GENERALIZED E-ALGEBRAS OVER VALUATION DOMAINS

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Abstract. Let R be a valuation domain. We investigate the notions of $E(R)$-algebra and generalized $E(R)$-algebra and show that for wide classes of maximal valuation domains R, all generalized $E(R)$-algebras have rank one. As a by-product we prove if R is a maximal valuation domain of finite Krull dimension, then the two notions coincide. We give some examples of $E(R)$-algebras of finite rank that are decomposable, but show that over Nagata domains of small degree, the $E(R)$-algebras are, with one exception, the indecomposable finite rank algebras.

Introduction.

If R is a commutative ring, M a torsion-free R-algebra and if $\mathrm{End}_R(M)$ denotes the R-algebra of endomorphisms of M, then M is said to be a *generalized* $E(R)$-*algebra*, if $M \cong E_R(M)$ as R-algebras. A related concept is that of an $E(R)$-algebra: an R-algebra M is an $E(R)$-algebra if the mapping $:M \to \mathrm{End}_R(M)$ which sends $m \in M$ to left scalar multiplication by m, is an algebra isomorphism. Clearly an $E(R)$-algebra is always a generalized $E(R)$-algebra; in fact it is well known that $E(R)$-algebras are exactly the commutative generalized $E(R)$-algebras. These notions have their origin in a paper by Schultz [12] – an $E(\mathbb{Z})$-algebra is just the original concept of an E-ring – and there has been a great deal of interest in them in the decades since the original paper. See, for example, [3] and the references therein. When R is the ring of integers $\mathbb{Z}$, the first examples of $E(\mathbb{Z})$-algebras were exhibited as pure subrings of the ring of p-adic integers. It was subsequently shown in [3] that arbitrarily large $E(\mathbb{Z})$-algebras exist; the technique used was based on Shelah's Black Box but made essential use of the fact that the underlying ring $\mathbb{Z}$ is not complete in its natural topology. In this first section of this paper we describe some methods of constructing $E(R)$-algebras when the base ring is a valuation ring which is not complete; in particular, when R is a so-called Nagata valuation domain, we construct finite rank indecomposable $E(R)$-algebras. Moreover we show that such algebras are not necessarily contained in the completion $\hat{R}$ of R. An example is also given which indicates that a maximal immediate extension of a Nagata domain may fail to be an $E(R)$-algebra.

Two obvious questions arise: (i) what happens when the base ring R is complete (or, more appropriately in this context, maximal)? (ii) do there exist generalized $E(R)$-algebras which are not $E(R)$-algebras? Surprisingly, there does not appear to be any reference to (i) in the literature, even when the base ring R is a complete discrete valuation ring. A query about this from Rüdiger Göbel to the first author was, in fact, our starting point. We shall explore (i) in Section 2 and show, for a wide class of maximal valuation domains – the valuation domains whose prime spectra are well-ordered by reverse inclusion – that the only generalized $E(R)$-algebras have rank 1. A positive answer to (ii) is promised in a forthcoming paper by Göbel and Shelah, using arguments based on the λ Calculus; the purported answer in [9] is

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flawed. It will follow from the results in Section 2 that in some circumstances this phenomenon is impossible: for example, an algebra over a maximal valuation domain R of finite Krull dimension is a generalized $E(R)$-algebra if, and only if it is an $E(R)$-algebra.

Our notation is standard and any undefined terms may be found in the texts [7] and [8]. For notions and results on valuation theory used throughout, we refer to [2], or [7], [8].

Throughout the sequel R will denote a valuation domain and Q its field of quotients; the maximal ideal of R will always be denoted by P. The symbols $\hat{R}$ and $\hat{Q}$ will denote the completions of R and Q in the topology of the valuation.

1. $E(R)$-algebras of finite rank.

For the convenience of the reader, we recall briefly the notion of a maximal valuation domain (for further information we refer to [7], [8]; see also [15]). For $R \subseteq S$ given valuation domains, we say that S is an *immediate extension* of R if R and S have the same value group, and the canonical embedding of the residue field of R into that of S is an isomorphism. A valuation domain R is said to be *maximal* if it does not admit proper immediate extensions. Every valuation domain R has a maximal immediate extension $\tilde{R}$. In general, a maximal immediate extension is not determined by R as a ring. However, $\tilde{R}$ is determined as an R-module, since it is isomophic to the pure-injective envelope of R. A valuation domain R is maximal if and only if the ring R/I is complete in the topology of its ideals, for every ideal I of R such that R/I is Hausdorff (equivalently, I is not isomorphic to P). Therefore a discrete valuation domain of rank one is maximal if and only if it is complete.

Let A be an R-algebra. An endomorphism ϕ of A is said to be *scalar* if ϕ is induced by the multiplication by an element $a \in A$. Then A is an $E(R)$-algebra if and only if every endomorphism of A is scalar.

The following lemma, valid over any ring R, is well known. For a proof, see for instance [10].

Lemma 1.1. *Let R be a ring and let $M = A \oplus B$ be an R-module, where $Hom_R(A, B) \neq 0$. Then $End_R(M)$ is not commutative.*

Recall that a family $\{A_i : i \in \Lambda\}$ of R-modules is said to be a *rigid system* if $\mathrm{Hom}_R(A_i, A_j) = 0$ whenever $i \neq j$.

