0\})$ and $k \in C^1(\mathbb{R}^N \setminus \{0\})$ if f, g, l are of class C^1 and $V \in C^3(\mathbb{R}^N \setminus \{0\})$. Moreover

$$(f + kg) \cdot DV + trace\left[\frac{1}{2}\sigma\sigma^T D^2 V\right] \le -\frac{l}{2}, \qquad \sigma \cdot DV = 0,$$

in $\mathbb{R}^N \setminus \{0\}$, so V is a strict Lyapunov function for (4.2) and the origin is a.s. asymptotically stable. In conclusion, k is a stabilizing feedback for the affine control system (4.1).

If the control must satisfy a hard constraint, say $\alpha \in [-1, 1]$, it is not hard to check that $k(x)$ can be used in a neighborhood of the origin provided that DV and $D^2 V$ are bounded near 0 and either $g(x) \to 0$ or $DV(x) \to 0$ as $x \to 0$.

Next we use the same idea for the more general system with both the drift and the diffusion terms affine in the control

$$dX_t = \left(f(X_t) + \sum_{i=1}^{P-1} \alpha_t^i g_i(X_t) \right) dt + \left(\sigma(X_t) + \alpha_t^P \tau(X_t) \right) dB_t, \tag{4.4}$$

where f, g_i, σ, τ are vector fields in $\mathbb{R}^N$, B_t is a standard one-dimensional Brownian motion, and the controls α_t^i, $i = 1, \dots, P$, are $\mathbb{R}$-valued. The existence of a strict control Lyapunov function V implies that for some real number r the vector $\sigma + r\tau$ is orthogonal to DV, so $\tau \cdot DV \neq 0$ at all points where $\sigma \cdot DV \neq 0$, and we can define for all $x \in \mathbb{R}^N \setminus \{0\}$,

$$h(x) := \begin{cases} 0 & \text{if } \sigma(x) \cdot DV(x) = 0, \\ -\frac{\sigma(x) \cdot DV(x)}{\tau(x) \cdot DV(x)} & \text{if } \sigma(x) \cdot DV(x) \neq 0. \end{cases}$$

PROPOSITION 4.1. *Assume system* (4.4) *has a strict control Lyapunov function* $V \in C^2(\mathbb{R}^N \setminus \{0\})$ *and the function* h *is continuous in* $\mathbb{R}^N \setminus \{0\}$*. Then there exists continuous functions* $k_i : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$, $i = 1, \dots, P-1$, *such that* $(k_1(x), \dots, k_{P-1}(x), h(x))$ *is an a.s. asymptotically stabilizing feedback for system* (4.4).

Moreover, $k_i(x) \in [-1, 1]$ *for* x *in a neighborhood of 0 if* DV *and* D^2V *are bounded near* 0, *and either* $DV(x) \to 0$ *or* $g_i(x) \to 0$ *for all* i *as* $x \to 0$.

Proof. We recall from [42] that the function $\phi(a, 0) := 0$ for $a < 0$, $\phi(a, b) := (a + \sqrt{a^2 + b^2})/b$ is real-analytic in the set $S := \{(a, b) \in \mathbb{R}^2 \, : \, b > 0 \text{ or } a < 0\}$. We set

$$\gamma(x) := f(x) \cdot DV(x) + trace \left[(\sigma(x) + h(x)\tau(x))(\sigma(x) + h(x)\tau(x))^T \frac{D^2V(x)}{2} \right] + \frac{l(x)}{2},$$

$$\beta(x) := \sum_{i=1}^{P-1} (g_i(x) \cdot DV(x))^2.$$

Since V is a strict control Lyapunov function,

$$\gamma(x) + \inf_{\alpha_i \in \mathbb{R}} \sum_{i=1}^{P-1} \alpha_i g_i(x) \cdot DV(x) \leq -\frac{l(x)}{2},$$

so, for $x \neq 0$,

$$\beta(x) = 0 \quad \Rightarrow \quad \gamma(x) \leq -l(x)/2 < 0.$$

Therefore $(\gamma(x), \beta(x)) \in S$. Now we define, for $i = 1, \dots, P-1$,

$$k_i(x) := -\phi(\gamma(x), \beta(x)) g_i(x) \cdot DV(x), \quad x \neq 0,$$

and $k(0) = 0$. Then $(k_1(x), \dots, k_{P-1}(x), h(x))$ is continuous in $\mathbb{R}^N \setminus \{0\}$ and satisfies

$$\left(f + \sum_{i=1}^{P-1} k_i g_i \right) \cdot DV + trace \left[(\sigma + h\tau)(\sigma + h\tau)^T \frac{D^2V}{2} \right] + \frac{l}{2}$$
$$= \gamma - \beta\phi(\gamma, \beta) = -\sqrt{\gamma^2 + \beta^2} < 0.$$

