Algebraic Connections vs. Algebraic $\mathcal{D}$-modules: inverse and direct images.

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Abstract. This paper is devoted to a new proof of the comparison between the derived direct image functor for $\mathcal{D}$-Modules and the construction of the Gauss-Manin connection for smooth morphisms.

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Introduction

In the dictionary between the language of (algebraic integrable) connections and that of (algebraic) $\mathcal{D}$-modules, the operations of direct and inverse images for a smooth morphism are very important. To compare the definitions of inverse images for connections and $\mathcal{D}$-modules is easy. But the comparison between direct images for connections (the classical construction of the Gauss-Manin connection for smooth morphisms, see [8], [7]) and for $\mathcal{D}$-modules, although known to specialists, has been explicitly proved only recently in a paper of Dimca, Maaref, Sabbah and Saito in 2000 (see [5]), where the authors' main technical tool was M. Saito's equivalence between the derived category of $\mathcal{D}$-modules and a localized category of differential complexes.

The aim of this short paper is to give a simplified summary of the [5] argument, and to propose an alternative proof of this comparison which is simpler, in the sense that it does not use Saito equivalence. Moreover, our alternative strategy of comparison works in a context which is a precursor to the Gauss-Manin connection (at the level of $f^{-1}(\mathcal{D}_Y)$-modules, for a morphism $f : X \to Y$), and may be of some intrinsic interest. In particular we prove that the usual left $f^{-1}(\mathcal{D}_Y)$-module structure of the transfer module $\mathcal{D}_{Y \leftarrow X}$ (used in the definition of derived direct image for $\mathcal{D}$-modules) coincides with the structure induced via a quasi-isomorphism from the relative De Rham complex of $\mathcal{D}_X$ (this structure is just the morphism, in derived category, which induces the Gauss-Manin connection after applying $\mathbf{R}f_*$). We will then deduce the comparison theorem.

In section 1 we recall some generalities on connections and $\mathcal{D}$-modules ([1], [3], [10]). In section 2 we compare the operations of "inverse image" for connections and $\mathcal{D}$-modules. Section 3 is devoted to the comparison of the Gauss-Manin connection (in the case of smooth morphisms) with the notion of direct image for $\mathcal{D}$-modules: we supply a simplified summary of the [5] argument. Finally, in the last section we propose our alternative proof of this comparison which does not use Saito equivalence.

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1. Generalities on connections and $\mathcal{D}$-modules

Let X be a smooth K-variety of pure dimension $d_X = \dim X$, where K is a field of characteristic 0. Following the terminology of [6], *IV*, §16 we denote by Ω^1_X the $\mathcal{O}_X$-module of differentials, by $\mathcal{P}^1_X$ the $\mathcal{O}_X$-algebra of principal parts of order one (1-jets): its two structures as $\mathcal{O}_X$-algebra (induced by the projections p_1, p_2 on $X \times X$) will be referred to as the "left" and "right" structures, and tensor products will be specified by the position of the $\mathcal{P}^1_X$ factor. Let us recall that the difference of the inclusions i_1, i_2 of $\mathcal{O}_X$ in $\mathcal{P}^1_X$ induced by p_1, p_2 gives the differential $d = i_2 - i_1 : \mathcal{O}_X \to \Omega^1_X$ (i.e. $d(x) = 1 \otimes x - x \otimes 1$).

We also use $\mathcal{D}er_X$ or Θ_X to denote the $\mathcal{O}_X$-module of derivations ($\mathcal{O}_X$-dual of Ω^1_X, endowed with the usual structure of Lie-algebra), and $\mathcal{D}_X$ to indicate the graded (left) $\mathcal{O}_X$-algebra of differential operators. On $\mathcal{D}_X$ we consider the increasing filtration F defined by the order of differential operators. Then the associated graded $\mathcal{O}_X$-algebra, denoted by $\mathrm{Gr}\mathcal{D}_X$, is commutative and it is generated (as $\mathcal{O}_X$-algebra) by $\mathcal{D}er_X \subseteq F^1\mathcal{D}_X$.

For any $\mathcal{O}_X$-module $\mathcal{E}$ we will use the standard notation $\mathcal{P}^1_X(\mathcal{E})$ for $\mathcal{P}^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$, where the tensor product involves the right $\mathcal{O}_X$-module structure of $\mathcal{P}^1_X$, while the $\mathcal{O}_X$-module structure is given by the left $\mathcal{O}_X$-module structure on $\mathcal{P}^1_X$.

1.1. Connections and $\mathcal{D}$-modules. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. The following supplementary structures on $\mathcal{E}$ are equivalent:

(i) a connection, that is a morphism of abelian sheaves $\nabla : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ which satisfies the Leibniz rule with respect to sections of $\mathcal{O}_X$, plus the integrability condition, that is $\nabla^2 = 0$ for the natural extension of ∇ to the De Rham sequence;

(ii) an $\mathcal{O}_X$-linear section $\delta : \mathcal{E} \to \mathcal{P}^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ of the canonical morphism $\pi : \mathcal{P}^1_X \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E}$ extending to a stratification in the sense of [2], 2.10;

(iii) an $\mathcal{O}_X$-linear Lie-algebra homomorphism $\Delta : \mathcal{D}er_X \to \mathcal{D}iff_X(\mathcal{E})$ (for the usual Lie-algebra structures), where $\mathcal{D}iff_X(\mathcal{E})$ is the sheaf of differential operators of $\mathcal{E}$;

(iv) a structure of left $\mathcal{D}_X$-module on $\mathcal{E}$.

The dictionary between these equivalent structures is well explained in [2], 2.9, 2.11, 2.15; let us give a sketch.

If $c = c_X(\mathcal{E}) : \mathcal{E} \to \mathcal{P}^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$ denotes the inclusion induced by i_2 (1-jets), then $\delta = c - \nabla$ and $\nabla = c - \delta$.

For any ∂ section of $\mathcal{D}er_X$ the morphism Δ is defined by $\Delta_\partial = (\partial \otimes \mathrm{id}) \circ \nabla$, i.e. $\Delta_\partial(e) = \langle \partial, \nabla(e) \rangle$. On the other hand, the reconstruction of ∇ from Δ involves a description using local coordinates x_i on X (dx_i and ∂_i are the dual bases of differentials and derivations): if e is a section of $\mathcal{E}$, then $\nabla(e) = \sum_i dx_i \otimes \Delta_{\partial_i}(e)$.

The morphism Δ is equivalent to the data of a left $\mathcal{D}_X$-module structure on $\mathcal{E}$ since it extends to a left action of $\mathcal{D}_X$ on $\mathcal{E}$ (see [3],VI,1.6).

In fact the datum of a connection (without the integrability condition) is equivalent to the datum of a section of π (without further conditions), as explained in [4],I,2.3, and in that correspondence, since K is of characteristic 0, integrable connections correspond to sections extending to stratifications (see [2], 2.15).

From now on, the word connection means integrable connection, that is connection satisfying the integrability condition.

1.2. Morphisms. A morphism of connections on X is an $\mathcal{O}_X$-linear morphism $h : \mathcal{E} \to \mathcal{E}'$ compatible with the data, that is, such that $\nabla' \circ h = (\mathrm{id} \otimes h) \circ \nabla$, or $\delta' \circ h = (\mathrm{id} \otimes h) \circ \delta$, or equivalently $\Delta'_\partial \circ h = h \circ \Delta_\partial$ for any section ∂ of $\mathcal{D}er_X$, or finally which is $\mathcal{D}_X$-linear.

2. Inverse image for connections and $\mathcal{D}$-modules

Let $f : X \to Y$ be a finite type morphism of smooth K-varieties. For any $\mathcal{O}_Y$-module $\mathcal{E}$, let $f^*(\mathcal{E}) = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{E})$ be its inverse image by f.

