On the Classes of Dense and Closed Subobjects

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Abstract

We provide analogous characterizations of the families of dense and of closed subobjects with respect to closure operators. The analogous behavior of hereditary and minimal closure operators with respect to the families of dense and of closed subobjects, respectively, is pointed out. We prove that, in the category of topological abelian groups, the total denseness cannot be described as denseness with respect to a closure operator.

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Introduction

Notions involving denseness or closedness are encountered in many fields of contemporary mathematics. The categorical notion of closure operator gives us the possibility of their unified treatment. The notion of closure operator used in this paper is the one gradually developed by Dikranjan, Giuli, Tholen and further by Strecker and coworkers (see [5], [7], [10], [4]). It has many predecessors throughout this century, most notably in the works of Birkhoff, Kuratowski, Čech, Lawvere and Tierney.
Every closure operator C gives rise to two classes of subobjects – C-dense and C-closed ones. The aim of this paper is to determine the properties which characterize a family of subobjects as a class of C-dense or C-closed subobjects for a suitable closure operator C. Translated in terms of factorization systems, the problem has a solution in the case of classes of closed subobjects: these are the classes of subobjects stable under pullback and intersection (see [12] and [19]). This result is not completely satisfactory in our setting: its categorical characterization does not give a characterization of dense subobjects since they need not be stable under pushout. However it can be formulated in lattice-theoretic terms and then leads to an analogous characterization of the classes of dense subobjects (see Theorem 2.1, announced in [20] and [18]). This characterization is used (see Example 3.3) to give a negative answer to the following question, set by Dikranjan: can *total denseness* ([16]) in the category of topological groups be obtained by means of a suitable closure operator? This notion of denseness plays a crucial role in topological groups: for example, among the dense subgroups of compact groups the totally dense ones are those satisfying the open-mapping theorem ([8], [9]).

1 The categorical setting

Consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of monomorphisms in $\mathcal{X}$ containing all isomorphisms and closed under composition. For every object X in $\mathcal{X}$, the class of all morphisms in $\mathcal{M}$ with codomain X is preordered by the relation " $\leq$ ", where $m \leq n$, with $m, n \in \mathcal{M}$, means that there exists a unique morphism $\frac{m}{n}$ such that $n \circ \frac{m}{n} = m$. Each equivalence class with respect to the relation " $\leq$ & $\geq$ " is called $\mathcal{M}$-*subobject* of X. The family of all $\mathcal{M}$-subobjects of X is denoted by $\mathcal{M}/X$; we do not distinguish between the equivalence classes and their representative morphisms. In the examples, when there is no danger of confusion, we will denote the $\mathcal{M}$-subobjects simply by their domains. Throughout the paper we assume $\mathcal{X}$ to be an *$\mathcal{M}$-complete category*, i.e.

- $\forall X \in \mathcal{O}b(\mathcal{X})$, $\mathcal{M}/X$ is closed under intersection (briefly $\mathcal{X}$ has $\mathcal{M}$-intersection);
- $\forall f : X \to Y \in \mathcal{X}$ and $\forall n \in \mathcal{M}/Y$, the pullback of n under f, belongs to $\mathcal{M}/X$ (briefly $\mathcal{X}$ has $\mathcal{M}$-pullback).

In such a category there is a (uniquely determined) class of morphisms $\mathcal{E}$ such that $(\mathcal{E}, \mathcal{M})$ is a factorization system. The class $\mathcal{E}$ contains all isomorphisms, it is closed under composition and is right-cancellable with respect to all morphisms (see definition below), while the class $\mathcal{M}$ is left-cancellable with respect to monomorphisms (see [13]). In particular if $m \le n \in \mathcal{M}$, then $\frac{m}{n} \in \mathcal{M}$. A *closure operator* on $\mathcal{X}$ (see [7], 2.10) is a family $C = (c_X)_{X \in \mathcal{O}b(\mathcal{X})}$ of maps

$$\begin{array}{ll} c_X : \mathcal{M}/X & \longrightarrow \mathcal{M}/X \\ m : M \to X & \longmapsto c_X(m) : c_X(M) \to X \end{array}$$

such that for every X in $\mathcal{X}$ the following properties are satisfied:

Expansion : $m \le c_X(m)$ for all $m \in \mathcal{M}/X$;

Monotonicity : $m \le m' \in \mathcal{M}/X \Rightarrow c_X(m) \le c_X(m')$;

Continuity : $f(c_X(m)) \le c_Y(f(m))$ for all $f : X \to Y$ in $\mathcal{X}$ and $m \in \mathcal{M}/X$.

