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ON BANG–BANG CONSTRAINED SOLUTIONS OF A CONTROL SYSTEM*

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Abstract. Given $\phi_1, \phi_2 \in L^1([0,T])$ and a function $x \in W^{2,1}([0,T])$ solving the control problem (P) $x''+a_1(t)x'+a_0(t)x \in [\phi_1(t), \phi_2(t)]$ a.e., $x(0)=x_0$, $x(T)=x_1$, $x'(0)=v_0$, $x'(T)=v_1$, there exists a bang–bang solution y to (P) satisfying $y \le x$; moreover there exists a finite union of intervals E such that $y''+a_1y'+a_0y = \phi_1\chi_E + \phi_2\chi_{[0,T]\setminus E}$. The reachable set of bang–bang constrained solutions is convex: an application to the calculus of variations.

Key words. bang-bang, linear control system, range of a vector measure, reachable set, calculus of variations

AMS subject classifications. 34H05, 49B10, 93C15

1. Introduction. We consider the family of bidimensional linear control systems (P) described by a generic second-order equation subject to a scalar control:

$$x''+a_1(t)x'+a_0(t)x \in \Phi(t) = [\phi_1(t), \phi_2(t)],\ (x(0), x'(0), x(T), x'(T)) = (x_0, v_0, x_1, v_1),$$

where $\phi_1 \le \phi_2 \in L^1([0,T])$ and $a_1, a_0 \in \mathcal{C}([0,T])$, $x_0, v_0, x_1, v_1 \in \mathbb{R}$, $x \in W^{2,1}([0,T])$.

The function y is said to be a bang–bang solution to (P) if it solves (P) and, moreover,

$$y''+a_1(t)y'+a_0(t)y \in \text{extr } \Phi(t) = \{\phi_1(t), \phi_2(t)\} \text{ a.e.} \tag{1.1}$$

Existence of bang–bang solutions has been proved, for instance, by Cesari [4, Thm. 16.3]. The purpose of this paper is to prove that, given an arbitrary solution x to (P), there exists a bang–bang solution y such that

$$\forall t \in [0,T] \qquad y(t) \le x(t) \tag{1.2}$$

and, in addition, $y''+a_1y'+a_0y$ steers from ϕ_1 to ϕ_2 only a finite number of times.

Motivation of such a problem was to study the reachable set

$$\mathcal{Y}_T^c = \{(y(T), y'(T)) : y \le c,\ y''+a_1(t)y'+a_0(t)y \in \text{extr } \Phi(t),\ (y(0), y'(0)) = (x_0, v_0)\},$$

where c is an arbitrary function. A consequence of Theorem 3.1 is that $\mathcal{Y}_T^c$ coincides with

$$\mathcal{X}_T^c = \{(y(T), y'(T)) : y \le c,\ y''+a_1(t)y'+a_0(t)y \in \Phi(t),\ (y(0), y'(0)) = (x_0, v_0)\}.$$

Notice that $\mathcal{X}_T^c$ is convex, so the above assumption implies that $\mathcal{Y}_T^c$ is convex too. Another motivation arises from nonconvex problems of the calculus of variations (see [1]).

A possible approach in finding bang–bang solutions is to use the Lyapunov Theorem on the range of a vector measure [4, §16.1].

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Here, the solution of $x'' + a_1(t)x' + a_0(t)x = \rho(t),\ x(0) = x'(0) = 0$ is given by

$$x(t) = \int_0^t h(t,s)\rho(s)\,ds,$$

where $h \in C^1([0,T] \times [0,T])$, and for each $s \in [0,T]$ the function $h_s(.) = h(.,s) \in C^2([0,T])$ is the solution to the associated homogeneous differential equation satisfying the initial conditions $h_s(s) = 0$, $h_s'(s) = 1$. The Lyapunov Theorem yields the existence of a measurable subset E of $[0,T]$ such that

$$\int_0^T h(T,s)\rho(s)\,ds = \int_0^T h(T,s)(\phi_1(s)\chi_E(s) + \phi_2(s)\chi_{[0,T]\setminus E}(s))\,ds, \tag{1.3}$$

$$\int_0^T \frac{\partial h}{\partial t}(T,s)\rho(s)\,ds = \int_0^T \frac{\partial h}{\partial t}(T,s)(\phi_1(s)\chi_E(s) + \phi_2(s)\chi_{[0,T]\setminus E}(s))\,ds. \tag{1.4}$$

Clearly, by differentiating under the integral sign, the function y defined by

$$y(t) = \int_0^t h(t,s)(\phi_1(s)\chi_E(s) + \phi_2(s)\chi_{[0,T]\setminus E}(s))\,ds \tag{1.5}$$

is a bang–bang solution. However, this approach does not give any information on the behaviour of y with respect to x on $[0,T]$.

Here we prove a new Lyapunov-type theorem concerning the range of a two-dimensional vector measure whose densities are such that their quotient is monotone; in this case, the set E can be chosen in the form $[\alpha,\beta]$. Note that this is not true in general; for instance, there are no $\alpha,\beta \in [0,3\pi]$ satisfying

$$\int_\alpha^\beta \sin t\,dt = \int_0^{3\pi} \sin t\chi_{[0,\pi]\cup[2\pi,3\pi]}(t)\,dt, \qquad \int_\alpha^\beta 1\,dt = \int_0^{3\pi} 1\chi_{[0,\pi]\cup[2\pi,3\pi]}(t)\,dt.$$

In our application, the equalities $h(s,s) = 0$ and $\frac{\partial h}{\partial t}(s,s) = 1$ imply that the monotonicity condition is locally fulfilled; this allows us to build a set E satisfying (1.3)–(1.4) as a finite union of intervals and, in the case where $\phi_1 < \rho < \phi_2$ are continuous, to choose E in such a way that neither 0 nor T belong to its closure.

These facts, together with a decomposition of the kernel $h(t,s)$ into a linear combination of linearly independent functions, are the main tools that we use to show that the bang–bang solution y defined by (1.5) satisfies the inequality $y \le x$.