Since $(\sigma + h\tau) \cdot DV = 0$ by definition of h, V is a strict Lyapunov function for the equation

$$dX_t = \left(f(X_t) + \sum_{i=1}^{P-1} k_i(X_t) g_i(X_t) \right) dt + (\sigma(X_t) + h(X_t)\tau(X_t))\, dB_t.$$

Therefore the origin is a.s. asymptotically stable for this equation.

Finally, we check the boundedness of k in a neighborhood of 0. This is trivial for $\beta(x) = 0$. If $\beta(x) \neq 0$, then

$$|k| \leq \frac{|\gamma + |\gamma| + \beta|}{\sqrt{\beta}}.$$

Since either $DV \to 0$ or $g_i \to 0$ for all i, $\beta(x) \to 0$ as $x \to 0$. We fix $\delta > 0$ such that $\beta(x) \leq \delta$ implies $\gamma(x) < 0$ and then choose a neighborhood of the origin where $\beta(x) \leq \delta$. In this set $|k(x)| \leq \sqrt{\beta(x)} \to 0$. ∎

Remark 7. The proof above gives an explicit formula for the stabilizing feedback in terms of the data and the Lyapunov function V only, which reduces to (4.3) if $\tau \equiv 0$ and $P = 2$. From the formula, one sees that the feedback is C^1 in $\mathbb{R}^N \setminus \{0\}$ if h, f, g, σ, τ, and l are C^1 in $\mathbb{R}^N \setminus \{0\}$ and $V \in C^3(\mathbb{R}^N \setminus \{0\})$.

Note also that the continuity assumption on h is automatically satisfied if $\tau \cdot DV$ is either always nonnull or identically 0.

Finally, it is straightforward to extend the proposition to the case of M-dimensional noise with independent Brownian components $B_t^1, \dots, B_t^M$ and a diffusion term of the form $\sum_{i=P}^{P+M-1} \left(\sigma_i + \alpha_t^i \tau_i\right) dB_t^i$, with σ_i, τ_i vector fields and α_t^i scalar controls.

5. Some variants and extensions. In this section we collect several remarks on other applications of our methods. We begin with the *almost sure exponential stabilizability*. It means that there exists a positive rate λ and $\gamma \in \mathcal{K}$ such that for every initial data x there exists an admissible control $\overline{\alpha}_\cdot \in \mathcal{A}_x$ whose corresponding trajectory $\overline{X}_\cdot$ satisfies

$$|\overline{X}_t| \leq e^{-\lambda t} \gamma(|x|) \quad \text{a.s.}$$

PROPOSITION 5.1 (almost sure exponential stabilizability). *Under assumptions* (2.1) *and* (2.2)*, the null state is a.s. exponentially stabilizable for (CSDE) if there exists a control Lyapunov function V satisfying conditions* (i), (ii), (iii) *of Definition* 2.3 *and, for some $\lambda > 0$,*

(iv)′ *for every $(p, Y) \in \mathcal{J}^{2,-}V(x)$ there exists $\overline{\alpha} \in A$ such that*

$$\sigma(x, \overline{\alpha})^T p = 0 \quad \textit{and} \quad -p \cdot f(x, \overline{\alpha}) - \textit{trace}\,[a(x, \overline{\alpha})Y] - \lambda V(x) \geq 0.$$

Proof. We consider the system

$$\begin{cases} dX_t = f(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dB_t, \\ dY_t = dt \end{cases}$$

with initial data $X_0 = x$ and $Y_0 = 0$, and the Lyapunov function $W(x, y) = e^{\lambda y} V(x)$. By applying Theorem 2.5 we obtain the existence of a control $\overline{\alpha}_\cdot$ such that the corresponding trajectory a.s. satisfies $V(\overline{X}_t) \leq V(x) e^{-\lambda t}$, which is the desired inequality. ∎

Next we extend the results of section 2 to the stabilizability of a general closed set $M \subseteq \mathbb{R}^N$. We denote by $d(x, M)$ the distance between a point $x \in \mathbb{R}^N$ and M.

