Barycentric rational interpolation at quasi-equidistant nodes

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Abstract

A collection of recent papers reveals that linear barycentric rational interpolation with the weights suggested by Floater and Hormann is a good choice for approximating smooth functions, especially when the interpolation nodes are equidistant. In the latter setting, the Lebesgue constant of this rational interpolation process is known to grow only logarithmically with the number of nodes. But since practical applications not always allow to get precisely equidistant samples, we relax this condition in this paper and study the Floater–Hormann family of rational interpolants at distributions of nodes which are only almost equidistant. In particular, we show that the corresponding Lebesgue constants still grow logarithmically, albeit with a larger constant than in the case of equidistant nodes.

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1   Introduction

Given a set of $n + 1$ data values $f_0, f_1, \ldots, f_n$ that are associated with the $n + 1$ real valued nodes $x_0 < x_1 < \cdots < x_n$, the classical univariate interpolation problem is to find a function g with $g(x_j) = f_j$ for $j = 0, 1, \ldots, n$, where g is taken from a finite-dimensional linear subspace of the Banach space $C^0[x_0, x_n]$ of continuous functions over $[x_0, x_n]$ with the maximum norm. A recently discovered generalization of polynomial interpolation is the family of rational interpolants

$$r(x) = \frac{\sum_{j=0}^{n-d} \lambda_j(x) p_j(x)}{\sum_{j=0}^{n-d} \lambda_j(x)}, \qquad 0 \le d \le n,$$

where p_j is the unique polynomial of degree at most d with $p_j(x_k) = f_k$ for $k = j, \ldots, j + d$, and

$$\lambda_j(x) = \frac{(-1)^j}{(x - x_j) \cdots (x - x_{j+d})}.$$

For $d = n$, the interpolant r is the unique interpolating polynomial, and in general the degrees of the numerator and denominator of r are at most n and $n - d$, respectively. Moreover, r has no real poles, as proved by Berrut [1] for $d = 0$ and by Floater and Hormann [6] for $d \ge 1$. The latter work also shows how to derive the *barycentric form* of these rational interpolants,

$$r(x) = \sum_{j=0}^{n} \frac{w_j}{x - x_j} f_j \Big/ \sum_{j=0}^{n} \frac{w_j}{x - x_j}, \tag{1}$$

with *weights*

$$w_k = (-1)^{k-d} \sum_{i \in J_k} \prod_{j=i, j \ne k}^{i+d} \frac{1}{|x_k - x_j|}, \tag{2}$$

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where

$$J_k = \{i \in \{0, 1, \ldots, n - d\} : k - d \leq i \leq k\}, \tag{3}$$

which is to be preferred for the numerical evaluation of r. We refer to [2] for more details on the implementation of linear barycentric rational interpolants.

Floater and Hormann [6] further demonstrate the favourable approximation properties of these rational interpolants. If the data values $f_j = f(x_j)$ are sampled from some sufficiently smooth function f, then the approximation error $\|r - f\|_\infty$ over the interpolation interval $[x_0, x_n]$ is on the order of h^{d+1}, where

$$h = \max_{0 \leq j \leq n-1} (x_{j+1} - x_j)$$

is the maximal distance between neighbouring interpolation nodes. Alternatively, the approximation error can be bounded from above by considering the *Lebesgue constant*

$$\Lambda_n = \max_{x_0 \leq x \leq x_n} \Lambda_n(x), \tag{4}$$

which is the maximum value of the *Lebesgue function*

$$\Lambda_n(x) = \sum_{j=0}^{n} \frac{|w_j|}{|x - x_j|} \Big/ \left| \sum_{j=0}^{n} \frac{w_j}{x - x_j} \right|. \tag{5}$$

Since the rational interpolants in (1) reproduce polynomials of degree at least d by construction, it is straightforward to show that

$$\|r - f\|_\infty \leq (1 + \Lambda_n)\|p^* - f\|_\infty,$$

where p^* is the best approximation of f among all polynomials of degree d. Bos et al. [3, 4] show that Λ_n grows only logarithmically with n, at least for equidistant nodes and fixed d.