As an application, we consider the problem of minimizing the integral functionals

$$I(x,u) = \int_0^T f(t,x(t),u(t))\,dt,$$

where $x : [0,T] \to \mathbb{R}^2$ is such that $x(0)$, $x'(0)$, $x(T)$, $x'(T)$ are fixed and u is a control belonging to $U(t,x) \subset R^2$. The classical approach to obtain existence of a minimum is to impose conditions in order to have the lower semicontinuity of I with respect to u (for instance convexity of $u \mapsto f(t,x,u)$).

Recently, in an effort to provide existence criteria other than convexity in u, some sufficient conditions have been given: for problems of the calculus of variations ($x' = u$ in the above setting) and for maps of the form $f(t,x,x') = g(t,x) + h(t,x')$, existence of solutions has been obtained by requiring that the real map $x \mapsto g(t,x)$ be monotone [5] or, for x in $\mathbb{R}^n$, that the same function be concave [2]. Optimal control problems escaping to convexity conditions have been handled in [6].

It has been proved further in [3] that there exists a dense subset $\mathcal{D}$ of $\mathcal{C}(\mathbb{R})$ such that, for g in it, the problem

$$\text{minimize} \int_0^T g(x(t))\,dt + \int_0^T h(x'(t))\,dt \quad : \quad x(0) = x_0,\, x(T) = x_1$$

admits a solution for every lower semicontinuous h satisfying growth conditions.

Our theorem gives a straightforward generalization of the above result.

2. Assumptions and preliminary results. Let $\phi_1, \phi_2 \in L^1[0,T]$, $\phi_1 \leq \phi_2$, and put $\Phi(t) = [\phi_1(t), \phi_2(t)] \subset \mathbb{R}$. We are interested in the solutions of the following control problem.

Problem P.

$$a_1, a_0 \in \mathcal{C}([0,T]), \quad x_0, x_1, v_0, v_1 \in \mathbb{R}, \quad x \in W^{2,1}([0,T]),$$

$$\text{(P)} \qquad x'' + a_1(t)x' + a_0(t)x \in \Phi(t) \text{ a.e.},$$

$$x(0) = x_0, \quad x'(0) = v_0, \quad x(T) = x_1, \quad x'(T) = v_1.$$

By extr Φ we mean the extreme points of Φ, i.e., extr $\Phi(t) = \{\phi_1(t), \phi_2(t)\}$.

DEFINITION 2.1. *A function $y \in W^{2,1}([0,T])$ is said to be a bang–bang solution to* (P) *if y solves* (P) *and, moreover,*

$$y'' + a_1(t)y' + a_0(t)y \in \text{ extr } \Phi(t) \ \ a.e.$$

The following representation formula of the solutions to (P) will be used later.

PROPOSITION 2.1. *There exists a function $h \in \mathcal{C}^1([0,T] \times [0,T])$ satisfying Property* S *below such that, for each function $\rho \in L^1([0,T])$, the solution of*

$$(\mathrm{P}_\rho) \qquad x'' + a_1(t)x' + a_0(t)x = \rho(t), \quad x(0) = x'(0) = 0$$

is given by the formula

$$(2.1) \qquad x(t) = \int_0^t h(t,s)\rho(s)\,ds\,.$$

Moreover, for each $s \in [0,T]$, the function $h(.,s)$ is of class $\mathcal{C}^2([0,T])$.

PROPERTY S.

(1) There exist $w_1, w_2 \in \mathcal{C}^2([0,T])$, $z_1, z_2 \in \mathcal{C}^1([0,T])$ such that

$$(2.2) \qquad \forall s,t \in [0,T] \qquad h(t,s) = w_1(t)z_1(s) + w_2(t)z_2(s)$$

$$\text{and} \quad W(w_1,w_2,t) = \det \begin{vmatrix} w_1(t) & w_2(t) \\ w_1'(t) & w_2'(t) \end{vmatrix} \neq 0.$$

For each t_0 in $[0,T]$ there exists $\delta > 0$ such that if we set $I_\delta = [t_0 - \delta, t_0 + \delta] \cap [0,T]$ then:

(2) $\forall s,t \in I_\delta \qquad h(t,s) > 0$ if $s < t$, $\quad h(t,s) < 0$ if $t < s \quad$ (whence $h(s,s) = 0$);

(3) $\forall s,t \in I_\delta \qquad \frac{\partial h}{\partial t}(t,s) > 0$;

(4) $\forall t \in I_\delta \qquad s \mapsto h(t,s)/\frac{\partial h}{\partial t}(t,s)$ is decreasing on I_δ.

Proof of Proposition 2.1. For each $s \in [0,T]$, let $h_s(.) = h(.,s) \in \mathcal{C}^2([0,T])$ be the solution to

$$h_s''(t) + a_1(t)h_s'(t) + a_0(t)h_s(t) = 0, \qquad h_s(s) = 0,\, h_s'(s) = 1.$$

Set $z(t) = \int_0^t h(t,s)\rho(s)\,ds$. Differentiation under the integral sign shows that z is a solution to (P_ρ) whence, by uniqueness, $z = x$.

