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Two Dimensional Zonoids and Chebyshev Measures

Stefano Bianchini*

S.I.S.S.A. (I.S.A.S.), Via Beirut 2/4, 34013, Trieste, Italy

Raphaël Cerf†

Mathématique, Université de Paris Sud, Bât. 425, 91405 Orsay Cedex, France

and

Carlo Mariconda‡

Dipartimento di Matematica Pura ed Applicata, Via Belzoni 7, 35131, Padova, Italy

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We give an alternative proof to the well known fact that each convex compact centrally symmetric subset of $\mathbb{R}^2$ containing the origin is a zonoid, i.e., the range of a two dimensional vector measure, and we prove that a two dimensional zonoid whose boundary contains the origin is strictly convex if and only if it is the range of a Chebyshev measure. We give a condition under which a two dimensional vector measure admits a decomposition as the difference of two Chebyshev measures, a necessary condition on the density function for the strict convexity of the range of a measure and a characterization of two dimensional Chebyshev measures.

1. INTRODUCTION

A well known Theorem of Lyapunov [10] states that the range of a non-atomic vector measures is compact and convex. Conversely (see for instance [1]) each compact convex centrally symmetric subset of $\mathbb{R}^2$ con-

* E-mail address: bianchin@gandalf.sissa.it.
† E-mail address: Raphael.Cerf@math.u-psud.fr.
‡ E-mail address: mariconda@math.unipd.it.

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taining the origin is the range of a two dimensional measure (such a set is called a zonoid).

Some problems related to the bang-bang principle in control theory led us to work with the class of the Chebyshev measures. Our definition of a Chebyshev measures is essentially a linear independence condition on some vectors of its range. In [4, 5] we proved that the range of an n-dimensional Chebyshev measure is strictly convex and its boundary contains the origin. Recently Schneider showed in [11] that the range of an n-dimensional measure is strictly convex if and only if for every set A with $\mu(A) \neq 0$ there exist n measurable subsets $A_1, \dots, A_n$ of A such that $\mu(A_1), \dots, \mu(A_n)$ are linearly independent. A result by Neyman [8] states that if the origin is an extreme point of the boundary of a zonoid Z and μ is a vector measure such that $\mathcal{R}(\mu) = Z$ then Z determines the m-range of μ, i.e., the set of m-uples $(\mu(A_1), \dots, \mu(A_m))$ where $A_1, \dots, A_m$ are a measurable partition of the space. An n-dimensional strictly convex zonoid whose boundary contains the origin is then naturally expected to be the range of a Chebyshev measure.

Here we prove that a strictly convex, centrally symmetric, compact subset of $\mathbb{R}^2$ whose boundary contains the origin is actually the range of a two dimensional Chebyshev measure. We give two different proofs: the first one involves the representation theorem for Chebyshev measures proved in [5]; the second one is based on a new simple representation result for convex sets in $\mathbb{R}^2$. Our technique allows also, given an arbitrary convex centrally symmetric compact set, to build explicitly a measure whose range coincides with it. Moreover, we give a condition under which a two dimensional vector measure admits a decomposition as the difference of two Chebyshev measures.

Further, for two dimensional measures, we state a necessary condition on the density function of μ with respect to its total variation for the strict convexity of the range $\mathcal{R}(\mu)$ of μ: as an application we show that μ is a Chebyshev measure on $[0, 1]$ if and only if the map θ defined by $\theta(\alpha, \beta) = \mu(\alpha, \beta)$ for $0 < \alpha < \beta < 1$ is a homeomorphism onto $\mathrm{int}(\mathcal{R}(\mu))$.

2. NOTATIONS AND PRELIMINARY RESULTS

Let $\mu = (\mu_1, \mu_2)$ be a two dimensional vector measure defined on the interval $[0, 1]$ equipped with a σ-field $\mathcal{M}$ and $|\mu|$ be its total variation. The determinant measure $\det \mu$ associated to μ is the two dimensional measure on $[0, 1]^2$ defined by

$$\det \mu = \mu_1 \otimes \mu_2 - \mu_2 \otimes \mu_1;$$

we point out that if A, B are measurable then $\det \mu(A \times B) = \det(\mu(A), \mu(B))$.

We assume that $\mathcal{M}$ contains the Borelians and we set $\Gamma = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le y \le 1\}$.

DEFINITION 1. The measure μ is a Chebyshev measure (or simply T-measure) with respect to the intervals $([0, \alpha])_{0 \le \alpha \le 1}$ if it is non-atomic and each $|\mu|^{\otimes 2}$-non-negligible measurable subset of Γ has a positive (or negative) $\det \mu$ measure.

Remark. In what follows we will always assume $\det \mu$ to be positive whenever μ is a Chebyshev measure; its properties do not change in the other case.