In this paper we show that Λ_n grows logarithmically for *quasi-equidistant* nodes, too (see Section 2, where we also improve a bound from [3]). That is, we assume that there exists some *global mesh ratio* $M \geq 1$, which does not depend on n, such that

$$\frac{h}{h_*} \leq M, \tag{6}$$

where

$$h_* = \min_{0 \leq j \leq n-1} (x_{j+1} - x_j)$$

is the minimal distance between neighbouring nodes. Such node distributions may arise from experiments, where the aim is to sample a signal at a constant rate, which in turn is perturbed by imprecision in the measurement locations (see Section 3). The behaviour of the derivatives of the rational interpolants for such nodes has been studied in [7].

2   Theoretical bounds

Let us begin by deriving bounds on the absolute value of the barycentric weights in (2). This result is then used to establish an upper and a lower bound on the Lebesgue constant.

Lemma 1. *If the interpolation nodes are quasi-equidistant, then the weights in (2) satisfy*

$$W_k \leq |w_k| \leq M^d W_k,$$

where

$$W_k = \frac{1}{h^d d!} \sum_{i \in J_k} \binom{d}{k - i}$$

for $k = 0, 1, \ldots, n$. Moreover,

$$W_k \leq \frac{2^d}{h^d d!} =: W,$$

with equality if and only if $d \leq k \leq n - d$.

Proof. By the definition of the minimal and maximal node distances h_* and h, the distance between two arbitrary nodes x_k and x_j satisfies

$$h_*|k - j| \leq |x_k - x_j| \leq h|k - j|. \tag{7}$$

This leads to the lower bound

$$|w_k| = \sum_{i \in J_k} \prod_{j=i, j \neq k}^{i+d} \frac{1}{|x_k - x_j|}$$

$$\geq \frac{1}{h^d} \sum_{i \in J_k} \prod_{j=i, j \neq k}^{i+d} \frac{1}{|k - j|} = \frac{1}{h^d} \sum_{i \in J_k} \frac{1}{(k - i)!(i + d - k)!} = \frac{1}{h^d d!} \sum_{i \in J_k} \binom{d}{k - i}$$

and similarly, using (6), to the upper bound

$$|w_k| \leq \frac{1}{h_*^d d!} \sum_{i \in J_k} \binom{d}{k - i} \leq \frac{M^d}{h^d d!} \sum_{i \in J_k} \binom{d}{k - i}.$$

The statement about the upper bound W of W_k follows directly from the definition of the index set J_k in (3). $\square$

If the nodes are equidistant, then $M = 1$ and $|w_k| = W_k$ for $k = 0, 1, \ldots, n$, as stated in [6].

Theorem 1. *If the interpolation nodes are quasi-equidistant, then the Lebesgue constant in (4) satisfies*

$$\Lambda_n \leq (2 + M \ln n) \cdot \begin{cases} \frac{3}{4} M, & \text{if } d = 0, \\ 2^{d-1} M^d, & \text{if } d \geq 1. \end{cases}$$

Proof. From (5) it follows that $\Lambda_n(x_k) = 1$ for $k = 0, 1, \ldots, n$. Therefore, let $x_k < x < x_{k+1}$ for some k with $0 \leq k \leq n - 1$, and rewrite the Lebesgue function in (5) as

$$\Lambda_n(x) = \frac{(x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{|w_j|}{|x - x_j|}}{(x - x_k)(x_{k+1} - x) \left| \sum_{j=0}^{n} \frac{w_j}{x - x_j} \right|} =: \frac{N_k(x)}{D_k(x)}.$$