2.1. Inverse image for connections. The easiest definition for the inverse image by f of a connection $\mathcal{E}$ on Y is given in terms of $\mathcal{O}_Y$-linear maps. If $\delta : \mathcal{E} \to \mathcal{P}^1_Y \otimes_{\mathcal{O}_Y} \mathcal{E}$ is the $\mathcal{O}_Y$-linear section defining the connection, let $f^*\delta$ be the composition of the inverse image of δ with the canonical morphism $f^*(\mathcal{P}^1_Y) \to \mathcal{P}^1_X$ (which is left $\mathcal{O}_X$-linear and right $f^{-1}(\mathcal{O}_Y)$-linear). Then we have a morphism

$$f^*\delta : f^*\mathcal{E} \to \mathcal{P}^1_X \otimes_{\mathcal{O}_X} f^*\mathcal{E} \cong \mathcal{P}^1_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{E}) \; ,$$

which is clearly an $\mathcal{O}_X$-linear section of the canonical map $\pi : \mathcal{P}^1_X \otimes_{\mathcal{O}_X} f^*\mathcal{E} \to f^*\mathcal{E}$.

An explicit description of the connection $f^*\nabla$ on $f^*\mathcal{E}$ can be given in the following way:

$$\begin{aligned}(f^*\nabla)(\alpha\otimes e) &= (c_X - f^*\delta)(\alpha\otimes e)\\ &= c_X(\alpha\otimes e) - \alpha f^*\delta(1\otimes e)\\ &= \mathbb{I}\otimes(\alpha\otimes e) - \alpha f^*((c_Y-\nabla)(e))\\ &= \mathbb{I}\otimes(\alpha\otimes e) - \alpha(\mathbb{I}\otimes e) + \alpha f^{-1}(\nabla(e))\\ &= 1\otimes\alpha\otimes e - \alpha\otimes 1\otimes e + \alpha\nabla(e)\\ &= d(\alpha)\otimes e + \alpha\nabla(e)\end{aligned}$$

(as usual, α is a section of $\mathcal{O}_X$ and e is a section of $\mathcal{E}$, or $f^{-1}(\mathcal{E})$).

2.2. Inverse image for $\mathcal{D}$-modules. Let $\mathcal{M}$ be a left $\mathcal{D}_Y$-module. The inverse image as $\mathcal{O}$-modules $f^*\mathcal{M} = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{M})$ locally admits an action of $\mathcal{D}_X$ defined by

$$\alpha'(\alpha\otimes m) = (\alpha'\alpha)\otimes m \quad\text{and}\quad \partial(\alpha\otimes m) = \partial(\alpha)\otimes m + \alpha\big(\textstyle\sum_i \partial(y_i)\otimes\eta_i(m)\big)$$

where ∂ is a section of $\mathcal{D}er_X$, m a section of $\mathcal{M}$ (or $f^{-1}(\mathcal{M})$), α, α' sections of $\mathcal{O}_X$ (y_i local coordinates on Y and η_i the dual basis of dy_i). These local definitions globalize to a $\mathcal{D}_X$-module structure on $f^*\mathcal{M}$ (see [3],VI.4).

In this way the $\mathcal{O}_X$-module $f^*\mathcal{D}_Y = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y)$ is endowed with a structure of left $\mathcal{D}_X$-module, compatible with the obvious structure of $f^{-1}(\mathcal{D}_Y)$-module (by right multiplication). With this structure, $f^*\mathcal{D}_Y$ is usually denoted by $\mathcal{D}_{X\to Y}$. Now, the inverse image of a left $\mathcal{D}_Y$-module $\mathcal{M}$ can be defined as $f^*\mathcal{M} = \mathcal{D}_{X\to Y}\otimes_{f^{-1}(\mathcal{D}_Y)} f^{-1}(\mathcal{M})$, taking account also of the $\mathcal{D}_X$-module structure.

We remark for future reference that the canonical morphism $\mathcal{D}_f : \mathcal{D}_X \to f^*\mathcal{D}_Y = \mathcal{D}_{X\to Y}$ (induced by restriction to $\mathcal{O}_X$ of the action on $\mathcal{D}_{X\to Y}$) is left $\mathcal{D}_X$-linear and right $f^{-1}(\mathcal{O}_Y)$-linear (the first one by definition, the second one by compatibility of the two actions).

2.3. Comparison. Let $\mathcal{M}$ be a $\mathcal{D}_Y$-module. We regard it as a connection on Y and consider its inverse image as a connection. The action of derivations is described in terms of local coordinates y_i on Y, by

$$\begin{aligned}(f^*\Delta)_\partial(\alpha\otimes m) &= \langle\partial, (f^*\nabla)(\alpha\otimes m)\rangle\\ &= \langle\partial, \alpha\nabla(m) + d(\alpha)\otimes m\rangle\\ &= \langle\partial, \alpha(\textstyle\sum_i dy_i\otimes\Delta_{\eta_i}(m)) + d(\alpha)\otimes m\rangle\\ &= \alpha\big(\textstyle\sum_i \partial(y_i)\otimes\Delta_{\eta_i}(m)\big) + \partial(\alpha)\otimes m\end{aligned}$$

where ∂ is a section of $\mathcal{D}er_X$, m is a section of $\mathcal{M}$ (or $f^{-1}(\mathcal{M})$), α is a section of $\mathcal{O}_X$ (and η_i is the dual basis of dy_i). Therefore the local descriptions make clear that for a connection $\mathcal{E}$ on Y, its inverse image as a connection induces the structure of $\mathcal{D}_X$-module given by the inverse image of the corresponding $\mathcal{D}_Y$-module. Notice moreover that, since f is smooth, the functor f^* is exact in the category of $\mathcal{D}_Y$-modules (see [3],VI.4.8), so that for any left $\mathcal{D}_Y$-module $\mathcal{M}$ we have $f^*(\mathcal{M}) = \mathbf{L}f^*(\mathcal{M})$ (and the comparison can be proved avoiding the derived categories).

3. Direct image for connections and $\mathcal{D}$-modules (and comparison following [5])

Let $f : X \to Y$ be a smooth morphism of smooth K-varieties. In order to compare the notions of (derived) direct images in the category of connections (the Gauss-Manin connections) and in the category of $\mathcal{D}$-modules, we need some preliminary materials, most concerning right $\mathcal{D}$-modules, De Rham functors, differential complexes (and the M.Saito equivalence).

3.1. Right and left $\mathcal{D}$-modules. We denote by $\mathcal{D}$-Mod the category of left $\mathcal{D}$-modules and by Mod-$\mathcal{D}$ the category of right $\mathcal{D}$-modules. It is well known that $\omega_X = \Omega_X^{\dim X}$ has a canonical structure of right $\mathcal{D}_X$-module (action of vector fileds through Lie derivative, see [3],VI,3.2). Let us define $\omega_X(\mathcal{D}_X) = \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$. It is endowed with two different structures of right $\mathcal{D}_X$-module (which commute): the first comes from the right multiplication on $\mathcal{D}_X$ and the other is induced by the tensor product over $\mathcal{O}_X$ of a right and a left $\mathcal{D}_X$-module (given for any vector field ∂ by the rule $m \otimes n \mapsto (m\partial) \otimes n - m \otimes (\partial n)$). There exists a unique involution $\iota : \omega_X(\mathcal{D}_X) \to \omega_X(\mathcal{D}_X)$ which is the identity on ω_X and exchanges these two right $\mathcal{D}_X$-module structures (see [11],1.7, using local coordinates x_i on X, the involution sends $\omega \otimes P$ to $\omega \otimes P^*$, where $\omega = dx_1 \wedge \cdots \wedge dx_{d_X}$, and P^* is the transposition of P, defined by $\alpha^* = \alpha$ for sections of $\mathcal{O}_X$, $\partial_i^* = -\partial_i$, and $(PQ)^* = Q^*P^*$). In the same way, we define $\omega_X^{-1}(\mathcal{D}_X) = \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1} = \mathcal{H}om_{\mathcal{D}_X}(\omega_X(\mathcal{D}_X), \mathcal{D}_X)$ and we notice that $\omega_X^{-1}(\mathcal{D}_X)$ has two compatible and "interchangeable" structures of left $\mathcal{D}_X$-module.

