

Perpendicular categories of infinite dimensional partial tilting modules and transfers of tilting torsion classes

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Abstract

Let R be a ring and P be an (infinite dimensional) partial tilting module. We show that the perpendicular category of P is equivalent to the full module category $\text{Mod-}S$ where $S = \text{End}(\ell R)$ and ℓR is the Bongartz complement of P modulo its P -trace. Moreover, there is a ring epimorphism $\varphi : R \rightarrow S$. We characterize the case when φ is a perfect localization. By [Riccardo Colpi, Alberto Tonolo, Jan Trlifaj, Partial cotilting modules and the lattices induced by them, *Comm. Algebra* 25 (10) (1997) 3225–3237], there exist mutually inverse isomorphisms μ' and ν' between the interval $[\text{Gen}P, P^{\perp_1}]$ in the lattice of torsion classes in $\text{Mod-}R$, and the lattice of all torsion classes in $\text{Mod-}S$. We provide necessary and sufficient conditions for μ' and ν' to preserve tilting torsion classes. As a consequence, we show that these conditions are always satisfied when R is a Dedekind domain, and if P is finitely presented and R is an artin algebra, then the conditions reduce to the trivial ones, namely that each value of μ' and ν' contains all injectives.

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0. Introduction

Tilting torsion classes and (infinite dimensional) tilting modules over general associative rings have a rather complex structure. Tilting classes have recently been shown to be of finite type [5], and a complete description is available in particular cases (e.g., over Prüfer domains, [13]). However, in general, it appears more accessible to study tilting classes only inside particular intervals of the lattice of all torsion classes.

In the case of finite-dimensional modules over finite-dimensional hereditary algebras, this approach was successfully used in [2]. The general case goes back to [8] where, given an (infinite dimensional) partial tilting module P , a lattice isomorphism μ (with inverse ν) was established between the lattice of all torsion classes in the perpendicular subcategory $P^+ = P^{\perp_1} \cap \text{Ker Hom}_R(P, -)$ of $\text{Mod-}R$ on one hand, and of all torsion classes of R -modules containing $\text{Gen}P$ and contained in P^{\perp_1} on the other.

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As a consequence of a general result of Geigle and Lenzing [11] on perpendicular categories, we get, for P which is not a tilting module, that the perpendicular category P^\perp is equivalent to the full module category $\text{Mod-}S$ where $S = \text{End}(\ell R)$, via mutually inverse category equivalences $F = - \otimes_S \ell R : \text{Mod-}S \rightarrow P^\perp$ and $G = \text{Hom}_R(\ell R, -) : P^\perp \rightarrow \text{Mod-}S$. Moreover, there is a ring epimorphism $\varphi : R \rightarrow S$. In this way, the study of torsion classes over R can be reduced to the analogous study over the ring S that usually has a simpler structure.

In Section 2, we investigate homological properties of φ in more detail. In Theorem 2.1, we completely characterize the case when φ is a perfect localization (that is, when $- \otimes_R S$ is flat). We also show that even for artin algebras of finite representation type φ need not be a homological epimorphism of rings (Example 2.4).

In Section 3, we deal with the question of the action of the mutually inverse lattice isomorphisms $\mu' = \mu \circ F$ and $\nu' = G \circ \nu$ restricted to tilting classes. First, tilting classes obviously contain all injective modules, and they are definable by [5]. So we first deal with preservation of these two properties by μ' and ν' . In Theorem 3.7, we show that over artin algebras, tilting classes are exactly the definable torsion classes containing all injectives. As a consequence, we obtain criteria for preservation of tilting classes by μ' and ν' in the artin algebra case. Roughly speaking, the criteria say that the image of a tilting class is again a tilting class if and only if the image contains all injective modules. However, this need not be the case even for algebras of finite representation type, see Examples 2.2 and 3.3.

In Theorem 3.9, we then prove that for any ring R and any tilting class $\mathcal{C} \in \text{Mod-}R$ such that $\text{Gen}P \subseteq \mathcal{C} \subseteq P^\perp$, the class $\mathcal{D} = \nu'(\mathcal{C}) \subseteq \text{Mod-}S$ is tilting iff \mathcal{D} contains all injective right S -modules.

Finally, we apply our results to the setting of Dedekind domains. In Theorem 3.13, we show that in this setting, both μ' and ν' preserve tilting classes.

1. Preliminaries

For a given $P \in \text{Mod-}R$, we denote by $\text{Gen}P$ the class of all the P -generated modules, i.e., all factors of direct sums of arbitrary copies of P . For any $M \in \text{Mod-}R$ there is a largest submodule of M which belongs to $\text{Gen}P$, namely $\text{tr}_P M = \sum \{\text{Im}f \mid f \in \text{Hom}_R(P, M)\}$. The endofunctor tr_P of $\text{Mod-}R$ is called the P -trace, and it is an idempotent preradical. It is a radical exactly when $\text{Gen}P$ is a torsion class. The class of all R -submodules of modules in $\text{Gen}P$ is denoted by $\overline{\text{Gen}P}$, and $\text{Add}P$ denotes the class of direct summands of arbitrary direct sums of copies of P . Moreover we set $P^{\perp_0} = \text{Ker Hom}_R(P, -)$, $P^{\perp_1} = \text{Ker Ext}_R^1(P, -)$ and $P^+ = P^{\perp_1} \cap P^{\perp_0}$. P^+ is called the perpendicular category of P .

Given a subclass \mathcal{C} and an object M of $\text{Mod-}R$, we say that a monomorphism

$$i : M \hookrightarrow M_0$$

is a special \mathcal{C} -preenvelope of M whenever $M_0 \in \mathcal{C}$ and $\mathcal{C} \subseteq (\text{Coker } i)^\perp$. In particular the group homomorphism $\text{Hom}_R(i, C) : \text{Hom}_R(M_0, C) \rightarrow \text{Hom}_R(M, C)$ is surjective, for any $C \in \mathcal{C}$.

Following [9, Section 1], an R -module T is called a tilting module provided that $\text{Gen}T = T^{\perp_1}$. Equivalently, T is tilting iff T has projective dimension at most one, $\text{Ext}_R^1(T, T^{(X)}) = 0$ for all sets X and there is an exact sequence $0 \rightarrow R \rightarrow T_0 \rightarrow T_1 \rightarrow 0$, where T_0 and T_1 are in $\text{Add}T$. The class $\text{Gen}T = T^{\perp_1}$ is then a torsion class in $\text{Mod-}R$: it is called the tilting (torsion) class generated by T .

Moreover an R -module P is partial tilting if $\text{Gen}P \subseteq P^{\perp_1}$ and P^{\perp_1} is a torsion class. In particular, P has no self-extensions, and P has projective dimension at most one. By [9, Lemma 2.4], $\text{Gen}P$ is a torsion class and P^{\perp_1} is a tilting torsion class, so that, according to [1, Theorem 2.1], every right R -module has a special P^{\perp_1} -preenvelope.

In the sequel P will always denote a partial tilting right R -module which is not tilting.

