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Pair Correlation of Zeros, Primes in Short Intervals and Exponential Sums over Primes

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1. INTRODUCTION

In a previous paper [7] we considered, in connection with the problem of existence of Goldbach numbers in short intervals, the asymptotic behavior as $X \to \infty$ of

$$S(X, \xi) = \int_{-\xi}^{\xi} |S(\alpha)|^2 \, d\alpha \qquad 0 \leqslant \xi \leqslant \tfrac{1}{2},$$

where $S(\alpha) = \sum_{n \leqslant X} \Lambda(n)\, e(n\alpha)$, $e(x) = e^{2\pi i x}$ and $\Lambda(n)$ is the von Mangoldt function. Writing $f(x) \asymp g(x)$ for $f(x) \ll g(x) \ll f(x)$, we found

$$S(x, \xi) \asymp \begin{cases} X^2 \xi & \text{if} \quad 0 \leqslant \xi \leqslant \dfrac{1}{X} \\ X & \text{if} \quad \dfrac{1}{X} \leqslant \xi \leqslant \dfrac{1}{\log X} \\ X\xi \log X & \text{if} \quad \dfrac{1}{\log X} \leqslant \xi \leqslant \dfrac{1}{2} \end{cases} \qquad (1)$$

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as $X \to \infty$, uniformly in ξ. We remark that (1) was proved in [7] with

$$\tilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n)\, e^{-n/X} e(n\alpha)$$

in place of $S(\alpha)$, but a similar argument proves (1) as well.

In this note we are mainly concerned with the behavior as $X \to \infty$ of the remainder term function

$$R(X, \xi) = \int_{-\xi}^{\xi} |R(\alpha)|^2\, d\alpha \qquad 0 \leqslant \xi \leqslant \tfrac{1}{2},$$

where $R(\alpha) = S(\alpha) - T(\alpha)$ and $T(\alpha) = \sum_{n \leqslant X} e(n\alpha)$.

It is known (see, e.g., Section 2 of Perelli [11]) that $R(X, \xi)$ is related with the mean-square of primes in short intervals, i.e., with

$$J(X, h) = \int_0^X |\psi(x+h) - \psi(x) - h|^2\, dx,$$

where $\psi(x) = \sum_{n \leqslant x} \Lambda(n)$. In turn (see, e.g., Goldson and Montgomery [5]) $J(X, h)$ is related with Montgomery's pair correlation function

$$F(X, T) = \sum_{0 < \gamma, \gamma' \leqslant T} X^{i(\gamma - \gamma')} w(\gamma - \gamma'),$$

where $w(u) = 4/(4 + u^2)$ and γ, γ' run over the imaginary part of the non-trivial zeros of the Riemann zeta function. In view of the above results, we may therefore expect that the quantities $R(X, \xi)$, $J(X, h)$ and $F(X, T)$ are closely related.

In fact, Goldston and Montgomery [5] proved that determining the asymptotic behavior of $J(X, h)$ or of $F(X, T)$, as $X \to \infty$ and h or T in suitable ranges, are, under the Riemann Hypothesis (*RH*), equivalent problems. Our first goal is to show that such an equivalence can be extended to $R(X, \xi)$ as well. We have

THEOREM 1. *Assume RH. As $X \to \infty$, the following statements are equivalent*:

(i) *for every* $\varepsilon > 0$, $R(X, \xi) \sim 2X\xi \log X\xi$ *uniformly for* $X^{-(1/2 - \varepsilon)} \leqslant \xi \leqslant \frac{1}{2}$

(ii) *for every* $\varepsilon > 0$, $J(X, h) \sim hX \log \frac{X}{h}$ *uniformly for* $1 \leqslant h \leqslant X^{1/2 - \varepsilon}$

(iii) *for every* $\varepsilon > 0$ *and* $A \geqslant 1$, $F(X, T) \sim \frac{T}{2\pi} \log \min(X, T)$ *uniformly for* $X^{1/2 + \varepsilon} \leqslant T \leqslant X^A$.