To prove the second part of the claim, let $w_1, w_2 \in \mathcal{C}^2([0,T])$ be two solutions of the differential equation

$$x'' + a_1(t)x' + a_0(t)x = 0 \tag{2.3}$$

such that their Wronskian

$$W(w_1, w_2, t) = \det \begin{vmatrix} w_1(t) & w_2(t) \\ w_1'(t) & w_2'(t) \end{vmatrix}$$

is nonzero for every t. Such functions exist because the set of the solutions of a second-order linear differential equation is a two-dimensional vector space. Since for each $s \in [0,T]$ the function h_s is a solution to (2.3), there exist z_1, z_2 defined on $[0,T]$ such that

$$\forall s,t \in [0,T] \qquad h_s(t) = w_1(t)z_1(s) + w_2(t)z_2(s). \tag{2.4}$$

Conditions on h_s at s and equation (2.4) yield

$$\begin{cases} h_s(s) = 0 = w_1(s)z_1(s) + w_2(s)z_2(s), \\ h_s'(s) = 1 = w_1'(s)z_1(s) + w_2'(s)z_2(s). \end{cases}$$

Since $W(w_1, w_2, s) \neq 0$ for each s, we find

$$z_1(s) = -\frac{w_2(s)}{W(w_1, w_2, s)}, \quad z_2(s) = \frac{w_1(s)}{W(w_1, w_2, s)},$$

so that $z_1, z_2 \in \mathcal{C}^1([0,T])$; hence $h(t,s) = h_s(t)$ belongs to $\mathcal{C}^1([0,T] \times [0,T])$. By construction

$$\forall s \in [0,T] \qquad h(s,s) = 0 \quad \text{and} \quad \frac{\partial h}{\partial t}(s,s) = 1$$

implying

$$\begin{aligned} \forall s \in [0,T] \qquad \frac{d}{ds}h(s,s) = 0 \Leftrightarrow \forall s \in [0,T] \qquad & \frac{\partial h}{\partial t}(s,s) + \frac{\partial h}{\partial s}(s,s) = 0 \\ \Leftrightarrow \forall s \in [0,T] \qquad & \frac{\partial h}{\partial s}(s,s) = -1. \end{aligned}$$

As a consequence,

$$\forall s \in [0,T] \qquad \frac{\partial}{\partial s}\left(\frac{h}{\frac{\partial h}{\partial t}}\right)(s,s) = -1.$$

By continuity for a fixed t_0 in $[0,T]$, there exists $\delta > 0$ such that

$$\forall s,t \in [t_0 - \delta, t_0 + \delta] \cap [0,T] \qquad \frac{\partial h}{\partial t}(t,s) > 0 \quad \text{and} \quad \frac{\partial}{\partial s}\left(\frac{h}{\frac{\partial h}{\partial t}}\right)(t,s) < 0;$$

for this δ (2), (3), and (4) in Property S are satisfied.

Assume, for instance, $\Phi(t) = [0, \phi(t)]$ and let $\rho \in L^1([0,T])$ be such that $0 \le \rho \le \phi$. For a solution x to (P_ρ) formula (2.1) yields, in particular,

$$x(T) = \int_0^T h(T,s)\rho(s)\,ds, \tag{2.5}$$

$$x'(T) = \int_0^T \frac{\partial h}{\partial t}(T,s)\rho(s)\,ds. \tag{2.6}$$

Let us point out that the classical Lyapunov Theorem on the range of a vector measure [4, §16.1] allows us to find a bang–bang solution. In fact, its application yields the existence of a measurable subset E of $[0,T]$ such that

$$\int_0^T h(T,s)\rho(s)\,ds = \int_0^T h(T,s)\phi(s)\chi_E(s)\,ds, \tag{2.7}$$

$$\int_0^T \frac{\partial h}{\partial t}(T,s)\rho(s)\,ds = \int_0^T \frac{\partial h}{\partial t}(T,s)\phi(s)\chi_E(s)\,ds, \tag{2.8}$$

so that the function $\bar{x}$ defined by

$$\bar{x}(t) = \int_0^t h(t,s)\phi(s)\chi_E(s)\,ds$$

is, by Proposition 2.1, a bang–bang solution to (P) (with $\phi_1 = 0$, $\phi_2 = \phi$, $x_0 = v_0 = 0$). However, for $0 < t < T$, the Lyapunov Theorem does not give any information on the relative positions of $\bar{x}$ and the original solution x.

The purpose of Proposition 2.2 below is to show that if $s \mapsto \left(h/\frac{\partial h}{\partial t}\right)(t,s)$ is monotone on $[0,T]$ then the measurable subset E can be chosen to be an interval $[\alpha,\beta]$ with $0 \le \alpha \le \beta \le T$. Taking into account Property S (4), this will allow us to define in §3 a bang–bang solution y satisfying $y(t) \le x(t)$ for each t.

In what follows $[a,b]$ is an interval of $\mathbb{R}$; ρ and ϕ are two functions belonging to $L^1([a,b])$ satisfying $0 \le \rho \le \phi$. We say that $r \in \mathbb{R}$ is positive (resp. negative) if $r \ge 0$ (resp. $r \le 0$).

We consider the following hypothesis.

Hypothesis H. The functions f, g belong to $L^\infty([a,b])$ and are positive almost everywhere. Moreover there exists a strictly monotone positive function k such that

$$g(t) = k(t)f(t) \text{ a.e.}$$

We have the following Lyapunov-type result.

PROPOSITION 2.2. *Let f, g satisfy Hypothesis* H. *Then there exist $\alpha, \beta \in \mathbb{R}$ such that, if we put $E = [\alpha,\beta]$, we have*

$$\int_a^b \rho(s)f(s)\,ds = \int_\alpha^\beta \phi(s)f(s)\,ds = \int_a^b \phi(s)f(s)\chi_E(s)\,ds, \tag{2.9}$$

$$\int_a^b \rho(s)g(s)\,ds = \int_\alpha^\beta \phi(s)g(s)\,ds = \int_a^b \phi(s)g(s)\chi_E(s)\,ds. \tag{2.10}$$

Moreover, α and β are unique if ρ, ϕ, f, g are continuous, and $0 < \rho < \phi$, $f > 0$, $g > 0$.

To prove Proposition 2.2, we need the following fundamental lemma.