The first result we recall here is a general form of the Bongartz lemma, see [4, 2.1] and [15, Lemma 6.9], applied to partial tilting modules.

Lemma 1.1. *Let M_R be an arbitrary right R -module, and c the minimal number of generators of $\text{Ext}_R^1(P, M)$ as a module over $\text{End}(P_R)$. Then there exists an exact sequence*

$$0 \rightarrow M \xrightarrow{i} M_0 \rightarrow P^{(c)} \rightarrow 0$$

with $M_0 \in P^{\perp_1}$. In particular, i is a special P^{\perp_1} -preenvelope of M .

For short, i , or simply M_0 as above, will be called the *Bongartz preenvelope* of M .

In the case M_R is the regular module R_R , the Bongartz preenvelope R_0 of R_R is also called the *Bongartz complement of P* : indeed, by [9, Theorem 1.9], $T = P \oplus R_0$ is a tilting module such that $\text{Gen} T = T^{\perp 1} = P^{\perp 1}$.

The construction of the Bongartz preenvelope is neither unique nor functorial. Nevertheless, there is a closely related functorial construction which will be very useful in the sequel. It will provide for a functor $\ell : \text{Mod-}R \rightarrow P^+$ which will be a left adjoint of the embedding of P^+ into $\text{Mod-}R$.

Lemma 1.2. *For any module M , we fix a Bongartz preenvelope $M \xrightarrow{i} M_0$ and set $\ell M = M_0/\text{tr}_P M_0$. For any morphism $f : M \rightarrow N$ in $\text{Mod-}R$, let $\eta_M : M \rightarrow \ell M$ and $\eta_N : N \rightarrow \ell N$ be the maps obtained by composing the preenvelope inclusions with projections modulo the P -trace. Then there exists a unique map $\ell(f) : \ell M \rightarrow \ell N$ which makes the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & \ell M \\ \downarrow f & & \downarrow \ell(f) \\ N & \xrightarrow{\eta_N} & \ell N \end{array}$$

commutative.

Proof. First note that $\ell M \in P^+$ since $P^{\perp 1}$ is closed under factors.

The morphism $\eta_N f$ factors through the embedding $M \hookrightarrow M_0$, since ℓN is in $P^{\perp 1}$ and M_0 is a $P^{\perp 1}$ -preenvelope of M . Moreover, since $\ell N \in P^{\perp 0}$, the map from M_0 to ℓN factors through the projection $M_0 \twoheadrightarrow \ell M$, proving the existence of the desired morphism $\ell(f)$.

For the uniqueness, let $g_1, g_2 : M_0 \rightarrow N_0$ be two maps making the diagrams

$$\begin{array}{ccc} M & \longrightarrow & M_0 \\ \downarrow f & & \downarrow g_i \\ N & \longrightarrow & N_0 \end{array}$$

commutative, for $i = 1, 2$, and let $h_1, h_2 : \ell M \rightarrow \ell N$ be the corresponding choices for $\ell(f)$. We claim that $h_1 = h_2$.

Let $h = h_1 - h_2$. Then the morphism $M_0 \rightarrow \ell M \xrightarrow{h} \ell N$ factors through the projection $M_0 \twoheadrightarrow P^{(c)}$. Since $\ell N \in P^{\perp 0}$, the induced map $P^{(c)} \rightarrow \ell N$ is zero, so that $h = 0$. This proves that $\ell(f)$ is unique. \square

Proposition 1.3. ℓ is a well defined functor from $\text{Mod-}R$ to P^+ , left adjoint to the full embedding $P^+ \hookrightarrow \text{Mod-}R$.

Proof. Applying the previous lemma to the identity map on M and to possibly different choices of M_0 , we see that the module ℓM is unique up to isomorphism. Moreover it is clear that for any $N \in P^+$ we have $\ell N \cong N$, so that, again from Lemma 1.2, we see that $\text{Hom}_R(M, N) \cong \text{Hom}_R(\ell M, N)$ for any $M \in \text{Mod-}R$. \square

In other words, the perpendicular category P^+ is a *reflective* subcategory of $\text{Mod-}R$ (see [14, Chapter X, Section 1]).

Proposition 1.4. *Let P_R be a partial tilting module. Then:*

(a) *if $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is any exact sequence in $\text{Mod-}R$, with $M \in P^+$, then*

$$L \in P^+ \iff L \in P^{\perp 1} \iff N \in P^{\perp 0} \iff N \in P^+.$$

(b) *if $f : L \rightarrow M$ is a morphism in $\text{Mod-}R$ with M and N in P^+ , then $\text{Ker } f, \text{Im } f$ and $\text{Coker } f$ are in P^+ .*

(c) *P^+ is an exact full abelian subcategory of $\text{Mod-}R$ which is closed under extensions, arbitrary direct sums and products, pure submodules and direct limits.*

Proof. The statement (a) is contained in [8, Lemma 4.1], and (b) is an easy consequence of (a). Thus (b) shows that P^+ is an exact full abelian subcategory of $\text{Mod-}R$, and it is clear that P^+ is closed under extensions, arbitrary direct sums and products. Moreover [5, Theorem 2.6] gives that $P^{\perp 1}$ is closed under pure submodules, so that the same

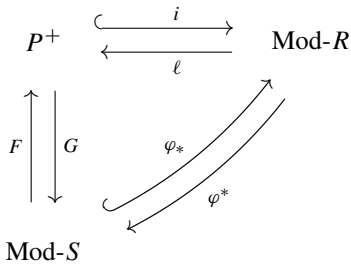
holds for P^+ . Finally, representing a direct limit of right R -modules in P^+ as a pure factor of their direct sum, using (a) we see that the direct limit is actually in P^+ . \square

Propositions 1.3 and 1.4 make it possible to apply a general result by Geigle and Lenzing [11, Proposition 3.8], and obtain thus the following description of P^+ and of its functorial connection with $\text{Mod-}R$.

Proposition 1.5. *The module ℓR is a progenerator of P^+ , so that P^+ is equivalent to $\text{Mod-}S$, where $S = \text{End}(\ell R)$. Moreover, denoting with $\lambda_r : R \rightarrow R$ the left multiplication by $r \in R$, the map*

$$\varphi : R \rightarrow S, \quad r \mapsto \ell(\lambda_r)$$

is a ring epimorphism making S_R and ℓR_R canonically isomorphic, and the diagram of categories and functors



commutative, where $F = - \otimes_S \ell R$ and $G = \text{Hom}_R(\ell R, -)$ are mutually inverse functors, $\varphi_* = - \otimes_S S_R$ is the canonical embedding, and $\varphi^* = - \otimes_R S_S$ is its left adjoint.

Note that the functor φ_* is exact. In general, this is not true of φ^* . Clearly, the functor φ^* is (left) exact if and only if ${}_R S$ is flat or, equivalently, $\text{Mod-}S$ is a Giraud subcategory of $\text{Mod-}R$ (see, for instance, [14, Sections 10.1 and 11.2]). In this case we will call φ a *perfect localization* (see [14, Chapter XI]); this case will be considered in more detail in Section 2.