In particular if μ is a Chebyshev measure and A, B are $|\mu|$-non-negligible subsets of $[0,1]$ such that $\sup A \le \inf B$ then $\det(\mu(A), \mu(B)) > 0$.

For Φ being an endomorphism of $\mathbb{R}^2$ and μ a two dimensional measure on $[0,1]$ we define the two dimensional measure $\Phi\mu$ by $\Phi\mu(A) = \Phi(\mu(A))$ for every measurable $A \subset [0,1]$. The next proposition is a straightforward consequence of the definitions.

PROPOSITION 1. *Let μ be a T-measure and Φ be a rotation; then $\Phi\mu$ is a T-measure.*

DEFINITION 2. A function f in $L^1_\nu([0,1], \mathbb{R}^2)$ is a Chebyshev system (or simply T-system) with respect to a prescribed measure ν whenever the determinant $\det(f(t_1), f(t_2))$ is positive for $\nu^{\otimes 2}$-almost every (t_1, t_2) in Γ.

PROPOSITION 2 [5, Theorem 3.4]. *A measure μ is a Chebyshev measure if and only if the density with respect to its total variation is a T-system.*

For $\mu = (\mu_1, \mu_2)$ being a two dimensional measure on $[0,1]$ we denote by $\mathcal{R}(\mu)$ the range of μ defined by $\mathcal{R}(\mu) = \{\mu(E) = (\mu_1(E), \mu_2(E)) : E \in \mathcal{M}\}$ and by $\theta : \Gamma \to \mathcal{R}(\mu)$ the map defined by $\theta(\alpha, \beta) = \mu(\alpha, \beta)$ for every (α, β) in Γ.

We denote by $\operatorname{int}(A)$ the interior of a set A, by $\operatorname{cl}(A)$ its closure, by ∂A its boundary, and by $\operatorname{co}(A)$ its convex hull; for L being a convex set in $\mathbb{R}^n$ we denote by $\operatorname{ri}(L)$ its relative interior. We refer to [9] for these definitions and the basic properties.

The peculiar properties of a Chebyshev measure rely on the following result.

THEOREM 1 [5]. *Let* μ *be a Chebyshev measure on* $[0,1]$. *Then the restriction of* θ *to* $\mathrm{int}(\Gamma)$ *induces a homeomorphism onto* $\mathrm{int}(R(\mu))$; *in particular* $R(\mu) = \{\mu(\alpha,\beta)\colon 0 \le \alpha \le \beta \le 1\}$ *and* $\partial R(\mu) = \{\mu(0,\alpha)\colon 0 \le \alpha \le 1\} \cup \{\mu(\beta,1)\colon 0 \le \beta \le 1\}$.

3. A CHARACTERIZATION OF PLANAR STRICTLY CONVEX ZONOIDS

THEOREM 2. *Let* K *be a subset of* $\mathbb{R}^2$. *We have the following equivalence*:

(i) *the set* K *is strictly convex, compact, centrally symmetric, and* $(0,0) \in \partial K$;

(ii) *there exists a two-dimensional Chebyshev measure* μ *such that* $K = R(\mu)$.

Proof. (ii) $\Rightarrow$ (i). The identity

$$\forall A \in M, \qquad \mu([0,1] \setminus A) = \mu(0,1) - \mu(A)$$

shows that $R(\mu)$ is symmetric with respect to $\frac{1}{2}\mu([0,1])$. We proved in [4, 5] that the range of a Chebyshev measure is strictly convex; the compactness of $R(\mu)$ follows from the Lyapunov Theorem. Moreover $(0,0) = \mu([0,0])$ so that Theorem 1 yields $(0,0) \in \partial K$.

(i) $\Rightarrow$ (ii). There exists a rotation Φ such that $\Phi(K) \subset \{(x,y) \in \mathbb{R}^2\colon x \ge 0\}$; assume for simplicity that $[0,1]$ is the projection of $\Phi(K)$ on the x-axis. For each x in $[0,1]$ let

$$y(x) = \inf\{y \in \mathbb{R}\colon (x,y) \in \Phi(K)\}.$$

Clearly $p = \frac{1}{2}(1, y(1))$ is the center of $\Phi(K)$ and the boundary of $\Phi(K)$ is the union of the graph of y and its symmetric with respect to p. Since y is strictly convex and $y(0) = 0$ there exists a strictly increasing function g such that

$$\forall x \in [0,1], \qquad y(x) = \int_0^x g(t)\,dt.$$

Let μ be the two dimensional vector measure on $[0,1]$ whose density function with respect to the Lebesgue measure is $f(t) = (1, g(t))$. Since g is strictly increasing, f is a T-system with respect to the Lebesgue measure: Proposition 2 then implies that μ is a T-measure. By Theorem 1 the boundary points of the range of μ are exactly the points $\mu([0,x])$ where x varies in $[0,1]$ together with their symmetric with respect to $\frac{1}{2}\mu([0,1])$. By the definition of μ we have $\mu([0,x]) = (x, y(x))$; it follows that the boundaries of $R(\mu)$ and of $\Phi(K)$ coincide: these sets being convex and closed we obtain $R(\mu) = \Phi(K)$ so that $K = R(\Phi^{-1}\mu)$. The conclusion follows from Proposition 1. ∎