We then derive an upper bound on the numerator $N_k(x)$, following the proof of Theorem 2 in [3]. By Lemma 1,

$$N_k(x) = (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{|w_j|}{|x - x_j|} \leq M^d W (x - x_k)(x_{k+1} - x) \sum_{j=0}^{n} \frac{1}{|x - x_j|},$$

and further, using the inequalities (6) and (7), and the fact that $x_k < x < x_{k+1}$,

$$\frac{N_k(x)}{M^d W} \leq (x_{k+1} - x_k) + (x - x_k)(x_{k+1} - x)\left(\sum_{j=0}^{k-1} \frac{1}{x - x_j} + \sum_{j=k+2}^{n} \frac{1}{x_j - x}\right)$$

$$\leq h + \left(\frac{h}{2}\right)^2 \left(\sum_{j=0}^{k-1} \frac{1}{x_k - x_j} + \sum_{j=k+2}^{n} \frac{1}{x_j - x_{k+1}}\right)$$

$$\leq h + \left(\frac{h}{2}\right)^2 \frac{M}{h}\left(\sum_{j=1}^{k} \frac{1}{j} + \sum_{j=1}^{n-k-1} \frac{1}{j}\right)$$

$$\leq h + \frac{Mh}{2} \ln n.$$

For establishing a lower bound on the denominator $D_k(x)$, we distinguish the cases $d = 0$ and $d \geq 1$. If $d = 0$, then the weights in (2) simplify to $w_k = (-1)^k$ and we have

$$D_k(x) = (x - x_k)(x_{k+1} - x)\left|\sum_{j=0}^{n} \frac{(-1)^j}{x - x_j}\right|$$

$$\geq (x - x_k)(x_{k+1} - x)\left(-\frac{1}{x - x_{k-1}} + \frac{1}{x - x_k} + \frac{1}{x_{k+1} - x} - \frac{1}{x_{k+2} - x}\right) \tag{8}$$

$$= (x_{k+1} - x_k) - (x - x_k)(x_{k+1} - x)\left(\frac{1}{x - x_{k-1}} + \frac{1}{x_{k+2} - x}\right) \geq \frac{2}{3}h_*, \tag{9}$$

where the inequality in (9) can be seen by first multiplying both sides with $3(x - x_{k-1})(x_{k+2} - x)$ and then verifying that the quadratic polynomial

$$Q_k(x) = \big(3(x_{k+1} - x_k) - 2h_*\big)(x - x_{k-1})(x_{k+2} - x) - 3(x - x_k)(x_{k+1} - x)(x_{k+2} - x_{k-1})$$

is non-negative. Substituting $a = x_k - x_{k-1}$, $b = x_{k+1} - x_k$, $c = x_{k+2} - x_{k+1}$, and $y = x - x_k$, we find

$$Q_k(y + x_k) = (3a + 2h_* + 3c)y^2 + (2ah_* - 2bh_* - 2ch_* - 6ab)y + a(b + c)(3b - 2h_*),$$

and it is easy to check that the minimal value of this quadratic expression is

$$\frac{a + b + c}{3a + 2h_* + 3c}\big(6ac(b - h_*) + a(bc - h_*^2) + b(ac - h_*^2) + c(ab - h_*^2)\big),$$

which is non-negative, because $0 < h_* \leq \min(a, b, c)$. Note that the inequality in (8) also holds if $k = 0$ or $k = n - 1$, for example by letting $x_{-1} = x_0 - h$ and $x_{n+1} = x_n + h$.

To handle the case $d \geq 1$, we use two results from [6] to get

$$D_k(x) = (x - x_k)(x_{k+1} - x)\left|\sum_{j=0}^{n} \frac{w_j}{x - x_j}\right| = (x - x_k)(x_{k+1} - x)\left|\sum_{j=0}^{n-d} \lambda_j(x)\right|$$

$$\geq (x - x_k)(x_{k+1} - x)|\lambda_i(x)| = \frac{(x - x_k)(x_{k+1} - x)}{\prod_{j=i}^{i+d}|x_j - x|},$$

for any $i \in J_k \setminus \{k - d\}$. It then follows from (7) that

$$D_k(x) \geq \frac{1}{\prod_{j=i}^{k-1}(x - x_j)\prod_{j=k+2}^{i+d}(x_j - x)} \geq \frac{1}{\prod_{j=i}^{k-1}(x_{k+1} - x_j)\prod_{j=k+2}^{i+d}(x_j - x_k)}$$

$$\geq \frac{1}{h^{d-1}(k+1-i)!(i+d-k)!} \geq \frac{1}{h^{d-1}d!}.$$

$\square$

For equidistant nodes (i.e., $M = 1$), the upper bound in Theorem 1 improves the one given in [3] for $d = 0$ by a factor of $3/4$, and it simplifies to the one given in [4] for $d \geq 1$.