We end this section by recalling the lattice isomorphism established in [8, Theorem 4.4]. If \mathcal{A}, \mathcal{B} are subclasses of $\text{Mod-}R$, we set

$$\mathbb{E}(\mathcal{B}, \mathcal{A}) = \{M \in \text{Mod-}R \mid \exists 0 \rightarrow A \rightarrow M \rightarrow B \rightarrow 0 \text{ exact with } A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Theorem 1.6. *Let P be a partial tilting right R -module, \mathcal{C} any torsion class in P^+ and \mathcal{D} any torsion class in $\text{Mod-}R$ such that $\text{Gen}P \subseteq \mathcal{D} \subseteq P^{\perp 1}$. Then the assignments*

$$\mathcal{C} \mapsto \mu(\mathcal{C}) = \mathbb{E}(\mathcal{C}, \text{Gen}P) \quad \text{and} \quad \mathcal{D} \mapsto \nu(\mathcal{D}) = \mathcal{D} \cap P^{\perp 0}$$

define mutually inverse lattice isomorphisms between torsion classes in P^+ and torsion classes in $\text{Mod-}R$ containing $\text{Gen}P$ and included in $P^{\perp 1}$.

In view of Proposition 1.5, these lattice isomorphisms naturally induce analogous isomorphisms $\mu' = \mu \circ F$ and $\nu' = G \circ \nu$ between torsion classes in $\text{Mod-}S$ and in $\text{Mod-}R$. In Section 3, we will deal with transfer of tilting torsion classes by these lattice isomorphisms.

2. Perfect localizations and homological epimorphisms of rings

We start with a necessary and sufficient condition for φ to be a perfect localization.

Theorem 2.1. *The left R -module ${}_R S$ is flat if and only if ℓ vanishes on $\overline{\text{Gen}P}$, i.e., any Bongartz preenvelope of a module in $\overline{\text{Gen}P}$ belongs to $\text{Gen}P$.*

Proof. Since $- \otimes_R S_S = \varphi^* = G \circ \ell$, the module ${}_R S$ is flat iff ℓ is left exact.

The necessity is clear, because ℓ vanishes on $\text{Gen}P$.

Take a monomorphism $\iota : L \rightarrow M$ in $\text{Mod-}R$ and consider the following commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\iota} & M \\ \downarrow & \searrow g & \downarrow \\ L_0 & \xrightarrow{f} & M_0 \end{array}$$

where the vertical maps are Bongartz preenvelopes and f is induced by g . Then we have the following diagram

$$\begin{array}{ccccccc} & & & \text{Ker } f & & & \\ & & & \downarrow j & & & \\ 0 & \longrightarrow & L & \longrightarrow & L_0 & \xrightarrow{p} & P^{(c)} \longrightarrow 0 \\ & & \downarrow g & \swarrow f & & & \\ & & M_0 & & & & \end{array}$$

First, let us prove that $\text{Ker } f \subseteq \text{tr}_P L_0$. Since $\text{Ker } f \cap L = 0$, $p \circ j$ is a monomorphism, and so $\text{Ker } f$ belongs to $\overline{\text{Gen}P}$. By assumption, its Bongartz preenvelope K_0 is P -generated. Moreover, since $L_0 \in P^{\perp_1}$, the map j extends to a map h

$$\begin{array}{ccc} \text{Ker } f \hookrightarrow & K_0 & \\ \downarrow j & \swarrow h & \\ & L_0 & \end{array}$$

proving that $\text{Ker } f \subseteq \text{tr}_P L_0$, because $K_0 \in \text{Gen}P$.

Note that $\text{tr}_P L_0 \subseteq f^{-1}(\text{tr}_P M_0)$, since tr_P is a radical. We prove that $\text{tr}_P L_0 = f^{-1}(\text{tr}_P M_0)$; then the map $\ell(\iota) : \ell L \rightarrow \ell M$ is monic, and ℓ is left exact.

Since $\text{Ker } f \subseteq \text{tr}_P L_0$, the map f induces an embedding of $X := f^{-1}(\text{tr}_P M_0)/\text{tr}_P L_0$ in $Y := \text{tr}_P M_0/f(\text{tr}_P L_0)$. Since Y is generated by P , the module X belongs to $\overline{\text{Gen}P}$. Consider the diagram

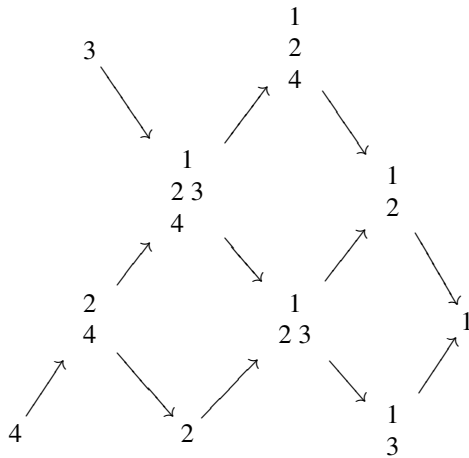
$$\begin{array}{ccc} X \hookrightarrow & X_0 & \\ \downarrow \eta & \swarrow \xi & \\ L_0/\text{tr}_P(L_0) & & \end{array}$$

where $X \hookrightarrow X_0$ is a Bongartz preenvelope and η is the canonical embedding. Since L_0 , and hence $L_0/\text{tr}_P L_0$, is in P^{\perp_1} , the morphism η extends to a morphism $\xi : X_0 \rightarrow L_0/\text{tr}_P L_0$. By the hypothesis, X_0 is generated by P ; then $\xi = 0$, and hence $X = 0$. \square

Example 2.2. We give an example of a ring R admitting partial tilting modules P and Q such that the ring epimorphism φ_P is a perfect localization but φ_Q is not. Let k be an algebraically closed field. Let R denote the k -algebra given by the quiver

$$3 \longrightarrow 1 \longleftarrow 2 \longleftarrow 4$$

The Auslander Reiten quiver associated to R is



Consider the right R -modules

$${}_R T = \begin{matrix} 1 \\ 2 \\ 4 \end{matrix} \oplus \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \oplus \begin{matrix} 1 \\ 3 \end{matrix} \oplus \begin{matrix} 1 \\ 2 \end{matrix}, \quad {}_R P = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \oplus \begin{matrix} 1 \\ 3 \end{matrix} \oplus \begin{matrix} 1 \\ 2 \end{matrix}, \quad \text{and} \quad {}_R Q = \begin{matrix} 1 \\ 3 \end{matrix} \oplus \begin{matrix} 1 \\ 2 \end{matrix}.$$

It is easy to verify that

$$\text{Gen} Q \subseteq \text{Gen} P \subseteq \text{Gen} T = T^{\perp 1} = P^{\perp 1} = Q^{\perp 1},$$

and all of them are torsion classes. Therefore P and Q are partial tilting modules and T is a tilting module. The simple modules 2 and 3 are the only indecomposable modules in $\overline{\text{Gen}} P \setminus \text{Gen} P$ and in $\overline{\text{Gen}} Q \setminus \text{Gen} Q$; the module $\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$ is their Bongartz preenvelope with respect to both P and Q . Clearly $\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$ belongs to $\text{Gen} P$, but it does not belong to $\text{Gen} Q$; therefore the functor ℓ_P is exact, while ℓ_Q is not exact.