Definition 5.2 (almost sure stabilizability at M). *The system* $(CSDE)$ *is* a.s. (stochastic open-loop) stabilizable *at M if there exists $\gamma \in \mathcal{K}$ such that, for every x in a neighborhood of M, there is an admissible control function $\overline{\alpha}_\cdot \in \mathcal{A}_x$ whose trajectory $\overline{X}_\cdot$ verifies*

$$d(\overline{X}_t, M) \le \gamma(d(x, M)) \quad \forall\, t \ge 0 \quad a.s.$$

If, in addition,

$$\lim_{t \to +\infty} d(\overline{X}_t, M) = 0 \quad a.s.,$$

the system is a.s. (stochastic open-loop) locally asymptotically stabilizable *at M.*

If these properties hold for all $x \in \mathbb{R}^N$, the system is a.s. (stochastic open-loop) globally asymptotically stabilizable *at M.*

Remark 8. If M is a.s. stabilizable, then it is viable for $(CSDE)$. In fact, the definition gives for $x \in M$ and $\varepsilon > 0$ an admissible control such that a.s. $d(X_t, M) \le \varepsilon$ for all $t \ge 0$.

Then for such control and any $\lambda > 0$ $\mathbf{E}_x \int_0^{+\infty} d(X_t, M) e^{-\lambda t} dt \le \frac{\varepsilon}{\lambda}$, and so

$$\inf_{\alpha_\cdot \in \mathcal{A}_x} \mathbf{E}_x \int_0^{+\infty} d(X_t, M) e^{-\lambda t} dt = 0.$$

The convexity assumption (2.2) and an existence theorem for optimal controls [27] imply that the inf is attained, and the minimizing control produces a trajectory remaining in M for all $t \ge 0$.

Definition 5.3 (control Lyapunov functions at M). *Let $\mathcal{O}$ be an open neighborhood of the closed set M. A function $V : \mathcal{O} \to [0, +\infty)$ is a control Lyapunov function at M for $(CSDE)$ if*

(i) *V is lower semicontinuous;*

(ii) *there exists $\gamma_1 \in \mathcal{K}_\infty$ such that $V(x) \le \gamma_1(d(x, M))$ for all $x \in \mathcal{O}$;*

(iii) *there exists $\gamma_2 \in \mathcal{K}_\infty$ such that $\gamma_2(d(x, M)) \le V(x)$ for all $x \in \mathcal{O}$;*

(iv) *for all $x \in \mathcal{O} \setminus M$ and $(p, Y) \in \mathcal{J}^{2,-}V(x)$ there exists $\overline{\alpha} \in A$ such that condition* (2.4) *holds.*

The function V is a strict control Lyapunov function *at M if it satisfies conditions* (i)–(iii) *and*

(iv)$'$ *for some Lipschitz continuous $l : \mathcal{O} \to \mathbb{R}$, $l > 0$ for all $x \in \mathcal{O} \setminus M$ and $(p, Y) \in \mathcal{J}^{2,-}V(x)$, there exists $\overline{\alpha} \in A$ such that condition* (2.7) *holds.*

Now we can state the analogues of the first and second Lyapunov theorems for the almost sure stabilizability at M. Their proofs are easily obtained from the arguments of Theorems 2.5 and 2.6 by using $d(x, M)$ instead of $|x|$ and noting that conditions (ii) and (iii) in the Definition 5.3 say that the sublevel sets of the Lyapunov function form a basis of neighborhoods of M.

Theorem 5.4. *Assume* (2.1), (2.2), *and the existence of a control Lyapunov function V at M. Then*

(i) *the system $(CSDE)$ is a.s. stabilizable at M*;

(ii) *if, in addition, the domain $\mathcal{O}$ of V is all $\mathbb{R}^N$, for all $x \notin M$ there exists $\overline{\alpha}_\cdot \in \mathcal{A}_x$ such that the corresponding trajectory $\overline{X}_\cdot$ satisfies*

$$d(\overline{X}_t, M) \le \gamma_1^{-1}(\gamma_2(d(x, M))) \quad \forall\, t \ge 0 \quad a.s.$$

with $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ from Definition 5.3*; in particular, if M is bounded, the system is also a.s. Lagrange stabilizable.*