We have an equivalence of categories between $\mathcal{D}_X$-Mod and Mod-$\mathcal{D}_X$ given by the quasi-inverse functors:

$$\begin{array}{rcl} \mathcal{D}_X\text{-Mod} & \longleftrightarrow & \text{Mod-}\mathcal{D}_X \\ \mathcal{M} & \longmapsto & \omega_X(\mathcal{M}) \cong \omega_X(\mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathcal{M} \\ \mathcal{N} \otimes_{\mathcal{D}_X} \omega_X^{-1}(\mathcal{D}_X) \cong \omega_X^{-1}(\mathcal{N}) & \longleftarrow\!\shortmid & \mathcal{N}\ . \end{array}$$

Notice that the first functor does not depend on the right $\mathcal{D}_X$-module structure of $\omega_X(\mathcal{D}_X)$ used in the tensor product (the other one inducing the right $\mathcal{D}_X$-module structure of $\omega_X(\mathcal{M})$), and similarly for its quasi-inverse functor.

3.2. De Rham functor for right and left $\mathcal{D}$-modules. Let $(\mathcal{E},\nabla)$ be a connection on X. By definition its De Rham complex is $\Omega^\bullet_X(\mathcal{E}) = \Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{E}$ where the differentials are induced by the connection ∇ as usual: $\nabla(\omega\otimes e) = d(\omega)\otimes e + (-)^{\deg\omega}\omega\wedge\nabla(e)$.

The De Rham functor for left $\mathcal{D}$-modules is defined to be compatible with the notion of De Rham complex for connections, up to a shift. Let us consider $\mathcal{D}_X$ as a left $\mathcal{D}_X$-module, then its De Rham complex as a connection is $\Omega^\bullet_X(\mathcal{D}_X) = \Omega^\bullet_X\otimes_{\mathcal{O}_X}\mathcal{D}_X$ (usual differentials). It is a resolution of $\omega_X[-\dim X]$ in Mod-$\mathcal{D}_X$. For this reason, it is usual to define $\mathrm{DR}_X(\mathcal{D}_X) = \Omega^\bullet_X(\mathcal{D}_X)[\dim X]$, so that $\mathrm{DR}_X(\mathcal{D}_X)$ is a resolution of ω_X in Mod-$\mathcal{D}_X$.

Now, if $\mathcal{M}$ is a left $\mathcal{D}_X$-module we define $\mathrm{DR}_X(\mathcal{M}) = \mathrm{DR}_X(\mathcal{D}_X)\otimes_{\mathcal{D}_X}\mathcal{M}$ which is a complex of K_X-vector spaces. This functor extends to complexes and gives in the derived categories the functor $\mathrm{DR}_X : \mathbf{D}(\mathcal{D}_X\text{-Mod}) \to \mathbf{D}(K_X)$ (where $\mathbf{D}(K_X)$ is the derived category of sheaves of K_X-vector spaces). Let us observe that for any $\mathcal{M}\in\mathbf{D}(\mathcal{D}_X\text{-Mod})$ we have $\mathrm{DR}_X(\mathcal{M}) \cong \omega_X\otimes^{\mathbf{L}}_{\mathcal{D}_X}\mathcal{M}$.

The De Rham functor for right $\mathcal{D}$-modules is defined to be compatible with the left/right equivalence. Let us consider $\mathcal{D}_X$ as a right $\mathcal{D}_X$-module, then its Spencer complex is $\Theta^\bullet_X(\mathcal{D}_X) = \mathcal{D}_X\otimes_{\mathcal{O}_X}\Theta^\bullet_X$ (where $\Theta^\bullet_X = \bigwedge^{-\bullet}\mathcal{D}er_X$ and the differentials are locally defined by a Koszul complex). It is a resolution of $\mathcal{O}_X$ in $\mathcal{D}_X$-Mod. Now, if $\mathcal{N}$ is a right $\mathcal{D}_X$-module, then $\mathrm{DR}_X(\mathcal{N}) = \mathcal{N}\otimes_{\mathcal{D}_X}\Theta^\bullet_X(\mathcal{D}_X)$ as a functor Mod-$\mathcal{D}_X\to\mathbf{C}(K_X)$. This definition naturally extends to the category of complexes of Mod-$\mathcal{D}_X$, and to its derived category as before.

The compatibility between De Rham functors is expressed by the relations $\mathrm{DR}_X(\mathcal{M}) = \mathrm{DR}_X(\omega_X(\mathcal{M}))$ and $\mathrm{DR}_X(\mathcal{N}) = \mathrm{DR}_X(\omega_X^{-1}(\mathcal{N}))$.

3.2.1. Relative De Rham functor. Let $f : X\to Y$ be a smooth morphism between smooth K-varieties. The morphism $f^*(\Omega^1_Y)\to\Omega^1_X$ induces a canonical short exact sequence

$$0\to f^*(\Omega^1_Y)\to\Omega^1_X\to\Omega^1_{X/Y}\to 0 \qquad (\mathbf{3.2.2})$$

where $\Omega^1_{X/Y}$ is the sheaf of relative differential forms of degree one. Moreover any $\mathcal{O}_X$-module in (3.2.2) is locally free of finite type so (3.2.2) locally splits. Let $\Theta_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/Y},\mathcal{O}_X)$ be the $\mathcal{O}_X$-dual of $\Omega^1_{X/Y}$ and let $\mathcal{D}_{X/Y}$ denote the $\mathcal{O}_X$-algebra generated by $\mathcal{O}_X$ and $\Theta_{X/Y}$. As in 1.1 for any $\mathcal{O}_X$-module $\mathcal{E}$ the following supplementary structures on $\mathcal{E}$ are equivalent:

(1) an integrable relative connection, that is a morphism of $f^{-1}(\mathcal{O}_Y)$-modules $\nabla_{X/Y} : \mathcal{E}\to\Omega^1_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{E}$ which satisfies the Leibniz rule with respect to sections of $\mathcal{O}_X$ and such that $\nabla^2_{X/Y} = 0$ for the natural extension of $\nabla_{X/Y}$;

(2) a structure of left $\mathcal{D}_{X/Y}$-module on $\mathcal{E}$.

For any $\mathcal{O}_X$-module $\mathcal{E}$ endowed with a relative integrable connection $\nabla_{X/Y}$ let us define its relative De Rham complex as $\Omega^\bullet_{X/Y}(\mathcal{E}) = \Omega^\bullet_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{E}$ where the differentials are induced by $\nabla_{X/Y}$.

The relative De Rham functor for left $\mathcal{D}_{X/Y}$-modules is defined to be compatible with the De Rham functor for connections up to a shift (as in the case of $\mathcal{D}_X$-modules), and it induces a functor of derived categories:

$$\begin{array}{rcl} \mathrm{DR}_{X/Y}: \ \mathbf{D}(\mathcal{D}_{X/Y}) & \longrightarrow & \mathbf{D}(f^{-1}(\mathcal{O}_Y)) \\ \mathcal{E} & \longmapsto & \mathrm{DR}_{X/Y}(\mathcal{E}) = \Omega^\bullet_{X/Y}(\mathcal{E})[d_{X/Y}] \end{array}$$

where $d_{X/Y}$ is the relative dimension $d_X - d_Y$. Let us recall that, in the case of a projection $f: X = Y \times Z \to Y$, the relative De Rham complex is $f^{-1}(\mathcal{D}_Y)$-linear.

3.3. Direct images for connections (the Gauss-Manin connections). The Leray filtration Ler on $\Omega^\bullet_X$ is defined (see [7], [8]) by

$$\mathrm{Ler}^p \Omega^\bullet_X = \mathrm{Im}(f^*\Omega^p_Y \otimes_{\mathcal{O}_X} \Omega^{\bullet-p}_X \to \Omega^\bullet_X)$$

and, since f is a smooth morphism, the associated graded $\mathcal{O}_X$-module has $\mathrm{Gr}^p_{\mathrm{Ler}} \Omega^\bullet_X \cong f^*\Omega^p_Y \otimes_{\mathcal{O}_X} \Omega^{\bullet-p}_{X/Y}$.

If $\mathcal{E}$ is a connection, we define the Leray filtration on its De Rham complex $\Omega^\bullet_X(\mathcal{E})$ by the tensor product: $\mathrm{Ler}^p \Omega^\bullet_X(\mathcal{E}) = \mathrm{Ler}^p \Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{E}$. Therefore the graded pieces are $\mathrm{Gr}^p_{\mathrm{Ler}} \Omega^\bullet_X(\mathcal{E}) = f^*\Omega^p_Y \otimes_{\mathcal{O}_X} \Omega^{\bullet-p}_{X/Y}(\mathcal{E})$.