Geigle and Lenzing introduced the notion of *homological epimorphism of rings*, that is, of a ring epimorphism $\varphi : R \rightarrow S$ such that $\text{Tor}_i^R(S, S) = 0$ for all $i \geq 1$, [11, Section 4]. Clearly, any perfect localization is a homological epimorphism of rings. In fact, homological epimorphism arises in more general situations:

Proposition 2.3. *Assume that $\text{fd}_R S \leq 1$ or $\text{fd}_R S \leq 1$. Then φ is a homological epimorphism.*

Proof. We have only to prove that $\text{Tor}_R^1(S, S) = 0$. Let W be an injective cogenerator in $\text{Mod-}S$. Then $\text{Tor}_R^1(S, S) = 0$ iff $\text{Ext}_R^1(S, S^*) = 0$, for $S^* = \text{Hom}_S({}_R S, W) = W_R \in \text{Mod-}R$ where W_R is W viewed as a right R -module via φ . Since $\text{Mod-}S$ is closed under extensions in $\text{Mod-}R$ (see Proposition 1.4(c)), we have $\text{Ext}_R^1(S, W) \cong \text{Ext}_S^1(S, W) = 0$. \square

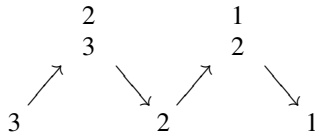
Note that in Example 2.2, the partial tilting module Q has projective dimension ≤ 1 hence φ_Q is a homological epimorphism of rings, but φ_Q is not a perfect localization.

On the other hand, the following example shows that in general, the ring epimorphism φ need not be a homological one.

Example 2.4. Let k be an algebraically closed field. Let R denote the k -algebra given by the quiver

$$1 \xleftarrow{a} 2 \xleftarrow{b} 3$$

with relation $ba = 0$. The Auslander Reiten quiver associated to R is



Let P be the projective module $\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$. Clearly $P^{\perp 1} = \text{Mod-}R$; therefore P is a partial tilting module. It is easy to see that $S_R = \ell(R) = 3 \oplus 1$ is a semisimple k -algebra. Consider the simple modules 1 and 3; clearly $\text{Ext}_S^2(1, 3) = 0$, while $\text{Ext}_R^2(1, 3) \cong \text{Ext}_R^1(2, 3) \neq 0$. Therefore by [11, Theorem 4.4 (5)] φ is not a homological epimorphism.

Remark 2.5. Observe that $\varphi : R \rightarrow S$ is a homological epimorphism if and only if $\text{Ext}_R^1(K, P^+) = 0$ for all syzygies K of S_R . Indeed, this is an easy consequence of [11, Theorem 4.4 (4)], because P^+ is a reflective subcategory of $\text{Mod-}R$ which is closed under extensions (Propositions 1.3 and 1.4).

3. Transfer of tilting torsion classes

In this section, we will denote by \mathfrak{I}_R and \mathfrak{I}_S the class of all the injective right R - and S -modules, respectively. Moreover, if $H : \mathcal{A} \rightarrow \mathcal{B}$ is a functor and B is an object of \mathcal{B} , we set $H^{-1}(B) = \{A \in \mathcal{A} \mid HA \cong B\}$.

Lemma 3.1. Let $N \in P^+$. Then $\ell^{-1}(N) = \{M \in \text{Mod-}R \mid M_0 \in \mathbb{E}(N, \text{Gen}P)\}$ where M_0 denotes a Bongartz preenvelope of M . In particular

$$\mathbb{E}(N, \text{Gen}P) = \ell^{-1}(N) \cap P^{\perp 1}.$$

Proof. If $\ell M \cong N$ and M_0 is a Bongartz preenvelope of M , then there is an exact sequence $0 \rightarrow \text{tr}_P M_0 \rightarrow M_0 \rightarrow \ell M \cong N \rightarrow 0$, so that $M_0 \in \mathbb{E}(N, \text{Gen}P)$. Conversely, let $M \in \text{Mod-}R$ such that $M_0 \in \mathbb{E}(N, \text{Gen}P)$, i.e., there is an exact sequence of the form $0 \rightarrow G \rightarrow M_0 \rightarrow N \rightarrow 0$ with $G \in \text{Gen}P$. Since $N \in P^+ \subseteq P^{\perp 0}$, we see that $G \cong \text{tr}_P M_0$ so that $N \cong \ell M$.

Finally, a right R -module M belongs to $P^{\perp 1}$ if and only if M coincides with its Bongartz preenvelope, and clearly $\mathbb{E}(P^+, \text{Gen}P) \subseteq P^{\perp 1}$. Therefore the second identity is an immediate consequence of the first one. \square

Now, it is easy to describe the lattice isomorphisms between torsion classes in $\text{Mod-}S$ and $\text{Mod-}R$ in terms of the functors φ^* and φ_* introduced in Proposition 1.5.

Corollary 3.2. Let \mathcal{C} be a torsion class in $\text{Mod-}S$, \mathcal{D} a torsion class in $\text{Mod-}R$ with $\text{Gen}P \subseteq \mathcal{D} \subseteq P^{\perp 1}$, and let $\mu' = \mu \circ F$ and $\nu' = G \circ \nu$. Then

$$\mu'(\mathcal{C}) = \{M_R \in P^{\perp 1} \mid \varphi^* M \in \mathcal{C}\} \quad \text{and} \quad \nu'(\mathcal{D}) = \{N_S \mid \varphi_* N \in \mathcal{D}\}.$$

In particular

$$\mathfrak{I}_R \subseteq \mu'(\mathcal{C}) \iff I \otimes_R S_S \in \mathcal{C} \forall I \in \mathfrak{I}_R \quad \text{and} \quad \mathfrak{I}_S \subseteq \nu'(\mathcal{D}) \iff J \otimes_S S_R \in \mathcal{D} \forall J \in \mathfrak{I}_S.$$

Proof. From 1.6 and 3.1 we have

$$\mu'(\mathcal{C}) = \mu \circ F(\mathcal{C}) = \mathbb{E}(F(\mathcal{C}), \text{Gen}P) = \ell^{-1}(F(\mathcal{C})) \cap P^{\perp 1} = (G \circ \ell)^{-1}(\mathcal{C}) \cap P^{\perp 1} = (\varphi^*)^{-1}(\mathcal{C}) \cap P^{\perp 1}$$

and

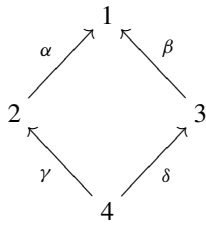
$$\nu'(\mathcal{D}) = G \circ \nu(\mathcal{D}) = G(\mathcal{D} \cap P^{\perp 0}) = G(i^{-1}(\mathcal{D})) = (i \circ F)^{-1}(\mathcal{D}) = \varphi_*^{-1}(\mathcal{D}).$$

This proves the first two identities.