For each X in $\mathcal{X}$, an $\mathcal{M}$-subobject m of X is called *C-dense* (resp. *C-closed*) if $c_X(m) = 1_X$ (resp. $m = c_X(m)$). A closure operator C is said to be *weakly hereditary* (resp. *idempotent*) if for each X in $\mathcal{X}$ and $m \in \mathcal{M}/X$, the $\mathcal{M}$-subobject $\frac{m}{c_X(m)}$ is C-dense (resp. $c_X(m)$ is C-closed); it is *hereditary* (resp. *minimal*) if for each X in $\mathcal{X}$ and $m \le n \in \mathcal{M}/X$, $c_N(\frac{m}{n}) = \frac{c_X(m) \wedge n}{n}$ with N the domain of n (resp. $c_X(n) = c_X(m) \vee n$).
A class $\mathcal{A} \subseteq \mathcal{M}$

is *right-cancellable* (resp. *left-cancellable*) with respect to $\mathcal{M}$ if $n \circ m \in \mathcal{A}$ with m, $n \in \mathcal{M}$ implies $n \in \mathcal{A}$ (resp. $m \in \mathcal{A}$);

is *stable under $\mathcal{M}$-union* if for all X in $\mathcal{X}$ one has $1_X \in \mathcal{A}$ and, given $m \le n_i \in \mathcal{M}/X$, with $i \in I \ne \emptyset$, if $\frac{m}{n_i} \in \mathcal{A}$ for all $i \in I$ then $\frac{m}{\vee n_i} \in \mathcal{A}$;

is *stable under $\mathcal{M}$-intersection* if for all X in $\mathcal{X}$ and $\{m_i\}_{i \in I} \subseteq \mathcal{M}/X$, with $m_i \in \mathcal{A}$ for all $i \in I$, also $\wedge m_i \in \mathcal{A}$ (in particular, one has $1_X \in \mathcal{A}$ with $I = \emptyset$);

has the *preservation property* (resp. *reflection property*) for morphisms if the following equivalent conditions are satisfied

a) for every $f : X \to Y$ in $\mathcal{X}$ and $n \in \mathcal{M}/Y$, $f^{-1}(n) \in \mathcal{A}$ implies $n \in \mathcal{A}$ (resp. $n \in \mathcal{A}$ implies $f^{-1}(n) \in \mathcal{A}$) whenever $f(1_X) \vee n = 1_Y$;

b) for every $f : X \to Y$ in $\mathcal{X}$ and $n \in \mathcal{M}/Y$, $f^{-1}(n) \in \mathcal{A}$ implies $\frac{n}{f(1_X) \vee n} \in \mathcal{A}$ (resp. $\frac{n}{f(1_X) \vee n} \in \mathcal{A}$ implies $f^{-1}(n) \in \mathcal{A}$).

2 The Main Theorem

Let $\mathcal{X}$ be an $\mathcal{M}$-complete category. It is known that a family of $\mathcal{M}$-subobjects is the class of all closed subobjects with respect to a closure operator if and only if it is a right factorization class (see [7], 2.4), i.e. if and only if it is stable under intersection and under pullback (see [12] and [19] for a proof of the dual result). To be stable under pullback is equivalent to being both left-cancellable with respect to $\mathcal{M}$ and to have the reflection property. Hence a subclass $\mathcal{C}$ of $\mathcal{M}$ is the family of closed $\mathcal{M}$-subobjects in $\mathcal{X}$ with respect to an appropriate closure operator if and only if *i)* $\mathcal{C}$ is left-cancellable with respect to $\mathcal{M}$,

ii) $\mathcal{C}$ is stable under $\mathcal{M}$-intersection,

iii) $\mathcal{C}$ has the reflection property.