Theorem 2. *If the interpolation nodes are quasi-equidistant, then the Lebesgue constant in* (4) *satisfies*

$$\Lambda_n \geq \frac{1}{2^{d+2}M^{d+1}} \binom{2d+1}{d} \cdot \begin{cases} (2 + \ln(2n+1)), & \text{if } d = 0, \\ \ln\left(\frac{n}{d} - 1\right), & \text{if } d \geq 1. \end{cases}$$

Proof. In order to establish this lower bound, we follow the idea of the proof of Theorem 2 in [4] and derive a lower bound for the numerator $N(x) = \sum_{j=0}^{n} \frac{|w_j|}{|x-x_j|}$ and an upper bound for the denominator $D(x) = \left|\sum_{j=0}^{n} \frac{w_j}{x-x_j}\right|$ of the Lebesgue function in (5) at the midpoint $x^* = (x_0 + x_1)/2$ between the first two interpolation nodes. According to (7), the distance between x^* and x_j satisfies

$$\frac{h_*}{2}|2j - 1| \leq |x^* - x_j| \leq \frac{h}{2}|2j - 1|. \tag{10}$$

For the numerator, we omit some terms of the sum and use the lower bound W from Lemma 1 as well as inequality (10) to get

$$N(x^*) = \sum_{j=0}^{n} \frac{|w_j|}{|x^* - x_j|} \geq \sum_{j=d}^{n-d} \frac{|w_j|}{|x^* - x_j|} \geq W \sum_{j=d}^{n-d} \frac{1}{|x^* - x_j|} \geq W \frac{2}{h} \sum_{j=d}^{n-d} \frac{1}{|2j - 1|}.$$

If $d = 0$, we then have

$$\frac{N(x^*)}{W} \geq \frac{2}{h}\left(1 + \sum_{j=1}^{n} \frac{1}{2j-1}\right) \geq \frac{2}{h}\left(1 + \int_1^{n+1} \frac{dx}{2x-1}\right) = \frac{1}{h}(2 + \ln(2n+1)),$$

and for $d \geq 1$ we find

$$\frac{N(x^*)}{W} \geq \frac{2}{h} \sum_{j=d}^{n-d} \frac{1}{2j-1} \geq \frac{2}{h} \int_d^{n-d+1} \frac{dx}{2x-1} = \frac{1}{h} \ln\left(\frac{2n - 2d + 1}{2d - 1}\right) \geq \frac{1}{h} \ln\left(\frac{n}{d} - 1\right).$$

To bound the denominator, we first rewrite it in terms of the functions λ_j and notice that $\lambda_0(x^*)$ and $\lambda_1(x^*)$ both have the same sign, and that the subsequent $\lambda_j(x^*)$ oscillate in sign and decrease in absolute value. Therefore,

$$D(x^*) = \left|\sum_{j=0}^{n} \frac{w_j}{x - x_j}\right| = \left|\sum_{j=0}^{n-d} \lambda_j(x^*)\right| \leq |\lambda_0(x^*)| + |\lambda_1(x^*)| = \frac{1}{\prod_{j=0}^{d}|x^* - x_j|} + \frac{1}{\prod_{j=1}^{d+1}|x^* - x_j|}.$$

We then conclude, using first (10) and then (6), that

$$D(x^*) \leq \frac{2^{d+1}}{h_*^{d+1} \prod_{j=0}^{d}|2j - 1|} + \frac{2^{d+1}}{h_*^{d+1} \prod_{j=1}^{d+1}|2j - 1|}$$

$$= \frac{2^{d+1}}{h_*^{d+1}}\left(\frac{2d+1}{\prod_{j=1}^{d+1}(2j-1)} + \frac{1}{\prod_{j=1}^{d+1}(2j-1)}\right) \leq \frac{2^{2d+2}M^{d+1}}{h^{d+1}d!\binom{2d+1}{d}}.$$