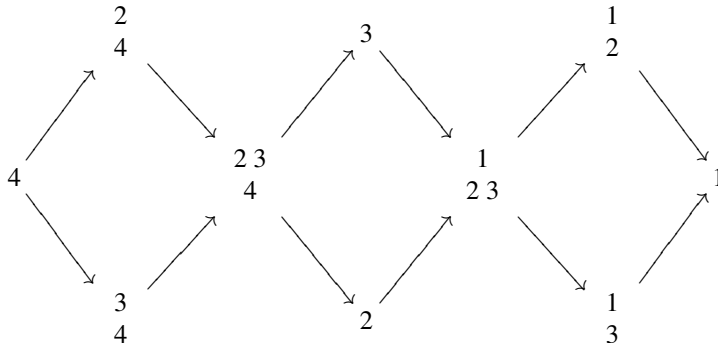
The second statement is now clear. \square

Let P be a partial tilting module such that $\text{Gen}P$ is a tilting class. Then $\nu'(\text{Gen}P) = \{0\}$, so $\nu'(\text{Gen}P)$ is certainly not a tilting class in $\text{Mod-}S$. The following example shows that, in general, the tilting property is not preserved by μ' either:

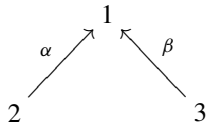
Example 3.3. Let k be an algebraically closed field. Let R denote the k -algebra given by the quiver



with relation $\alpha\gamma = 0 = \beta\delta$. The Auslander Reiten quiver associated to R is



Consider the simple projective module $P = 4$. Clearly $P^{\perp 1} = \text{Mod-}R$ is a tilting class. Moreover $P^+ = P^{\perp 0}$ is equivalent to the category of modules over the k -algebra S given by the quiver



Since S is hereditary, the class \mathfrak{I}_S of all injective S -modules is a tilting class; the class $\mu'(\mathfrak{I}_S) = \mathbb{E}(F(\mathfrak{I}_S), \text{Gen}P)$ does not contain the injective module associated to vertex 4, $E(4) = \begin{smallmatrix} 2 & 3 \\ 4 \end{smallmatrix}$, and hence it is not a tilting class in $\text{Mod-}R$.

The trivial necessary condition (that fails in Example 3.3) for a class \mathcal{D} to be tilting is that \mathcal{D} contains all injective modules. We will now express this condition in a different way from Corollary 3.2, and show that it always holds when R is right hereditary:

Lemma 3.4. Let \mathcal{C} be a torsion class in $\text{Mod-}S$ such that $\mathfrak{I}_S \subseteq \mathcal{C}$. Then the following are equivalent:

- (1) The class $\mu'(\mathcal{C})$ contains \mathfrak{I}_R ;
- (2) $\text{Ext}_R^1(S_R, \text{tr}_P I) \in \mathcal{C}$ for each $I \in \mathfrak{I}_R$.

These conditions are always satisfied in the case R is a right hereditary ring.

Proof. Since ${}_S S$ is flat, the right S -module $\text{Hom}_R(S_R, I)$ is injective for any $I \in \mathfrak{I}_R$. Applying $\text{Hom}_R(S_R, -)$ to the exact sequence in $\text{Mod-}R$

$$0 \rightarrow \text{tr}_P I \rightarrow I \rightarrow \ell I \rightarrow 0$$

we get the exact sequence in $\text{Mod-}S$

$$\text{Hom}_R(S_R, I) \rightarrow \text{Hom}_R(S_R, \ell I) \rightarrow \text{Ext}_R^1(S_R, \text{tr}_P I) \rightarrow 0.$$

Since $\text{Hom}_R(S_R, I) \in \mathfrak{I}_S$ and \mathcal{C} is closed under extensions and factors, the last exact sequence yields that $\text{Ext}_R^1(S_R, \text{tr}_P I) \in \mathcal{C}$ if and only if $\text{Hom}_R(S_R, \ell I) \in \mathcal{C}$. Now, Corollary 3.2 shows that $\text{Hom}_R(S_R, \ell I) =$

$\text{Hom}_R({}_S\ell R, \ell I) = G \circ \ell(I) = I \otimes_R S_S \in \mathcal{C}$ for all $I \in \mathfrak{T}_R$ if and only if $\mathfrak{T}_R \subseteq \mu'(\mathcal{C})$. This proves that (1) and (2) are equivalent.

Finally, if R is right hereditary, then for any $I \in \mathfrak{T}_R$ its factor ℓI is injective, therefore $\text{Hom}_R(S_R, \ell I) = I \otimes_R S_S$ is injective as well, and so it belongs to \mathcal{C} . \square

Note that even for artin algebras, the assumption of S being right hereditary may not suffice to ensure that $\mu'(\mathcal{C})$ contains all injective right R -modules (see Example 3.3).

Following [6, Section 2.3], we call a class of modules $\mathcal{D} \subseteq \text{Mod-}R$ *definable* provided that \mathcal{D} is closed under arbitrary direct products, direct limits and pure submodules. Equivalently, $\mathcal{D} \subseteq \text{Mod-}R$ is definable iff \mathcal{D} is the kernel of a class of coherent functors from $\text{Mod-}R$ to $\text{Mod-}\mathbb{Z}$. Clearly the intersection of definable classes is definable. Moreover, since the composition of coherent functors is coherent, if $H : \text{Mod-}S \rightarrow \text{Mod-}R$ is a coherent functor and \mathcal{D} is a definable class in $\text{Mod-}R$, then $H^{-1}(\mathcal{D})$ is a definable class in $\text{Mod-}S$.

By [5, Theorem 2.6], all tilting torsion classes are definable. So it is natural to look at preservation of definability by the maps μ' and ν' :

Lemma 3.5. *Let R be a ring.*

- (1) $\text{Mod-}S$ is (via φ_*) a definable class in $\text{Mod-}R$.
- (2) Assume that $\mathcal{D} \subseteq \text{Mod-}R$ is definable. Then $\varphi_*^{-1}(\mathcal{D}) \subseteq \text{Mod-}S$ is definable.
- (3) Assume that $\mathcal{C} \subseteq \text{Mod-}S$ is definable, and that S is a finitely presented left R -module. Then $\mu'(\mathcal{C}) \subseteq \text{Mod-}R$ is definable.

Proof. The statement (1) is an immediate consequence of the fact that P^+ is a definable class in $\text{Mod-}R$, see Proposition 1.4(c).

For (2) it is enough to remark that $\varphi_* = - \otimes_S S_R$ is a coherent functor.