LEMMA 2.1. *Assume that f, g satisfy Hypothesis* H *and let $\alpha, \beta \in [a,b]$ be such that*

$$\int_\alpha^b \phi(s)f(s)\,ds = \int_a^b \rho(s)f(s)\,ds, \tag{2.11}$$

$$\int_a^\beta \phi(s)f(s)\,ds = \int_a^b \rho(s)f(s)\,ds. \tag{2.12}$$

Then, if k is increasing, we have

$$\int_\alpha^b \phi(s)g(s)\,ds \geq \int_a^b \rho(s)g(s)\,ds, \tag{2.13}$$

$$\int_a^\beta \phi(s)g(s)\,ds \leq \int_a^b \rho(s)g(s)\,ds. \tag{2.14}$$

If k is decreasing on $[a,b]$, inequalities (2.13) *and* (2.14) *are reversed. Moreover, inequalities* (2.13)–(2.14) *are strict if $0 < \rho < \phi$ and $f > 0$, $g > 0$ a.e.*

Proof of Lemma 2.1. Assume for instance that k is increasing. To prove (2.14) let f_ϕ, f_ρ be the monotone functions defined by

$$f_\phi(t) = \int_a^t \phi(s)f(s)\,ds, \qquad f_\rho(t) = \int_a^t \rho(s)f(s)\,ds.$$

The Lebesgue–Stieltjes formula for integration by parts yields

$$\begin{aligned}\int_a^b \rho(s)g(s)\,ds &= \int_a^b \rho(s)k(s)f(s)\,ds\\ &= \int_a^b k(s)\,df_\rho(s)\\ &= k(b)f_\rho(b) - k(a)f_\rho(a) - \int_a^b f_\rho(s)\,dk(s);\end{aligned}$$

analogously we have

$$\int_a^\beta \phi(s)g(s)\,ds = k(\beta)f_\phi(\beta) - k(a)f_\phi(a) - \int_a^\beta f_\phi(s)\,dk(s).$$

Taking into account that $f_\phi(a) = f_\rho(a) = 0$ and that by (2.12) $f_\rho(b) = f_\phi(\beta)$, we are thus led to show that

$$\int_a^b f_\rho(s)\,dk(s) - \int_a^\beta f_\phi(s)\,dk(s) \leq (k(b) - k(\beta))f_\rho(b). \tag{2.15}$$

By our assumptions we have

$$\forall t \in [a,b] \qquad f_\phi(t) \geq f_\rho(t); \tag{2.16}$$

therefore,

$$\int_a^b f_\rho(s)\,dk(s) - \int_a^\beta f_\phi(s)\,dk(s) \leq \int_\beta^b f_\rho(s)\,dk(s). \tag{2.17}$$

Furthermore, since functions f_ρ and k are increasing we have

$$\int_\beta^b f_\rho(s)\,dk(s) \leq (k(b) - k(\beta))f_\rho(b),$$

which, together with (2.17), gives (2.15).

To prove the final part of the lemma, it is enough to remark that if $f > 0$ and $\rho > 0$ then, by (2.12), $\beta \neq a$; if, moreover, $0 < \rho < \phi$ a.e., then inequality (2.16) is strict for every $t > a$ so that (2.17) is strict too (k being increasing). Similar arguments prove (2.13). ⌉

Proof of Proposition 2.2.

(i) *Existence.* (a) Assume first $0 < \rho < \phi$ and $f > 0$, $g > 0$ a.e. Let α_1, α_2, β_1, $\beta_2 \in [a, b]$ be such that

$$\int_{\alpha_1}^{b} \phi(s)f(s)\,ds = \int_a^b \rho(s)f(s)\,ds, \tag{2.18}$$

$$\int_{\alpha_2}^{b} \phi(s)g(s)\,ds = \int_a^b \rho(s)g(s)\,ds, \tag{2.19}$$

$$\int_{a}^{\beta_1} \phi(s)f(s)\,ds = \int_a^b \rho(s)f(s)\,ds, \tag{2.20}$$

$$\int_{a}^{\beta_2} \phi(s)g(s)\,ds = \int_a^b \rho(s)g(s)\,ds. \tag{2.21}$$

Assume for instance that k is decreasing on $[a, b]$. In this situation Lemma 2.1 yields

$$\beta_2 \le \beta_1, \qquad \alpha_2 \le \alpha_1. \tag{2.22}$$

The function v defined by

$$v(x) = \int_a^x \phi(s)f(s)\,ds$$

is continuous and increasing with values in $[0, v(b)]$: let v^{-1} denote its inverse function. Set

$$m = \int_a^b \rho(s)f(s)\,ds.$$

Since, by (2.18), $v(b) = v(\alpha_1) + m$, then $v(\alpha) + m \in [0, v(b)]$ if and only if $a \le \alpha \le \alpha_1$; this allows us to introduce the continuous function ξ_1 defined by the formula

$$\forall \alpha \in [a, \alpha_1] \qquad \xi_1(\alpha) = v^{-1}(v(\alpha) + m).$$

By definition, we have

$$\forall \alpha \in [a, \alpha_1] \qquad \int_\alpha^{\xi_1(\alpha)} \phi(s)f(s)\,ds = v(\xi_1(\alpha)) - v(\alpha) = m = \int_a^b \rho(s)f(s)\,ds \tag{2.23}$$

so that, by (2.20) and (2.22), we deduce

$$\forall \alpha \in [a, \alpha_1] \qquad \xi_1(\alpha) \ge \beta_1 \ge \beta_2. \tag{2.24}$$

Similarly, (2.21) allows us to define a continuous function $\xi_2 : [\beta_2, b] \to \mathbb{R}$ such that we have

$$\forall \beta \ge \beta_2 \qquad \int_{\xi_2(\beta)}^{\beta} \phi(s)g(s)\,ds = \int_a^b \rho(s)g(s)\,ds, \tag{2.25}$$

from which, together with (2.19) and (2.22), we deduce

$$\forall \beta \ge \beta_2 \qquad \xi_2(\beta) \le \alpha_2 \le \alpha_1. \tag{2.26}$$

We deduce from (2.24) and (2.26) that the composed application

$$\xi_2 \circ \xi_1 : [a, \alpha_1] \xrightarrow{\xi_1} [\beta_2, b] \xrightarrow{\xi_2} [a, \alpha_1]$$

is defined and continuous from $[a, \alpha_1]$ into itself and, therefore, admits a fixed point $\bar{\alpha}$. Thus, if we set $\bar{\beta} = \xi_1(\bar{\alpha})$ we have $\bar{\alpha} = \xi_2(\bar{\beta})$. Equalities (2.23) and (2.25) with α, β replaced by $\bar{\alpha}, \bar{\beta}$ yield the conclusion.