$\square$

Note that the lower bound in Theorem 2 simplifies to the one given in [4] for $d \geq 1$ and equidistant nodes with $M = 1$.
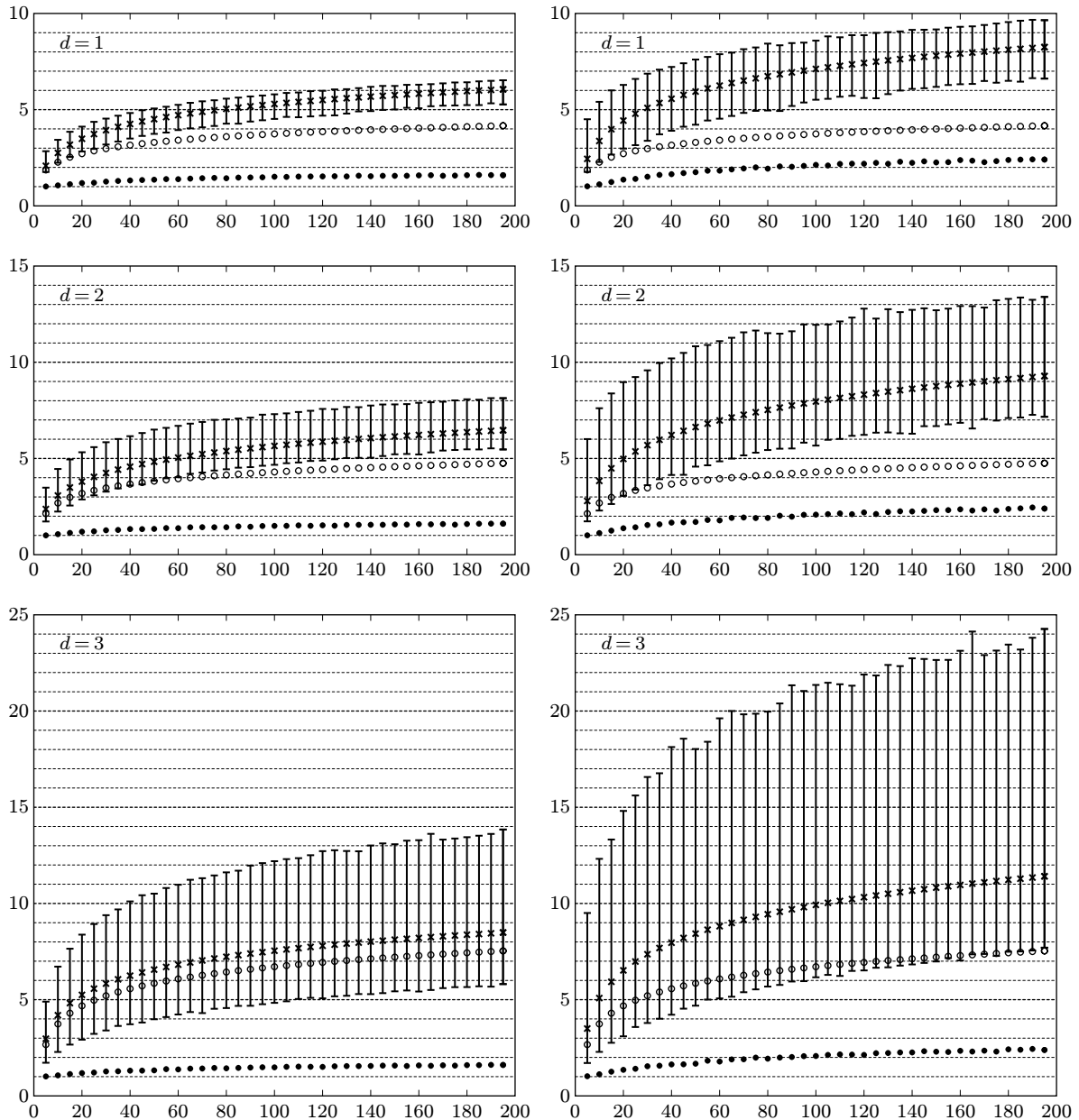
Figure 1: Minimum, maximum, and average Lebesgue constants (vertical bars with crosses) for quasi-equidistant nodes with a random perturbation of at most $\delta = 1/6$ (left) and $\delta = 3/10$ (right), that is, with global mesh ratios at most $M = 2$ (left) and $M = 4$ (right). Circles indicate the Lebesgue constants for equidistant nodes, bullets mark the minimal global mesh ratios. The plots for $d = 0$ are not shown because they are very similar to those for $d = 1$.