The Leray filtration induces the Leray spectral sequence for the direct image functor by f:

$$\textbf{(3.3.1)} \qquad E_1^{p,q} = \Omega^p_Y(\mathbf{R}^{p+q} f_* \Omega^{\bullet-p}_{X/Y}(\mathcal{E})) \Longrightarrow \mathbf{R}^n f_* \Omega^\bullet_X(\mathcal{E})$$

in the category of $\mathcal{O}_Y$-modules with differential operators (the complexes appearing in the spectral sequences are differential complexes on Y, see 3.5 below). For $p = 0$ the differentials $d_1^{p,q}$:

$$\textbf{(3.3.2)} \quad E_1^{p,q} = \Omega^p_Y(\mathbf{R}^{p+q} f_* \Omega^{\bullet-p}_{X/Y}(\mathcal{E})) \longrightarrow E_1^{p+1,q} = \Omega^{p+1}_Y(\mathbf{R}^{p+q+1} f_* \Omega^{\bullet-p-1}_{X/Y}(\mathcal{E}))$$

define the Gauss-Manin connections on the $\mathcal{O}_Y$-modules $R^q f_* \Omega^\bullet_{X/Y}(\mathcal{E})$ (it is explicitly given by the connecting homomorphism for the direct image functor of the short exact sequence of complexes

$$\textbf{(3.3.3)} \quad 0 \longrightarrow \mathrm{Gr}^{p+1}_{\mathrm{Ler}} \Omega^\bullet_X(\mathcal{E}) \longrightarrow \mathrm{Ler}^p \Omega^\bullet_X(\mathcal{E}) / \mathrm{Ler}^{p+2} \Omega^\bullet_X(\mathcal{E}) \longrightarrow \mathrm{Gr}^p_{\mathrm{Ler}} \Omega^\bullet_X(\mathcal{E}) \longrightarrow 0$$

which gives a piece of E_1). Additional details and computations will be given in section 4.1.

3.4. Direct images for $\mathcal{D}$-modules. The direct image for $\mathcal{D}$-modules is defined using the following transfer modules:

(3.4.1) $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y)$ which is in $\mathcal{D}_X$-Mod-$f^{-1}(\mathcal{D}_Y)$ (the left $\mathcal{D}_X$-module structure is induced by (the tensor with) that of $\mathcal{O}_X$, the right $f^{-1}(\mathcal{D}_Y)$-module structure is induced by that of $f^{-1}(\mathcal{D}_Y)$, and the compatibility is obvious);

(3.4.2) $\mathcal{D}_{Y\leftarrow X} = \omega_X(\mathcal{D}_X)\otimes_{\mathcal{D}_X}\mathcal{D}_{X\to Y}\otimes_{f^{-1}(\mathcal{D}_Y)} f^{-1}\omega_Y^{-1}(\mathcal{D}_Y)$ which is an object of $f^{-1}(\mathcal{D}_Y)$-Mod-$\mathcal{D}_X$ since it is obtained from $\mathcal{D}_{X\to Y}$ by a double left/right exchange.

For a right $\mathcal{D}_X$-module $\mathcal{N}$, the direct image by f is defined by $f_+\mathcal{N} = \mathbf{R}f_*(\mathcal{N}\otimes^{\mathbf{L}}_{\mathcal{D}_X}\mathcal{D}_{X\to Y})$ as a right $\mathcal{D}_Y$-module. For a left $\mathcal{D}_X$-module $\mathcal{M}$, the direct image by f is defined by $f_+\mathcal{M} = \mathbf{R}f_*(\mathcal{D}_{Y\leftarrow X}\otimes^{\mathbf{L}}_{\mathcal{D}_X}\mathcal{M})$ as a left $\mathcal{D}_Y$-module. The compatibility of the two definitions is expressed by the following relations:

$$\omega_Y^{-1}(f_+(\mathcal{N}))\cong f_+(\omega_X^{-1}(\mathcal{N}))\quad\text{and}\quad \omega_Y(f_+(\mathcal{M}))\cong f_+(\omega_X(\mathcal{M}))\ .$$

3.5. Differential complexes and M.Saito equivalence. Following M. Saito let $\mathcal{O}_X$-Diff be the category of $\mathcal{O}_X$-modules with differential operators as morphisms and let $\mathbf{C}(\mathcal{O}_X\text{-Diff})$ be its category of complexes (see [5]). Objects in $\mathbf{C}(\mathcal{O}_X\text{-Diff})$ are called differential complexes on X. Any object in $\mathbf{C}(\mathcal{O}_X\text{-Diff})$ could be regarded as a complex of K_X-vector spaces so that there is a functor $F:\mathbf{C}(\mathcal{O}_X\text{-Diff})\to\mathbf{C}(K_X)$.

In [11],1.3.2, M.Saito defined the linearization functor $\widetilde{\mathrm{DR}}_X^{-1}:\mathbf{C}(\mathcal{O}_X\text{-Diff})\to\mathbf{C}(\text{Mod-}\mathcal{D}_X)$ acting on the differential complex $\mathcal{L}$ by $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X$ (the differentials being extended to $\mathcal{D}_X$-linear maps). By localization with respect to the multiplicative system of quasi-isomorphisms on the right hand side and with respect to their pull-back on the left hand side (that is, the multiplicative system of $\mathcal{D}_X$-quasi-isomorphisms: morphisms of differential complexes whose linearization is a quasi-isomorphism in the category of right $\mathcal{D}_X$-modules) one obtains the functor $\widetilde{\mathrm{DR}}_X^{-1}:\mathbf{D}(\mathcal{O}_X\text{-Diff})\to\mathbf{D}(\text{Mod-}\mathcal{D}_X)$.

It is clear that for any right (resp. left) $\mathcal{D}_X$-module $\mathcal{N}$ (resp. $\mathcal{M}$), its De Rham complex is an object in $\mathbf{C}(\mathcal{O}_X\text{-Diff})$. In particular the functor DR_X factors through $\mathbf{C}(\mathcal{O}_X\text{-Diff})$ and we denote this factorization by $\widetilde{\mathrm{DR}}_X:\mathbf{C}(\text{Mod-}\mathcal{D}_X)\to\mathbf{C}(\mathcal{O}_X\text{-Diff})$. We have the following commutative diagram of functors

$$\begin{array}{ccc}
\mathbf{C}(\text{Mod-}\mathcal{D}_X) & \xrightarrow{\widetilde{\mathrm{DR}}_X} & \mathbf{C}(\mathcal{O}_X\text{-Diff}) \\
 & {\scriptstyle \mathrm{DR}_X}\searrow & \downarrow{\scriptstyle F} \\
 & & \mathbf{C}(K_X)\ .
\end{array}$$

M.Saito also proved that $\widetilde{\mathrm{DR}}_X$ localizes with respect to quasi-isomorphisms on the left hand side and with respect to $\mathcal{D}_X$-quasi-isomorphisms on the right hand side, so it induces a functor $\widetilde{\mathrm{DR}}_X:\mathbf{D}(\text{Mod-}\mathcal{D}_X)\to\mathbf{D}(\mathcal{O}_X\text{-Diff})$.

Proposition 3.5.1 (M.Saito's equivalence). *The functor* $\widetilde{\mathrm{DR}}_X^{-1}$ *is an equivalence of categories whose quasi-inverse is* $\widetilde{\mathrm{DR}}_X$. *In particular we have canonical quasi-isomorphisms*

$$\widetilde{\mathrm{DR}}_X^{-1}\widetilde{\mathrm{DR}}_X(\mathcal{N})\cong\mathcal{N}\qquad\textit{and}\qquad\widetilde{\mathrm{DR}}_X^{-1}\widetilde{\mathrm{DR}}_X(\mathcal{M})\cong\omega_X(\mathcal{M})\ ,$$

both in $\mathbf{C}(\text{Mod-}\mathcal{D}_X)$ *(see* [11]*,1.8).*

Remark 3.5.2. In the case of right $\mathcal{D}_X$-modules, there is also a compatibility with the direct image of differential complexes (which is induced by the usual direct image for abelian sheaves), via the linearization functor:

$$\widetilde{\mathrm{DR}}_Y^{-1}\mathbf{R}f_*(\mathcal{L}) = f_+\widetilde{\mathrm{DR}}_X^{-1}(\mathcal{L})$$

(see [5],1.3.2).