Finally, from Corollary 3.2 we have that $\mu'(\mathcal{C}) = (\varphi^*)^{-1}(\mathcal{C}) \cap P^{\perp 1}$. Since by assumption the functor $\varphi^* = - \otimes_R S_S$ is coherent, and by [5, Theorem 2.6] $P^{\perp 1}$ is definable, we get (3). \square

Remark 3.6. Note that S is always a finitely presented left R -module in the case R is an artin algebra and $P \in \text{Mod-}R$ is finitely generated. Indeed, by the Bongartz construction, R_0 can be taken finitely generated, hence $\ell(R) \in \text{Mod-}R$ is finitely generated. Then $S = \text{End}(\ell(R))$ is finitely generated as a left or right R -module.

We continue with a characterization of tilting classes over artin algebras in terms of their closure properties.

Theorem 3.7. *Let R be an artin algebra and \mathcal{C} a class of right R -modules. Then the following are equivalent:*

- (1) \mathcal{C} is a tilting class;
- (2) \mathcal{C} is a definable torsion class containing all injective modules.

Proof. Assume (1). By [5, Theorem 2.6], \mathcal{C} is of finite type, that is, $\mathcal{C} = \mathcal{S}^{\perp 1}$ where \mathcal{S} is a set of finitely presented modules of projective dimension ≤ 1 . Since $\mathcal{S}^{\perp 1}$ is definable for any finitely presented module S , and it is closed under homomorphic images iff S has projective dimension ≤ 1 , we get (2).

Assume (2). Let $\mathcal{T} = \mathcal{C} \cap \text{mod-}R$. Then \mathcal{T} is a torsion class in $\text{mod-}R$ with the corresponding torsion-free class \mathcal{F} ($\subseteq \text{mod-}R$). Let $\mathcal{D} = \text{Ker Hom}_R(-, \mathcal{F})$. By the Auslander-Reiten formula, we get $\mathcal{D} = \text{Ker Ext}_R^1(\tau^{-1}(\mathcal{F}), -)$ where τ is the Auslander-Reiten translation, $\tau^{-1}(\mathcal{F})$ consists of modules of projective dimension ≤ 1 , and $\mathcal{T} = \mathcal{D} \cap \text{mod-}R$ (see [12, Theorem 2.1]). In particular, \mathcal{D} is a tilting class, and the finitely generated modules in the (definable) classes \mathcal{C} and \mathcal{D} coincide.

We claim that also the pure-injective modules in \mathcal{C} and \mathcal{D} coincide. Indeed, let M be a pure-injective module in \mathcal{C} and consider a representative set \mathcal{H} of all R -epimorphisms from M to elements of \mathcal{T} , $\mathcal{H} = \{f_i \mid f_i : M \rightarrow T_i; T_i \in \mathcal{T}\}$. Then any R -homomorphism $g : M \rightarrow F$ where F is finitely presented has a factorization through the map $f : M \rightarrow \prod_i T_i$. Note that f is monic: for each $0 \neq x \in M$, consider the embedding $xR \hookrightarrow E(xR)$ and let $h : M \rightarrow E(xR)$ be its extension to M . Then there is i such that $f_i(m) = h(m)$ for each $m \in M$, so $x \notin \text{Ker } f_i$. From [6, 2.2(c)] it follows that f is a pure embedding of M into $\prod_i T_i \in \mathcal{D}$, so f splits, and $M \in \mathcal{D}$. Similarly, any pure-injective module in \mathcal{D} is also in \mathcal{C} .

In order to prove (1), it suffices to show that $\mathcal{C} = \mathcal{D}$. Let $M \in \mathcal{C}$. We claim that also $P \in \mathcal{C}$ where P is the pure-injective envelope of M . Since M and P are elementarily equivalent [6, 2.3], P is a pure submodule (and hence

a direct summand) in an ultrapower U of M . However, U is isomorphic to a direct limit of products of copies of M , so $U \in \mathcal{C}$ because \mathcal{C} is definable. So $P \in \mathcal{C}$, and being pure-injective, $P \in \mathcal{D}$. Since \mathcal{D} is closed under pure submodules, $M \in \mathcal{D}$. Similarly, $\mathcal{D} \subseteq \mathcal{C}$. \square

As remarked above, definable classes are kernels of coherent functors from $\text{Mod-}R$ to abelian groups. Theorem 3.7 says that in our particular setting, we can restrict ourselves to the coherent functors of the form $\text{Ext}_R^1(S, -)$ where S is a finitely presented module of projective dimension ≤ 1 .

Theorem 3.8. *Let R be an artin algebra.*

- (1) *Assume that S is an artin algebra. Let \mathcal{D} be a tilting torsion class in $\text{Mod-}R$ such that $\text{Gen}P \subseteq \mathcal{D} \subseteq P^{\perp 1}$. Then $v'(\mathcal{D})$ is a tilting torsion class in $\text{Mod-}S$ iff $v'(\mathcal{D})$ contains all injective right S -modules.*
- (2) *Assume that R is right hereditary and ${}_R S$ is a finitely generated left R -module. Then $\mu'(\mathcal{C})$ is a tilting torsion class in $\text{Mod-}S$ provided that \mathcal{C} is a tilting torsion class in $\text{Mod-}R$.*
- (3) *Assume P is a finitely generated right R -module. Then the image $v'(\mathcal{D})$ of a tilting torsion class \mathcal{D} in $\text{Mod-}R$ such that $\text{Gen}P \subseteq \mathcal{D} \subseteq P^{\perp 1}$ is a tilting torsion class in $\text{Mod-}S$ iff $v'(\mathcal{D})$ contains all injective right S -modules. Conversely, the image $\mu'(\mathcal{C})$ of a tilting torsion class $\mathcal{C} \subseteq \text{Mod-}S$ is a tilting torsion class in $\text{Mod-}R$ iff \mathcal{C} contains all injective right S -modules.*

Proof. (1) follows by Lemma 3.5(2), Theorems 1.6 and 3.7.

(2) follows by Lemma 3.5(3) and 3.4, Theorems 1.6 and 3.7.

(3) In view of Remark 3.6, the first statement follows by part (1), and the second from Lemma 3.5(3) and Theorem 3.7. \square

Using a different argument, we can actually prove the claim of Theorem 3.8(1) for any ring R :

Theorem 3.9. *Let R be a ring and \mathcal{D} a tilting torsion class in $\text{Mod-}R$ such that $\text{Gen}P \subseteq \mathcal{D} \subseteq P^{\perp 1}$, so $\mathcal{D} = \text{Gen}V = V^{\perp 1}$ for a tilting module $V \in \text{Mod-}R$. Let $\bar{V} = \varphi^*V \in \text{Mod-}S$. Then \bar{V} is finendo, $v'(\mathcal{D}) = \text{Gen}\bar{V} \subseteq \bar{V}^{\perp 1}$, and the following conditions are equivalent:*

- (a) *$v'(\mathcal{D})$ is tilting in $\text{Mod-}S$;*
- (b) *$\mathfrak{I}_S \subseteq v'(\mathcal{D})$;*
- (c) *the right S -module \bar{V} is faithful.*

Proof. Using the fact that both $\varphi_* = - \otimes_S S_R$ and $\varphi^* = - \otimes_R S_S$ are right exact and preserve direct sums, and, for any $M \in \text{Mod-}S$, there is a natural isomorphism

$$\text{Hom}_S(\bar{V}, M) = \text{Hom}_S(\varphi^*V, M) \cong \text{Hom}_R(V, \varphi_*M)$$

induced by the adjointness $\langle \varphi^*, \varphi_* \rangle$, we obtain from Corollary 3.2 that $v'(\mathcal{D}) = \varphi_*^{-1}(\text{Gen}V) = \text{Gen}\bar{V}$.