Proposition 1.2. *Let R be a commutative ring, and let $\{A_i : 1 \leq i \in\leq n\}$ be a rigid system of $E(R)$-algebras. Then $A = \bigoplus_{i=1}^n A_i$ is an $E(R)$-algebra.*

We remark that, if $\{A_i : i \in \Lambda\}$ is a family of R-algebras and Λ is infinite, then the R-algebra $A = \bigoplus_{i\in\Lambda} A_i$ does not have an identity element. Therefore, in no such case is A an $E(R)$-algebra, since $\mathrm{End}_R(A)$ always contains an identity.

We have a natural source of $E(R)$-algebras (not necessarily of finite rank), whenever R is not complete.

Proposition 1.3. *Let R be a non-complete valuation domain, and D any R-algebra contained in $\hat{R}$ as a pure R-submodule. Then D is an $E(R)$-algebra.*

Proof. Let $\rho : D \to D$ be any endomorphism of D. Since D is pure in $\hat{R}$, then it is dense in the R-topology and therefore ρ extends uniquely to an $\hat{R}$-endomorphism $\hat{\rho}$ of $\hat{R}$. Thus we may assume that $\hat{\rho} \in \hat{R}$ is a scalar endomorphism. Since $1 \in D$ we get $\rho(1) = \hat{\rho} \in D$, and therefore $\rho = \hat{\rho}$ is a scalar endomorphism of D, as desired. □

Note that every pure subalgebra of $\hat{R}$ must be indecomposable, as an R-module.

A large part of the discussion in the present section will involve the important class of discrete valuation domains called "Nagata valuation domains" in [14]. These are discrete valuation rings R of rank one such that $\hat{Q} = Q[u_1, \ldots, u_k]$. Here the u_i are elements of $\hat{R}$ which are p-independent and satisfy $u_i^p \in R$, where $p > 0$ is the characteristic of Q. By the definition of p-independence (see [11]) it follows that $[\hat{Q} : Q] = p^k$. These type of discrete valuation domains were first constructed in Nagata's book [11], Example E33, page 207. Since they are not complete, they are, of course, not maximal. For a thorough study of them and their generalizations, see [4].

The following result and its proof are based on Proposition 1.7 of Faticoni's paper [5]. It allows us to construct decomposable $E(R)$-algebras of finite rank.

Theorem 1.4. *Let R be a non-complete valuation domain. Assume that $\hat{R}$ contains a family $\{u_i : i \in \lambda\}$ of units such that $u_i \notin Q(u_j)$ whenever $i \neq j$, and for $i \in \Lambda$, let A_i be the R-purification of $R[u_i, u_i^{-1}]$ in $\hat{R}$. Then the R-algebras A_i form a rigid system. In particular, when Λ is finite, $M = \bigoplus_{i\in\Lambda} A_i$ is an $E(R)$-algebra.*

Proof. Note firstly that from the definition of the A_i it follows that $u_i A_i = A_i$ for all indices i. Moreover, if $i \neq j$, we have $u_i A_j \cap A_j = 0$. Indeed, since $A_j \subseteq Q(u_j)$, the relation $0 \neq u_i f = g$ with $f, g \in A_j$ implies $u_i \in Q(u_j)$, contrary to our hypothesis.

Let $\varphi : A_i \to A_j$ be any R-homomorphism, where $i \neq j$. Since A_i, A_j are pure and hence dense in $\hat{R}$, φ extends to an $\hat{R}$-endomorphism $\hat{\varphi}$ of $\hat{R}$. It follows that $\varphi(A_i) = \varphi(u_i A_i) = u_i \hat{\varphi}(A_i) \subseteq u_i A_j \cap A_j = 0$, that is, $\varphi = 0$. We conclude that the A_i form a rigid system, and so Proposition 1.2 shows that $M = \bigoplus_{i\in\Lambda} A_i$ is an $E(R)$-algebra, when Λ is finite. □

Note that, from any given family $\{w_i : i \in \Lambda\}$ of elements of $\hat{Q}$ such that $w_i \notin Q(w_j)$ whenever $i \neq j$, we readily get a family $\{u_i : i \in \lambda\}$ of units of $\hat{R}$ with the analogous property. For $i \in \Lambda$, it suffices to choose $t_i \in P$ such that $t_i w_i \in P\hat{R}$ and to set $u_i = 1 + t_i w_i$. Thus the requirement in the hypothesis of the preceding theorem is not as restrictive as it might seem.

Corollary 1.5. *(1) For every $n \geq 1$, there exist indecomposable $\mathbb{Z}_p$-algebras $A_1, \ldots, A_n$ of finite rank such that $M = \bigoplus_{i=1}^n A_i$ is an $E(\mathbb{Z}_p)$-algebra.*

(2) Let R be a Nagata valuation domain with $[\hat{Q} : Q] = p^m$. Then there exist indecomposable R-algebras $A_1, \ldots, A_m$ of finite rank such that $M = \bigoplus_{i=1}^m A_i$ is an $E(R)$-algebra.