THEOREM 5.5. *Assume* (2.1), (2.2)*, and the existence of a strict control Lyapunov function V at M. Then*

(i) *the system* $(CSDE)$ *is a.s. locally asymptotically stabilizable at M;*

(ii) *if, in addition, the domain $\mathcal{O}$ of V is all $\mathbb{R}^N$, the system is a.s. globally asymptotically stabilizable at M.*

Remark 9 (stochastic target problems and absorbing sets). A stochastic target problem consists of steering the state of the system $(CSDE)$ in finite time into a given closed set $\mathcal{T}$ (the target) by an appropriate choice of the control. One of the objects of interest is the set of initial positions from which this goal can be achieved a.s. in a given time t. We define these reachability sets for $t > 0$ as

$$\mathcal{R}(t) = \{x \in \mathbb{R}^N \mid \exists \alpha_{\cdot} \in \mathcal{A}_x \, : \, X_t \in \mathcal{T} \text{ a.s.}\}.$$

We consider a target $\mathcal{T}$ containing 0 and being invariant for the stochastic system and we assume there exists a global strict control Lyapunov function V as defined in (2.4) such that

$$\inf_{\mathbb{R}^N \setminus \mathcal{T}} l(x) = L > 0.$$

We are going to show that each reachability set $\mathcal{R}(t)$ lies between two sublevel sets of the Lyapunov function V. The arguments in the proof of Theorem 2.6 show that for every initial point $x \notin \mathcal{T}$ there exists a control $\overline{\alpha}_{\cdot} \in \mathcal{A}_x$ such that the first entry time $\overline{\tau}_x$ of the corresponding trajectory in the target is a.s. bounded by

$$\overline{\tau}_x \leq \left(V(x) - \inf_{\partial \mathcal{T}} V(y) \right) \Big/ L. \tag{5.1}$$

In particular, since the target $\mathcal{T}$ is invariant, it is reached a.s. in a finite time, and as such time is also uniformly bounded, $\mathcal{T}$ is an *absorbing set* for the system according to the terminology in [5]. Next, from the assumptions and inequality (5.1) we get

$$\left\{ x \in \mathbb{R}^N \mid V(x) \leq Lt + \inf_{\partial \mathcal{T}} V(y) \right\} \subseteq \mathcal{R}(t).$$

Using Chebyshev inequality and estimates of the same kind as in the proof of Theorem 2.6 we can find also for every $t > 0$ a positive number $k(t)$ depending continuously on t such that

$$\mathcal{R}(t) \subseteq \{x \in \mathbb{R}^N \mid V(x) \leq k(t)\}.$$

Let us mention that Soner and Touzi [39] developed recently a PDE approach to stochastic target problems; see also [40] and the references therein for some interesting applications to geometric PDEs and front propagation problems.

6. Examples. We begin with an example of an uncontrolled system that does not have a continuous Lyapunov function but has an l.s.c. Lyapunov function and therefore is a.s. stable. It shows that allowing V to be merely l.s.c. in Theorem 2.5 really increases the range of the applications. Our example is a variant of a deterministic one by Krasowski [30], namely,

$$\begin{cases} \dot{X}_t = Y_t, \\ \dot{Y}_t = -X_t + Y_t (X_t^2 + Y_t^2)^3 \sin^2 \left(\frac{\pi}{X_t^2 + Y_t^2} \right); \end{cases}$$

see [6] for a discussion of this and other deterministic examples.

Example 1. We transform the previous system into polar coordinates and perturb it with a white noise tangential to the circles $C_n := \{(x,y) : |(x,y)| = \frac{1}{\sqrt{n}}\}$ and nondegenerate between two consecutive circles:

$$\begin{cases} d\rho_t &= \left[\rho_t^7 \sin^2(\theta_t)\sin^2(\frac{\pi}{\rho_t^2})\right] dt + \left[\sigma(\rho_t,\theta_t)\sin^2(\frac{\pi}{\rho_t^2})\right] dB_t, \\ d\theta_t &= \left[-1 + \rho_t^6 \sin(\theta_t)\cos(\theta_t)\sin^2(\frac{\pi}{\rho_t^2})\right] dt, \end{cases}$$