In such a case there exists a uniquely determined idempotent closure operator ${}_{\mathcal{C}}C$ such that $\mathcal{C}$ is the family of all ${}_{\mathcal{C}}C$-closed $\mathcal{M}$-subobjects in $\mathcal{X}$. For $X \in \mathcal{X}$ and $m \in \mathcal{M}/X$, ${}_{\mathcal{C}}C$ is defined by

$$({}_{\mathcal{C}}c)_X(m) = \bigwedge\{l \in \mathcal{M}/X : m \leq l \ \&\ l \in \mathcal{C}\}.$$

The analogous behavior of the notions of denseness and closedness becomes clear, thanks to the following

Theorem 2.1 *A subclass $\mathcal{D}$ of $\mathcal{M}$ is the family of dense $\mathcal{M}$-subobjects in $\mathcal{X}$ with respect to an appropriate closure operator if and only if*

i) $\mathcal{D}$ is right-cancellable with respect to $\mathcal{M}$,

ii) $\mathcal{D}$ is stable under $\mathcal{M}$-union,

iii) $\mathcal{D}$ has the preservation property.

In such a case there exists a uniquely determined weakly hereditary closure operator $C_{\mathcal{D}}$ such that $\mathcal{D}$ is the family of all $C_{\mathcal{D}}$-dense $\mathcal{M}$-subobjects in $\mathcal{X}$. For $X \in \mathcal{X}$ and $m \in \mathcal{M}/X$, $C_{\mathcal{D}}$ is defined by

$$(c_{\mathcal{D}})_X(m) = \bigvee\{l \in \mathcal{M}/X : m \leq l \ \&\ \tfrac{m}{l} \in \mathcal{D}\}.$$

Proof. Necessity: let $\mathcal{D}$ be the class of C-dense $\mathcal{M}$-subobjects with respect to a closure operator C. The right-cancellation property is obviously satisfied. To verify the stability under $\mathcal{M}$-union, let us consider an object X of $\mathcal{X}$ and a family $m \leq n_i$, $i \in I$, of $\mathcal{M}$-subobjects. Suppose the $\mathcal{M}$-subobjects $\frac{m}{n_i}$ are C-dense for each $i \in I$. Denoting by N_i the domain of n_i, $i \in I$, for each $k \in I$ we have

$$\tfrac{n_k}{\vee n_i} = \tfrac{n_k}{\vee n_i} \circ 1_{N_k} = \tfrac{n_k}{\vee n_i}(c_{N_k}(\tfrac{m}{n_k})) \leq c_{(\vee N_i)}(\tfrac{m}{\vee n_i});$$

then the $\mathcal{M}$-subobject $\frac{m}{\vee n_i}$ is C-dense. To check the preservation property take a morphism $f : X \to Y$ in $\mathcal{X}$ and an $\mathcal{M}$-subobject n of Y with $f(1_X) \vee n = 1_Y$. Suppose $f^{-1}(n)$ is C-dense and let $f(1_X) \circ e$ be the $(\mathcal{E}, \mathcal{M})$-factorization of f. The $\mathcal{M}$-subobject $e(f^{-1}(n))$ is C-dense and $f(1_X) \circ e(f^{-1}(n)) = f(f^{-1}(n))$. Then, since

$$f(1_X) = f(1_X) \circ 1_{f(X)} = f(1_X)[c_{f(X)}(e(f^{-1}(n)))] \leq c_Y(f(f^{-1}(n))) \leq c_Y(n)$$