(b) Let $\rho_n = \rho + \frac{1}{n}$, $\phi_n = \phi + \frac{2}{n}$, $f_n = f + \frac{1}{n}$ so that $0 < \rho_n < \phi_n$ and $f_n > 0$ a.e., and set $g_n = kf_n$ so that the monotonicity of k implies that $g_n > 0$ a.e. and f_n, g_n satisfy H. By (a) there exist α_n, β_n such that

$$\int_a^b \rho_n(s) f_n(s)\, ds = \int_{\alpha_n}^{\beta_n} \phi_n(s) f_n(s)\, ds, \tag{2.27}$$

$$\int_a^b \rho_n(s) g_n(s)\, ds = \int_{\alpha_n}^{\beta_n} \phi_n(s) g_n(s)\, ds. \tag{2.28}$$

By compactness we may assume $\alpha_n \to \alpha$, $\beta_n \to \beta$. The conclusion follows by passing through the limit in (2.27) and (2.28).

(ii) *Uniqueness.* Assume that $0 < \rho < \phi$, $f > 0$, $g > 0$ are continuous and that, for instance, k is decreasing. By (i(a)) the points α, such that there exists β satisfying (2.11) and (2.12), are the fixed points of the composed map $\xi_2 \circ \xi_1$. By definition the functions ξ_1, ξ_2 are differentiable and we have

$$\forall \alpha \in [a, \alpha_1] \qquad \xi_1'(\alpha) = \frac{v'(\alpha)}{v'(\xi_1(\alpha))} = \frac{\phi(\alpha) f(\alpha)}{\phi(\xi_1(\alpha)) f(\xi_1(\alpha))},$$

$$\forall \beta \in [\beta_2, b] \qquad \xi_2'(\beta) = \frac{\phi(\beta) g(\beta)}{\phi(\xi_2(\beta)) g(\xi_2(\beta))}.$$

To prove the claim we notice that if α satisfies $\xi_2 \circ \xi_1(\alpha) = \alpha$ then

$$(\xi_2 \circ \xi_1)'(\alpha) = \xi_2'(\xi_1(\alpha)) \xi_1'(\alpha) = \frac{k(\xi_1(\alpha))}{k(\alpha)}. \tag{2.29}$$

By (2.23) we have $\xi_1(\alpha) > \alpha$ so that the strict monotonicity of k implies $k(\xi_1(\alpha)) < k(\alpha)$ and thus $(\xi_2 \circ \xi_1)'(\alpha) < 1$ whenever $\xi_2 \circ \xi_1(\alpha) = \alpha$. Let $S = \{\alpha \in [a, b] : \xi_2 \circ \xi_1(\alpha) = \alpha\}$. Clearly, S is compact and nonempty by (i); moreover, taking (2.29) into account, for each $\alpha \in S$ there exists η such that

$$\begin{aligned} &\forall t \in]\alpha - \eta, \alpha[\qquad \xi_2 \circ \xi_1(t) > t, \\ &\forall t \in]\alpha, \alpha + \eta[\qquad \xi_2 \circ \xi_1(t) < t. \end{aligned} \tag{2.30}$$

As a consequence, the set S has no accumulation points and is therefore finite.

Let $\alpha_1 = \min S$ and assume $S \neq \{\alpha_1\}$; let $\alpha_2 = \min S \setminus \{\alpha_1\}$. Then by (2.30) there exist $t_1 < t_2 \in [\alpha_1, \alpha_2]$ such that $\xi_2 \circ \xi_1(t_1) < t_1$ and $\xi_2 \circ \xi_1(t_2) > t_2$. Therefore there exists $\bar{t} \in [t_1, t_2]$ such that $\xi_2 \circ \xi_1(\bar{t}) = \bar{t}$, a contradiction. □

3. Main result.

THEOREM 3.1. *Let $x \in W^{2,1}([0,T])$ be a solution to* (P). *Then there exists a bang–bang solution y to* (P) *satisfying*

$$\forall t \in [0, T] \qquad y(t) \le x(t).$$

Moreover, there exists a set E which is a finite union of intervals such that

$$y'' + a_1(t) y' + a_0(t) y = \phi_1(t) \chi_E(t) + \phi_2(t) \chi_{[0,T] \setminus E}(t) \text{ a.e.}$$

COROLLARY 1. *Under the above assumption, there exists a bang–bang solution y satisfying*

$$\forall t \in [0,T] \qquad y(t) \geq x(t).$$

Proof of Corollary 1. Let $-\Phi$ be defined by the equality $(-\Phi)(t) = -\Phi(t)$. Clearly, $\tilde{x} = -x$ solves

$$\tilde{x}'' + a_1(t)\tilde{x}' + a_0(t)\tilde{x} \in -\Phi(t) \text{ a.e.}$$

By Theorem 3.1 there exists a bang–bang solution $\tilde{y}$ satisfying the same boundary conditions as $\tilde{x}$ and satisfying

$$\forall t \in [0,T] \qquad \tilde{y}(t) \leq \tilde{x}(t).$$

Then the function y defined by

$$\forall t \in [0,T] \qquad y(t) = -\tilde{y}(t)$$

is a solution to our problem. $\square$

Proof of Theorem 3.1. Let h be the function defined in Proposition 2.1.