Proof. (1) We deal firstly with the case where $p \geq 3$. Pick pairwise distinct prime numbers $q_1, \ldots, q_n$ of the form $q_i = 1 + a_i p$. Since J_p is Henselian and the polynomials $X^2 - q_i$ have distinct roots modulo p, for $1 \leq i \leq n$ there exist units $u_i \in J_p$ such that $u_i^2 = q_i$. Then the condition $u_i \notin \mathbb{Q}(u_j)$ is satisfied for $i \neq j$. Now define the $E(\mathbb{Z}_p)$-algebras A_i as in Theorem 1.4. Then the A_i all have finite rank, since the u_i are algebraic over $\mathbb{Q}$, and they are indecomposable, since $\mathrm{End}_{\mathbb{Z}_p}(A_i) \cong A_i$ contains no non-trivial idempotents. The $\mathbb{Z}_p$-algebra $M = \bigoplus_{i=1}^n A_i$ fulfills our requirements.

When $p = 2$, we take the u_i to be roots in J_p of the polynomials $X^3 - q_i$ (which have distinct roots modulo 2). Now we may argue as above, since again the condition $u_i \notin \mathbb{Q}(u_j)$ is satisfied for $i \neq j$.

(2) By the definition of Nagata valuation domains, we have $\hat{Q} = Q(u_1, \ldots, u_m)$, for suitable units $u_i \in \hat{R}$ such that $u_i^p \in R$. Then the condition $u_i \notin Q(u_j)$ is satisfied for $i \neq j$, and we get the result by a similar argument to that in part (1). □

Part (2) of the preceding corollary acquires further interest in the light of the following result, valid for Nagata valuation domains when the degrees of the field extensions are small.

Theorem 1.6. *Let R be a Nagata valuation domain, with $[\hat{Q} : Q] = 2$ or 3. Then*

(1) every $E(R)$-algebra of finite rank is indecomposable:

(2) every indecomposable R-module of finite rank admits an $E(R)$-algebra structure, with, up to isomorphism, one exception. The exceptional case is provided by indecomposable R-modules of rank 2 when $[\hat{Q} : Q] = 3$; all such modules are isomorphic.

Proof. Let M be any $E(R)$-algebra of finite rank; recall that M is then necessarily commutative.

In the case $[\hat{Q} : Q] = 2$, it was proved in [14] that the indecomposable R-modules of finite rank are isomorphic to either R, or $\hat{R}$, or Q. It readily follows that there are no non-trivial rigid systems of finite rank R-algebras. Therefore M must be indecomposable, since otherwise $M \cong \mathrm{End}_R(M)$ cannot be commutative, by Lemma 1.1. Clearly R, $\hat{R}$ and Q are $E(R)$-algebras. So (1) and (2) hold in the case of degree 2.

Assume for the remainder of the proof that $[\hat{Q} : Q] = 3$. It was proved in [1] that every indecomposable R-module of finite rank has rank ≤ 3, and a description, up to isomorphism, of indecomposables of rank ≤ 3 is provided (see Lemma 2.1 and Theorem 2.3 of [1]). That description is recalled below in the proof. Using Proposition 2.4 (b) of [1], we easily prove that there are no non-trivial rigid systems of finite rank R-algebras. Thus again M must be indecomposable, and so (1) is valid in this case.

The hard part of the proof is to show that (2) holds when the extension is of degree 3. So, R is now a Nagata valuation domain with $\hat{Q} = Q[u]$, where u is a unit of $\hat{R}$ such that $u^3 = \lambda \in R$. Let $t \in R$ be a generator of the maximal ideal of R – (note that in [1] the maximal ideal of R was denoted by pR; we have changed their notation to avoid confusion with the characteristic of Q).

Let A be any indecomposable torsion-free module of finite rank. Then A has rank ≤ 3, by [1]. More precisely, if the rank is one, then A is isomorphic to either R, or Q, and these clearly are $E(R)$-algebras. If A has rank 2, then it is isomorphic to $A[u] = (Q \oplus Qu) \cap \hat{R}$ (see [14] and [1]). Now $A[u]$ is our exception, since it cannot be endowed with an $E(R)$-algebra structure, due to the following claim.

CLAIM. The endomorphism ring of $A[u]$ is isomorphic to R.

Since $A[u]$ is a pure submodule of $\hat{R}$, every endomorphism of $A[u]$ is given by the multiplication by a suitable $a = a_0 + a_1 u + a_2 u^2 \in \hat{R}$. Since $1, u \in A[u]$, from $a \cdot 1 = a \in A[u]$ it follows $a_2 = 0$, and then $au = a_0 u + a_1 u^2 \in A[u]$ implies $a_1 = 0$. Then $a = a_0 \in R$, as desired.

Finally consider the case when A has rank 3. If A is isomorphic to $\hat{R}$, then it is an $E(R)$-algebra. Otherwise A must be isomorphic to the module $A_i = A[u, t^i u^2]$, for a suitable $i \geq 1$, where

$$A_i = (Q(1,0) \oplus Q(0,1) \oplus Q(u, t^i u^2)) \cap (\hat{R} \oplus \hat{R})$$

(see [1, Theorem 3.3]).

We want to show that $A_i \cong \mathrm{End}_R(A_i)$. We begin by describing a generic endomorphism of A_i. Firstly observe that an element $\eta \in \hat{R} \oplus \hat{R}$ lies in A_i if and only if it has the form $\eta = (x + zu, y + zt^i u^2)$, for suitable $x, y, z \in Q$.