where B_t is a one-dimensional Brownian motion and σ satisfies the hypotheses for the existence and uniqueness of the solution of the stochastic differential equation. As in the undisturbed case, the circles C_n are a.s. invariant and any point in C_n is eventually reached a.s. by any trajectory starting in C_n. Then any Lyapunov function V is constant on C_n because $V(\rho_t,\theta_t) \le V(\rho_0,\theta_0)$ a.s., and $c_n := V_{|C_n} \neq c_{n-1} := V_{|C_{n-1}}$ at least on a subsequence. By property (iv) in Definition 2.3 of the Lyapunov function, for every (ρ,θ) in the interior of $C_{n-1} \setminus C_n$ and every $(p,X) \in \mathcal{J}^{2,-}V(\rho,\theta)$, we get $(\sigma(\rho,\theta)\sin^2(\frac{\pi}{\rho^2}),0)\cdot p = 0$. Since the diffusion is nondegenerate in the ρ direction in the interior of $C_{n-1}\setminus C_n$, from the previous equality we deduce that, for such (ρ,θ), every element in $\mathcal{J}^{2,-}V(\rho,\theta)$ is of the form $((0,p_2),X)$. This implies that the function V is constant in the ρ direction in the interior of $C_{n-1}\setminus C_n$ and cannot be continuous.

Now we check that the Lyapunov function of the undisturbed system in the unit ball does the job also for our perturbed stochastic system. We take

$$V(\rho,\theta) := \frac{1}{\sqrt{n}} \quad \text{for } \frac{1}{\sqrt{n}} < \rho \le \frac{1}{\sqrt{n-1}} \quad \forall\theta.$$

This is a positive definite function, l.s.c. and continuous at 0. We calculate its second order subjets and plug them into (2.4). If $\rho \neq \frac{1}{\sqrt{n}}$ for all n, $(p,\mathbf{X}) \in J^{2,-}V(\rho,\theta)$ if and only if $p=0$ and $\mathbf{X} \le 0$, so condition (2.4) is trivially satisfied. On the other hand, $(p,\mathbf{X}) \in J^{2,-}V(\frac{1}{\sqrt{n}},\theta)$ if and only if

$$p = \begin{pmatrix} s \\ 0 \end{pmatrix},\ s \ge 0, \quad \text{and} \quad \mathbf{X} = \begin{pmatrix} a & b \\ b & c \end{pmatrix},\ c \le 0.$$

At the points with $\rho = \frac{1}{\sqrt{n}}$ the drift f of the system is $(0,-1)$ and the dispersion vector σ is $(0,0)$. Then

$$f\cdot p + \frac{1}{2} trace\left[\sigma\sigma^T \mathbf{X}\right] = 0, \qquad \sigma\cdot p = 0,$$

and condition (2.4) is satisfied. Therefore Theorem 2.5 applies and the system is a.s. Lyapunov stable at the origin.

The next two examples are about *stochastic perturbations of stabilizable systems.* We consider a deterministic controlled system in $\mathbb{R}^N$,

$$\dot{X}_t = f(X_t,\alpha_t), \tag{6.1}$$

globally asymptotically (open-loop) stabilizable at the origin, i.e., asymptotically controllable in the terminology of deterministic systems [43, 44]. By the converse Lyapunov theorem of Sontag [41, 44], there exists a strict continuous control Lyapunov function for the system, i.e., for some positive definite continuous function L, a proper function V satisfying, in $\mathbb{R}^N \setminus \{0\}$,

$$\max_{\alpha \in A} \{-f(x,\alpha) \cdot DV\} - L(x) \geq 0 \tag{6.2}$$

in the viscosity sense. (This is perhaps not explicitly stated in the literature; the original result of Sontag [41] interprets this inequality in the sense of Dini derivatives of V along relaxed trajectories; the paper of Sontag and Sussmann [44] interprets it in the sense of directional Dini subderivatives; and both these senses are known to be equivalent to the viscosity one; see, e.g., [47, 7]).

In the following examples we perturb (6.1) in two different ways and give a condition under which V remains a control Lyapunov function for the almost sure stabilizability of the new stochastic system.