and $1_Y = f(1_X) \vee n$, the $\mathcal{M}$-subobject n is C-dense.
Sufficiency: let $\mathcal{D}$ satisfy the conditions *i)*, *ii)*, *iii)*. For each X in $\mathcal{X}$ and $m \in \mathcal{M}/X$, set $\mathcal{F}_X(m) = \{l \in \mathcal{M}/X : m \leq l \ \&\ \frac{m}{l} \in \mathcal{D}\}$. Let us prove that $c_X(m) := \bigvee \mathcal{F}_X(m)$ defines a closure operator $C = (c_X)_{X \in \mathcal{O}b(\mathcal{X})}$. For $m : M \to X$ in $\mathcal{M}$, $1_M = \frac{m}{m} \in \mathcal{D}$ gives $m \leq c_X(m)$. To show the monotonicity it suffices to see that for $m \leq m' \in \mathcal{M}/X$ and $n \in \mathcal{F}_X(m)$ one has $n \vee m' \in \mathcal{F}_X(m')$. By the right-cancellation property, $\frac{m}{n} \in \mathcal{D}$ implies $\frac{m' \wedge n}{n} \in \mathcal{D}$. Since $\frac{m' \wedge n}{n} = n^{-1}(m')$, the preservation property implies $\frac{m'}{n \vee m'} \in \mathcal{D}$, i.e. $n \vee m' \in \mathcal{F}_X(m')$. Now $c_X(m) \leq c_X(m')$ is obvious. To conclude, for $f : X \to Y$ and $m \in \mathcal{M}/X$, we must show $f(c_X(m)) \leq c_Y(f(m))$. Since f preserves joins, it is sufficient to show that for each $n \in \mathcal{F}_X(m)$, $f(n) \in \mathcal{F}_Y(f(m))$ holds. By definition, $n \in \mathcal{F}_X(m)$ implies $\frac{m}{n} \in \mathcal{D}$; since $\frac{m}{n} \leq (f \circ n)^{-1}(f(m))$, by the right-cancellation property $(f \circ n)^{-1}(f(m))$ belongs to $\mathcal{D}$. Hence, again the preservation property implies $\frac{f(m)}{(f \circ n)(1_N) \vee f(m)} = \frac{f(m)}{f(n)} \in \mathcal{D}$, i.e. $f(n) \in \mathcal{F}_Y(f(m))$.
To prove that C is weakly hereditary note that for $m \in \mathcal{M}/X$

$$c_{(c_X(m))}(\tfrac{m}{c_X(m)}) = \vee\mathcal{F}_{(c_X(m))}(\tfrac{m}{c_X(m)}) = \tfrac{\vee\mathcal{F}_X(m)}{c_X(m)} = 1_{(c_X(m))}.$$

Finally, $\mathcal{D}$ is the class of C-dense $\mathcal{M}$-subobjects. In fact, if $\frac{m}{1_X} = m \in \mathcal{D}$, then $1_X \in \mathcal{F}_X(m)$; hence m is C-dense. Conversely, $c_X(m) = 1_X$ means $\bigvee \mathcal{F}_X(m) = 1_X$. Hence the stability of $\mathcal{D}$ under $\mathcal{M}$-union yields $m = \frac{m}{\vee \mathcal{F}_X(m)} \in \mathcal{D}$. ∎

Given two closure operators C_1 and C_2 we set $C_1 \leq C_2$ if for all X in $\mathcal{X}$, $m \in \mathcal{M}/X$, $(c_1)_X(m) \leq (c_2)_X(m)$ holds. The above proof shows that $C_{\mathcal{D}}$ is the smallest closure operator which has $\mathcal{D}$ as class of dense $\mathcal{M}$-subobjects. For a closure operator C denote by $\mathcal{D}_C$ the class of all C-dense $\mathcal{M}$-subobjects. Then the closure operator $C_{\mathcal{D}_C}$ associated to $\mathcal{D}_C$ (see Theorem 2.1) is the *weakly hereditary core* $\check{C}$ of C, i.e. the largest weakly hereditary closure operator below C (see [7], 2.11).