(i) We show that it is not restrictive to assume

$$\Phi(t) = [0, \phi(t)] \quad (\phi \in L^1([0,T]),\ \phi > 0 \text{ a.e.}) \quad \text{and} \quad x_0 = v_0 = 0.$$

In fact, let $\Phi(t) = [\phi_1(t), \phi_2(t)]$ and x satisfy

$$x'' + a_1(t)x' + a_0(t)x \in \Phi(t) \text{ a.e.}$$

Then the function $\tilde{x}$ defined by

$$\tilde{x}(t) = x(t) - x'(0)t - x(0)$$

satisfies $\tilde{x}(0) = \tilde{x}'(0) = 0$ and

$$\tilde{x}'' + a_1(t)\tilde{x}' + a_0(t)\tilde{x} \in [\psi_1(t), \psi_2(t)] \text{ a.e.,}$$

where

$$\psi_1(t) = \phi_1(t) - a_0(t)x'(0)t - a_1(t)x'(0) - a_0(t)x(0),$$
$$\psi_2(t) = \phi_2(t) - a_0(t)x'(0)t - a_1(t)x'(0) - a_0(t)x(0).$$

Moreover, by Proposition 2.1, the function $\bar{x}$ defined by

$$\bar{x}(t) = \tilde{x}(t) - \int_0^t h(t,s)\psi_1(s)\,ds$$

satisfies $\bar{x}(0) = 0$, $\bar{x}'(0) = 0$ and

$$\bar{x}'' + a_1(t)\bar{x}' + a_0(t)\bar{x} \in [0, \psi_2(t) - \psi_1(t)] \text{ a.e.}$$

If we assume that Theorem 3.1 holds for such an interval and initial boundary conditions, there exists a function $\bar{y}$ satisfying

$$\bar{y}(0) = \bar{x}(0), \quad \bar{y}'(0) = \bar{x}'(0), \quad \bar{y}(T) = \bar{x}(T), \quad \bar{y}'(T) = \bar{x}'(T),$$
$$\bar{y}'' + a_1(t)\bar{y}' + a_0(t)\bar{y} \in \{0, \psi_2(t) - \psi_1(t)\} \text{ a.e.,}$$
$$\forall t \in [0,T] \qquad \bar{y}(t) \leq \bar{x}(t).$$

It is now easy to check that the function y defined by

$$y(t) = \bar{y}(t) + \int_0^t h(t,s)\psi_1(s)\,ds + x'(0)t + x(0)$$

is a solution to our problem.

(ii) Assume first that the δ of Property (S) can be chosen in such a way that $I_\delta = [0,T]$. In this case, if we set

$$\rho = x'' + a_1 x' + a_0 x$$

then by Proposition 2.1 we can write

$$x(t) = \int_0^t h(t,s)\rho(s)\,ds, \tag{3.1}$$

where h satisfies Property (S(1)) and, in addition,

$$\forall s,t \in [0,T] \qquad h(t,s) > 0 \text{ if } s < t, \quad h(t,s) < 0 \text{ if } t < s, \tag{3.2}$$

$$\forall s,t \in [0,T] \qquad \frac{\partial h}{\partial t}(t,s) > 0, \tag{3.3}$$

$$\forall t \in [0,T] \qquad s \mapsto h(t,s)/\frac{\partial h}{\partial t}(t,s) \text{ is decreasing on } [0,t]. \tag{3.4}$$

In particular, the functions f and g defined on $[0,T]$ by

$$g(s) = h(T,s), \qquad f(s) = \frac{\partial h}{\partial t}(T,s)$$

verify Hypothesis H with $k(.) = h(T,.)/\frac{\partial h}{\partial t}(T,.)$.

By Proposition 2.1, each bang–bang solution y such that $x(0) = x'(0) = 0$ is given by the formula $y(t) = \int_0^t h(t,s)\nu(s)\,ds$ for some measurable function ν with values in $\{0, \phi(t)\}$.

We are thus led to show that there exists such a ν satisfying

$$\int_0^T h(T,s)\rho(s)\,ds = \int_0^T h(T,s)\nu(s)\,ds, \tag{3.5}$$

$$\int_0^T \frac{\partial h}{\partial t}(T,s)\rho(s)\,ds = \int_0^T \frac{\partial h}{\partial t}(T,s)\nu(s)\,ds, \tag{3.6}$$

and for each t in $[0,T]$,

$$\int_0^t h(t,s)\rho(s)\,ds \geq \int_0^t h(t,s)\nu(s)\,ds. \tag{3.7}$$

(a) Assume $0 < \rho < \phi$ a.e.

By Proposition 2.2 there exist $\alpha, \beta \in [0,T]$ such that

$$\int_0^T h(T,s)\rho(s)\,ds = \int_\alpha^\beta h(T,s)\phi(s)\,ds, \tag{3.8}$$

$$\int_0^T \frac{\partial h}{\partial t}(T,s)\rho(s)\,ds = \int_\alpha^\beta \frac{\partial h}{\partial t}(T,s)\phi(s)\,ds. \tag{3.9}$$

It is clear that if we set

$$\nu(s) = \phi(s)\chi_{[\alpha,\beta]}(s) \tag{3.10}$$

then (3.5) and (3.6) are satisfied. In order to prove (3.7) we first show that under our assumptions on ρ and ϕ we have

$$0 < \alpha < \beta < T. \tag{3.11}$$

Notice first that the equalities $(\alpha, \beta) = (0, T)$ or $\alpha = \beta$ cannot hold otherwise by (3.8), $\rho = \phi$ or $\rho = 0$ a.e., a contradiction. Assume, for instance, $\alpha = 0$ and $\beta < T$, the case $\alpha > 0$ and $\beta = T$ being similar. Under this assumption, equalities (3.8) and (3.9) become

$$\int_0^T h(T,s)\rho(s)\,ds = \int_0^\beta h(T,s)\phi(s)\,ds, \tag{3.12}$$

$$\int_0^T \frac{\partial h}{\partial t}(T,s)\rho(s)\,ds = \int_0^\beta \frac{\partial h}{\partial t}(T,s)\phi(s)\,ds. \tag{3.13}$$

Property (3.4) and the assumption $0 < \rho < \phi$ a.e. allow us to apply Lemma 2.1, from which we deduce

$$\int_0^T h(T,s)\rho(s)\,ds < \int_0^\beta h(T,s)\phi(s)\,ds,$$

contradicting (3.12).