4. BIDIMENSIONAL ZONOIDS

A point P of a convex set C is said to be exposed (see [9]) if there exists an hyperplane whose intersection with C is reduced to P. If C is compact in $\mathbb{R}^n$ there exists at least one exposed point: let $[P_1, P_2]$ be a diameter of C (i.e., $|P_1 - P_2| = \max\{|P - Q|: P, Q \in C\}$) and let H be its orthogonal hyperplane at P_1; if Q is in $C \cap H$ then the segment $[Q, P_2]$ is contained in C and if $Q \neq P_1$ we have $|Q - P_2| > |P_1 - P_2|$, a contradiction: thus P_1 is exposed. Strszewicz–Klee Theorem ensures the existence of exposed points in compact convex subsets of arbitrary normed spaces.

Let C be a compact, convex, centrally symmetric subset of $\mathbb{R}^2$. In the following Lemma 1 and Proposition 3 we will assume that O is an exposed point of C and that

$$C \subset \{(x, y) \in \mathbb{R}^2: x \geq 0\}, \qquad C \cap \{(0, y): y \in \mathbb{R}\} = \{O\};$$

notice that, in general, this can always be obtained applying an isometry of $\mathbb{R}^2$.

Let $L > 0$, $M \in \mathbb{R}$ be such that $(L/2, M/2)$ are the coordinates of the center I of C; clearly C is contained in $[0, L] \times \mathbb{R}$. Let y be the function defined by

$$\forall x \in [0, L], \qquad y(x) = \min\{y \in \mathbb{R}: (x, y) \in C\}$$

and let Graph(y) be its graph. Clearly y is convex, bounded, and (thus) continuous on its domain.

We set $\partial^- C = \partial C \cap \{(x, y) \in \mathbb{R}^2: y \leq (M/L)x\}$.

LEMMA 1. $\partial^- C = \text{Graph}(y)$.

Proof. Let $P = (M, L)$; the central symmetry implies that $C \cap \{(L, y): y \in \mathbb{R}\} = \{P\}$. Therefore $\partial^- C \cap \{(L, y): y \in \mathbb{R}\} = \{P\}$ and $y(L) = M$.

If $\text{int}(C) = \emptyset$ the result is trivial; in what follows we assume that $\text{int}(C) \neq \emptyset$: remark that by [9, Theorem 6.3], C is the closure of its interior and thus each relative interior point of $[O, P]$ belongs to the interior of C.

Let $x \in]0, L[$ and $(x, y) \in C$ be such that $y \leq (M/L)x$. If $y > y(x)$ then (x, y) belongs to the relative interior of the segment joining $(x, y(x)) \in C$ and $(x, (M/L)x) \in \text{int}(C)$; Theorem 6.1 in [9] then implies that $(x, y) \in \text{int}(C)$. Conversely if $(x, y(x)) \notin \partial C$ for some $x \in [0, L]$ there exists $y_1 < y(x)$ such that $(x, y_1) \in C$, contradicting the definition of $y(x)$. ∎

Let $G\colon [0, L] \to \partial^- C$ be the surjective map defined by

$$\forall x \in [0, L], \qquad G(x) = (x, y(x)).$$

Remark that for x in $[0, L]$ the symmetric point of $G(L - x)$ with respect to I is the point $(x, M - y(L - x))$ of the boundary of C. It follows that

$$\forall (x, y) \in [0, L] \times \mathbb{R}, \qquad (x, y) \in C \Leftrightarrow y(x) \leq y \leq M - y(L - x). \tag{$\circ$}$$

We will widely use the next representation result.

PROPOSITION 3. *The following identity holds*:

$$C = \{G(x_2) - G(x_1): x_1, x_2 \in [0, L], x_1 \leq x_2\}. \tag{$*$}$$

Proof. Let $x_1 \leq x_2$; if $x_1 = x_2$ then $O = G(x_1) - G(x_1) \in C$.

Assume that $x_1 < x_2$; since $0 \leq x_1$ and $x_2 - x_1 \leq x_2$ then by convexity we have

$$\frac{y(x_2 - x_1)}{x_2 - x_1} \leq \frac{y(x_2) - y(x_1)}{x_2 - x_1};$$

similarly since $x_1 \leq L - (x_2 - x_1)$ and $x_2 \leq L$ then

$$\frac{y(x_2) - y(x_1)}{x_2 - x_1} \leq \frac{y(L) - y(L - (x_2 - x_1))}{L - (L - (x_2 - x_1))}.$$

It follows that $y(x_2 - x_1) \leq y(x_2) - y(x_1) \leq M - y(L - (x_2 - x_1))$; thus by ($\circ$) the point $(x_2 - x_1, y(x_2) - y(x_1)) = G(x_2) - G(x_1))$ belongs to C.