Example 2. Consider the controlled diffusion process

$$dX_t = f(X_t, \alpha)dt + \sigma(X_t)dB_t, \tag{6.3}$$

where B_t is an M-dimensional Brownian motion and σ a Lipschitzean $N \times M$ matrix. Then V *is a Lyapunov function for* (6.3) *if, for some open set* $\mathcal{O} \ni 0$ *and some continuous* $l : \mathcal{O} \to [0, +\infty)$, V *satisfies in viscosity sense in* $\mathcal{O} \setminus \{0\}$,

$$-trace\left[\frac{1}{2}\sigma\sigma^T D^2 V\right] + L - l \geq 0, \quad \sigma_i \cdot DV = 0 \quad \forall i, \tag{6.4}$$

and it is a strict Lyapunov function if l *is positive definite.*

In fact, this inequality and (6.2) give, for any $(p, \mathbf{X}) \in J^{2,-}V(x)$,

$$\max_{\alpha \in A} \{-f(x,\alpha) \cdot p\} - trace\left[\frac{1}{2}\sigma\sigma^T \mathbf{X}\right] - l \geq 0,$$

so V satisfies the inequality in condition (2.7), whereas the equality in condition (2.7) reduces to $\sigma_i \cdot p = 0$.

In the classical special case of $V(x) = |x|^2$ and $M = 1$, the sufficient condition (6.4) for V to be a Lyapunov function of (6.3) reads

$$l(x) := L(x) - |\sigma(x)|^2 \geq 0, \quad \sigma(x) \cdot x = 0.$$

For a noise of dimension $M = N$ an example of σ satisfying the orthogonality condition in (6.4) is

$$\sigma(x) = k\left(\mathbf{I} - \frac{DV(x) \otimes DV(x)}{|DV(x)|^2}\right)$$

for any constant k.

Example 3. Here we consider the perturbation of the deterministic system (6.1) by a function g of a K-dimensional diffusion process Y_t:

$$\begin{cases} \dot{X}_t = f(X_t, \alpha_t) + g(X_t, Y_t), \\ dY_t = b(Y_t, X_t, \alpha_t)dt + \tau(Y_t, X_t, \alpha_t)dB_t, \end{cases} \tag{6.5}$$

where the function $g : \mathbb{R}^n \times \mathbb{R}^K \to \mathbb{R}^n$ is Lipschitz continuous with $g(0, y) = 0$ for all y, B_t is a one-dimensional Brownian motion, and b, τ are vector fields in $\mathbb{R}^K$

with the usual assumptions. We are still assuming that (6.1) has a strict control Lyapunov function V, i.e., (6.2) holds with L positive definite. We are interested in the stabilizability of the perturbed system at the set $M := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^K \,:\, x = 0\}$, which corresponds to the origin of the unperturbed system (6.1); see Definition 5.2. Note that the assumption on g implies the viability of M for (6.5).

We claim that *the function* V*, defined by* $V(x,y) := V(x)$ *for all* y*, is a Lyapunov function at* M *for* (6.5) *(see Definition* 5.3*) if, for some open set* $\mathcal{O} \ni 0$ *and some continuous* $l : \mathcal{O} \times \mathbb{R}^K \to [0,+\infty)$*,* V *satisfies in viscosity sense in* $\mathcal{O} \setminus \{0\}$*,*

$$\inf_{y \in \mathbb{R}^K} \{-g(x,y) \cdot DV(x) - l(x,y)\} + L(x) \geq 0, \tag{6.6}$$

and V *is a strict Lyapunov function if* $l(x,y) > 0$ *for all* $x \neq 0$ *and all* y*.*

In fact, since $d((x,y), M) = |x|$, V satisfies conditions (i)–(iii) of Definition 5.3. By (6.2) and (6.6) V is also a viscosity supersolution in $\mathcal{O} \times \mathbb{R}^K \setminus M$ of

$$\sup_{a \in A} \{-f(x,a) \cdot DV(x)\} - g(x,y) \cdot DV(x) - l(x,y) \geq 0,$$

which is the inequality in (2.7) in this case, because V is constant in y. Finally, for the same reason, the condition in (2.7) of orthogonality of the diffusion vector to the level sets of V is trivially satisfied.