Remark 2.2 For each subclass $\mathcal{L}$ of $\mathcal{M}$, the family of classes of $\mathcal{M}$-subobjects containing $\mathcal{L}$ and satisfying the conditions of Theorem 2.1 is not empty (it contains $\mathcal{M}$) and closed with respect to intersection. Let $\overline{\mathcal{L}}$ be its minimum. Setting $\mathcal{L} \mapsto C_{\overline{\mathcal{L}}}$, $C \mapsto \mathcal{D}_C$ we define an order-preserving Galois correspondence ([1], 6.25), between the subclasses of $\mathcal{M}$ and the closure operators of the $\mathcal{M}$-complete category $\mathcal{X}$. This correspondence is a Galois equivalence between the subclasses of $\mathcal{M}$ satisfying the conditions of Theorem 2.1 and the weakly hereditary closure operators of $\mathcal{X}$. □

Proposition 2.3 *An operator C is hereditary if and only if it is weakly hereditary and the class $\mathcal{D}$ of all C-dense $\mathcal{M}$-subobjects is left-cancellable with respect to $\mathcal{M}$. Dually, C is minimal if and only if it is idempotent and the class $\mathcal{C}$ of all C-closed $\mathcal{M}$-subobjects is right-cancellable with respect to $\mathcal{M}$.*

Proof. If C is hereditary, then it is clearly weakly hereditary. Let $m \leq n \in \mathcal{M}$ with $m = n \circ \frac{m}{n} \in \mathcal{D}$. By the hereditariness of C we have, with N the domain of n, $c_N(\frac{m}{n}) = \frac{n \wedge c_X(m)}{n} = \frac{n \wedge 1_X}{n} = 1_N$, i.e. $\frac{m}{n} \in \mathcal{D}$. Conversely, let C be weakly hereditary and let $\mathcal{D}$ have the left-cancellation property. Given X in $\mathcal{X}$ and $m \leq n \in \mathcal{M}/X$, $\frac{m}{c_X(m)} = \frac{n \wedge c_X(m)}{c_X(m)} \circ \frac{m}{n \wedge c_X(m)}$ is C-dense. By our hypothesis on $\mathcal{D}$, the $\mathcal{M}$-subobject $\frac{m}{n \wedge c_X(m)}$ belongs to $\mathcal{D}$. Since $C = C_{\mathcal{D}}$ (see Theorem 2.1) we have $c_N(\frac{m}{n}) = \frac{n \wedge c_X(m)}{n}$ and C is hereditary. Next, if C is minimal, obviously C is idempotent. Let X be in $\mathcal{X}$ and $m \leq n \in \mathcal{M}/X$ with $m = n \circ \frac{m}{n} \in \mathcal{C}$. By the minimality of C we have $c_X(n) = c_X(m) \vee n = n$, i.e. $n \in \mathcal{C}$. Conversely, let C be idempotent. Pick X in $\mathcal{X}$ and $m \leq n \in \mathcal{M}/X$, then $c_X(m)$ is C-closed and $c_X(m) = (c_X(m) \vee n) \circ \frac{c_X(m)}{c_X(m) \vee n}$. By the right-cancellation property of $\mathcal{C}$, the $\mathcal{M}$-subobject $c_X(m) \vee n$ belongs to $\mathcal{C}$. Then we have $c_X(n) = n \vee c_X(m)$ and C is minimal. ∎

3 Comments and Applications

Clearly, the conditions to be stable under $\mathcal{M}$-union and to have the preservation property in Theorem 2.1 are independent. The following example guarantees also the independence of the right-cancellation property with respect to $\mathcal{M}$ from the others.

Example 3.1 The set $\{0, 1, 2\}$ with the natural order is a small category $\mathcal{X}$. It is a $\mathcal{M}$-complete category with $\mathcal{M} := \mathrm{Mor}(\mathcal{X})$. Let $m : 0 \to 2$ and $n : 1 \to 2$ be the (unique) morphisms with the indicated domain and codomain. The class $\mathcal{A} := \mathcal{M} \setminus \{n\}$ is stable under $\mathcal{M}$-union and has the preservation property; but, since $m \in \mathcal{A}$ and $n \notin \mathcal{A}$, it is not right-cancellable. □

The following examples show that the left- and the right-cancellation property with respect to $\mathcal{M}$ of the class of C-dense and of C-closed subobjects alone does not imply the hereditariness and the minimality, respectively, of the operator C. Let r be a preradical on the category R-Mod of left modules on a ring R and their homomorphisms. It is possible to define two closure operators C_r and C^r, respectively called the *minimal* and the *maximal* closure operators associated to r (see [6]). The minimal closure operator C_r is defined, for any submodule L of M, by $(c_r)_M(L) := L + r(M)$. The maximal closure operator C^r is defined, for any submodule L of M, by $c^r_M(L) = \pi^{-1}(r(M/L))$, where π is the canonical projection.