Now let us show that $\text{Gen}\bar{V} \subseteq \bar{V}^{\perp 1}$. Since $\varphi_*\bar{V} = \varphi_*\varphi^*V \cong i\ell V = V/\text{tr}_P V \in P^+$, because $V \in P^{\perp 1}$, we have an exact sequence

$$0 \rightarrow \text{tr}_P V \rightarrow V \rightarrow \varphi_*\bar{V} \rightarrow 0. \tag{+}$$

If $M \in \text{Gen}\bar{V}$, applying $\text{Hom}_R(-, \varphi_*M)$ to (+) we derive an exact sequence

$$\text{Hom}_R(\text{tr}_P V, \varphi_*M) \rightarrow \text{Ext}_R^1(\varphi_*\bar{V}, \varphi_*M) \rightarrow \text{Ext}_R^1(V, \varphi_*M)$$

where $\text{Hom}_R(\text{tr}_P V, \varphi_*M) = 0$ since $\varphi_*M \in P^+ \subseteq P^{\perp 0}$, and $\text{Ext}_R^1(V, \varphi_*M) = 0$ since $\varphi_*M \in \text{Gen}V = V^{\perp 1}$. Therefore $\text{Ext}_R^1(\varphi_*\bar{V}, \varphi_*M) = 0$. Using the fact that $\text{Mod-}S$ is extension closed in $\text{Mod-}R$, see Propositions 1.4 and 1.5(c), we conclude that $\text{Ext}_S^1(\bar{V}, M) \cong \text{Ext}_R^1(\varphi_*\bar{V}, \varphi_*M) = 0$, i.e., $M \in \bar{V}^{\perp 1}$.

Since $\varphi_* = - \otimes_S S_R$ clearly preserves products, we see that $\varphi_*(\bar{V}^\alpha) = \varphi_*(\bar{V})^\alpha \cong (V/\text{tr}_P V)^\alpha \in \text{Gen}V$ for any cardinal α , because $\text{Gen}V$ is tilting. This means that $\bar{V}^\alpha \in v'(\mathcal{D}) = \text{Gen}\bar{V}$, and so \bar{V} is finendo by [7, Lemma 1.5].

That (a) implies (b) is clear. If (b) holds true, then there is an S -epimorphism $\bar{V}^{(\alpha)} \twoheadrightarrow E(S)$, so that the embedding of the projective module S into its injective envelope $E(S)$ lifts to a monomorphism $S \hookrightarrow \bar{V}^{(\alpha)}$, proving (c).

Finally, let us assume (c) and prove (a). Since \bar{V} is finendo, the map f sending the unit of S into a vector of \bar{V}^n formed by a finite spanning set over its endomorphism ring is a $\text{Gen}\bar{V}$ -preenvelope of S by the proof of [1, Proposition 1.2]. Since \bar{V} is faithful, f is monic. Moreover, $\text{Ext}_S^1(\bar{V}^n, M) = 0$ for any $M \in \text{Gen}\bar{V}$, since $\text{Gen}\bar{V} \subseteq \bar{V}^{\perp 1}$. So f is a special $\text{Gen}\bar{V}$ -preenvelope of S . By [1, Theorem 2.1] we conclude that $\text{Gen}\bar{V}$ is a tilting torsion class. \square

In the case φ is a perfect localization, the condition $\mathcal{T}_S \subseteq v'(\mathcal{D})$ in **Theorem 3.9(b)** is automatically guaranteed by the assumptions on \mathcal{D} :

Corollary 3.10. *Assume that ${}_R S$ is flat. Then $v'(\mathcal{D})$ is a tilting torsion class in $\text{Mod-}S$ whenever \mathcal{D} is a tilting torsion class in $\text{Mod-}R$ such that $\text{Gen}P \subseteq \mathcal{D} \subseteq P^{\perp 1}$.*

Proof. If ${}_R S$ is flat then φ^* is (left) exact, so that $\text{Mod-}S$ is (equivalent to) a Giraud subcategory of $\text{Mod-}R$. Therefore [14, Chapter X, Proposition 1.4] shows that a module I in $\text{Mod-}S$ is injective iff $\varphi_* I$ is injective in $\text{Mod-}R$. So, if \mathcal{D} is a tilting torsion class in $\text{Mod-}R$ we have that $\varphi_* I \in \mathcal{D}$ for any $I \in \mathcal{T}_S$, i.e., $\mathcal{T}_S \subseteq \varphi_*^{-1}(\mathcal{D}) = v'(\mathcal{D})$, and **Theorem 3.9** applies. \square

Remark 3.11. The previous result cannot be extended to the case when $\varphi : R \rightarrow S$ is a homological epimorphism of rings. In **Example 2.2**, the class \mathcal{T}_R of all injective R -modules is a tilting class in $\text{Mod-}R$. Consider the partial tilting module Q_R . Clearly $\text{Gen}Q \subseteq \mathcal{T}_R \subseteq Q^{\perp 1}$. Moreover the only indecomposable modules in the perpendicular category Q^{\perp} are $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ and $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$; therefore $S = \text{End}(\ell_Q R)$ is a semisimple k -algebra. Now, $v'(\mathcal{T}_R)$ is not a tilting class in $\text{Mod-}S$, since it does not contain the injective S -module $\varphi^*(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix})$.

Now, we consider a case when μ' and v' restrict to inverse isomorphisms between the lattice of tilting torsion classes in $\text{Mod-}S$ and the lattices of tilting torsion classes \mathcal{D} in $\text{Mod-}R$ such that $\mathcal{T}_R \subseteq \mathcal{D} \subseteq P^{\perp 1}$. First, we need a lemma:

Lemma 3.12. *Let \mathcal{C} be a tilting torsion class in $\text{Mod-}S$. Let V be a tilting module generating \mathcal{C} and $0 \rightarrow S \rightarrow V_1 \rightarrow V_2 \rightarrow 0$ be the corresponding short exact sequence with $V_1, V_2 \in \text{Add}V$. Assume that $\text{Ext}_R^2(\varphi_* V_2, \text{Gen}P) = \text{Ext}_R^1(\varphi_* V_2, \text{Gen}P) = 0$. Then $\mu'(\mathcal{C})$ is a tilting torsion class in $\text{Mod-}R$.*

Proof. Consider the following short exact sequences in $\text{Mod-}R$: $0 \rightarrow R \rightarrow R_0 \rightarrow P^{(c)} \rightarrow 0$, $0 \rightarrow \text{tr}_P R_0 \rightarrow R_0 \rightarrow \ell R \rightarrow 0$, and $0 \rightarrow \ell R \rightarrow \varphi_* V_1 \rightarrow \varphi_* V_2 \rightarrow 0$.