Conversely let $z = (a, b) \in C$. Let $\varphi\colon [0, L - a] \to \mathbb{R}$ be the map defined by

$$\forall x \in [0, L - a], \qquad \varphi(x) = y(x + a) - y(x) - b.$$

Clearly φ is continuous; moreover by ($\circ$) we have $y(a) \leq b \leq M - y(L - a)$. Therefore $\varphi(0) = y(a) - b \leq 0$ and $\varphi(L - a) = M - y(L - a) - b \geq 0$: it follows that there exists x_1 such that $\varphi(x_1) = 0$. Then if we put $x_2 = x_1 + a$ we obtain $y(x_2) = b + y(x_1)$ implying that $G(x_2) = z + G(x_1)$, which is the desired conclusion. ∎

The construction in Theorem 2 suggests an alternative proof (and an improvement) to the well-known fact that C is the range of a measure (see, for instance, [1]).

For I, J being intervals in $\mathbb{R}$ we write that $I < J$ if $i < j$ for every i in I and j in J; we shall denote by λ the Lebesgue measure in $\mathbb{R}$.

THEOREM 3. *Let K be a non-empty, compact, centrally symmetric, convex subset of $\mathbb{R}^2$ containing the origin. Then there exists a non-atomic measure μ on the Borelians of $[0,1]$ such that $K = \mathsf{R}(\mu)$ and for every x in K there exist $\alpha, \beta, \gamma, \delta$ in $[0,1]$ such that $x = \mu(\alpha, \beta) - \mu(\gamma, \delta)$. Moreover if the origin is an exposed point of K then*

$$\mathsf{R}(\mu) = \{\mu(\alpha, \beta): 0 \le \alpha \le \beta \le 1\}.$$

Proof. Let e be an exposed point of K; then O is an exposed point of $-e + K$. Let T be a rotation such that

$$T(-e + K) \subset \{(x, y) \in \mathbb{R}^2 : x \ge 0\},$$

$$T(-e + K) \cap \{(0, y) : y \in \mathbb{R}\} = \{O\}$$

and let $I = (L/2, M/2)$ be the center of $T(-e + K)$; we will assume that $L = 1$ and set $C = T(-e + K)$. Correspondingly let y and G be the function defined above.

By [9, Corollary 24.2.1] there exists an increasing function $g: [0,1] \to \mathbb{R}$ such that

$$\forall x_1, x_2 \in [0,1], \qquad y(x_2) - y(x_1) = \int_{x_1}^{x_2} g(t)\, dt.$$

Let ν be the measure whose density function with respect to the Lebesgue measure is $(1, g)$. Proposition 3 then yields

$$C = \{\nu(x_1, x_2) : x_1, x_2 \in [0,1],\ x_1 \le x_2\} \tag{$**$}$$

so that, in particular, $C \subset \mathsf{R}(\nu)$. To prove the opposite inclusion let $I_1 < \cdots < I_m$ be m disjoint non-trivial open intervals and set $V = I_1 \cup \cdots \cup I_m$. Let

$$0 = x_0 < x_1 < \cdots < x_m \le 1 \qquad \text{and} \qquad 1 = y_0 > y_1 > \cdots > y_m \ge 0$$

be such that $J_i =]x_{i-1}, x_i[$ and $L_i =]y_{m-i+1}, y_{m-i}[$ are translates of I_i. Then

$$J_1 < \cdots < J_m, \qquad L_1 < \cdots < L_m, \qquad x_m = \lambda(V), \qquad y_m = 1 - x_m.$$

Clearly for each i J_i, I_i and L_i have the same length and

$$\inf J_i \le \inf I_i \le \inf L_i, \qquad \sup J_i \le \sup I_i \le \sup L_i.$$

The function g being increasing we obtain

$$\forall i \in \{1, \ldots, m\}, \qquad \int_{J_i} g(t)\, dt \leq \int_{I_i} g(t)\, dt \leq \int_{L_i} g(t)\, dt$$

and thus

$$\int_0^{x_m} g(t)\, dt \leq \int_V g(t)\, dt \leq \int_{y_m}^1 g(t)\, dt.$$

Now by $(*)$ we have

$$p = \left(x_m, \int_0^{x_m} g(t)\, dt \right) = G(x_m) - G(0) \in C$$

$$\text{and} \qquad q = \left(x_m, \int_{y_m}^1 g(t)\, dt \right) = G(1) - G(y_m) \in C;$$

by convexity we obtain

$$\nu(V) = \left(x_m, \int_V g(t)\, dt \right) \in \operatorname{co}(\{p, q\}) \subset C.$$