Set $y(t) = \int_0^t h(t,s)\nu(s)\,ds$ so that (3.8) and (3.9) become $y(T) = x(T)$ and $y'(T) = x'(T)$.

The purpose of what follows is to show (3.7), i.e., that $y(t) \le x(t)$ for each t. We consider the cases $t \in [0, \alpha]$, $t \in [\beta, T]$, $t \in [\alpha, \beta]$ separately.

Inequality (3.7) is trivial if $t \le \alpha$; in fact we have

$$y(t) = 0 \le \int_0^t h(t,s)\rho(s)\,ds = x(t),$$

the inequality being strict for $t \in]0, \alpha]$. In particular

$$y(\alpha) < x(\alpha). \tag{3.14}$$

Assume $t \in [\beta, T]$.

Since, taking (3.2) into account, $h(t,s) \le 0$ whenever $s \ge t$, we have

$$\forall t \ge \beta \qquad \int_t^T h(t,s)\rho(s)\,ds \le 0 = \int_t^T h(t,s)\nu(s)\,ds \tag{3.15}$$

or, equivalently,

(3.16)
$$\forall t \ge \beta \int_0^T h(t,s)\rho(s)\,ds - \int_0^t h(t,s)\rho(s)\,ds \le \int_0^T h(t,s)\nu(s)\,ds - \int_0^t h(t,s)\nu(s)\,ds.$$

Therefore, in order to prove that $y(t) \le x(t)$ for $t \in [\beta, T]$ it is enough to show that

$$\forall t \in [\beta, T] \qquad \int_0^T h(t,s)\rho(s)\,ds = \int_0^T h(t,s)\nu(s)\,ds. \tag{3.17}$$

For this purpose, we use Property (S(1)). Equalities (3.8) and (3.9) become

$$\begin{cases} w_1(T)\displaystyle\int_0^T z_1(s)(\rho(s)-\nu(s))\,ds + w_2(T)\int_0^T z_2(s)(\rho(s)-\nu(s))\,ds = 0,\\ w_1'(T)\displaystyle\int_0^T z_1(s)(\rho(s)-\nu(s))\,ds + w_2'(T)\int_0^T z_2(s)(\rho(s)-\nu(s))\,ds = 0.\end{cases}$$

The condition on the Wronskian of w_1, w_2 at T implies

$$\int_0^T z_1(s)(\rho(s)-\nu(s))\,ds = 0, \tag{3.18}$$

$$\int_0^T z_2(s)(\rho(s)-\nu(s))\,ds = 0. \tag{3.19}$$

Multiplying (3.18) by $w_1(t)$, (3.19) by $w_2(t)$, and adding the two equations we obtain

$$\int_0^T (w_1(t)z_1(s)+w_2(t)z_2(s))\rho(s)\,ds = \int_0^T (w_1(t)z_1(s)+w_2(t)z_2(s))\nu(s)\,ds,$$

which, together with Property (S(1)), gives (3.17). Moreover, note that since inequality (3.15) is strict for $t \neq T$,

$$y(\beta) < x(\beta). \tag{3.20}$$

At this stage, we only need to prove that (3.7) holds for $t \in [\alpha, \beta]$.

Assume by contradiction that there exists $t \in [\alpha,\beta]$ such that $x(t) = y(t)$. Let

$$\bar t = \sup\{t \in [\alpha,\beta] : x(t) = y(t)\}.$$

Then $\alpha < \bar t < \beta$ and, by the very definition of $\bar t$, $x(\bar t) = y(\bar t)$ so that

$$y'(\bar t) - x'(\bar t) = \lim_{t \to \bar t^+} \frac{y(t)-x(t)}{t-\bar t} \le 0.$$

It follows that

$$\int_\alpha^{\bar t} h(\bar t, s)\phi(s)\,ds = \int_0^{\bar t} h(\bar t, s)\rho(s)\,ds, \tag{3.21}$$

$$\int_\alpha^{\bar t} \frac{\partial h}{\partial t}(\bar t, s)\phi(s)\,ds \le \int_0^{\bar t} \frac{\partial h}{\partial t}(\bar t, s)\rho(s)\,ds. \tag{3.22}$$

For each $s \in [0, \bar t[$ let $f(s) = h(\bar t, s)$, $g(s) = \frac{\partial h}{\partial t}(\bar t, s)$, and $k = g/f$ so that by (3.2)–(3.4) the function k is increasing and $f > 0$, $g > 0$. If we replace (a, b) by $(0, \bar t)$, Lemma 2.1 together with (3.21) implies that

$$\int_\alpha^{\bar t} \frac{\partial h}{\partial t}(\bar t, s)\phi(s)\,ds > \int_0^{\bar t} \frac{\partial h}{\partial t}(\bar t, s)\rho(s)\,ds,$$

thus contradicting (3.22).

(b) Assume, in general, $0 \le \rho \le \phi$ a.e. and let ϕ_n, $\rho_n \in L^1([0,T])$ be such that

$$0 < \rho_n < \phi_n \text{ a.e.} \quad \text{and} \quad \rho_n \to \rho,\ \phi_n \to \phi \text{ in } L^1([0,T])$$

(for instance, $\rho_n = \rho + \frac{1}{n}$, $\phi_n = \phi + \frac{2}{n}$).

Corresponding to each n, there exist α_n, $\beta_n \in [0,T]$ such that, if we set $\nu_n = \phi_n \chi_{[\alpha_n,\beta_n]}$, we have

$$\int_0^T h(T,s)\rho_n(s)\,ds = \int_0^T h(T,s)\nu_n(s)\,ds, \tag{3.23}$$

$$\int_0^T \frac{\partial h}{\partial t}(T,s)\rho_n(s)\,ds = \int_0^T \frac{\partial h}{\partial t}(T,s)\nu_n(s)\,ds, \tag{3.24}$$

and, for each t in $[0,T]$,

$$\int_0^t h(t,s)\rho_n(s)\,ds \geq \int_0^t h(t,s)\nu_n(s)\,ds. \tag{3.25}$$

Because the interval $[0,T]$ is compact, we may assume $\alpha_n \to \alpha$, $\beta_n \to \beta$ for some $\alpha \leq \beta \in [0,T]$.