3.6. Comparison for direct images, following [5].

Theorem. *Let* $f : X \to Y$ *be a smooth morphism of smooth* K*-varieties. For any left* $\mathcal{D}_X$*-module* $\mathcal{M}$ *(identified with a connection on* X*) and for any* q *we have natural isomorphisms* $\mathbf{R}^q f_* \mathrm{DR}_{X/Y}(\mathcal{M}) \cong \mathcal{H}^q(f_+\mathcal{M})$ *in the category of left* $\mathcal{D}_Y$*-modules, where the left hand side has the structure of the Gauss-Manin connection.*

Proof. Let consider the Leray spectral sequence E of $\mathcal{M}$ with respect to f. Since $\mathcal{D}_Y$ is a flat $\mathcal{O}_Y$-module, we may apply the linearization functor $\widetilde{\mathrm{DR}}_Y^{-1}$ to obtain the spectral sequence

$$\widetilde{\mathrm{DR}}_Y^{-1} E_1^{p,q} = \widetilde{\mathrm{DR}}_Y^{-1} \Omega_Y^p(\mathbf{R}^{p+q} f_* \Omega^{\bullet - p}_{X/Y}(\mathcal{M})) \Longrightarrow \widetilde{\mathrm{DR}}_Y^{-1} \mathbf{R}^n f_* \Omega^\bullet_X(\mathcal{M})$$

in the category of right $\mathcal{D}_Y$-modules. Now (by 3.5.1) the complex

$$\widetilde{\mathrm{DR}}_Y^{-1} E_1^{\bullet,q} = \widetilde{\mathrm{DR}}_Y^{-1}\widetilde{\mathrm{DR}}_Y[-\dim Y](\mathbf{R}^q f_* \mathrm{DR}_{X/Y}[-d_{X/Y}](\mathcal{M}))$$

is quasi-isomorphic to

$$\omega_Y(\mathbf{R}^q f_* \mathrm{DR}_{X/Y}(\mathcal{M}))[-\dim Y - d_{X/Y}]$$

(where $d_{X/Y} = \dim X - \dim Y$ is the relative dimension) so that the spectral sequence degenerates at E_2; while the limit is quasi-isomorphic (by 3.5.2, 3.5.1 and 3.4) to

$$\begin{aligned}\widetilde{\mathrm{DR}}_Y^{-1}\mathbf{R}f_*\widetilde{\mathrm{DR}}_X[-\dim X](\mathcal{M}) &\cong f_+(\widetilde{\mathrm{DR}}_X^{-1}\widetilde{\mathrm{DR}}_X[-\dim X](\mathcal{M})) \\ &\cong f_+(\omega_X(\mathcal{M})[-\dim X]) \\ &\cong \omega_Y(f_+(\mathcal{M})[-\dim X])\ .\end{aligned}$$

So we have the isomorphisms $\mathbf{R}^q f_* \mathrm{DR}_{X/Y}(\mathcal{M}) \cong \mathcal{H}^q(f_+(\mathcal{M}))$ in the category of left $\mathcal{D}_Y$-modules. □

4. Alternative proof of the comparison between direct images

We present here an alternative proof of the comparison theorem for direct images, which is in some sense more elementary, since Saito's equivalence is not used. The strategy we discuss here has also the advantage of clarifying the structure of the Gauss-Manin connection, taking account of one of its avatars before the application of the derived direct image functor. In fact we construct a distinguished triangle, involving the relative De Rham complex, whose connecting morphism induces the Gauss-Manin connection (after applying the derived direct image) on one side, and the connection associated to the usual left $f^{-1}(\mathcal{D}_Y)$-module structure of the transfer module on the other side. From this fact, we will deduce the comparison theorem. The main technical tool is the commutativity of a diagram in the derived category of right $\mathcal{D}_X$-modules (see Proposition 4.3.2), for which the homotopy lemma 4.3.1 is used.

4.1. The distinguished triangle for the Gauss-Manin connection. Let us consider the exact sequence of complexes of right $\mathcal{D}_X$-modules (3.3.3) for $p = 0$ and $\mathcal{E} = \mathcal{D}_X$:

(**4.1.0**)

$$0 \longrightarrow f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\Omega^{\bullet-1}_{X/Y}(\mathcal{D}_X) \xrightarrow{i} \mathrm{Ler}^0\Omega^\bullet_X(\mathcal{D}_X)/\mathrm{Ler}^2\Omega^\bullet_X(\mathcal{D}_X) \xrightarrow{\pi} \Omega^\bullet_{X/Y}(\mathcal{D}_X) \longrightarrow 0$$

(recall that $\mathrm{Gr}^1_{\mathrm{Ler}}(\Omega^\bullet_X(\mathcal{D}_X)) \cong f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\Omega^{\bullet-1}_{X/Y}(\mathcal{D}_X)$ and $\mathrm{Gr}^0_{\mathrm{Ler}}(\Omega^\bullet_X(\mathcal{D}_X)) \cong \Omega^\bullet_{X/Y}(\mathcal{D}_X)$) which gives the distinguished triangle:

(**4.1.1**)
$$\begin{array}{ccc} f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\Omega^{\bullet-1}_{X/Y}(\mathcal{D}_X) & \xrightarrow{i} & \mathrm{Ler}^0\Omega^\bullet_X(\mathcal{D}_X)/\mathrm{Ler}^2\Omega^\bullet_X(\mathcal{D}_X) \\ & \nwarrow^{[+1]}_{\delta(\mathcal{D}_X)} & \downarrow \pi \\ & & \Omega^\bullet_{X/Y}(\mathcal{D}_X) \end{array}$$

in $\mathbf{D}^b(\text{Mod-}\mathcal{D}_X)$ (derived category of right $\mathcal{D}_X$-modules). Since $\Omega^\bullet_{X/Y}(\mathcal{D}_X)$ is quasi-isomorphic to the mapping cone $MC(i)$ of i, the connecting morphism $\delta(\mathcal{D}_X)$ is represented in the derived category by the following diagram:

(**4.1.2**)
$$\begin{array}{ccc} MC(i) = (f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\Omega^\bullet_{X/Y}(\mathcal{D}_X)) \oplus \mathrm{Ler}^0\Omega^\bullet_X(\mathcal{D}_X)/\mathrm{Ler}^2\Omega^\bullet_X(\mathcal{D}_X) & & \\ \text{qis}\downarrow (0,\pi) & \searrow^{-p_1=(-1,0)} & \\ \Omega^\bullet_{X/Y}(\mathcal{D}_X) & \xrightarrow{\delta(\mathcal{D}_X)} & f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\Omega^\bullet_{X/Y}(\mathcal{D}_X)\ . \end{array}$$

Notice moreover that for any left $\mathcal{D}_X$-module $\mathcal{E}$ we can apply the derived functor

$-\otimes^{\mathbf{L}}_{\mathcal{D}_X}\mathcal{E}$ to the distinguished triangle (4.1.1) so we obtain the distinguished triangle:

$$\begin{array}{ccc} f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\Omega^{\bullet-1}_{X/Y}(\mathcal{E}) & \longrightarrow & \mathrm{Ler}^0\Omega^\bullet_X(\mathcal{E})/\mathrm{Ler}^2\Omega^\bullet_X(\mathcal{E}) \\ & {}_{\delta(\mathcal{E})}\nwarrow^{[+1]} & \downarrow \\ & & \Omega^\bullet_{X/Y}(\mathcal{E}) \end{array}$$

(really, we do not need to take the derived functor $-\otimes^{\mathbf{L}}_{\mathcal{D}_X}\mathcal{E}$ but simply $-\otimes_{\mathcal{D}_X}\mathcal{E}$ because any complex in (4.1.1) is acyclic for $-\otimes_{\mathcal{D}_X}\mathcal{E}$). This is the distinguished triangle induced by the short exact sequence (3.3.3) for $p=0$.