Let A be a measurable subset of $[0, 1]$; the measure ν being regular there exists a G_δ subset E such that $\nu(A) = \nu(E)$. We may write $E = \bigcap_m V_m$ where $(V_m)_m$ is a decreasing sequence of countable unions of disjoint open intervals. Since $\nu(E) = \lim_m \nu(V_m)$ then the previous remarks and the closure of C imply that $\nu(A) = \nu(E) \in C$. It follows that

$$C = R(\nu) \qquad (***)$$

and therefore $K = e + T^{-1} R(\nu) = e + R(T^{-1}\nu)$. If O is an exposed point of K we may take $e = 0$, proving the claim. Otherwise since $O \in TK$ there exists a set E such that $\nu(E) = -Te$; let ν' be the measure on the Borelians of $[0, 1]$ defined by

$$\nu'(B) = \nu(B \setminus E) - \nu(B \cap E).$$

It is well known [1, Lemma 1.3] (and easy to check) that the range of ν' is given by

$$R(\nu') = R(\nu) - \nu(E)$$

so that $\mathcal{R}(\nu') = TK$ and therefore $K = T^{-1}(\mathcal{R}(\nu')) = \mathcal{R}(\mu)$ where $\mu = T^{-1}\nu'$. Now let A be a measurable subset of $[0, 1]$. Then

$$\mu(A) = T^{-1}\nu(A \setminus E) - T^{-1}\nu(A \cap E);$$

$(**)$ and $(***)$ yield the conclusion. ∎

Remark. A generalized version of the integral inequalities that we use to show that $\mathcal{R}(\nu)$ is contained in C was stated in [2]; their proof in this less general context is simpler and it is given here for the convenience of the reader.

The above arguments yield an alternative proof of Theorem 2.

COROLLARY. *Let K be a non-empty, compact, centrally symmetric, strictly convex subset of $\mathbb{R}^2$ such that O belongs to ∂K. Then there exists a Chebyshev measure μ on the Borelians of $[0, 1]$ such that $K = \mathcal{R}(\mu)$.*

Proof. Since K is strictly convex and O belongs to ∂K then O is exposed: with the notations of the proof of Theorem 3 we may take $e = O$ and thus no translation is needed. Then $C = T(K)$ so that by $(***)$ we obtain $K = \mathcal{R}(T^{-1}\nu)$ where ν is the measure whose density with respect to λ is the vector $(1, g)$. Since the function y is strictly convex then g is strictly monotonic and therefore $(1, g)$ is a T-system. Proposition 2 then shows that ν is a Chebyshev measure; Proposition 1 yields the result. ∎

Remark. The main difference between the two proofs is that, in Theorem 3, the representation result for convex sets (Proposition 3) is used a substitute of the representation Theorem 1 for Chebyshev measures.

5. DECOMPOSITION OF MEASURES

Let $(X, \mathcal{M})$ be a measurable space and μ be a non-atomic positive measure on X. There exists a family $(M_i)_{i \in [0,1]}$ of sets of $\mathcal{M}$ such that μ is a Chebyshev measure with respect to μ and to $(M_i)_{i \in [0,1]}$ (we refer to [5] for the definition of T-measure in this more general setting). In fact Lyapunov Theorem on the range of measures yields the existence of an increasing family $(M_i)_{i \in [0,1]}$ such that $\mu(M_i) = i\mu(X)$ for each i in $[0, 1]$.

More generally, if μ is a signed measure on X, by the Hahn decomposition theorem we may decompose X into a disjoint union $X^- \cup X^+$ such that $\mu = \mu^+ - \mu^-$ and

$$\mu^+(\cdot) = \mu(X^+ \cap \cdot), \qquad \mu^-(\cdot) = -\mu(X^- \cap \cdot).$$

The measures μ^+ and μ^- being positive, there exist two increasing families $(M_i^+)_{i \in [0,1]}$ and $(M_i^-)_{i \in [0,1]}$ such that μ^+ (resp. μ^-) is a Chebyshev measure with respect to $(M_i^+)_{i \in [0,1]}$ (resp. $(M_i^-)_{i \in [0,1]}$). Thus μ is the difference of two Chebyshev measures.

We give now a condition under which the same conclusion holds for two dimensional vector measures. For a vector v of $\mathbb{R}^2 \setminus \{(0,0)\}$ we denote by $\arg v$ its principal argument in $]-\pi, \pi]$. Let f be a measurable function with values in $\mathbb{R}^2$.