Examples 3.2 *1)* Let us consider, on the category $\mathcal{A}b$ of abelian groups and their homomorphisms, the preradicals socle and maximal divisible subgroup, denoted by *Soc* and d respectively. Fix $\mathcal{M}$ to be the class of monomorphisms of $\mathcal{A}b$. For $r = \mathrm{Soc} \circ d$ the class $\mathcal{D}$ of all isomorphisms of

$\mathcal{A}b$ is the class of C^r-dense $\mathcal{M}$-subobjects since $G = \mathrm{Soc} \circ d(G)$ always implies $G = 0$. Obviously $\mathcal{D}$ is left-cancellable. Nevertheless C^r is not hereditary: consider the subgroups $\mathbf{Z}(p)$, $\mathbf{Z}(p^2)$ of the Prüfer group $\mathbf{Z}(p^\infty)$. Then $(c^r)_{\mathbf{Z}(p^2)}(\mathbf{Z}(p)) = \mathbf{Z}(p)$, while $(c^r)_{\mathbf{Z}(p^\infty)}(\mathbf{Z}(p)) = \mathbf{Z}(p^2)$.
2) Let us consider a totally ordered set $(X, \leq)$ with bottom and top elements as a small category. It is $\mathcal{M}$-complete with $\mathcal{M} := \mathrm{Mor}(\mathcal{X})$. Given a pair $a \leq b$ of elements of X, setting $(c_{a,b})_y(x) = \begin{cases} a \wedge y & \text{if } a > x \\ x \vee (b \wedge y) & \text{if } a \leq x \end{cases}$ for each $x \leq y$ in X, we define a closure operator $C_{a,b}$. It is easy to see that $C_{a,b}$ and $C_{b,b}$ have the same class $\mathcal{C}$ of closed $\mathcal{M}$-subobjects. Since $C_{b,b}$ is minimal, $\mathcal{C}$ is right-cancellable, while, if $|X| \geq 3$ and a is not the bottom element of X, $C_{a,b}$ is not idempotent, hence certainly minimal. □

Let $\mathcal{TA}b$ be the category of topological abelian groups and their continuous homomorphisms. Fixing $\mathcal{M}$ to be the class of topological embeddings, $\mathcal{TA}b$ is an $\mathcal{M}$-complete category. A subgroup J of $G \in \mathcal{O}b(\mathcal{TA}b)$ is called *totally dense* in G if for each closed subgroup H of G, the intersection $J \cap H$ is dense in H. The following example answers a question of D. Dikranjan and shows that the class of totally dense topological embeddings is not the class of dense $\mathcal{M}$-subobjects with respect to any closure operator.

Example 3.3 Let $\mathbf{Z}$ and $\mathbf{Z}_p$ be the additive groups of integers and of p-adic integers endowed with their p-adic topologies. Consider the subgroups $N = \mathbf{Z} \times \mathbf{Z}$ and $X = \mathbf{Z}_p \times 0$ of $\mathbf{Z}_p \times \mathbf{Z}_p$. Then the topological subgroup $X \cap N$ is totally dense in X, while N is not totally dense in $Y = N + X = \mathbf{Z}_p \times \mathbf{Z}$, so that the class of totally dense topological embeddings does not have the preservation property. To see that N is not totally dense in Y pick $\xi \in \mathbf{Z}_p$ such that $m\xi \notin \mathbf{Z}$ for each $0 \neq m \in \mathbf{Z}$. Then the cyclic subgroup L of Y generated by $(\xi, 1)$ is closed: for $(\eta, n) \in Y \setminus L$, one has $[(\eta, n) + p^{s+1}(H + K)] \cap L = \emptyset$, with s the maximum integer such that p^s divides $\eta - n\xi \neq 0$. By $L \cap N = 0$, N is not totally dense in Y. □

