CONTINUATION OF HOLOMORPHIC SOLUTIONS OF MICROHYPERBOLIC DIFFERENTIAL EQUATIONS

Andrea D'Agnolo   Giuseppe Zampieri

Abstract. Let M be a real analytic manifold, X a complexification of M, Ω an open subset of M with $N^*_{x_0}(\Omega) \neq T^*_{x_0}M$, $x_0 \in \partial\Omega$. Let γ (resp γ') be an open set of $\overline{\Omega} \times_M T_M X$ with convex conic fibers and with $\Omega \times_M \gamma \supset \Omega \times_M T_M X$ (resp $\gamma' = \overline{\Omega} \times_M T_M X$); denote by $\mathcal{U}$ (resp $\mathcal{W}$) the Ω-tuboids in X with profile γ (resp γ') (cf [Z]) and by $\mathcal{S}$ the neighborhoods of x_0.

Let $P = P(x, D)$ be a differential operator at x_0 with C^ω-coefficients which is microhyperbolic to each $-\theta \in N^*_{x_0}(\Omega)^a$ in $\gamma^{*a}_{x_0}$ relative to $\overline{\Omega}$ (in the sense of (2.3)). We prove that for every U, W, S there exist W', S' such that

$$f \in \mathcal{O}_X(U \cap S), Pf \in \mathcal{O}_X(W \cap S) \quad \text{implies} \quad f \in \mathcal{O}_X(W' \cap S').$$

A similar result is obtained for $\overline{\Omega}$-microhyperbolic operators in the sense of [S-Z] and for semihyperbolic operators in the sense of [Kan], [Kat].

We aim to refine the above conclusions and show (cf [D'A-Z]) that in the preceding hypotheses P is an isomorphism of the sheaf $(\mathcal{C}_{\Omega|X})_{T^*_M X}$ (cf [S]) at any $p \in \gamma^{*a}_{x_0}$.

1. Preliminaries. Let X be a complex manifold, P a differential operator with holomorphic coefficients, and let $\sigma(P)$ be the principal symbol of P. First we introduce a lemma which will be our main tool in proving propagation theorems.

Lemma 1.1. *Let* $\{V_\alpha\}_\alpha$ $(0 \leq \alpha \leq 1)$ *and* V *be open sets in* X *such that:*

(i) $V_0 \subset V$, $V_\alpha \subset V_\beta$, *for* $\beta > \alpha$,

(ii) $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$, $\overline{V}_\alpha = \bigcap_{\beta'>\alpha} V_{\beta'}$,

(iii) $\partial V_\alpha \cap \overline{V_1 \setminus V} \subset\subset V_1$,

(iv) $N^*_x(V_\alpha) \neq T^*_x X$ *for every* $x \in \partial V_\alpha \cap \overline{V_1 \setminus V}$,

(v) $\sigma(P)(z, \zeta) \neq 0$ *for every* $z \in \partial V_\alpha \cap \overline{V_1 \setminus V}$ *and for every* ζ *conormal to* V_α *at* z *(cf. par. 1).*

Then:

$$f \in \mathcal{O}_X(V), Pf \in \mathcal{O}_X(V \cup V_1) \quad \textit{implies} \quad f \in \mathcal{O}_X(V \cup V_1). \tag{1.2}$$

Proof. For f as in the left hand side of (1.2) set $\mathcal{V} = \{V \cup V_\alpha; f \in \mathcal{O}_X(V \cup V_\alpha)\}$, endowed with the natural order relation; this is an inductive family. Let $V \cup V_{\alpha_0}$ be a maximal element for $\mathcal{V}$ and suppose by absurd that $\alpha_0 < 1$.
Note that $f \in \mathcal{O}_X(V_{\alpha_0})$ and, by (iii), $Pf \in (\mathcal{O}_X)_z$ $\forall z \in \partial V_{\alpha_0} \cap \overline{V_1 \setminus V}$. Using (iv) and the refined version of the theorem of Cauchy-Kowalevsky-Leray given in [B-S

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1], we conclude that f extends holomorphically to a neighborhood of $\partial V_{\alpha_0} \cap \overline{V_1 \setminus V}$. By (iii) $(V_1 \setminus V) \cap \overline{V}_\alpha \subset\subset V_1 \setminus V$, hence from (ii) we get that, in $V_1 \setminus V$, $\{V_\beta\}_\beta$ $(\beta > \alpha_0)$ is a fundamental system of neighborhood of $\overline{V}_\alpha$; it follows that each open set containing $(\partial V_{\alpha_0} \cap \overline{V_1 \setminus V}) \cup (V \cup V_{\alpha_0})$ contains also $V \cup V_\beta$ for some $\beta > \alpha_0$. Hence $f \in \mathcal{O}_X(V \cup V_\beta)$ which is a contradiction. □

Remark 1.3. This result is a variant of a wider principle by Kashiwara concerning the "propagation of cohomology of a complex" (cf. [K-S 1,theorem 1.4.3]).