THEOREM 4. *Let μ be a two dimensional measure on $(X, \mathcal{M})$ and let $f = (f_1, f_2)$ be its density function with respect to $|\mu|$. If $|\mu|(\{x: \arg f(x) = \theta\}) = 0$ for each θ in $]-\pi, \pi]$ then there exist two T-measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$.*

Proof. We define $X^+ = \{x \in X: \arg f(x) \geq 0\}$, $X^- = \{x \in X: \arg f(x) < 0\}$ and, for every i in $[0,1]$,

$$M_i^+ = \{x \in X^+: \arg f(x) \leq i\pi\},$$
$$M_i^- = \{x \in X^-: \arg f(x) \geq -i\pi\}.$$

Let f^+ and f^- be the functions $f^+ = f\mathbf{1}_{X^+}$ and $f^- = f\mathbf{1}_{X^-}$. Then f^+ (resp. f^-) is a T-system on X^+ (resp. X^-) with respect to $|\mu|$ and $(M_i^+)_{i \in [0,1]}$ (resp. $(M_i^-)_{i \in [0,1]}$). Then setting $d\mu^+ = f^+ d|\mu|$ and $d\mu^- = f^- d|\mu|$ we obtain a decomposition of μ as the difference of two Chebyshev measures. ∎

Remark. Under the above assumptions Theorem 5.1 in [5] then implies that for every $A \in \mathcal{M}$ there exist i_1, i_2, j_1, j_2 in $[0,1]$ such that $\mu(A) = \mu^+(M_{i_2}^+ \setminus M_{i_1}^+) - \mu^-(M_{j_2}^- \setminus M_{j_1}^-)$. This results looks similar to the one stated in Theorem 3; however, here the measure μ is imposed whereas in Theorem 3, given a zonoid, the measure is built.

6. A CHARACTERIZATION OF TWO DIMENSIONAL CHEBYSHEV MEASURES

Let μ be a two dimensional vector measure on $([0,1], \mathcal{M})$ and let f be its density with respect to the total variation $|\mu|$. We denote by $\langle u: u \in E \rangle$ the vector subspace of $\mathbb{R}^2$ spanned by the vectors u belonging to a set E and by "$\cdot$" the usual scalar product in $\mathbb{R}^2$. The next result will be applied later and has an interest in itself.

THEOREM 5. *If $\mathcal{R}(\mu)$ is strictly convex then the determinant* $\det(f(x), f(y))$ *of the vectors $f(x), f(y)$ is not zero $|\mu|^{\otimes 2}$-a.e. on* $[0,1]^2$.

Proof. Let A, Z, A_1 be the sets defined by

$$A = \{(x, y): \det(f(x), f(y)) = 0\}, \qquad Z = \{x: f(x) = 0\}$$
$$A_1 = \{(x, y): f(x) \neq 0, f(y) \in \langle f(x) \rangle\};$$

clearly we have $A = (Z \times [0,1]) \cup A_1$. Let τ be the map defined by $\tau(a, b) = (-b, a)$; then $A_1 = \{(x, y): f(x) \neq 0, f(y) \cdot \tau(f(x)) = 0\}$ so that A_1 is measurable.

Moreover Fubini's Theorem gives

$$|\mu|^{\otimes 2}(A_1) = \int_{[0,1] \setminus Z} \left\{ \int_{D_x} d|\mu|(y) \right\} d|\mu|(x),$$

where, for x in $[0,1]$, $D_x = \{y: f(y) \cdot \tau(f(x)) = 0\}$.

If $|\mu|^{\otimes 2}(A_1) \neq 0$ there exists x in $[0,1] \setminus Z$ such that $|\mu|(D_x) \neq 0$. The very definition of D_x implies that for every measurable subset B of D_x we have

$$\mu(B) \cdot \tau(f(x)) = \int_B f(y) \cdot \tau(f(x)) \, d|\mu|(y) = 0$$

and thus the vector space $\langle \mu(B): B \in \mathsf{M}, B \subset D_x \rangle$ is at most one dimensional: Theorem 3.1.2 in [11] then implies that $\mathsf{R}(\mu)$ is not strictly convex, a contradiction.

Obviously we have $|\mu|(Z) = 0$; therefore $|\mu|^{\otimes 2}(A) \leq |\mu|^{\otimes 2}(Z \times [0,1]) + |\mu|^{\otimes 2}(A_1) = 0$, proving the claim. ∎

LEMMA 2. *Let A be a non-empty open convex bounded subset of $\mathbb{R}^2$ and assume that ∂A contains a non-trivial segment L. Then* $\mathrm{ri}(L)$ *is open in* ∂A.