An endomorphism ϕ of A_i is determined by a matrix $T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with entries in $\hat{R}$. Write $a = a_0 + a_1 u + a_2 u^2$, for suitable $a_i \in Q$, $b = b_0 + b_1 u + b_2 u^2$, etc. A direct check of the requirements $(1,0)T \in A_i$ and $(0,1)T \in A_i$ implies the following conditions hold for the coefficients

$$0 = a_2 = b_2 = c_1 = d_1; \quad c_2 = a_1 t^i; \quad d_2 = b_1 t^i.$$

Then the matrix T has the following form

$$T = \begin{pmatrix} a_0 + a_1 u & c_0 + a_1 t^i u^2 \\ b_0 + b_1 u & d_0 + b_1 t^i u^2 \end{pmatrix}$$

The requirement that $(u, t^i u^2)T \in A_i$ implies the following equalities

$$\begin{cases} a_1 + b_0 t^i & = 0 \\ c_0 + b_1 t^{2i} \lambda & = 0 \\ a_0 t^i = d_0 t^i \end{cases}$$

Therefore, to ensure that T is the matrix of an endomorphism of A_i, it is necessary that T has the form

$$T = \begin{pmatrix} a_0 - b_0 t^i u & -b_1 t^{2i} \lambda - b_0 t^{2i} u^2 \\ b_0 + b_1 u & a_0 + b_1 t^i u^2 \end{pmatrix}$$

Conversely, suppose that a matrix T with entries in $\hat{R}$ may be written in the above form. For any $\eta = (x + zu, y + zt^i u^2) \in A_i$, direct calculation shows that $(x + zu, y + zt^i u^2)T = (x' + z'u, y' + z't^i u^2)$, for suitable $x', y', z' \in Q$. Since both $x + zu$, $y + zt^i u^2$ and the entries of T lie in $\hat{R}$, we also have $x' + z'u, y' + z't^i u^2 \in \hat{R}$, and therefore $(x' + z'u, y' + z't^i u^2) \in A_i$. We conclude that T is the matrix of an endomorphism of A_i. Thus we have necessary and sufficient conditions for a matrix T to be the matrix of an endomorphism of A_i.

The above calculation also shows that the matrix T associated to a generic endomorphism ϕ is a linear combination of three matrices

$$T = b_0 T_1 + a_0 T_2 + b_1 T_3,$$

where

$$T_1 = \begin{pmatrix} -t^i u & -t^{2i} u^2 \\ 1 & 0 \end{pmatrix}; T_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; T_3 = \begin{pmatrix} -t^i u & -t^{2i} u^2 \\ 1 & 0 \end{pmatrix}$$

We conclude that the following identification holds

$$\mathrm{End}_R(A_i) = (QT_1 \oplus QT_2 \oplus QT_3) \cap M_{2\times 2}(\hat{R}),$$

where $M_{2\times 2}(\hat{R})$ are the 2×2 matrices with entries in $\hat{R}$. Note that the matrices T_i satisfy the crucial relation

$$uT_1 + t^i u^2 T_2 = T_3$$

that is, the same relation that holds for $(1,0), (0,1), (u, t^i u^2)$. (The above formula is readily verified using the fact that $u^3 = \lambda$.)

Now consider the R-linear map

$$f : Q(1,0) \oplus Q(0,1) \oplus Q(u, t^i u^2) \to QT_1 \oplus QT_2 \oplus QT_3$$

that extends the assignments

$$(1,0) \mapsto T_1; \quad (0,1) \mapsto T_2; \quad (u, p^i u^2) \mapsto T_3.$$

Clearly f is an isomorphism. We will show that $f(A_i) = \mathrm{End}_R(A_i)$, and therefore f gives rise to an isomorphism between A_i and $\mathrm{End}_R(A_i)$.

Let $\eta = (x + zu, y + zt^i u^2) \in A_i$, for suitable $x, y, z \in Q$. We have

$$f(\eta) = xT_1 + yT_2 + zT_3 = \begin{pmatrix} y - xt^i u & -zt^{2i}\lambda - xt^{2i} u^2 \\ x + zu & y + zt^i u^2 \end{pmatrix}$$

Then $f(\eta) \in \mathrm{End}_R(A_i)$ if and only if all its entries lie in $\hat{R}$. Since $(x+zu, y+zt^iu^2) \in \hat{R}\oplus\hat{R}$, trivially the entries of the second row and also $-zt^{2i}\lambda - xt^{2i}u^2 = -t^{2i}u^2(zu+x)$ lie in $\hat{R}$. Moreover $y - xt^iu = y + zt^iu^2 - t^iu(x+zu) \in \hat{R}$. Thus we have that $f(A_i) \subseteq \mathrm{End}_R(A_i)$. Let f^{-1} be the inverse map of f and let $T = xT_1 + yT_2 + zT_3$ be an element of $\mathrm{End}_R(A_i)$. Then

$$f^{-1} : T = \begin{pmatrix} y - xt^iu & -zt^{2i}\lambda - xt^{2i}u^2 \\ x + zu & y + zt^iu^2 \end{pmatrix} \mapsto (x + zu, y + zt^iu^2)$$

Since T has entries in $\hat{R}$, then $f^{-1}(T) \in (Q(1,0) \oplus Q(0,1) \oplus Q(u, t^iu^2)) \cap (\hat{R} \oplus \hat{R}) = A_i$. We conclude that $f(A_i) = \mathrm{End}_R(A_i)$, as desired.