2. Statement of the results. Let M be a C^ω-manifold, X a complexification of M. We denote by T^*M, T^*X the cotangent bundles to M, X, and $T^*_M X$ the conormal bundle to M in X; in particular we denote by $T^*_X X$ the zero section of T^*X. We set $\dot{T}^*X = T^*X \setminus T^*_X X$.

For subsets $S, V \subset X$ one denotes by $C(S, V)$ the normal cone to S along V (cf [K-S 1]) and by $N(S)$ the normal cone to S in X; these are objects of TX . The same notation will be used to denote the normal cone to a subset S of the manifold M, which is, of course, an object of TM.

Let $\Omega \subset M$ be an open set verifying for a fixed $x_0 \in \partial\Omega$

$$N^*_{x_0}(\Omega) \neq T^*_{x_0}M. \tag{2.1}$$

Let γ be an open set of $\overline{\Omega} \times_M T_M X$ with convex conic fiber. A domain $U \subset X$ is said to be an Ω-*tuboid* with profile γ iff $C(X \setminus U, \overline{\Omega}) \cap \gamma_1 = \emptyset$ for some open set $\gamma_1 \subset TX$ with convex conic fiber such that $\gamma_1 \supset \sigma(N(\Omega))$, $\rho(\gamma_1) \supset \gamma$ (cf [Z]).
Here

$$T_M X \xleftarrow{\rho} M \times_X TX \xleftarrow{\sigma} TM$$

are the canonical maps.

Remark 2.2. Let $X \cong \mathbf{R}^n + \sqrt{-1}\,\mathbf{R}^n \ni x + \sqrt{-1}\,y$, $M \cong \mathbf{R}^n \ni x$. We recall that U is an Ω-tuboid with profile γ iff $\forall \gamma' \subset\subset \gamma$, $\exists \varepsilon = \varepsilon_{\gamma'}$ such that $U \supset \{(x, y) \in \Omega \times_M \gamma' : |y| < \varepsilon(dist(x, \partial\Omega) \wedge 1)\}$.

Let $q \in \partial\Omega \times_M \dot{T}^*_M X$, set $x_0 = \pi(q)$ (where π is the projection $T^*X \longrightarrow X$) and let P be a differential operator with holomorphic coefficients in a neighborhood of x_0.

Choose a system of coordinates $(x; \sqrt{-1}\,\eta) \in T^*_M X$ and $(z, \zeta) \in T^*X$ ($z = x + \sqrt{-1}\,y$, $\zeta = \xi + \sqrt{-1}\,\eta$), and assume that

$$\begin{aligned} &\sigma(P)(z,\zeta) \neq 0 \quad \text{for} \\ &\quad -c_1|\eta| < \langle \xi, \theta \rangle < -c_2[|y||\eta| + |\xi - \langle \xi, \theta \rangle \theta|] \\ &\quad \forall (x, \sqrt{-1}\,\eta) \in (\overline{\Omega} \cap S) \times \sqrt{-1}\,\Lambda, \quad \forall \theta \in \dot{N}^*_{x_0}(\Omega), \end{aligned} \tag{2.3}$$

where Λ is a closed cone of $\dot{\mathbf{R}}^n$ and c_1, c_2 are constants independent of x, η, θ.

Remark 2.4. Since condition (2.3) is not C^1-coordinate-invariant, no propagation theorem involving the notion of micro-support of a sheaf (as in [K-S 1]) could be applied.

Remark 2.5. It is obvious that if (2.3) is satisfied by θ then it is even satisfied by any θ' in a neighborhood of θ. It follows that we can replace $\dot{N}^*_{x_0}(\Omega)$ of (2.3) by $(\dot{N}^*_{x_0}(\Omega))_\varepsilon$ for a suitable ε. Here, for a cone $A \subset \dot{\mathbf{R}}^n$, we denote by A_ε the conic ε-neighborhood of A:

$$A_\varepsilon = \{\theta \in \dot{\mathbf{R}}^n : \sup_{\eta \in A} \langle \frac{\theta}{|\theta|}, \frac{\eta}{|\eta|} \rangle > 1 - \varepsilon\}.$$

We shall now introduce a slight modification of condition (2.3) which is coordinate invariant.