Since $\text{Ext}_R^2(\varphi_* V_2, \text{tr}_P R_0) = 0$, the map

$$\text{Ext}_R^1(\varphi_* V_1, \text{tr}_P R_0) \rightarrow \text{Ext}_R^1(\ell R, \text{tr}_P R_0)$$

is surjective. So there exists a module $M \in \text{Mod-}R$ and a commutative diagram in $\text{Mod-}R$

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 & \text{tr}_P R_0 & \xlongequal{\quad} & \text{tr}_P R_0 & & & \\
 & \downarrow & & \downarrow & & & \\
 0 & \longrightarrow & R_0 & \longrightarrow & M & \longrightarrow & \varphi_* V_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \ell R & \longrightarrow & \varphi_* V_1 & \longrightarrow & \varphi_* V_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

The middle vertical column shows that $M \in P^{\perp 1}$ and that $\varphi^* M \cong \varphi^* \varphi_* V_1 \cong V_1$, so $M \in \mathcal{D} = \mu'(\mathcal{C})$ by **Corollary 3.2**. Moreover, we have the short exact sequences $0 \rightarrow R \rightarrow M \rightarrow N \rightarrow 0$ and $0 \rightarrow P^{(c)} \rightarrow N \rightarrow \varphi_* V_2 \rightarrow 0$. We also have $P^{(c)} \in {}^{\perp 1}(P^{\perp 1}) \subseteq {}^{\perp 1} \mathcal{D}$. Since P^+ is closed under extensions, $0 = \text{Ext}_S^1(V_2, \mathcal{C}) \cong \text{Ext}_R^1(\varphi_* V_2, \varphi_*(\mathcal{C}))$. By assumption, $\text{Ext}_R^1(\varphi_* V_2, \text{Gen}P) = 0$, so $\varphi_* V_2 \in {}^{\perp 1} \mathcal{D}$, and hence $N \in {}^{\perp 1} \mathcal{D}$. This proves that the inclusion $0 \rightarrow R \rightarrow M$ is a special \mathcal{C} -preenvelope of R .

Finally, by **Theorem 1.6**, \mathcal{D} is a torsion class providing for a special \mathcal{D} -preenvelope of the regular module R . By [1, **Theorem 2.1**], this just says that \mathcal{D} is a tilting torsion class in $\text{Mod-}R$. \square

Theorem 3.13. *Let R be a Dedekind domain. Then μ' and ν' restrict to lattice isomorphisms between the lattice of all tilting torsion classes in $\text{Mod-}S$ and the lattice of all tilting torsion classes \mathcal{D} in $\text{Mod-}R$ such that $\mathfrak{T}_R \subseteq \mathcal{D} \subseteq P^{\perp 1}$.*

Proof. First, we recall the structure of tilting modules and torsion classes in $\text{Mod-}R$. By [3, Theorem 4.3], tilting classes in $\text{Mod-}R$ are parametrized by subsets, A , of the maximal spectrum of R . If $A \subseteq \text{Mspec}(R)$, then the corresponding tilting torsion class is $\mathcal{C}_A = \{M \in \text{Mod-}R \mid pM = M \ \forall p \in A\}$ (the class of all p -divisible modules, for all maximal ideals $p \in A$), and the corresponding tilting module is $T_A = R_A \oplus \bigoplus_{p \in A} E(R/p)$ where R_A is the overring of R such that $R_A/R \cong \bigoplus_{p \in A} E(R/p)$. So tilting torsion classes in $\text{Mod-}R$ form a lattice anti-isomorphic to the lattice of all subsets of $\text{Mspec}(R)$ (with \mathfrak{T}_R as the least element).

Assume that P is a partial tilting module that is not tilting. Then $P \oplus P_0 = T$ where T is tilting and $T^{\perp 1} = P^{\perp 1}$. Since T is equivalent to T_A for some $A \subseteq \text{Mspec}(R)$, w.l.o.g., we assume that $T = T_A^{(\kappa)}$ for a cardinal $\kappa > 0$.

If $A \subsetneq \text{Mspec}(R)$, then $P = D \oplus E$ and $P_0 = D_0 \oplus E_0$ where D, D_0 are divisible (=injective) and E, E_0 are reduced. Then $D \oplus D_0 = \bigoplus_{p \in A} E(R/p)^{(\kappa)}$, so $E \oplus E_0 \cong R_A^{(\kappa)}$.

We claim that $D = \bigoplus_{p \in A} E(R/p)^{(\kappa_p)}$ where $0 < \kappa_p (\leq \kappa)$ for each $p \in A$. Indeed, if $\kappa_p = 0$ then $R/p \in D^{\perp 1}$, and since E is flat and R/p pure-injective, also $R/p \in E^{\perp 1}$, so $R/p \in P^{\perp 1} \setminus T^{\perp 1}$, a contradiction.

Assume $E \neq 0$. Then the nucleus of E contains (the nucleus of) R_A by [16, Lemma 7], so E is a projective R_A -module, hence a progenerator in $\text{Mod-}R_A$ (cf. [10, Chapter VI, Exer.1.10]). Then $\text{Gen}(P) \supseteq \text{Gen}(E) = \text{Gen}(R_A) = T^{\perp 1}$, a contradiction. This proves that $E = 0$, that is, P is torsion and divisible.

If $A = \text{Mspec}(R)$, then $R_A = Q$, so T_A is divisible, and again, P is torsion and divisible.

Anyway, P is isomorphic to a direct sum of (possibly multiple) copies of $E(R/p)$ where p runs over all of A . So $P^{\perp 1}$ is the class of all p -divisible modules for all $p \in A$, $\text{Gen}P = \text{Add}P$ is the class of all direct sums of copies of the modules $E(R/p)$ where $p \in A$, and P^+ is the class of all p -divisible modules M for all $p \in A$ such that the divisible part of M contains no elements annihilated by any $p \in A$. Moreover, $\ell(R) = R_A$ so $S = \text{End}(R_A) \cong R_A$. Now, R_A is again a Dedekind domain and its maximal ideals are qR_A where q runs over all maximal ideals of R not in A (see [3, Theorem 4.3]). The lattice of all torsion pairs in $\text{Mod-}S$ is thus anti-isomorphic to the lattice of all subsets of $\text{Mspec}(R) \setminus A$.

Finally, since R_A is a flat R -module, ν' preserves tilting torsion classes by Corollary 3.10. On the other hand, μ' preserves tilting torsion classes by Lemma 3.12: the assumptions of $\text{Ext}_R^2(\varphi_* V_2, \text{Gen}P) = \text{Ext}_R^1(\varphi_* V_2, \text{Gen}P) = 0$ are trivially satisfied since $\text{Gen}P$ consists of divisible modules. \square

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