Note that $\mathrm{End}_R(A_i)$ is a commutative R-algebra. This fact may be verified directly and was also observed in Lemma 2.5 of [1].

Using the isomorphism f, we define a commutative multiplication on A_i by setting $\eta_1 \cdot \eta_2 = \eta_3$ if and only if $f(\eta_1)f(\eta_2) = f(\eta_3)$. Thus $(A_i, \cdot)$ is a commutative generalized $E(R)$-algebra; it is well known and follows easily from Lemma 1.1 – or see [12, Lemma 6] – that $(A_i, \cdot)$ is then an $E(R)$-algebra. □

It is natural to ask whether an indecomposable $E(R)$-algebra of finite rank is necessarily contained in $\hat{R}$. This is not the case, in general. We start with a lemma which is an adaptation of Theorem 1.2 of Faticoni's paper [5].

Lemma 1.7. *Let A be a free R-algebra of rank ≥ 2, say $A = R1_A \oplus Rw_1 \oplus \cdots \oplus Rw_n$. Let $d, c_1, \ldots c_n$ be elements of $\hat{R}$, where d is a unit, and consider the R-subalgebra $A[c_i1_A + dw_i : 1 \leq i \leq n]$ of $\hat{A} = \hat{R}1_A \oplus \hat{R}w_1 \oplus \cdots \oplus \hat{R}w_n$. Let E be the R-purification of $A[c_i1_A + dw_i : 1 \leq i \leq n]$ in $\hat{A}$. Then $E \cap dE = 0$ implies that E is an $E(R)$-algebra.*

Proof. In order to show that E is an $E(R)$-algebra, it suffices to show that $f(1_A) = 0$ implies $f = 0$, for every $f \in \mathrm{End}_R(E)$. Such an endomorphism $f : E \to E$ extends to a $\hat{R}$-endomorphism $\hat{f} : \hat{A} \to \hat{A}$, since the R-purity implies $\hat{A} = \hat{E}$. Then for any $i \leq n$ we have

$$f(c_i1_A + dw_i) = c_i\hat{f}(1_A) + d\hat{f}(w_i) = df(w_i) \in E \cap dE = 0.$$

It follows that $\hat{f}(w_i) = 0$, for all $i \leq n$. Since $\hat{f}(1_A) = 0$ and $\hat{A} = \hat{R}1_A + \hat{R}w_1 + \cdots + \hat{R}w_n$, we get $\hat{f}(\hat{A}) = 0$. In particular, $f = 0$, as desired. □

Using the preceding lemma, we can produce examples of indecomposable $E(R)$-algebras of finite rank not contained in $\hat{R}$.

Example 1.8. Let R be a Nagata valuation domain with maximal ideal πR and field of fractions Q such that $[\hat{Q} : Q] = p^2$. Let $\hat{Q} = Q[c, d]$, where c, d are units of $\hat{R}$ (thus $c^p, d^p \in R$, by the definition of Nagata valuation domains). We add to Q a square root $w = \sqrt{\pi}$ of π, and consider the R-algebra $A = R[w]$. Then A is free of rank 2, and its completion $\hat{A} = \hat{R} + \hat{R}w$ is not contained in $\hat{R}$. In the notation of the preceding lemma, the purification E of $A[c + dw]$ is an $E(R)$-algebra whenever $E \cap dE = 0$. Observe that both c and d have degree p over the field $K = Q[w]$, since $Q[c, w]$ and $Q[d, w]$ are extensions of Q of degree $2p$. Assume now for a contradiction that $0 \neq dz_1 = z_2$, where $z_1, z_2 \in E$. Then $d \in K[c + dw]$, and therefore $K[d] = K[c + dw]$, since d and $c + dw$ have both degree p over K. It follows readily that $K[c] = K[d]$, which implies $Q[c] = Q[d]$, a contradiction.

We conclude that E is an $E(R)$-algebra of finite rank, not contained in $\hat{R}$. Moreover, E is an indecomposable R-module, since $\mathrm{End}_R(E) \cong E$ contains no non-trivial idempotents.

The final example of this section, still based on Nagata valuation domains, shows that a maximal immediate extension $\tilde{R}$ of R fails in general to be an $E(R)$-algebra, even when $\tilde{R}$ has finite rank as an R-module.

Example 1.9. Let V be a Nagata valuation domain with maximal ideal πV and field of fractions K such that $[\hat{K} : K] = 2$. We may assume that $\hat{K} = K(u)$, where $u \in \hat{V}$ is such that $u^2 \in V$. Let t be an indeterminate and consider the valuation domain

$$R = V + tK[[t]]$$

consisting of formal power series in t with coefficients in K and constant term in V. The results in [4] show that a maximal immediate extension of R is

$$\tilde{R} = \hat{V} + t\hat{K}[[t]].$$

As a matter of fact, it is readily seen that $\tilde{Q} = Q[u]$ is the field of fractions of $\tilde{R}$. Note that R is a complete valuation domain, and $\tilde{R}$ has rank two as an R-module. Using the definitions, it is also immediate to verify that $a + u \notin t\tilde{R}$ for every $a \in Q$.

We want to show that $\tilde{R}$ is not an $E(R)$-algebra. We will prove that there exists $\varphi \in \mathrm{End}_R(\tilde{R})$ which is not a scalar endomorphism..