Clearly $\nu_n = \phi_n \chi_{[\alpha_n,\beta_n]}$ converges to $\phi\chi_{[\alpha,\beta]}$ in $L^1([0,T])$; therefore, if we pass through the limit in (3.23), (3.24), and (3.25) and we set $\nu = \phi\chi_{[\alpha,\beta]}$, we obtain (3.5), (3.6), and (3.7).

(iii) In the general case, using Property (S) and the compactness of $[a,b]$, there exists a subdivision $a_0 = 0 < a_1 < \cdots < a_l < T = a_{l+1}$ of $[0,T]$ such that, if we put $I_j = [a_j, a_{j+1}]$, we have

- $\forall s,t \in I_j \quad h(t,s) > 0$ if $s < t$, $\quad h(t,s) < 0$ if $t < s$;
- $\forall s,t \in I_j \quad \frac{\partial h}{\partial t}(t,s) > 0$;
- $\forall t \in I_j \quad s \mapsto h(t,s)/\frac{\partial h}{\partial t}(t,s)$ is decreasing on I_j.

By (ii), on each interval I_j there exist α_j, β_j such that the solution y_j to the problem

$$y'' + a_1(t)y' + a_0(t)y = \phi_1(t)\chi_{[a_j,\alpha_j]\cup[\beta_j,b_j]}(t) + \phi_2(t)\chi_{[\alpha_j,\beta_j]}(t) \text{ a.e. on } I_j$$

with the initial conditions

$$y_j(a_j) = x(a_j), \quad y_j'(a_j) = x'(a_j)$$

satisfies the equalities

$$y_j(a_{j+1}) = x(a_{j+1}), \quad y_j'(a_{j+1}) = x'(a_{j+1}),$$

and, moreover, $y_j(t) \leq x(t)$ for each $t \in I_j$.

Clearly the function $y \in W^{2,1}([0,T])$ obtained by glueing together the functions y_j is a solution to our problem. □

Remark 3.1. The proof of Theorem 3.1, part (ii(a)) shows in fact that when $0 < \rho < \phi$, we have $y(t) < x(t)$ on $]0,T[$.

Remark 3.2. With the notations introduced in Proposition 2.1, the proof of Theorem 3.1, part (ii) shows that if $T = \delta$ then, given a solution x to (P), there exists a bang–bang solution $y \leq x$ satisfying

$$\begin{aligned} y'' + a_1(t)y' + a_0(t)y &= \min \Phi(t) \text{ on } [0,\alpha] \cup [\beta,T], \\ y'' + a_1(t)y' + a_0(t)y &= \max \Phi(t) \text{ on } [\alpha,\beta]. \end{aligned}$$

Because the number δ depends only on the function h, it can happen that $\delta = +\infty$.

This is the case when a_1 and a_0 are constant and the equation $\lambda^2 + a_1\lambda + a_0 = 0$ admits two real roots λ_1, λ_2. In fact, under this assumption we have either

$$h(t,s) = \frac{1}{\lambda_2 - \lambda_1}(e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}) \text{ if } \lambda_1 \neq \lambda_2, \text{ or}$$
$$h(t,s) = (t-s)e^{\lambda(t-s)} \text{ if } \lambda_1 = \lambda_2 = \lambda.$$

4. Applications. Our first application concerns the reachable set of bang–bang constrained solutions. Let c be an arbitrary function defined on $[0,T]$ and consider the reachable sets $\mathcal{X}_T^c$ and $\mathcal{Y}_T^c$ associated with (P) defined by

$$\mathcal{X}_T^c = \{(y(T), y'(T)) : y \leq c,\ y'' + a_1(t)y' + a_0(t)y \in \Phi(t),\ (y(0), y'(0)) = (x_0, v_0)\},$$

$$\mathcal{Y}_T^c = \{(y(T), y'(T)) : y \leq c,\ y'' + a_1(t)y' + a_0(t)y \in \operatorname{extr}\Phi(t),\ (y(0), y'(0)) = (x_0, v_0)\}.$$

Then Theorem 3.1 claims $\mathcal{X}_T^c = \mathcal{Y}_T^c$, whence $\mathcal{Y}_T^c$ is convex.

Finally, we give an application to the calculus of variations.

THEOREM 4.1. *Let a_0, $a_1 \in C([0,T])$, $\phi_1, \phi_2 \in L^1([0,T])$ verify $\phi_1(t) \leq \phi_2(t)$. Let x_0, v_0, x_1, v_1 be 4 fixed real numbers. Then there exists a dense subset $\mathcal{D}$ of $C(\mathbb{R})$ for the uniform convergence such that for g in $\mathcal{D}$ the problem*

$$\textit{minimize} \quad \left\{ \int_0^T g(x(t))\,dt + \int_0^T h(\rho(t))\,dt \right\}$$

on the subset of $W^{2,1}([0,T]) \times L^1([0,T])$ of those functions (x, ρ) satisfying

$$(x(0), x'(0), x(T), x'(T)) = (x_0, v_0, x_1, v_1),\ x'' + a_1(t)x' + a_0(t)x = \rho(t) \in [\phi_1(t), \phi_2(t)]$$

admits at least one solution for every lower semicontinuous function h satisfying the growth condition $h(u) \geq c\psi(|u|)$, ψ being lower semicontinuous and convex, $\lim_{r\to+\infty} \psi(r)/r = +\infty$.

Proof. With Theorem 3.1 and the preceding application, the proof is a direct adaptation of the proof given in [3]. □

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