The equivalence between (ii) and (iii) is due to Goldston and Montgomery [5]. In fact, [5] obtains such equivalence in the wider range $1 \leqslant h \leqslant X^{1-\varepsilon}$ for $J(X, h)$ and $X^{\varepsilon} \leqslant T \leqslant X^{A}$ for $F(X, T)$. Our restriction comes from the fact that we compare $R(X, \xi)$ and $J(X, h)$ by Lemma 1 below, and such comparison apparently requires the restricted range of Theorem 1. We remark that the Abelian/Tauberian arguments in [5] play an important role in our proof as well. We also remark that, in analogy with [5], we may prove a "localized" form of Theorem 1 as well.

The asymptotic behavior of $R(X, \xi)$ in (i) of Theorem 1 allows us to replace (1) by the following conditional asymptotic formula.

Corollary. *Assume* (i) *of Theorem* 1. *Then as* $X \to \infty$,

$$S(X, \xi) \sim X(1 + 2\xi \log X\xi) \qquad \textit{uniformly for} \quad \frac{\log^{10} X}{X} \leqslant \xi \leqslant \frac{1}{2}.$$

We remark that the asymptotic behavior of $S(X, \xi)$ can also be obtained, by a standard and unconditional argument in prime number theory, in the remaining range $0 \leqslant \xi \leqslant \frac{\log^{10} X}{X}$. Moreover, the asymptotic formula in the corollary holds *unconditionally* in the ranges

$$\frac{\log^{10} X}{X} \leqslant \xi \leqslant X^{-1/6-\varepsilon} \qquad \text{and} \qquad \frac{F(X)}{\log X} \leqslant \xi \leqslant \frac{1}{2}; \tag{2}$$

see (10) and (28) below. Further, under *RH* the first range in (2) can be enlarged to

$$\frac{\log^{10} X}{X} \leqslant \xi \leqslant \frac{f(X)}{\log^2 X};$$

see [7]. Here and throughout the paper we denote by $f(x)$ (resp. $F(x)$) a function $f(x) \searrow 0$ (resp. $F(x) \nearrow \infty$) arbitrarily slowly.

We remark that we can actually prove the asymptotic formulae in Theorem 1 unconditionally in some restricted ranges; *RH* is needed in case (iii). We have

Theorem 2. *Let* $A \geqslant 1$ *be any fixed constant. Then, as* $X \to \infty$,

(i) $R(X, \xi) \sim 2X\xi \log X\xi$ *uniformly for* $\frac{F(X)}{\log X} \leqslant \xi \leqslant \frac{1}{2}$

(ii) $J(X, h) \sim hX \log \frac{X}{h}$ *uniformly for* $1 \leqslant h \leqslant f(X) \log X$

(iii) *assuming RH*, $F(X, T) \sim \frac{T}{2\pi} \log \min(X, T)$ *uniformly for* $\frac{F(X)\, X}{\log X} \leqslant T \leqslant X^{A}$.

The asymptotic formula in (iii) is due, in the range $X \leqslant T \leqslant X^A$, to the important work of Montgomery [10]. After we obtained Theorem 2, we found that Goldston [3], see Lemma B of [3], already proved (iii) in the remaining range, by a very similar argument. We remark that the proof of (i) and (ii) is also based on a similar argument. However, we give a sketch of the proof since (i) and (ii) are unconditional and, in view of the hypothesis (1.3) in Friedlander and Goldston [2], may have some interest.

We thank Prof. D. A. Goldston for a useful discussion on this subject, and in particular for the following remark concerning the restricted range of the asymptotic formulae in Theorem 1. Let $\tilde{S}(\alpha)$ be as above and write

$$\tilde{T}(\alpha) = \sum_{n=1}^{\infty} e^{-n/X} e(n\alpha) \qquad \tilde{R}(\alpha) = \tilde{S}(\alpha) - \tilde{T}(\alpha)$$

$$\tilde{R}(X, \xi) = \int_{-\xi}^{\xi} |\tilde{R}(\alpha)|^2 \, d\alpha \qquad \tilde{J}(X, h) = \int_0^{\infty} |\psi(x+h) - \psi(x) - h|^2 \, e^{-2x/X} \, dx.$$