Extend the assignments

$$1 \mapsto u \ ; \ u \mapsto u^2 + t$$

to an R-endomorphism φ of $\tilde{Q} = Q \oplus Qu$.

To reach our desired conclusion, it suffices to prove that $\varphi(\tilde{R}) \subseteq \tilde{R}$, since then φ cannot be identified with a multiplication in $\tilde{R}$. Let $0 \neq z = a + bu \in \tilde{R}$, where a, b are suitable elements of Q. Observe that necessarily $bt \in R$; in fact, $bt \notin R$ implies $(bt)^{-1} \in R$, whence $1/b \in tR$. But this would yield $a/b + u \in t\tilde{R}$, which is impossible, as observed above. Finally we have

$$\varphi z = au + b(u^2 + t) = uz + bt \in \tilde{R},$$

since $uz \in \tilde{R}$ and $bt \in R$.

2. Generalized $E(R)$-algebras over maximal valuation domains.

Throughout this section R will be a maximal valuation domain. The main property of R we shall invoke, is that uniserial R-modules are pure-injective (for instance, see [8], Ch. XIII, Theorem 5.2).

The following easy result justifies our interest here in generalized $E(R)$-algebras, rather than simply in $E(R)$-algebras.

Proposition 2.1. *Let R be a maximal valuation domain. Then every $E(R)$-algebra has rank one.*

Proof. Assume for a contradiction that M is an $E(R)$-algebra of rank ≥ 2. We know that M has to be commutative. Now, since R is maximal, the purifications of elements of M are direct summands. Then, since M has rank ≥ 2, we may write $M = U \oplus V \oplus N$, where U, V are nonzero uniserial R-modules. We may also assume that $\mathrm{Hom}_R(U, V) \neq 0$ (actually, $\mathrm{Hom}_R(U, V) = 0$ happens only if $U \cong Q$ and $V \cong I \subset R$). In view of Lemma 1.1 we conclude that $\mathrm{End}_R(M)$ is non-commutative, and hence $M \ncong \mathrm{End}_R(M)$, impossible. □

Recall some notions that can be found in Ch. XI and Ch. X of [8].

Let M be a torsion-free R-module. A basic submodule B of M is any submodule which is maximal with respect to the properties:

(1) B is a direct sum of uniserial modules;
(2) B is a pure submodule of M.

In the torsion-free setting, basic submodules always exist and are unique, up to isomorphism

For any uniserial R-module U, the set $U^{\#} = \{r \in R : rU \neq U\}$ is an ideal of R, since R is a valuation domain. Moreover $U^{\#}$ is a prime ideal, since $R \setminus U^{\#}$ is multiplicatively closed.

Lemma 2.2. *Let M be a torsion-free R-module and B a basic submodule of M. For every ideal I of R there is an epimorphism*

$$Hom_R(M, I) \to Hom_R(B, I) \to 0.$$

Proof. From the exact sequence

$$0 \to B \to M \to M/B \to 0$$

we get the sequence

$$0 \to \mathrm{Hom}_R(M/B, I) \to \mathrm{Hom}_R(M, I) \to \mathrm{Hom}_R(B, I) \to \mathrm{Ext}_R(M/B, I) = 0$$

where the last equality occurs since R maximal, implies each ideal I is pure-injective. □

Proposition 2.3. *Let M be a torsion-free generalized $E(R)$-algebra which is not reduced. Then M is isomorphic to Q.*

Proof. We know that the divisible part D of M is a summand (see, e.g. [8], IX, Proposition 1.1 page 306). So we may write $M = D \oplus N$. Denote by κ the rank of D; this is well known to be an invariant of M. Now $\mathrm{Hom}_R(M, M)$ must contain $\mathrm{Hom}_R(D, D)$ as a direct summand, and this latter has rank either κ^2 or 2^κ according as κ is finite or not. Since M is isomorphic to $\mathrm{Hom}_R(M, M)$, κ must be one. Moreover, if $N \neq 0$, then N contains a nonzero uniserial direct summand, J say. Then $M \cong \mathrm{Hom}_R(M, M)$ contains a summand isomorphic to $\mathrm{Hom}_R(Q, Q) \oplus \mathrm{Hom}_R(J, Q) \cong Q \oplus Q$, which is impossible. Thus $N = 0$, as required. □

In view of the preceding proposition, we restrict our attention to reduced torsion-free R-modules in the sequel.

Proposition 2.4. *Let M be a generalized $E(R)$-algebra of finite rank. Then M has rank one.*

Proof. A torsion-free module M of finite rank n over a maximal valuation domain is a direct sum of n nonzero uniserial submodules (see [7], XIV, Theorem 3.3, page 278). In view of Proposition 2.3 we may assume that M is reduced, and so none of its uniserial summands is isomorphic to Q. Since $\mathrm{Hom}_R(U, V) \neq 0$ for all uniserial nonzero R-modules U, V not isomorphic to Q, it follows that $\mathrm{Hom}_R(M, M)$ has rank n^2, hence $n = 1$. □

Let P_1 be a (nonzero) prime ideal of R. We say that P_1 is *associated* to M if M has a uniserial pure submodule (equivalently a direct summand) J such that $J^{\#} = P_1$. In that case, we have $\mathrm{Hom}_R(J, J) \cong R_{P_1}$, by [8], Ch. II, Lemma 4.4. Of course, if B is any basic submodule of M, then B has a direct summand isomorphic to J.