Proof. Let p in $\mathbb{R}^2 \setminus \{O\}$ and c in $\mathbb{R}$ be such that

$$L \subset \{x \in \mathrm{cl}(A): p \cdot x = c\}, \qquad A \subset \{x \in \mathbb{R}^2: p \cdot x < c\}$$

and $\{a, b\}$ $(a \neq b)$ be the relative boundary of L. Let U_a, U_b be two disjoint neighbourhoods of a and b; since a, b are in $\mathrm{cl}(A)$ there exist a_1 in $U_a \cap A$ and b_1 in $U_b \cap A$: let

$$C = \mathrm{co}(\{a, a_1, b_1, b\}).$$

Remark that $p \cdot a_1 < c$ and $p \cdot b_1 < c$ so that $L \subset \partial C$ and $L \cap \mathrm{int}(C) = \emptyset$; furthermore since $C \subset \mathrm{cl}(A)$ then Theorem 6.3 in [9] implies that $\mathrm{int}(C) \subset A$. As a consequence if we put $D = \mathrm{int}(C) \cup \mathrm{ri}(L)$ we have

$$D \cap \partial A = \mathrm{ri}(L).$$

Finally if we denote by B the open set defined by

$$B = \operatorname{int}\big(\operatorname{co}\big(\{a_1, b_1, a + (a - a_1), b + (b - b_1)\}\big)\big)$$

then clearly we have $D = B \cap \operatorname{cl}(A)$ so that D is open in $\operatorname{cl}(A)$; the conclusion follows. ∎

LEMMA 3. *Let A, B be open bounded subsets of $\mathbb{R}^n$ and $\psi: \operatorname{cl}(A) \to \mathbb{R}^n$ be a continuous map inducing a homeomorphism from A onto B. Then $\psi(\partial A) = \partial B$.*

Proof. Clearly we have $B \subset \psi(\operatorname{cl}(A)) \subset \operatorname{cl}(\psi(A)) = \operatorname{cl}(B)$; moreover $\operatorname{cl}(A)$ is compat so that $\psi(\operatorname{cl}(A))$ is closed. It follows that $\psi(\operatorname{cl}(A)) = \operatorname{cl}(B)$. Clearly we have $\partial B \subset \psi(\partial A)$. Conversely let $x_0 \in \partial A$ and assume that $z_0 = \psi(x_0) \in B$. Let U (resp. W) be open neighborhoods of x_0 in $\operatorname{cl}(A)$ (resp. z_0 in B) such that $\psi(U) \subset W$. There exists y_0 in A satisfying $\psi(y_0) = z_0$; we may assume that ψ is a homeomorphism from an open neighborhood V of y_0 in A onto W and that $U \cap V = \emptyset$. Since $x_0 \in \operatorname{cl}(A)$ there exists x_1 in $U \cap A$. Now $\psi(x_1) \in W = \psi(V)$: let $y_1 \in V$ be such that $\psi(y_1) = \psi(x_1)$; the injectivity of ψ on A implies that $x_1 = y_1 \in U \cap V$, a contradiction. ∎

We recall that we denote by λ the Lebesgue measure on $[0, 1]$; in what follows we assume that there exists a strictly positive function $h \in L^1_\lambda(0, 1)$ such that $d|\mu| = h\, d\lambda$; in particular $|\mu|$ is absolutely continuous with respect to λ.

We prove here that Theorem 1 characterizes the Chebyshev measures.

THEOREM 6. *Let θ be the map defined in Section 2. If θ induces a homeomorphism from $\operatorname{int}(\Gamma)$ onto $\operatorname{int}(\mathsf{R}(\mu))$ then μ is a Chebyshev measure.*

Proof. Since Γ and $\mathsf{R}(\mu)$ are convex and closed then Theorem 6.3 in [9] yields $\Gamma = \operatorname{cl}\operatorname{int}(\Gamma))$ and $\mathsf{R}(\mu) = \operatorname{cl}(\operatorname{int}(\mathsf{R}(\mu)))$: applying Lemma 3 with $\psi = \theta$, $A = \operatorname{int}(\Gamma)$, $B = \operatorname{int}(\mathsf{R}(\mu))$ we obtain $\theta(\partial\Gamma) = \partial\mathsf{R}(\mu)$; in particular

$$\partial\mathsf{R}(\mu) = \{\mu(0, \alpha) : 0 \le \alpha \le 1\} \cup \{\mu(\beta, 1) : 0 \le \beta \le 1\}.$$

Assume that the boundary of $\mathsf{R}(\mu)$ contains a non-trivial segment L; let (for instance) $\alpha \in [0, 1]$ be such that $x = \mu(0, \alpha)$ belongs to the relative interior of L. By Lemma 2 there exists an open neighbourhood V of x such that $V \cap \partial\mathsf{R}(\mu) = V \cap \operatorname{ri}(L)$. By continuity there exist α_1, α_2 in $]0, 1[$ such that $\alpha_1 < \alpha < \alpha_2$ and

$$\{\mu(0, t) : t \in [\alpha_1, \alpha_2]\} \subset V.$$

Lemma 3 then implies that $\mu(0, t) \in V \cap \operatorname{ri}(L)$ for every $t \in (\alpha_1, \alpha_2)$.