Assume that

$$\theta \notin C_{q'}(char(P), \overline{\Omega} \times_M T^*_M X) \quad \forall q' \in \lambda, \forall \theta \in \dot{N}^*_{x_0}(\Omega)^a, \tag{2.6}$$

where $char(P)$ is the characteristic variety of P, λ is a closed neighborhood of q with conic fiber and where the exponent a denotes the antipodal map. Finally note that we have used the identification

$$T^*_{x_0}M \underset{j}{\hookrightarrow} T^*_{x_0}X \underset{\pi^*}{\hookrightarrow} T^*_q T^*X \underset{-H}{\overset{\sim}{\to}} T_q T^*X,$$

where j is due to the complex structure of X, π^* is the map associated to the projection $\pi : T^*X \to X$, and H denotes the Hamiltonian isomorphism.

As in [S-Z], we shall refer to (2.6) as the condition of $\overline{\Omega}$-*micro-hyperbolicity* in λ with respect to each $\theta \in \dot{N}^*_{x_0}(\Omega)^a$; this is a weaker condition then microhyperbolicity.

Remark 2.7. Note that one proves that if $\Lambda \subset\subset (int\,\lambda)_{x_0}$ then (2.6) implies (2.3). (Here, for A, B cones in $\dot{\mathbf{R}}^n$, one says that A is a proper subcone of B, and writes $A \subset\subset B$, whenever $A \cap \{y : |y| = 1\} \subset\subset int\, B$.)

Theorem 2.8. *Let Ω verify (2.1), take $q \in \partial\Omega \times_M \dot{T}^*_M X$, and let P be a differential operator at $x_0 = \pi(q)$ which verifies (2.3) in some system of coordinates (resp (2.6)). Denote by $\mathcal{U}$ the family of tuboids whose profile γ verifies:*

$$\Omega \times_M \gamma \supset \Omega \times_M T_M X, \quad \gamma^{*a}_{x_0} \subset \Lambda \tag{2.9}$$

(resp

$$\gamma^{*a}_{x_0} \subset (int\,\lambda)_{x_0}), \tag{2.9'}$$

and by $\mathcal{W}$ those with profile γ' verifying

$$\gamma' \supset \overline{\Omega} \times_M T_M X \tag{2.10}$$

(where the exponent $*$ denotes the polar). Let $\mathcal{S}$ be the family of neighborhoods of x_0. Then:

$$\begin{aligned} f &\in \varinjlim_{U \in \mathcal{U}, S \in \mathcal{S}} \Gamma(U \cap S, \mathcal{O}_X), \\ Pf &\in \varinjlim_{W \in \mathcal{W}, S \in \mathcal{S}} \Gamma(W \cap S, \mathcal{O}_X) \qquad \textit{implies} \\ f &\in \varinjlim_{W \in \mathcal{W}, S \in \mathcal{S}} \Gamma(W \cap S, \mathcal{O}_X). \end{aligned}$$

Remark 2.11. Let Γ be an open convex cone of $\dot{\mathbf{R}}^n$ with $\Gamma^{*a} \subset \Lambda$, fix $\eta \in \dot{\mathbf{R}}^n$ and let:

$$\gamma = (\overline{\Omega} \times \Gamma) \cup (\Omega \times c.h.(\Gamma, \{-\eta\}),$$
$$\gamma' = \overline{\Omega} \times c.h.(\gamma, \{-\eta\}).$$

Then the same conclusion of Theorem 2.8 holds. (here $c.h.$ denotes the convex hull.)
In fact in subsequent Theorem 2.15 the assumption $\eta \in int\Gamma^* \cap \Gamma$ is unessential. (It is only used in the conclusion to get $c.h.(\Gamma, \{-\eta\}) = \dot{\mathbf{R}}^n$.)

Theorem 2.12. *Let $\Omega = \{x = (x_1, x') : x_1 > 0\}$ and assume that $\sigma(P)(z, \zeta) \neq 0$ when (z, ζ) satisfies the conditions in (2.3) with $\Lambda = \mathbf{R} \times \Lambda'$ $(\Lambda' \subset \dot{\mathbf{R}}^{n-1})$, and when in addition $y_1 = 0$. Then the conclusion of Theorem 2.8 still holds.*

Note by the way that the condition for P expressed in this statement is a refinement of the hypothesis of semi-hyperbolicity in the sense of [Kan].