The inequality (6.6) is a smallness condition of the component of g in the direction of DV with respect to L in the set $\mathcal{O}$, uniformly in y. For $l \equiv 0$ and V smooth in $\mathcal{O} \setminus \{0\}$, it becomes

$$\sup_{y \in \mathbb{R}^K} g(x,y) \cdot DV(x) \leq L(x) \quad \text{in } \mathcal{O} \setminus \{0\}, \tag{6.7}$$

which is satisfied, in particular, if

$$\sup_{y \in \mathbb{R}^K} |g(x,y)| \leq L(x)/LipV,$$

where $LipV$ denotes the Lipschitz constant of V in $\mathcal{O}$. We recall that, under our assumption that the deterministic system (6.1) be asymptotically controllable, although V may not be smooth, it can be chosen semiconcave in $R^N \setminus \{0\}$ and therefore locally Lipschitz [38]. If we make this choice, it is enough that inequality (6.7) holds for all points $x \in \mathcal{O}$ where V is differentiable, and the last inequality is guaranteed for all perturbations g with small sup-norm with respect to y.

In the next two examples we give conditions on a radial function to be a Lyapunov function for almost sure stability.

Example 4. We consider as a candidate Lyapunov function for the general controlled system $(CSDE)$ the function $V(x) = v(|x|)$, for some smooth $v : [0,+\infty) \to [0,+\infty)$ with $v'(r) > 0$ for $r > 0$. Since $DV(x) = xv'(|x|)/|x|$, in view of the orthogonality condition in (2.4), we restrict ourselves to controls $\alpha \in A$ such that

$$\sigma_i(x,\alpha) \cdot x = 0 \quad \forall\, i = 1, \dots, M. \tag{6.8}$$

We compute

$$trace\left[a(x,\alpha) D^2 V(x)\right] = \frac{v'(|x|)}{|x|} trace\ a(x,\alpha) + \left(v''(|x|) - \frac{v'(|x|)}{|x|}\right) \frac{|\sigma(x,\alpha)^T x|^2}{|x|^2}$$

and use (6.8) to obtain that V is a Lyapunov function if and only if, in a neighborhood $\mathcal{O}$ of 0,

$$l(x) := \max_{\alpha\in A,\, \sigma(x,\alpha)^T x=0} [-f(x,\alpha)\cdot x - trace\, a(x,\alpha)]\, \frac{v'(|x|)}{|x|} \geq 0,$$

i.e.,

$$\min_{\alpha\in A,\, \sigma(x,\alpha)^T x=0} [f(x,\alpha)\cdot x + trace\, a(x,\alpha)] \leq 0. \tag{6.9}$$

This condition is independent of the choice of v. Moreover, if $l > 0$ and Lipschitz in $\mathcal{O}\setminus\{0\}$ and $l \to 0$ as $x \to 0$, then V is a strict Lyapunov function. Note that, although the radial component of the diffusion must be null by (6.8), its rotational component still plays a destabilizing role. In fact, $trace\, a(x,\alpha) \geq 0$ and whenever it is nonnull it must be compensated by a negative radial component of f.

In particular, a single-input affine system with uncontrolled diffusion and one-dimensional noise B_t,

$$dX_t = (f(X_t) + \alpha_t g(X_t))\, dt + \sigma(X_t) dB_t, \quad \alpha_t \in [-1,1],$$

has a radial Lyapunov function in $\mathcal{O}$ if and only if

$$\sigma(x)\cdot x = 0 \quad \text{and} \quad |g(x)\cdot x| \geq f(x)\cdot x + \frac{|\sigma(x)|^2}{2} \quad \text{in } \mathcal{O},$$

and $V(x) = |x|^2/2$ is a strict Lyapunov function in $\mathcal{O}$ if and only if

$$l(x) := |g(x)\cdot x| - f(x)\cdot x - \frac{|\sigma(x)|^2}{2} > 0 \quad \text{in } \mathcal{O}\setminus\{0\}.$$

Moreover, $k(x) := -sign(g(x)\cdot x)$ is a stabilizing feedback if $g(x)\cdot x$ does not change sign; if it does, k is discontinuous, and then a continuous stabilizing feedback in a neighborhood of 0 is given by the formula (4.3) in section 4.

Example 5. Here we study a system in $\mathbb{R}^2$ written in polar coordinates (ρ,θ) and look for radial Lyapunov functions, i.e., of the form $V(\rho,\theta) = v(\rho)$. Consider the stochastic controlled system:

$$(CSDE) \begin{cases} d\rho_t = f(\rho_t,\theta_t,\alpha_t)dt + \sigma(\rho_t,\theta_t,\alpha_t)dB_t, \\ d\theta_t = g(\rho_t,\theta_t,\alpha)dt + \tau(\rho_t,\theta_t,\alpha_t)dB_t, \end{cases}$$