In this case, the arguments in Lemma 1 and Theorem 1 show that the asymptotic formulae for $\tilde{R}(X, \xi)$ and $\tilde{J}(X, h)$ are equivalent in the *full* range for ξ and h, i.e., under *RH*

$$\tilde{R}(X, \xi) \sim X\xi \log X\xi \qquad \text{uniformly for} \quad X^{-(1-\varepsilon)} \leqslant \xi \leqslant \tfrac{1}{2} \tag{3}$$

and

$$\tilde{J}(X, h) \sim \frac{1}{2} hX \log \frac{X}{h} \qquad \text{uniformly for} \quad 1 \leqslant h \leqslant X^{1-\varepsilon} \tag{4}$$

are equivalent. Moreover, a standard Abelian/Tauberian argument, see Section 7.12 of Titchmarsh [13], shows that (4) is equivalent with

$$J(X, h) \sim hX \log \frac{X}{h} \qquad \text{uniformly for} \quad 1 \leqslant h \leqslant X^{1-\varepsilon}, \tag{5}$$

and from Goldston and Montgomery [5] we have that (5) is equivalent with

$$F(X, T) \sim \frac{T}{2\pi} \log \min(X, T) \qquad \text{uniformly for} \quad X^{\varepsilon} \leqslant T \leqslant X^A. \tag{6}$$

Therefore, a stronger version of Theorem 1 holds in the smoothed case, in the sense that Theorem 1 holds with (i), (ii) and (iii) replaced by (3), (4) and (6), respectively.

2. PROOF OF THEOREM 2

Let

$$\hat{f}(t)=\int_{-\infty}^{+\infty} f(x)\, e(-tx)\, dx$$

be the Fourier transform of $f(x)$ and, for $h>0$, write

$$K(\alpha, h)=\sum_{-h\leqslant n\leqslant h} (h-|n|)\, e(n\alpha) \qquad \text{and} \qquad U(\alpha, h)=\left(\frac{\sin \pi h\alpha}{\pi\alpha}\right)^2.$$

The following lemma forms the basis of our subsequent arguments; see also Lemma 1 of Brüdern and Perelli [1].

Lemma 1. *Let $h>0$. Then*

$$\int_0^1 |R(\alpha)|^2\, K(\alpha, h)\, d\alpha=\int_{-\infty}^{+\infty} |R(\alpha)|^2\, U(\alpha, h)\, d\alpha=J(X, h)+E(X, h),$$

where, as $X\to\infty$,

$$E(X, h)\ll\begin{cases}(h+1)^3\log^2 X & \textit{uniformly for} \quad 0<h\leqslant X^{1/100}\\ h^3 & \textit{uniformly for} \quad X^{1/100}\leqslant h\leqslant \frac{X}{2}\end{cases}$$

and, under RH, also

$$E(X, h)\ll (h+1)\, X\log^4 X \qquad \textit{uniformly for} \quad 0<h\leqslant\frac{X}{2}.$$

Proof. The argument in the proof of Gallagher's lemma, see Lemma 1.9 of Montgomery [9], gives

$$\int_{-\infty}^{+\infty} |R(\alpha)|^2\, U(\alpha, h)\, d\alpha=\int_{-\infty}^{+\infty} \Bigg|\sum_{\substack{|n-x|<h/2\\ 1\leqslant n\leqslant X}} (\Lambda(n)-1)\Bigg|^2 dx. \tag{7}$$

By periodicity we have

$$\int_{-\infty}^{+\infty} |R(\alpha)|^2\, U(\alpha, h)\, d\alpha=\int_0^1 |R(\alpha)|^2\left(\sum_{n=-\infty}^{+\infty} U(n+\alpha, h)\right) d\alpha.$$