If $P_0 > P_1$ are prime ideals of R, it is easy to check that $\mathrm{Hom}_R(R_{P_0}, R_{P_1}) = R_{P_1}$. This equality will be used in the following lemma.

Lemma 2.5. *Assume that M is a generalized $E(R)$-algebra of infinite rank. Let P_1 be associated to M and let B be a basic submodule of M.*

(1) B contains a direct summand isomorphic to R_{P_1}, and it contains as a direct summand an infinite direct sum of copies of R_{P_1} if P_1 is the only prime ideal associated to M.

(2) If P_1 is not maximal in the set of associated prime ideals, then B contains as a direct summand an infinite direct sum of copies of R_{P_1}.

Proof. To simplify the notation, we let $R_1 = R_{P_1}$.

(1) Let J be a summand of B with $J^\# = P_1$. Then $\mathrm{Hom}_R(M, M)$ contains as a summand $\mathrm{Hom}_R(J, J)$, which is isomorphic to R_1, as observed above. Since M is a generalized $E(R)$-algebra, R_1 is isomorphic to a summand of M, and, as it is uniserial, it is also isomorphic to a summand of B. Now suppose that P_1 is the only prime ideal associated to M. Assume for a contradiction that B contains only a finite direct sum W of n copies of R_1 as a direct summand. Since B has infinite rank we can write $M = W \oplus U \oplus M_1$, where U is a uniserial summand of B. Thus $U^\# = P_1$. Then $\mathrm{Hom}_R(M, M)$ contains $\mathrm{Hom}_R(W, W)$ as a summand. Therefore M, and hence B, contains a summand isomorphic to a direct sum of n^2 copies of R_1, and so $n = 1$ and $W \cong R_1$. However $\mathrm{Hom}_R(M, M)$ contains $\mathrm{Hom}_R(W, W) \oplus \mathrm{Hom}_R(U, U)$ as a summand, and $\mathrm{Hom}_R(U, U) = R_1$, since $U^\# = P_1$. Thus M and B must contain as a summand a direct sum of at least two copies of R_1, that is $n \geq 2$ – contradiction.

(2) Let P_0 be any prime ideal associated with M strictly larger than P_1. It follows from (1) that $R_0 = R_{P_0}$ is isomorphic to a summand of B and hence of M. Assume for a contradiction that B contains only a finite direct sum W of n copies of R_1 as a direct summand. Since $R_0 \not\cong R_1$ we can write $M = W \oplus U \oplus M_1$, where $U \cong R_0$. Then, as in (1), we can prove that $W \cong R_1$. However $\mathrm{Hom}_R(M, M)$ contains $\mathrm{Hom}_R(W, W) \oplus \mathrm{Hom}_R(U, W) = N$ as a summand. Since $P_0 > P_1$, we have $\mathrm{Hom}_R(R_0, R_1) = R_1$, whence $N \cong R_1 \oplus R_1$ and, as N is isomorphic to a direct summand of M, we get a contradiction. □

We will need another technical result.

Lemma 2.6. *Let $M = W \oplus M_1$ where W is a direct sum of uniserial submodules. Let B be a basic submodule of M of the form $B = W \oplus B_1$, where B_1 is basic in M_1. Assume that M_1 has a largest associated prime ideal, say P_1. Then $M = B + P_1 M$.*

Proof. It is clearly enough to prove that $M_1 = B_1 + P_1 M_1$. Take any $a \in M_1$ and let U be the purification of Ra in M_1. In order to show that $a \in B_1 + P_1 M_1$, we may assume $a \notin B_1$. Since B_1 is pure and M_1 is torsion-free, we have $U \cap B_1 = 0$, and so, by the definition of basic submodules, $U \oplus B_1$ cannot be pure in M_1. Thus there exists $t \in R$ such that

$$u + b = tz, \quad u \in U, \ b \in B_1, \ z \in M_1,$$

and $u + b \notin t(U \oplus B_1)$. Note that $u \notin tU$, otherwise $b \in tB_1$, since B_1 is pure in M_1. We claim that we may assume, without loss of generality, that $a \in Ru$. In fact, if $a \notin Ru$, then $u = sa$ for some $s \in R$, since U is uniserial. Then $u \notin tU$ implies $a \notin (t/s)U$; in particular, $t/s \in R$. From the preceding relation we get

$$a + b/s = (t/s)z$$

where $b/s \in B_1$, since B_1 is pure, and $a + b/s \notin (t/s)(U \oplus B_1)$ (since $a \notin (t/s)U$). Therefore in this case we may safely replace u by a.

Now $U \cong J$, where J is an ideal of R such that $J^\# \subseteq P_1$, since P_1 is the largest prime ideal associated to M_1. Therefore, for every $r \notin P_1$ we have $rJ = J$, whence $rU = U$. In particular, since $tU \neq U$, we have $t \in P_1$. Thus $u \in B_1 + P_1 M_1$ and so $a \in Ru$ implies $a \in B_1 + P_1 M_1$, as required. □

The next result, the main one of the present section, involves valuation domains that satisfy the property that their prime spectra are well-ordered by the reverse inclusion. We remark that this natural property holds for various classes of valuation domains, and, specifically, for the important *strongly discrete* valuation domains (see [6], [8] for definitions and results; these domains and their modules were investigated in [13] under the name of totally branched discrete valuation domains).