The fundamental fact here is that, by applying the derived functor $\mathbf{R}f_*$ to the connecting morphism

$$\delta(\mathcal{E}):\Omega^\bullet_{X/Y}(\mathcal{E}) \dashrightarrow f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\Omega^\bullet_{X/Y}(\mathcal{E})$$

and the projection formula (the isomorphism $\eta:\mathbf{R}f_*(f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\Omega^\bullet_{X/Y}(\mathcal{E}))\cong \Omega^1_Y\otimes_{\mathcal{O}_Y}\mathbf{R}f_*\Omega^\bullet_{X/Y}(\mathcal{E})$), one obtain a morphism in the derived category of K_Y-modules

$$GM(\mathcal{E})=\eta\circ\mathbf{R}f_*(\delta(\mathcal{E})):\mathbf{R}f_*\Omega^\bullet_{X/Y}(\mathcal{E})\dashrightarrow \Omega^1_Y\otimes_{\mathcal{O}_Y}\mathbf{R}f_*\Omega^\bullet_{X/Y}(\mathcal{E})$$

which for any degree $p\geq 0$ it gives an actual K-linear morphism

$$\mathbf{R}^pf_*\Omega^\bullet_{X/Y}(\mathcal{E})\longrightarrow \Omega^1_Y\otimes_{\mathcal{O}_Y}\mathbf{R}^pf_*\Omega^\bullet_{X/Y}(\mathcal{E})$$

which is the classical Gauss-Manin connection.

4.2. The Gauss-Manin morphism of $\mathcal{D}_{Y\leftarrow X}$. We prove now that the connecting morphism $\delta(\mathcal{D}_X)$ induces an actual K_X-linear morphism $\delta:\mathcal{D}_{Y\leftarrow X}\to f^*(\Omega^1_X)\otimes_{\mathcal{O}_X}\mathcal{D}_{Y\leftarrow X}$, called the Gauss-Manin morphism of $\mathcal{D}_{Y\leftarrow X}$. Next, we will prove that this morphism coincides with the connection induced by the left $f^{-1}(\mathcal{D}_Y)$-module structure of the transfer module.

Lemma 4.2.1 ([9], 5.2.3.4). *Let $f:X\to Y$ be a smooth morphism of smooth K-varieties. There exists a canonical morphism of complexes of right $\mathcal{D}_X$-modules $\lambda:\mathrm{DR}_{X/Y}(\mathcal{D}_X)\to\mathcal{D}_{Y\leftarrow X}$ which is a quasi-isomorphism. In particular, $\mathrm{DR}_{X/Y}(\mathcal{D}_X)$ is a left resolution of $\mathcal{D}_{Y\leftarrow X}$ in the category $f^{-1}(\mathcal{O}_Y)$-Mod-$\mathcal{D}_X$, with locally free right $\mathcal{D}_X$-modules.*

Proof. The canonical morphism λ is defined by the composition

$$\omega_{X/Y}\otimes_{\mathcal{O}_X}\mathcal{D}_X\xrightarrow{\ i\ }\mathcal{D}_{Y\leftarrow X}\otimes_{\mathcal{O}_X}\mathcal{D}_X\xrightarrow{\ \cdot\ }\mathcal{D}_{Y\leftarrow X}$$

where the first map comes from the canonical inclusion of $\omega_{X/Y}$ into $\mathcal{D}_{Y\leftarrow X}=\omega_{X/Y}\otimes_{\mathcal{O}_X}f^*\mathcal{D}_Y$, and the second one uses the canonical structure of right $\mathcal{D}_X$-module of $\mathcal{D}_{Y\leftarrow X}$. This composition is clearly right $\mathcal{D}_X$-linear, and it is also left

$f^{-1}(\mathcal{O}_Y)$-linear since it is obtained by a double left/right exchange of structures starting from the morphism $\mathcal{D}_f$ of 2.2.

A local computation using the canonical filtrations by the order of differential operators shows that the graded pieces are Koszul complexes, so that the assertion follows (see [9], 5.2.3.4 for details). □

Corollary 4.2.2. *Let $f : X \to Y$ be a smooth morphism of smooth K-varieties. For any left $\mathcal{D}_X$-module $\mathcal{M}$ there is a canonical quasi-isomorphism*

$$\lambda(\mathcal{M}) : \mathrm{DR}_{X/Y}(\mathcal{M}) \to \mathcal{D}_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M} \ ,$$

in the derived category of $f^{-1}(\mathcal{O}_Y)$-modules.

Applying the derived direct image functor to the above morphism, one obtains a canonical quasi-isomorphism $\mathbf{R}f_*\mathrm{DR}_{X/Y}(\mathcal{M}) \to f_+(\mathcal{M})$ in the derived category of $\mathcal{O}_Y$-modules.

Notice that the connecting morphism $\delta(\mathcal{D}_X)$ and the quasi-isomorphism λ define a commutative square in the derived category of K_X-modules:

(4.2.3)
$$\begin{array}{ccc}
\mathrm{DR}_{X/Y}(\mathcal{D}_X) & \overset{\delta(\mathcal{D}_X)}{\dashrightarrow} & f^{-1}\Omega^1_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathrm{DR}_{X/Y}(\mathcal{D}_X) \\
{\scriptstyle \mathrm{qis}}\downarrow{\scriptstyle \lambda} & & {\scriptstyle \mathrm{qis}}\downarrow{\scriptstyle 1\otimes\lambda} \\
\mathcal{D}_{Y\leftarrow X} & \xrightarrow{\quad\delta\quad} & f^{-1}\Omega^1_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{D}_{Y\leftarrow X}
\end{array}$$

where δ is an actual K_X-linear morphism (the last row contains complexes concentred in one degree, and the canonical inclusion of a category in its derived category is fully faithful). We call δ the Gauss-Manin morphism of $\mathcal{D}_{Y\leftarrow X}$.

We want to compare δ with $\nabla_{Y\leftarrow X}$, where

$$\nabla_{Y\leftarrow X} : \mathcal{D}_{Y\leftarrow X} \to f^{-1}\Omega^1_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{D}_{Y\leftarrow X}$$

is the connection (Leibniz with respect to section of $f^{-1}(\mathcal{O}_Y)$) induced by the $f^{-1}(\mathcal{D}_Y)$-module structure of $\mathcal{D}_{Y\leftarrow X}$.

4.3. Comparison for $\boldsymbol{\mathcal{D}_{Y\leftarrow X}}$. The following two results are the kernel of our comparison argument; we have to prove that the Gauss-Manin morphism of $\mathcal{D}_{Y\leftarrow X}$ coincides with the connection $\nabla_{Y\leftarrow X}$, that is that $\nabla_{Y\leftarrow X}$ make commutative the diagram (4.2.3). To do that, we re-write the projection p_1 of (4.1.2) up to homotopy.

Lemma 4.3.1 (homotopy lemma). *Let*

$$i : f^*(\Omega^1_Y) \otimes_{\mathcal{O}_X} \Omega^{\bullet-1}_{X/Y}(\mathcal{D}_X) \to \mathrm{Ler}^0\Omega^\bullet_X(\mathcal{D}_X)/\mathrm{Ler}^2\Omega^\bullet_X(\mathcal{D}_X)$$

be the canonical inclusion.

(i) *the identity morphism of the mapping cone of i is homotopic to the morphism $\Psi^\bullet$ defined by $\Psi^{d_{X/Y}} = \begin{pmatrix} 0 & -\phi^{-1}d \\ 0 & 1 \end{pmatrix}$ and $\Psi^q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for $q \neq d_{X/Y}$;*

(ii) *the connecting morphism $-p_1 = (-1, 0)$:*

$$(f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\Omega^\bullet_{X/Y}(\mathcal{D}_X))\oplus \mathrm{Ler}^0\Omega^\bullet_X(\mathcal{D}_X)/\mathrm{Ler}^2\Omega^\bullet_X(\mathcal{D}_X) \to (f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\Omega^\bullet_{X/Y}(\mathcal{D}_X))$$

of the distinguished triangle generated by i is homotopic to the morphism $\psi^\bullet = -p_1\circ\Psi^\bullet$, that is $\psi^{d_{X/Y}} = (0, \phi^{-1}d)$ and $\psi^q = (-1, 0)$ for $q \neq d_{X/Y}$;

where ϕ is the canonical isomorphism $f^*\Omega^1_Y\otimes_{\mathcal{O}_X}\omega_{X/Y}(\mathcal{D}_X) \longrightarrow \frac{\mathrm{Ler}^0}{\mathrm{Ler}^2}(\Omega_X^{d_{X/Y}+1}(\mathcal{D}_X))$ induced by i.