Trading denseness for C-denseness and closed subgroups for C-closed subobjects, we can define total C-denseness in any category endowed with a closure operator C. Sometimes it can be a denseness with respect to a suitable closure operator (see [21] for more details; examples include the b-closure for topological spaces and total denseness for an appropriate subcategory of $\mathcal{TA}b$).
Let $\mathcal{X}$ be an $\mathcal{M}$-complete category with equalizers in $\mathcal{M}$. The class $\mathcal{M}_{epi} := \{m \in \mathcal{M} : m$ is epimorphism$\}$ satisfies the three conditions of Theorem 2.1. The unique weakly hereditary closure operator which has $\mathcal{M}_{epi}$ as the class of dense $\mathcal{M}$-subobjects is called the *epi-closure operator* and denoted by C_{epi} (see [2] for the case of topological spaces). Since $\mathcal{M}_{epi}$ is closed under composition, C_{epi} is also idempotent (see [7], Corollary 2.9). The class $\mathcal{M}_{eq}$ of equalizers of morphisms of $\mathcal{X}$ is stable under intersection and pullback. The unique idempotent closure operator which has $\mathcal{M}_{eq}$ as the class of closed $\mathcal{M}$-subobjects is called the *regular operator* and denoted by C_{reg} (see [14]). The operators C_{epi} and C_{reg} have the same class of dense $\mathcal{M}$-subobjects; then, by Theorem 2.1, C_{epi} is equal to $\check{C}_{reg}$, the weakly hereditary core of C_{reg} and, by Proposition 2.3, C_{epi} is hereditary if and only if C_{reg} is hereditary. In general, the two operators do not coincide (see [14]), not even when $\mathcal{X}$ is the torsionfree class of a radical of R-modules:

Example 3.4 Let r be a radical of $Mod\text{-}R$, i.e. a preradical such that, for each module M in $Mod\text{-}R$, $r(M/r(M)) = 0$ holds. Let us consider the category $\mathcal{F}_r := \{M \in Mod\text{-}R : r(M) = 0\}$. Setting $\mathcal{M} := \mathrm{Mor}(\mathcal{F}_r)$, $\mathcal{F}_r$ is an $\mathcal{M}$-complete category with equalizers in $\mathcal{M}$. It is easy to prove that in $\mathcal{F}_r$ the operator C_{reg} coincides with the maximal operator C^r associated to r. Then we have $C_{epi} = \check{C}_{reg} = \check{C}^r = C^{r_*}$ where r_* is the idempotent core of r (Proposition 2.4, [6]). Then to have $C_{epi} \neq C_{reg}$ it is necessary to choose a non idempotent radical r. The Jacobson radical J is not idempotent; let us see that C^J and C^{J_*} do not coincide on $\mathcal{F}_J$. Consider $p^2\mathbf{Z}$, $p\mathbf{Z}$ and $\mathbf{Z}$ in $\mathcal{F}_J$. Clearly the C^J-closure of $p^2\mathbf{Z}$ in $\mathbf{Z}$ is equal to $p\mathbf{Z}$; while $p^2\mathbf{Z}$ is C^{J_*}-closed in $\mathbf{Z}$, since $J(J(\mathbf{Z}/p^2\mathbf{Z})) = J(p\mathbf{Z}/p^2\mathbf{Z}) = 0$. □

In the category of Hausdorff topological spaces (or Tychonoff spaces, 0-dimensional T_1 spaces etc.) and continuous maps the epi-closure coincides with the usual topological closure. Recently

Uspenskiĭ (see [22]), resolving a long-standing question, showed that this is not true in the category of Hausdorff topological groups. Moreover he proved (see [23]) that in this category C_{epi} (and hence C_{reg}, as observed above) is not hereditary.

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