Since $\hat{U}(\alpha, h) = \max(h - |\alpha|, 0)$, by Poisson summation formula we get

$$\sum_{n=-\infty}^{+\infty} U(n+\alpha, h) = \sum_{n=-\infty}^{+\infty} \hat{U}(n, h)\, e(n\alpha) = K(\alpha, h),$$

and hence

$$\int_0^1 |R(\alpha)|^2 K(\alpha, h)\, d\alpha = \int_{-\infty}^{+\infty} |R(\alpha)|^2 U(\alpha, h)\, d\alpha$$
$$= \int_{-\infty}^{+\infty} \Bigg| \sum_{\substack{|n-x|<h/2 \\ 1 \leqslant n \leqslant X}} (\Lambda(n) - 1) \Bigg|^2 dx. \tag{8}$$

The right hand side of (8) is easily reduced to $J(X, h)$. In fact, we have

$$\int_{-\infty}^{+\infty} \Bigg| \sum_{\substack{|n-x|<h/2 \\ 1 \leqslant n \leqslant X}} (\Lambda(n) - 1) \Bigg|^2 dx = J(X, h) + E(X, h),$$

where $E(X, h)$ is the sum of the four integrals, each integrated over an interval of length $\ll h+1$, of functions of the form

$$g(y, k) = \Bigg| \sum_{y \leqslant n \leqslant y+k} (\Lambda(n) - 1) \Bigg|^2$$

with $y \leqslant 2X$ and $0 \leqslant k \leqslant h$.

The result follows now, in the unconditional case, using the trivial estimate $g(y, k) \ll (k+1)^2 \log^2 X$ when $0 \leqslant k \leqslant X^{1/200}$ and the estimate $g(y, k) \ll k^2$, coming from Brun–Titchmarsch's theorem, when $X^{1/200} \leqslant k \leqslant X$. Assuming *RH*, we may also use the classical estimate

$$\psi(y) = y + O(y^{1/2} \log^2 y) \tag{9}$$

to bound $g(y, k)$. Hence $g(y, k) \ll X \log^4 X$ uniformly for $y, k \leqslant 2X$, and Lemma 1 follows. ∎

We first sketch the proof of (i) of Theorem 2. Clearly

$$R(X, \xi) = S(X, \xi) + O(T(X, \xi)) + O(S(X, \xi)^{1/2}\, T(X, \xi)^{1/2}),$$

where

$$T(X, \xi) = \int_{-\xi}^{\xi} |T(\alpha)|^2\, d\alpha \ll X.$$

Moreover, the argument in the proof of Theorem 2 of [7] in this case gives

$$S(X, \xi) = 2X\xi \log X\xi + O(X(\xi \log X)^{1/3}) + O(X), \tag{10}$$

and (i) follows at once.

In order to prove (ii), we observe that Lemma 1 implies, for $1 \leqslant h \ll \log X$, that

$$J(X, h) = \sum_{-h \leqslant n \leqslant h} (h - |n|) \int_0^1 |R(\alpha)|^2 e(-n\alpha)\, d\alpha + o(hX \log X). \tag{11}$$

By (i), the term with $n = 0$ in the right hand side of (11) contributes

$$hR\left(X, \frac{1}{2}\right) \sim hX \log \frac{X}{h}. \tag{12}$$

The terms with $n \neq 0$ contribute

$$\begin{aligned}
&\sum_{\substack{-h \leqslant n \leqslant h \\ n \neq 0}} (h - |n|) \sum_{\substack{1 \leqslant n_1, n_2 \leqslant X \\ n_1 - n_2 = n}} (\Lambda(n_1) - 1)(\Lambda(n_2) - 1) \\
&\quad = \sum_{\substack{-h \leqslant n \leqslant h \\ n \neq 0}} (h - |n|) \left\{ \psi(X, n) + O\left(X + \sum_{\substack{1 \leqslant n_1, n_2 \leqslant X \\ n_1 - n_2 = n}} \Lambda(n_1) \right) \right\} \\
&\quad = \sum_{\substack{-h \leqslant n \leqslant h \\ n \neq 0}} (h - |n|)\, \psi(X, n) + O(h^2 X)
\end{aligned}$$

by the prime number theorem, where $\psi(X, n)$ is the n-twin primes counting function weighted by the von Mangoldt function. By a well known sieve estimate, for $n \neq 0$ we have

$$\psi(X, n) \ll \mathfrak{S}(n)\, X,$$

where $\mathfrak{S}(n)$ is the singular series of the n-twin primes problem, see Chap. 17 of [9]. Moreover,

$$\sum_{n \leqslant x} \mathfrak{S}(n) \ll x,$$

see again Chap. 17 of [9].