Theorem 2.7. *Let R be a maximal valuation domain whose prime spectrum is well-ordered by reverse inclusion. Then every generalized $E(R)$-algebra M has rank one. In particular, M is isomorphic to an overring of R.*

Proof. Suppose that M has rank > 1. Then by Propositions 2.3 and 2.4 we may assume that M is reduced of infinite rank. Since the prime spectrum of R is well-ordered by reverse inclusion, there exists a largest prime ideal associated to M, say P_0. Let B be a basic submodule of M.

We have to distinguish two cases.

CASE 1. B contains as a summand an infinite direct sum of uniserial modules J_i, $i \in \Lambda$, all isomorphic to $R_{P_0} = R_0$.

Let $B = B_1 \oplus B_2 \oplus B_3$, where $B_1 = \bigoplus_{i\in\Lambda} J_i$, B_2 is a direct sum of uniserials U_k where $U_k^{\#} = P_0$ but $U_k \not\cong R_0$ and B_3 is a direct sum of uniserials V_j each of which is associated to a prime ideal strictly contained in P_0; note that $P_0 V_j = V_j$ for each such V_j in B_3. Moreover it follows from [7] I, Lemma 4.8, page 16, that $P_0 U_k = U_k$ for each U_k in B_2. Thus $B/P_0B = B_1/P_0B_1 \cong \bigoplus_\Lambda R_0/P_0$.

Since M has a summand isomorphic to R_0 we have canonical epimorphisms $M \cong \mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(M, R_0)$ and $\mathrm{Hom}_R(B, R_0) \to \mathrm{Hom}_R(B_1, R_0)$; moreover it follows from Lemma 2.2 that there is an epimorphism $\mathrm{Hom}_R(M, R_0) \to \mathrm{Hom}_R(B, R_0)$. The composition of these epimorphisms gives an epimorphism ϕ from M onto $\mathrm{Hom}_R(B_1, R_0) \cong \prod_\Lambda R_0$. Applying Lemma 2.6 we see that $M = B + P_0M$ and hence $M/P_0M \cong B/P_0B \cong \bigoplus_\Lambda R_0/P_0$, as noted above. However, the epimorphism ϕ induces a mapping from $M/P_0M \cong \bigoplus_\Lambda R_0/P_0$ onto $\mathrm{Hom}_R(B_1, R_0)/P_0\mathrm{Hom}_R(B_1, R_0) \cong \prod_\Lambda R_0/P_0$. Now we have $N = \prod_\Lambda R_0/P_0 = \bigoplus_\Gamma R_0/P_0$, where $|\Gamma| \geq 2^{|\Lambda|}$, since $|\Lambda|$ is infinite. Hence, tensoring B/P_0B and N by $F = R_0/P_0$, we get an epimorphism of F-vector spaces $\bigoplus_\Lambda F \to \bigoplus_\Gamma F$, which is impossible.

CASE 2. B contains as a summand only a finite direct sum of copies of R_0.

Note that the proof of Lemma 2.5 (1) readily implies that in this case exactly one copy W, say, of R_0 appears as a summand of B. Thus $B = W \oplus B_1$ and $M = W \oplus M_1$, where $M_1 \supseteq B_1$ and $M_1 \neq 0$, since M has rank > 1. Also note that the prime ideals associated to M_1 are all strictly smaller than P_0. Let P_1 be the largest prime associated to M_1, which exists by the hypothesis on $\mathrm{Spec}(R)$. By Lemma 2.5 (2), B, and hence B_1, contains a summand which is a direct sum of Λ copies of $R_{P_1} = R_1$, where $|\Lambda|$ is infinite. From Lemma 2.6 it follows that $M = B + P_1M$, and therefore, by the same argument as in Case 1, $M/P_1M \cong B/P_1B \cong W/P_1W \oplus \bigoplus_\Lambda R_1/P_1$. Since R_1 is a summand of M, an identical argument as in Case 1, replacing R_0 with R_1, yields an epimorphism ψ from M onto $\mathrm{Hom}_R(W, R_1) \oplus \mathrm{Hom}_R(B_1, R_1) \cong R_1 \oplus \prod_\Lambda R_1 \cong \prod_\Lambda R_1$. Then ψ induces an epimorphism from M/P_1M onto $\prod_\Lambda R_1/P_1 \prod_\Lambda R_1 \cong \prod_\Lambda R_1/P_1 \cong \bigoplus_\Gamma R_1/P_1$, where, as in Case 1, $|\Gamma| \geq 2^{|\Lambda|}$. Since $M/P_1M \cong W/P_1W \oplus \bigoplus_\Lambda R_1/P_1$, tensoring with the field $F' = R_1/P_1$, we get $M/P_1M \otimes F' \cong \bigoplus_\Lambda F'$. As in Case 1, we get an epimorphism $\bigoplus_\Lambda F' \to \bigoplus_\Gamma F'$, which is impossible.

Thus in both cases we have established a contradiction to the assumption that M has rank > 1. The desired conclusion follows. □

Corollary 2.8. *If R is a maximal valuation domain of finite Krull dimension (in particular, if R is a complete discrete valuation ring), then every generalized $E(R)$-algebra has rank one (and hence is an $E(R)$-algebra).*

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