For example in $T^*X \ni (z, \zeta)$, $z = (z_1, z')$ consider $\sigma(P)(z, \zeta) = \zeta_1^2 - z_1\zeta_2^2 - Q(z, \zeta')$, Q homogeneous of degree 2 and $Q|_{T^*_M X} \leq 0$. This is semihyperbolic but neither $\overline{\Omega}$-hyperbolic nor it satisfies (2.3).

The proof of Theorems 2.8, 2.12 will be given in the next section; it will follow from a statement which fully describes the shape of the sets U and V.

Let $\Omega \subset M$ be an open set verifying (2.1). Then we can write Ω on S, neighborhood of x_0, as $\Omega = \{x : x_1 > \varphi(x')\}$ for a Lipschitz-continuous function φ. We set

$$\rho(x) = x_1 - \varphi(x')$$

and remark that for suitable constants $k', k'' > 0$ we have:

$$k'\, dist(x, \partial\Omega) < \rho(x) < k''\, dist(x, \partial\Omega), \quad x \in \Omega; \tag{2.13}$$

hence we will use the function ρ as a substitute of the distance to $\partial\Omega$ in our arguments. Moreover, we can find $l', l'' > 0$ so that on S:

$$|\rho(\widetilde{x}) - \rho(x)| \leq l''|\widetilde{x} - x|, \tag{2.14}$$

$$\inf_{\{v \in (N^*_{x_0}(\Omega))_\varepsilon : |v| = 1\}} |\rho(x + av) - \rho(x)| \geq l'a, \quad 0 < a << 1. \tag{2.14'}$$

(As for (2.14)' we have to notice that we can choose coordinates at x_0 so that $\dot{N}^*_{x_0}(\Omega) \subset\subset N_{x_0}(\Omega)$; here we identify $T_{x_0}M \cong T^*_{x_0}M \cong M \cong \mathbf{R}^n$.)

Let Λ, Γ be open *convex* cones of $\dot{\mathbf{R}}^n$ with $\Lambda \supset\supset \Gamma^{*a}$ and take $\eta \in int\,\Gamma^* \cap \Gamma$.

Theorem 2.15. *Let P verify (2.3). Let*

(2.16)
$$U = \left[(\Omega + \sqrt{-1}\,\Gamma) \cup \left\{ z : t' < \rho(x) < t, y \in r\frac{\rho(x) - t'}{t - t'}\eta + \Gamma \right\} \right] \cap$$
$$\cap \{z : |y| < \frac{\delta}{t}\rho(x)\} \cap S,$$

where $\delta \geq r$ *and* S *is a suitable neighborhood of* x_0. *Then for every convex cone* $\Gamma' \subset\subset \Gamma$ $(\Gamma' \ni \eta, \quad \Gamma'^{*a} \subset \Lambda)$, *there exists* $k = k_{\Gamma'} < 1$ *such that if* t *verifies*

$$t < k c_2^{-1} l' \tag{2.17}$$

and if c *verifies*

$$\frac{crl''}{t - t'} < c_1, \quad crk^{-1} < \delta, \quad c < 1 \tag{2.18}$$

(l', l'' *being the constants of (2.14)), it follows that setting*

$$V = \{z : 0 < \rho(x) < t, y \in -cr\rho(x)\,\eta + \Gamma'\} \cap \{z : |y| < \frac{\delta}{t}\rho(x)\} \cap S, \tag{2.19}$$

then for a suitable $S' \subset S$, *depending on* t, l', l'' *and the* ε *of Remark 2.5, the following holds:*

$$f \in \mathcal{O}_X(U), Pf \in \mathcal{O}_X(V) \qquad \textit{implies} \qquad f \in \mathcal{O}_X(V \cap S').$$

Remark 2.20. Since $\eta \in \Gamma'$ then for a suitable $c' = c'_{\Gamma',\eta} : V \supset \{z : \rho(x) < t, |y| < c'cr\rho(x)\} \cap S'$.