Hence the contribution of the terms with $n \neq 0$ is

$$O(h^2 X), \tag{13}$$

and (ii) follows from (11)–(13). ∎

3. PROOF OF THEOREM 1 AND COROLLARY

In view of Theorem 2, throughout the proof we will assume that $X^{-1/2+\varepsilon} \leqslant \xi \leqslant (\log X)^{-1/2}$ in (i) and $(\log X)^{1/2} \leqslant h \leqslant X^{1/2-\varepsilon}$ in (ii) of Theorem 1. This is not strictly necessary, but simplifies the argument.

Assume (i). We use Lemma 1 in the form

$$J(X, h) = \int_0^1 |R(\alpha)|^2 K(\alpha, h)\, d\alpha + o\left(hX \log \frac{X}{h} \right), \tag{14}$$

uniformly for $1 \leqslant h \leqslant X^{1/2-\varepsilon}$. Observe that both $|R(\alpha)|^2$ and $K(\alpha, h)$ are even functions of α, and hence we may restrict our attention to $\alpha \in [0, \frac{1}{2}]$. Let $V = F(X)$. Since

$$K(\alpha, h) \ll \min\left(h^2, \frac{1}{\alpha^2} \right), \tag{15}$$

by partial integration we have

$$\left(\int_0^{1/hV} + \int_{V/h}^{1/2} \right) |R(\alpha)|^2 K(\alpha, h)\, d\alpha \ll h^2 R\left(X, \frac{1}{hV} \right) + \int_{V/h}^{1/2} |R(\alpha)|^2 \frac{d\alpha}{\alpha^2}$$

$$\ll \frac{hX \log X}{V}. \tag{16}$$

In the remaining range $\frac{1}{hV} \leqslant \alpha \leqslant \frac{V}{h}$ we write

$$|R(\alpha)|^2 = X \log \frac{X}{h} + X \log h\alpha + (|R(\alpha)|^2 - X \log X\alpha).$$

By (15) we have

$$\begin{aligned}\int_{1/hV}^{V/h} |R(\alpha)|^2 K(\alpha, h)\, d\alpha &= X \log \frac{X}{h} \int_0^{1/2} K(\alpha, h)\, d\alpha + X \int_{1/hV}^{V/h} \log(h\alpha)\, K(\alpha, h)\, d\alpha \\ &\quad + \int_{1/hV}^{V/h} (|R(\alpha)|^2 - X \log X\alpha)\, K(\alpha, h)\, d\alpha \\ &\quad + O\left(\frac{hX \log X}{V} \right) \\ &= I_1 + I_2 + I_3 + O\left(\frac{hX \log X}{V} \right),\end{aligned} \tag{17}$$

say. Clearly,

$$I_1 = \frac{hX}{2} \log \frac{X}{h}, \tag{18}$$

while

$$I_2 \ll h^2 X \int_0^{V/h} |\log h\alpha| \, d\alpha \ll hXV \log V. \tag{19}$$

By partial summation we have that

$$K'(\alpha, h) \ll \frac{h^2}{\alpha} \qquad \text{for} \quad \frac{1}{hV} \leqslant \alpha \leqslant \frac{V}{h}, \tag{20}$$

and hence by partial integration we get

$$I_3 \ll o(hXV \log X). \tag{21}$$

Choosing $F(X)$ in a suitable way with respect to the infinitesimal function implicit in (21), we see that (ii) follows from (14) and (16)–(21).

Now assume (ii). Since we adapt the proof of Lemma 4 of Goldston and Montgomery [5] to our case, we only sketch the argument.