3   Numerical experiments

We finally present some results of the extensive numerical experiments that we carried out. We first considered the situation where a set of equidistant interpolation nodes is perturbed randomly with a fixed maximum relative perturbation $\delta \in (0, 1/2)$. Since Lebesgue constants are invariant with respect to translations and uniform scalings of the interpolation nodes, due to the invariance of Lagrange
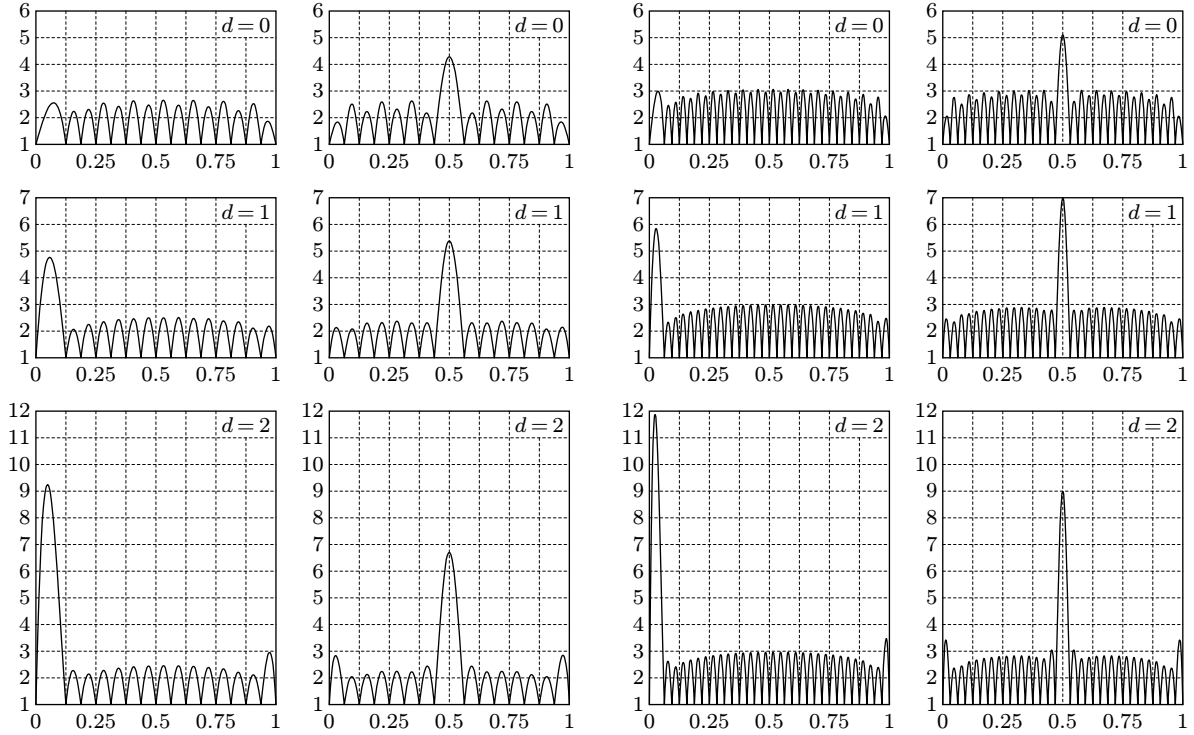
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Figure 2: Lebesgue functions for two specific sets of $n = 15$ (left) and $n = 31$ (right) quasi-equidistant nodes with fixed global mesh ratio $M = 2$.