To handle also the case when Pf does not extend to a convex set we introduce the following

Theorem 2.21. *Let* P *verify (2.3) and let* c, t *verify (2.17), (2.18), let* U *be defined by (2.16). For every* $g_1(x) > 0$ *with* $\inf_{\{x;\rho(x)=t\}} g_1(x) = r$, *there exists* $h(s)$, $s \in \mathbf{R}$ *with* $h(0) = 0$, $h(t) = c\,r$, h' *increasing and* $0 < h' \leq cr/(t - t')$ *for* $s > 0$, *such that if we set* $g_2(x) = h(\rho(x))$ *and*

$$V_1 = \{z : y \in -g_1(x)\,\eta + \Gamma,\ |y| < \frac{\delta}{t}\rho(x),\ 0 < \rho(x) < t\},$$

(resp

$$V_2 = \{z : y \in -g_2(x)\,\eta + \Gamma',\ |y| < \frac{\delta}{t}\rho(x),\ 0 < \rho(x) < t\}),$$

we get

$$f \in \mathcal{O}_X(U),\ Pf \in \mathcal{O}_X(V_1 \cap S) \qquad \textit{implies} \qquad f \in \mathcal{O}_X(V_2 \cap S').$$

Remark 2.22. Let $g_1(x) = h_1(\rho(x))$ for a C^1-function h_1 with h_1' increasing. Then one can show that the function h of Theorem 2.21 verifies $h_1' \wedge cr/t \leq h' \leq cr/(t - t')$. In particular for $g_1(x) = r\,\rho(x)$ one recovers Theorem 2.15.

3. Proofs. We will divide the proof of the theorems in some lemmas.

Lemma 3.1. *Let U be as in (2.16). For every open convex cone $\Gamma' \subset\subset \Gamma$, $\Gamma' \ni \eta$, there exists $k = k_{\Gamma'} < 1$ such that if one sets for $0 \le \alpha \le 1$ and for $\rho(x) < t$:*

$$\Phi_\alpha(x) = \frac{cr}{t - t'(1-\alpha)}(\rho(x) - t'(1-\alpha)), \tag{3.2}$$

and

$$U_\alpha = \{z : \rho(x) < t, y \in -\Phi_\alpha(x)\,\eta + \Gamma'\} \cap \{z : |y| < \frac{\delta}{t}\rho(x)\} \cap S, \tag{3.3}$$

then:

$$U_0 \subset U, \tag{3.4}$$

$$\emptyset \neq (U_\alpha)_x \cap \{y : k^{-1}\Phi_\alpha(x) < |y| < \frac{\delta'}{t}\rho(x)\} \subset\subset \Gamma, \qquad \forall \delta' < \delta, \tag{3.5}$$

and moreover the following holds. Whenever

$$\begin{cases} z \in \partial U_\alpha \cap \{z : t'(1-\alpha) \le \rho(x) < t, |y| < k^{-1}\Phi_\alpha(x)\} \\ \zeta \in N_z^*(U_\alpha), \end{cases} \tag{3.6}$$

we have

$$\begin{aligned} &\textit{(i)} \quad \xi \in (N_{x_0}^*(\Omega)^a)_\varepsilon \\ &\textit{(ii)} \quad \frac{|\xi|}{|\eta|} < c_1 \\ &\textit{(iii)} \quad \frac{|\xi|}{|\eta||y|} > c_2 \\ &\textit{(iv)} \quad \eta \in \Gamma'^{*a}. \end{aligned}$$

Proof. The relation in (3.4) is obvious.

For proving (3.5) let us first remark that there exists $k = k_{\Gamma'}$ such that for $a \in \mathbf{R}$:

$$(-a\eta + \Gamma') \cap \{y : k^{-1}a < |y| < d\} \subset\subset \Gamma \qquad \forall d > 0. \tag{3.7}$$

Putting $a = \Phi_\alpha(x)$ in (3.7) and observing that we have

$$\Phi_\alpha(x) \le \frac{cr}{t}\rho(x) < \frac{\delta}{t}\rho(x)$$

(owing to the second inequality of (2.18)) (3.5) follows.

As for (3.6), the point (i) is an easy consequence of the upper semicontinuity of the map $x \mapsto N_x^*(\Omega)$.

As for (ii),(iii) we first note that, on account of (2.14), $\Phi_\alpha(x)$ is a Lipschitz-continuous function with:

$$|\Phi_\alpha(\widetilde{x}) - \Phi_\alpha(x)| \leq l'' \frac{cr}{t} |\widetilde{x} - x|,$$
$$\inf_{\{v \in (N^*_{x_0}(\Omega))_\varepsilon : |v|=1\}} |\Phi_\alpha(x + av) - \Phi_\alpha(x)| \geq l' \frac{cr}{t} a \qquad 0 < a << 1.$$

(ii) is then a consequence of the first inequality of (2.18). As for (iii) we have, if $|y| < k^{-1}\Phi_\alpha(x)$ and $\rho(x) < t$, then clearly $|y| < k^{-1}cr$ and therefore

$$\frac{|\xi|}{|\eta||y|} > \frac{l'cr}{t} \frac{1}{k^{-1}cr} > c_2$$

(due to (2.17)).