As in Lemma 4 of [5], let $\eta > 0$ and

$$k_\eta(x) = \frac{\sin 2\pi x + \sin 2\pi(1+\eta)\, x}{2\pi x(1 - 4\eta^2 x^2)},$$

so that

$$\hat{k}_\eta(t) = \begin{cases} 1 & \text{if} \quad |t| \leqslant 1 \\ \cos^2\left(\dfrac{\pi(|t|-1)}{2\eta}\right) & \text{if} \quad 1 \leqslant |t| \leqslant 1+\eta \\ 0 & \text{if} \quad |t| \geqslant 1+\eta. \end{cases}$$

Hence, again considering only positive values of α, for $\eta > 0$ we have

$$\int_0^\infty |R(\alpha)|^2 \, \hat{k}_\eta\left(\frac{\alpha}{\xi}(1+\eta)\right) d\alpha \leqslant \frac{1}{2} R(X, \xi) \leqslant \int_0^\infty |R(\alpha)|^2 \, \hat{k}_\eta\left(\frac{\alpha}{\xi}\right) d\alpha. \tag{22}$$

By Lemma 3 of [5] we have

$$\hat{k}_\eta(t) = \int_0^\infty k''_\eta(x)\, U(t, x)\, dx,$$

and hence, writing

$$|R(\alpha)|^2 = X \log X\alpha + (|R(\alpha)|^2 - X \log X\alpha)$$

and observing that $U(\frac{\alpha}{\xi}, x) = \xi^2 U(\alpha, \frac{x}{\xi})$, we get

$$\begin{aligned}
&\int_0^\infty |R(\alpha)|^2 \hat{k}_\eta\left(\frac{\alpha}{\xi}\right) d\alpha \\
&\quad = X \int_0^\infty \log(X\alpha)\, \hat{k}_\eta\left(\frac{\alpha}{\xi}\right) d\alpha \\
&\qquad + \xi^2 \int_0^\infty k''_\eta(x) \left(\int_0^\infty (|R(\alpha)|^2 - X \log X\alpha)\, U\left(\alpha, \frac{x}{\xi}\right) d\alpha\right) dx \\
&\quad = J_1 + J_2,
\end{aligned} \tag{23}$$

say. A direct computation shows that

$$J_1 = (1 + O(\eta))(1 + o(1))\, X\xi \log X\xi. \tag{24}$$

In order to estimate J_2 we first observe that

$$\int_0^\infty U(\alpha, 1)\, d\alpha = \tfrac{1}{2},$$

therefore again by a direct computation, based on the substitution $\alpha = \frac{\tau\xi}{x}$, we get

$$\int_0^\infty X \log(X\alpha)\, U\left(\alpha, \frac{x}{\xi}\right) d\alpha = \frac{Xx}{2\xi} \log \frac{X\xi}{x} + O\left(\frac{Xx}{\xi}\right). \tag{25}$$

We need the following

Lemma 2. *Assume RH and* (ii) *of Theorem* 1. *Then, as* $X \to \infty$,

$$\int_0^\infty |R(\alpha)|^2\, U(\alpha, h)\, d\alpha$$

$$= \begin{cases} O(X \log X) & \textit{uniformly for} \quad 0 < h \leqslant 1 \\ \dfrac{1}{2}(1 + o(1))\, hX \log \dfrac{X}{h} & \textit{uniformly for} \quad 1 \leqslant h \leqslant X^{1/2-\varepsilon} \\ O(hX \log^4 X) & \textit{uniformly for} \quad h \geqslant X^{1/2-\varepsilon} \end{cases}$$

Proof. For $0<h\leqslant 1$ we have $U(\alpha, h) \ll \min(1, |\alpha|^{-2})$ and hence by periodicity

$$\int_0^{\infty} |R(\alpha)|^2\, U(\alpha, h)\, d\alpha \ll \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{n-1}^{n} |R(\alpha)|^2\, d\alpha$$
$$\ll T\left(X, \frac{1}{2}\right) + S\left(X, \frac{1}{2}\right) \ll X \log X,$$

while for $1\leqslant h\leqslant X^{1/2-\varepsilon}$ the assertion follows immediately from Lemma 1 and (ii).