Proof. Let us consider the exact sequence of complexes (4.1.0) in degrees $d_{X/Y} - 1, d_{X/Y}, d_{X/Y}+1$:

$$\begin{array}{ccccccccc}
0 \longrightarrow & f^*\Omega^1_Y\otimes_{\mathcal{O}_X}\Omega^{d_{X/Y}-2}_{X/Y}(\mathcal{D}_X) & \xrightarrow{i} & \frac{\mathrm{Ler}^0}{\mathrm{Ler}^2}(\Omega^{d_{X/Y}-1}_X(\mathcal{D}_X)) & \xrightarrow{\pi} & \Omega^{d_{X/Y}-1}_{X/Y}(\mathcal{D}_X) & \longrightarrow 0 \\
 & \downarrow d & & \downarrow d & & \downarrow d & \\
0 \longrightarrow & f^*\Omega^1_Y\otimes_{\mathcal{O}_X}\Omega^{d_{X/Y}-1}_{X/Y}(\mathcal{D}_X) & \xrightarrow{i} & \frac{\mathrm{Ler}^0}{\mathrm{Ler}^2}(\Omega^{d_{X/Y}}_X(\mathcal{D}_X)) & \xrightarrow{\pi} & \omega_{X/Y}(\mathcal{D}_X) & \longrightarrow 0 \\
 & \downarrow d & & \downarrow d & & \downarrow & \\
0 \longrightarrow & f^*\Omega^1_Y\otimes_{\mathcal{O}_X}\omega_{X/Y}(\mathcal{D}_X) & \underset{\cong}{\xrightarrow{\phi}} & \frac{\mathrm{Ler}^0}{\mathrm{Ler}^2}(\Omega^{d_{X/Y}+1}_X(\mathcal{D}_X)) & \longrightarrow & 0 & \longrightarrow 0
\end{array}$$

from the third line of which one deduces that ϕ is an isomorphism.

(i) Using the homotopy map of the mapping cone of i which is zero for degrees different from $d_{X/Y}+1$, and $\begin{pmatrix} -\phi^{-1} \\ 0 \end{pmatrix}$ in degree $d_{X/Y}+1$, we have that the identity map of the mapping cone is homotopic to the morphism having the following expression in degree $d_{X/Y}$: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\phi^{-1} \\ 0 \end{pmatrix}(\phi, d) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & -\phi^{-1}d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\phi^{-1}d \\ 0 & 1 \end{pmatrix}$ and unchanged otherwise, as stated.

(ii) This follows from the previous point, since $\mathrm{id} \sim \Psi^\bullet$ implies $p_1 \sim p_1\circ\Psi^\bullet = \psi^\bullet$. Explicitly, we may use the homotopy map of the connecting morphism which is zero for degrees different from $d_{X/Y}+1$, and ϕ^{-1} in degree $d_{X/Y}+1$. Then we have that $-p_1$ is homotopic to the morphism having the following expression in degree $d_{X/Y}$: $(-1, 0) + (\phi^{-1})(\phi, d) = (0, \phi^{-1}d)$ and unchanged otherwise, as stated. □

Proposition 4.3.2. *The following diagram*

$$\begin{array}{ccc} (f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X} \mathrm{DR}_{X/Y}(\mathcal{D}_X)) \oplus \mathrm{Ler}^0\mathrm{DR}_X(\mathcal{D}_X)/\mathrm{Ler}^2\mathrm{DR}_X(\mathcal{D}_X)[-d_Y] & & \\ \mathrm{qis}\downarrow (0,\pi) & \searrow^{-p_1=(-1,0)} & \\ \mathrm{DR}_{X/Y}(\mathcal{D}_X) & \overset{\delta(\mathcal{D}_X)}{\dashrightarrow} & f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\mathrm{DR}_{X/Y}(\mathcal{D}_X) \\ \mathrm{qis}\downarrow\lambda & & \mathrm{qis}\downarrow 1\otimes\lambda \\ \mathcal{D}_{Y\leftarrow X} & \xrightarrow{\nabla_{Y\leftarrow X}} & f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\mathcal{D}_{Y\leftarrow X}\end{array}$$

commutes in the derived category of right $\mathcal{D}_X$-modules.

Since this diagram is just the definition of δ, the commutativity proves that $\delta = \nabla_{Y\leftarrow X}$.

Proof. By the homotopy lemma, we may use ψ instead of $-p_1$, and since any object of the last row is a complex concentrated in degree zero, we need only to prove the commutativity of the following diagram:

$$\begin{array}{ccc} \frac{\mathrm{Ler}^0}{\mathrm{Ler}^2}(\Omega_X^{d_{X/Y}}(\mathcal{D}_X)) & & \\ \downarrow\pi & \searrow^{\phi^{-1}d} & \\ \omega_{X/Y}(\mathcal{D}_X) & & f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\omega_{X/Y}(\mathcal{D}_X) \\ \downarrow\lambda & \overset{\alpha}{\searrow} & \downarrow 1\otimes\lambda \\ \mathcal{D}_{Y\leftarrow X} & \xrightarrow{\nabla_{Y\leftarrow X}} & f^*(\Omega^1_Y)\otimes_{\mathcal{O}_X}\mathcal{D}_{Y\leftarrow X}\end{array}$$

(the first object of the mapping cone does not appear, because it is sent to zero using both morphisms).

Notice, first of all, that there exists a unique morphism $\alpha : \omega_{X/Y}(\mathcal{D}_X) \to f^*\Omega^1_Y \otimes_{\mathcal{O}_X} \mathcal{D}_{Y\leftarrow X}$ making the upper part ("parallelogram") of the diagram commutative, since (with reference to the diagram in 4.3.1) we have that $(1\otimes\lambda)\phi^{-1}di = (1\otimes\lambda)d = 0$, so that $(1\otimes\lambda)\phi^{-1}d$ factors as $\alpha\pi$ (unicity of α follows because π is an epimorphism). As a consequence, it is enough to prove that $\nabla_{Y\leftarrow X}\lambda = \alpha$. One has two different possibilities for the proof. The first one is a local computation. Using local coordinates x_i ($i = 1,\dots,d_X$) on X such that $dx_1,\dots,dx_{d_{X/Y}}$ are generators of the relative differentials, and $\omega = dx_1\wedge\cdots\wedge dx_{d_{X/Y}}$, for the morphism α we have the following expression

$$\begin{aligned}\alpha(\omega\otimes\partial) &= (1\otimes\lambda)\phi^{-1}d([\omega\otimes\partial]) \\ &= (1\otimes\lambda)\phi^{-1}\left[\sum\nolimits_i dy_i\wedge\omega\otimes\eta_i\partial\right] \\ &= (1\otimes\lambda)\left(\sum\nolimits_i dy_i\otimes\omega\otimes\eta_i\partial\right) \\ &= \sum\nolimits_i dy_i\otimes\omega\otimes(\eta_i\partial)^*\end{aligned}$$

where local coordinates y_i on Y are used (dy_i and η_i are dual bases for Ω^1_Y and Θ_Y) and ∂ is a local section of $\mathcal{D}_X$. Therefore, the action of the derivative η_i (dual bases of dy_i) is given by $\omega\otimes(\eta_i\partial)^*$. On the other hand, the structure of $f^{-1}(\mathcal{D}_Y)$-module of $\mathcal{D}_{Y\leftarrow X}$ is given by the twist of the right structure (by multiplication) of $f^*\mathcal{D}_Y$ with the right structure of $f^{-1}\omega_Y$ (by the action of $-\mathrm{Lie}_\eta$, which is trivial on ω), and has therefore the local expression $\eta_i(\omega\otimes\partial) = -\omega\otimes\partial\eta_i$; composing with λ, the action of η_i sends $\omega\otimes\partial$ to $\omega\otimes\partial^*\eta_i^*$ and coincides with that given above.