Therefore, if $p \in \mathbb{R}^2 \setminus \{0\}$, $c \in \mathbb{R}$ are such that $L \subset \{x \in \mathbb{R}^2 : p \cdot x = c\}$ we have

$$\forall t \in (\alpha_1, \alpha_2), \qquad p \cdot \mu(0, t) = c.$$

Let U be the open subset of $\operatorname{int}(\Gamma)$ defined by

$$U = \{(\alpha, \beta) \in \Gamma : \alpha_1 < \alpha < \beta < \alpha_2\}.$$

Our assumption implies that $\theta(U)$ is an open subset of $\mathbb{R}^2$; however, we have

$$\forall (\alpha, \beta) \in U,$$
$$p \cdot \theta(\alpha, \beta) = p \cdot \mu(\alpha, \beta) = p \cdot \mu(0, \beta) - p \cdot \mu(0, \alpha) = 0,$$

a contradiction; it follows that $R(\mu)$ is strictly convex. Theorem 5 then implies that

$$\det(f(\alpha), f(\beta)) \neq 0, \qquad |\mu|^{\otimes 2}\text{-a.e. in } [0,1]^2.$$

By [12, Corollary 10.50] we have

$$\lim_{x \to 0} \frac{\mu(\alpha, \alpha + x)}{|\mu|(\alpha, \alpha + x)} = f(\alpha), |\mu|\text{-a.e.}$$

$$\text{and} \qquad \lim_{x \to 0} \frac{|\mu|(\alpha, \alpha + x)}{\lambda(\alpha, \alpha + x)} = h(\alpha), \lambda\text{-a.e.}$$

so that

$$\lim_{x \to 0} \frac{\mu(\alpha, \alpha + x)}{x} = f(\alpha) h(\alpha), \qquad |\mu|\text{-a.e.}$$

Therefore the map θ is differentiable $|\mu|^{\otimes 2}$-a.e. on $[0, 1]^2$ and its Jacobian is given by

$$\operatorname{Jac}(\theta)(\alpha, \beta) = (-f(\alpha)h(\alpha), f(\beta)h(\beta)), \qquad |\mu|^{\otimes 2}\text{-a.e.}$$

so that in particular the determinant of the Jacobian vanishes only on a negligible set. The map θ is a homeomorphism on $\operatorname{int}(\Gamma)$ and Γ is connected; as a consequence the degree $\deg(\operatorname{int}(\Gamma), \theta, p)$ is constantly equal to 1 or -1 for every p in $\operatorname{int}(R(\mu))$ [7, Theorem 3.35], assume for instance that it equals -1. Then by [7, Lemma 5.9] we have

$$\operatorname{sgn} \det(-f(\alpha), f(\beta)) = \operatorname{sgn} \det \operatorname{Jac}(\theta)(\alpha, \beta) = \deg(\operatorname{int}(\Gamma), \theta, p)$$
$$= -1, \qquad |\mu|^{\otimes 2}\text{-a.e. in } \Gamma$$

and therefore f is a T-system; Proposition 2 yields the conclusion. ∎

APPENDIX: $(n-1)$ DIMENSIONAL FACES OF CONVEX SETS IN $\mathbb{R}^n$

The above Lemma 2 can be generalized to higher dimensions using some similar arguments. We thank G. De Marco for suggesting the following alternative proof.

THEOREM 7. *Let A be an open convex bounded subset of $\mathbb{R}^n$ and assume that ∂A contains a relatively open subset L of an hyperplane. Then L is open in ∂A.*

Proof. It is not restrictive to assume that $O \in A$ and that $L \subset H$ where (for some $\lambda > 0$)

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_n = \lambda\}.$$

Clearly H is a supporting hyperplane so that $x_n \leq \lambda$ for every $x = (x_1, \dots, x_n) \in \mathrm{cl}(A)$. We denote by $\|\cdot\|$ the norm of $\mathbb{R}^n$ defined by $\|(x_1, \dots, x_n)\| = \max_i |x_i|$; we recall that the map $\pi \colon x \mapsto x/\|x\|$ is a homeomorphism from ∂A onto the unit sphere S (in the $\|\cdot\|$-norm) of $\mathbb{R}^n$ (see for instance [6]; the elementary proof is based on the fact that the relative interior points of a segment joining an interior point with another point of a convex set C belong to the interior of C). It is not restrictive to assume that

$$L \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_n > \max\{|x_1|, \dots, |x_{n-1}|\}\};$$

in fact it is enough to transform A and H with the map $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, rx_n)$ for a sufficiently large r. Then in particular we have $\|x\| = \lambda$ for every x in L. It follows that $K = \pi(L) \subset S \cap Q$ where $Q = \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_n = 1\} = (1/\lambda)H$ and that $\pi(x) = x/\lambda$ for every x in L so that K is homothetic to L and is thus open in Q. Moreover K is contained in the open set $B = \{(x_1, \dots, x_n) \in \mathbb{R}^n \colon \ x_n > 0, \ 1 > \max\{|x_1|, \dots, |x_{n-1}|\}\}$ and $Q \cap B = S \cap B$: therefore K is open in S. ∎

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