elementary polynomials to affine transformations, we simply used

$$x_j = j + \delta_j, \qquad j = 0, \ldots, n,$$

with randomly chosen $\delta_j \in [-\delta, \delta]$. Hence, the global mesh ratio is at most $M = (1 + 2\delta)/(1 - 2\delta)$. For fixed d and n, we generated $N = 1,000,000$ different sets of such interpolation nodes and computed the corresponding Lebesgue constants. The vertical bars in Figure 1 show the ranges of these N Lebesgue constants and the crosses mark their averages. We observe that the variance of the Lebesgue constants grows rapidly with both d and δ and only slowly with n. Moreover, the average values are not too far from the Lebesgue constants in the equidistant setting with the same number of interpolation nodes, which are indicated by the circles in Figure 1, and the relative distance decreases with d. The case $d = 3$ also reveals that the random perturbations can actually improve the Lebesgue constant, and that this becomes less likely as n increases. In principle, one could expect the circles to always lie within the range of Lebesgue constants, as the perfectly uniform distribution is just a special case of the randomly perturbed nodes. The fact that this is not always the case is explained by the bullets in Figure 1, which mark the smallest global mesh ratio M that we found among the N randomly created sets of nodes; the greatest M is not shown because it was close to $(1 + 2\delta)/(1 - 2\delta)$ in all examples.

Figure 2 shows the Lebesgue functions for some sets of quasi-equidistant nodes with global mesh ratio $M = 2$. We tested many such sets of nodes and found two cases to give particularly large Lebesgue constants: in the first case, we let the first subinterval $[x_0, x_1]$ be of length M and all others of length 1, in the second case we let the central subinterval $[x_{\lfloor n/2 \rfloor}, x_{\lfloor n/2 \rfloor + 1}]$ be of length M and again all others of length 1. For a better comparison, we scaled the nodes so that $[x_0, x_n] = [0, 1]$. Except for $d = 0$ and the first set of nodes, the maximum of the Lebesgue function is always found in the single interval of length M, and for $d \geq 2$, the first set gives larger Lebesgue constants than the second set. This can also be seen in Figure 3, where the bullets mark the Lebesgue constants for the first set of nodes and the circles correspond to the second set. We observe that these are still quite far from the theoretical
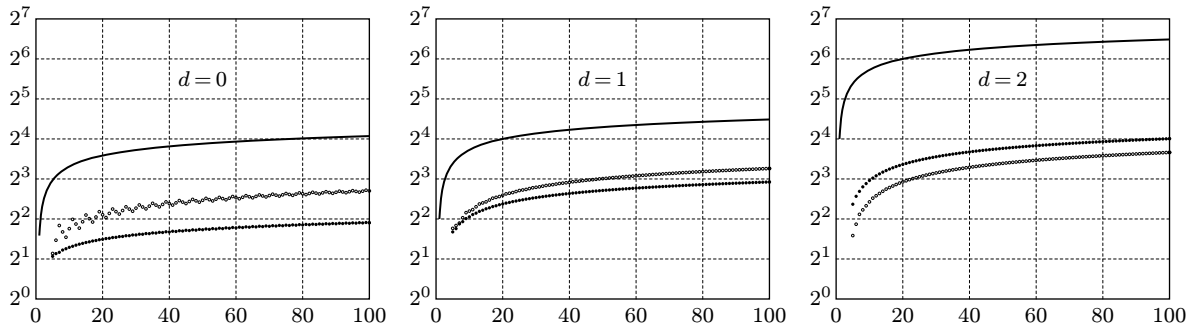
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Figure 3: Lebesgue constants (bullets and circles) for the two specific sets of quasi-equidistant nodes with global mesh ratio $M = 2$ that are shown in Figure 2, compared with the upper bound in Theorem 1.

upper bound that we derive in Theorem 1, and it remains to be figured out, as discussed in [5] in the case of polynomial interpolation, if there are node configurations with a fixed global mesh ratio that lead to yet larger Lebesgue constants than those displayed here.

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