In order to treat the last range, we recall the well known bound

$$J(X, h) \ll hX \log^2 X,$$

obtained by Selberg [12] under *RH*. Hence the result follows at once by Lemma 1, for $X^{1/2-\varepsilon}\leqslant h\leqslant \frac{X}{2}$. Finally, for $h\geqslant \frac{X}{2}$ we use the classical estimate (9) and our assertion follows from (7). ∎

By Lemma 2 with $h=\frac{x}{\xi}$ and (25) we have

$$\int_0^{\infty} (|R(\alpha)|^2 - X \log X\alpha)\, U\left(\alpha, \frac{x}{\xi}\right) d\alpha$$
$$= \begin{cases} O(X \log X) & \text{if } \ 0<x\leqslant \xi \\ o\left(\dfrac{x}{\xi} X \log X\right) & \text{if } \ \xi\leqslant x\leqslant \xi X^{1/2-\varepsilon} \\ O\left(\dfrac{x}{\xi} X \log^4 X\right) & \text{if } \ x\geqslant \xi X^{1/2-\varepsilon} \end{cases} \tag{26}$$

uniformly in x. Observe that since the ε's in (i) and (ii) are arbitrarily small, we may assume that $\xi X^{1/2-\varepsilon} \gg X^{\varepsilon}$ in the range of ξ under consideration. Since from Lemma 3 of [5] we have

$$k''_\eta(x) \ll_\eta \min(1, x^{-3}),$$

from (23) and (26) we get

$$J_2 = o_\eta(X\xi \log X\xi). \tag{27}$$

Choosing $\eta \searrow 0$ in a suitable way with respect to the infinitesimal function implicit in (27), from (22)–(24) and (27) we see that

$$R(X, \xi) \leqslant 2(1+o(1))\, X\xi \log X\xi.$$

In a similar way we also get that

$$R(X, \xi) \geqslant 2(1+o(1))\, X\xi \log X\xi,$$

and (i) follows.

Since the equivalence between (ii) and (iii) can be obtained by the same argument of [5], the proof of Theorem 1 is now complete.

The proof of Corollary is not difficult. We first observe that the argument in [11], coupled with Huxley's density estimate [6], gives

$$S(X, \xi) \sim X \qquad \text{uniformly for} \quad \frac{\log^{10} X}{X} \leqslant \xi \leqslant X^{-1/6-\varepsilon}. \tag{28}$$

In the remaining range we write

$$|S(\alpha)|^2 = |R(\alpha)|^2 + 2\operatorname{Re} S(\alpha)\, \overline{T(\alpha)} - |T(\alpha)|^2 \tag{29}$$

and use (i) of Theorem 1. Hence

$$R(X, \xi) \sim 2X\xi \log X\xi \qquad \text{uniformly for} \quad X^{-1/6-\varepsilon} \leqslant \xi \leqslant \tfrac{1}{2} \tag{30}$$

and clearly

$$\int_{-\xi}^{\xi} |T(\alpha)|^2\, d\alpha \sim X \qquad \text{uniformly for} \quad X^{-1/6-\varepsilon} \leqslant \xi \leqslant \tfrac{1}{2}. \tag{31}$$

Moreover, for $X^{-1/6-\varepsilon} \leqslant \xi \leqslant \frac{1}{2}$ we have

$$\begin{aligned} \int_{-\xi}^{\xi} S(\alpha)\, \overline{T(\alpha)}\, d\alpha &= \sum_{n \leqslant X} \int_0^1 S(\alpha)\, e(-n\alpha)\, d\alpha \\ &\quad + O\left(\left(\int_{\xi}^{1/2} |S(\alpha)|^2\, d\alpha \right)^{1/2} \left(\int_{\xi}^{1/2} |T(\alpha)|^2\, d\alpha \right)^{1/2} \right) \\ &= \psi(X) + O\left(\left(\frac{X \log X}{\xi} \right)^{1/2} \right) = X(1+o(1)) \end{aligned} \tag{32}$$

by the prime number theorem.

Our Corollary follows at once from (28)–(32). ∎

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