where all functions f,σ,g,τ are 2π-periodic and B_t is (for simplicity) a one-dimensional Brownian motion. The conditions for a function $V = v(\rho)$ to be a Lyapunov function of this system at the set $M := \{(0,\theta) : \theta \in \mathbb{R}\}$ are the following. The orthogonality condition in (2.7) requires that for every (ρ,θ) there exists a subset $A(\rho,\theta) \neq \emptyset$ of the control set A such that

$$\sigma(\rho,\theta,\alpha) = 0 \qquad \forall \alpha \in A(\rho,\theta).$$

Then the condition (2.7) is satisfied if v is a viscosity supersolution of the ordinary differential inequality

$$\sup_{\alpha\in A(\rho,\theta)} \{-f(\rho,\theta,\alpha)\cdot v'(\rho)\} \geq 0$$

for $\rho > 0$ and for each fixed $\theta \in [0, 2\pi]$. Of course the same result can be obtained from the previous example with some calculations based on the Ito chain rule.

The last two examples are about the stabilization of systems to sets M different from the origin, namely, the complement of a ball and a periodic orbit.

Example 6. We consider the general system $(CSDE)$ and the set

$$M := \{x \mid |x| \geq R\} = \mathbb{R}^N \setminus B_R.$$

We assume M is viable for the system. We take the radial function V

$$V(x) := \begin{cases} R^2 - |x|^2 & |x| < R, \\ 0 & |x| \geq R \end{cases}$$

and use the calculations of Example 4 to see that V is a Lyapunov function at M if and only if for every x with $|x| < R$ there exists $\overline{\alpha} \in A$ such that

$$\sigma_i(x, \overline{\alpha}) \cdot x = 0 \quad \forall i \quad \text{and} \quad f(x, \overline{\alpha}) \cdot x + \mathit{trace}\ a(x, \overline{\alpha}) \geq 0.$$

Contrary to Example 4, here the rotational component of the diffusion has a stabilizing effect. In fact, the drift $f(x, a)$ is allowed also to point away from M if its negative radial component is compensated by the positive term $\mathit{trace}\ a(x, \overline{\alpha})$.

If $K \subset B_R$ is a compact set and

$$l(x) := \max_{\alpha \in A,\, \sigma(x,\alpha)^T x = 0} [f(x, \alpha) \cdot x + \mathit{trace}\, a(x, \alpha)] > 0 \quad \text{in } B_R \setminus K,$$

then M is locally asymptotically stable by Theorem 5.5, and for all initial points $x \notin K$ there is a control whose trajectories tend a.s. to M as $t \to +\infty$. In this case we can say that K can be made a.s. repulsive by a suitable choice of the controls. In particular, we have a criterion of instability of an equilibrium point.

Note also that if $l > 0$ on $\partial M = \partial B_R$, then for some control the trajectories starting in a suitable neighborhood of ∂M reach M in finite time a.s., as we observed in the last remark of section 5. In particular, if $l > 0$ in $\overline{B_R}$, then for every $x \in B_R$ there exists a control $\overline{\alpha}_\cdot$ such that the exit time of the corresponding trajectory $\overline{X}_\cdot$ from B_R is a.s. bounded by $(R^2 - |x|^2)/\min_{B_R} l$.

Example 7. Consider $(CSDE)$ in $\mathbb{R}^2$ and assume the circle $\gamma := \{x : |x| = R\}$ is a viable set. By the results of [11] this occurs if for all $x \in \gamma$ there exists $\overline{\alpha} \in A$ such that

$$\sigma(x, \overline{\alpha}) \cdot x = 0 \quad \text{and} \quad f(x, \overline{\alpha}) \cdot x + \mathit{trace}\, a(x, \overline{\alpha}) = 0.$$

Then γ is locally asymptotically stabilizable if, in a neighborhood $\{x : R - \varepsilon \leq |x| \leq R + \varepsilon\}$,

$$\max_{\alpha \in A,\, \sigma(x,\alpha)^T x = 0} [f(x, \alpha) \cdot x + \mathit{trace}\ a(x, \alpha)] > 0 \quad \text{if } |x| < R,$$

$$\min_{\alpha \in A,\, \sigma(x,\alpha)^T x = 0} [f(x, \alpha) \cdot x + \mathit{trace}\ a(x, \alpha)] < 0 \quad \text{if } |x| > R.$$

This follows immediately from the arguments of Examples 4 and 6.

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