The second possibility is more abstract and relies on the compatibility of De Rham functors. Since the morphism α is induced by the differential of the absolute De Rham complex of $\mathcal{D}_X$, and it factors uniquely through λ (see again the diagram in 4.3.1, since $(\pi)d\alpha = 0$, π being an epimorphism, and λ gives a cokernel for d in the last column), it is sufficient to prove that the morphism $\nabla_{Y\leftarrow X}$ is compatible with that complex. We observe that $\omega_X(f^*\mathcal{D}_Y)$ has two compatible structures, as a right $\mathcal{D}_X$-module and as a right $f^{-1}(\mathcal{D}_Y)$-module. We define $\mathrm{DR}_Y(\omega_X(f^*\mathcal{D}_Y)) := \omega_X(f^*\mathcal{D}_Y)\otimes_{\mathcal{O}_X}\bigwedge^{-\bullet} f^*\Theta^1_Y$ (the De Rham complex as right $f^{-1}(\mathcal{D}_Y)$-module) which is isomorphic (by left/right exchange on the $f^{-1}(\mathcal{D}_Y)$-module structure) to $\mathrm{DR}_Y(\omega_{X/Y}(f^*\mathcal{D}_Y)) := f^{-1}\Omega^\bullet_Y\otimes_{f^{-1}(\mathcal{O}_Y)}\omega_{X/Y}(f^*\mathcal{D}_Y)[d_Y]$. Now, we have the following canonical morphisms of De Rham complexes:

$$\mathrm{DR}_X(\mathcal{D}_X)\cong\mathrm{DR}_X(\omega_X(\mathcal{D}_X))\xrightarrow[\sim]{\iota}\mathrm{DR}_X(\omega_X(\mathcal{D}_X))\longrightarrow\mathrm{DR}_Y(\omega_X(f^*\mathcal{D}_Y))\cong\mathrm{DR}_Y(\omega_{X/Y}(f^*\mathcal{D}_Y))$$

where the isomorphisms $\cong$ are given by left/right exchanges, the first morphism comes from the involution ι, the second comes from $\mathcal{D}_f : \mathcal{D}_X\to f^*\mathcal{D}_Y$. In degrees $-d_Y$ and $-d_Y+1$ we can read the following compatibilities:

$$\begin{array}{ccccccc}\Omega_X^{d_{X/Y}}(\mathcal{D}_X) & \underset{\sim}{\to} & \omega_X(\mathcal{D}_X)\otimes\bigwedge^{d_Y}\Theta^1_X & \to & \omega_X(f^*\mathcal{D}_Y)\otimes\bigwedge^{d_Y} f^*\Theta^1_Y & \cong & \omega_{X/Y}(f^*\mathcal{D}_Y) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow\nabla_{Y\leftarrow X} \\ \Omega_X^{d_{X/Y}+1}(\mathcal{D}_X) & \underset{\sim}{\to} & \omega_X(\mathcal{D}_X)\otimes\bigwedge^{d_Y-1}\Theta^1_X & \to & \omega_X(f^*\mathcal{D}_Y)\otimes\bigwedge^{d_Y-1} f^*\Theta^1_Y & \cong & f^{-1}\Omega^1_Y\otimes\omega_{X/Y}(f^*\mathcal{D}_Y)\end{array}$$

from which the result follows. □

4.4. The comparison.

Theorem. *For any left $\mathcal{D}_X$-module $\mathcal{M}$, the canonical quasi-isomorphism*

$$\mathbf{R}f_*(\lambda(\mathcal{M})) : \mathbf{R}f_*\mathrm{DR}_{X/Y}(\mathcal{M})\longrightarrow\mathbf{R}f_*(\mathcal{D}_{Y\leftarrow X}\otimes^{\mathbf{L}}_{\mathcal{D}_X}\mathcal{M}) = f_+(\mathcal{M})$$

makes the following diagram

$$\begin{array}{ccc}
\mathbf{R}f_*\mathrm{DR}_{X/Y}(\mathcal{M}) & \overset{GM(\mathcal{M})}{\dashrightarrow} & \Omega^1_Y \otimes_{\mathcal{O}_Y} \mathbf{R}f_*\mathrm{DR}_{X/Y}(\mathcal{D}_X) \\
{\scriptstyle \mathbf{R}f_*\lambda(\mathcal{M})}\downarrow{\scriptstyle \mathrm{qis}} & & {\scriptstyle \mathrm{qis}}\downarrow{\scriptstyle 1\otimes\mathbf{R}f_*\lambda(\mathcal{M})} \\
f_+(\mathcal{M}) & \xrightarrow{\nabla_{f_+(\mathcal{M})}} & \Omega^1_Y \otimes_{\mathcal{O}_Y} f_+(\mathcal{M})
\end{array}$$

commutative in the derived category of K_X-modules, where $\nabla_{f_+(\mathcal{M})}$ is the connection induced by the left $\mathcal{D}_Y$-module structure of $f_+(\mathcal{M})$ (and $GM(\mathcal{M}) = \eta \circ \mathbf{R}f_*\delta(\mathcal{M})$, as in 4.1, induces the Gauss-Manin connection in cohomology). As a consequence, for any q we have natural isomorphisms

$$\mathbf{R}^q f_*\mathrm{DR}_{X/Y}(\mathcal{M}) \cong \mathcal{H}^q(f_+\mathcal{M})$$

in the category of left $\mathcal{D}_Y$-modules, where the left hand side has the structure of the Gauss-Manin connection.

Proof. Notice that $f_+(\mathcal{M}) = \mathbf{R}f_*(\mathcal{D}_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M})$ is a complex of left $\mathcal{D}_Y$-modules, so that the associated connections induce a morphism of complexes $\nabla_{f_+(\mathcal{M})} : f_+(\mathcal{M}) \to \Omega^1_Y \otimes_{\mathcal{O}_Y} f_+(\mathcal{M})$ used in the diagram. Moreover it is clear that $\nabla_{f_+(\mathcal{M})}$ is obtained starting with $\nabla_{Y\leftarrow X}$ (connection associated to the transfer module) by derived tensor product with $\mathcal{M}$, then applying the functor $\mathbf{R}f_*$ and the projection formula, that is $\nabla_{f_+(\mathcal{M})} = \eta \circ \mathbf{R}f_*(\nabla_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M})$ (this follows directly from the definition of the $\mathcal{D}_Y$-module structure of the derived direct image). Now, from the commutative diagram (4.2.3), using that $\delta = \nabla_{Y\leftarrow X}$, and applying the derived tensor product (over $\mathcal{D}_X$) with $\mathcal{M}$ and the functor $\mathbf{R}f_*$, we obtain the following commutative diagram

$$\begin{array}{ccc}
\mathbf{R}f_*\mathrm{DR}_{X/Y}(\mathcal{M}) & \overset{\mathbf{R}f_*\delta(\mathcal{M})}{\dashrightarrow} & \mathbf{R}f_*(f^{-1}\Omega^1_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathrm{DR}_{X/Y}(\mathcal{M})) \\
{\scriptstyle \mathbf{R}f_*\lambda(\mathcal{M})}\downarrow{\scriptstyle \mathrm{qis}} & & {\scriptstyle \mathrm{qis}}\downarrow{\scriptstyle \mathbf{R}f_*(1\otimes\lambda(\mathcal{M}))} \\
\mathbf{R}f_*(\mathcal{D}_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M}) & \xrightarrow{\mathbf{R}f_*(\nabla_{Y\leftarrow X}\otimes^{\mathbf{L}}\mathcal{M})} & \mathbf{R}f_*(f^{-1}\Omega^1_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{D}_{Y\leftarrow X} \otimes^{\mathbf{L}}_{\mathcal{D}_X} \mathcal{M})\ .
\end{array}$$

Using the projection formula on the right side, we obtain the diagram of the theorem. □

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