Last, (iv) is obvious. $\square$

The family $\{U_\alpha\}_\alpha$ can be modified as follows. Let $T' = \mathbf{R} \times \{x' : |x'| < \sigma\}, T'' = \mathbf{R} \times \{x' : |x'| < \sigma + \sigma'\}$, let $\widetilde{N}$ be an open cone in $\dot{\mathbf{R}}^n$, and set

$$\widetilde{\Omega} = \widetilde{\Omega}_{\widetilde{N}, T', T''} = \bigcup_{x \in \partial\Omega \cap T'} (x + \widetilde{N}) \cap T''.$$

For a suitable choice of $\widetilde{N}, T', T''$ we have

$$\text{(3.8)} \qquad \begin{aligned} &\text{(i)} \quad \widetilde{\Omega} \cap S \subset \Omega, \quad \partial\widetilde{\Omega} \cap T' = \partial\Omega \cap T', \\ &\text{(ii)} \quad \emptyset \neq \widetilde{\Omega} \cap \{x : \rho(x) = t\} \subset\subset T'', \\ &\text{(iii)} \quad N^*_x(\widetilde{\Omega}) \subset (N^*_{x_0}(\widetilde{\Omega}))_\varepsilon, \qquad \forall x \in \partial\widetilde{\Omega} \cap T''. \end{aligned}$$

Similarly to Ω, such an $\widetilde{\Omega}$ can be represented as $\widetilde{\Omega} = \{x : x_1 > \widetilde{\varphi}(x')\}$ for a Lipschitz-continuous function $\widetilde{\varphi}$ so that the corresponding conditions to (i)-(iii)'s of (3.8) hold, i.e.:

$$\text{(3.8)'} \qquad \begin{aligned} &\text{(i)'} \quad \widetilde{\varphi}(x') \leq \varphi(x') \quad \text{and} \quad \widetilde{\varphi}(x') = \varphi(x'), \quad \text{for} \quad |x'| < \sigma, \\ &\text{(ii)'} \quad \widetilde{\varphi}(x') < \varphi(x') + t, \quad \text{for} \quad x' \geq \sigma + \sigma' \\ &\text{(iii)'} \quad \text{-the same as in (iii)-.} \end{aligned}$$

Let $\widetilde{\rho}(x) = x_1 - \widetilde{\varphi}(x')$ and observe that we could choose $\widetilde{\varphi}$ so that $\widetilde{\rho}$ still verifies the assumptions (2.14) with new constants l', l''. Let U be as in (2.16) on T'', let $\Gamma' \subset\subset \Gamma$, let t, c verify (2.17),(2.18). Define

$$\widetilde{\Phi}_\alpha(x) = cr \frac{\widetilde{\rho}(x) - t'(1-\alpha)}{\widetilde{\rho}(x) - \rho(x) + t - t'(1-\alpha)},$$

and

$$\text{(3.9)} \qquad \widetilde{U}_\alpha = \{z : \rho(x) < t, y \in -\widetilde{\Phi}_\alpha(x)\,\eta + \Gamma'\} \cap \{z : |y| < \frac{\delta'}{t}\rho(x)\} \cap S$$

for some $k^{-1}cr < \delta' < \delta$.

We then have the following

Lemma 3.10. *For a P verifying (2.3) the sets $\{\widetilde{U}_\alpha\}_\alpha$ and U verify the hypotheses of Lemma 1.1.*

Proof. (i) and (ii) of Lemma 1.1 are obvious. As for (iii) it is enough to show that for every $x \in \overline{\pi(\widetilde{U}_\alpha)} \left(= \pi\left(\overline{\widetilde{U}_\alpha}\right)\right)$ we have $\overline{(\widetilde{U}_\alpha)_x} \subset U_x \cup (\widetilde{U}_1)_x$. To prove it, we will distinguish three cases.

If $\widetilde{\rho}(x) < t'(1-\alpha)$ we get, for some $a > 0$, $\overline{(\widetilde{U}_\alpha)_x} = (\sqrt{-1}\, a\eta + \sqrt{-1}\,\overline{\Gamma'}) \cap \{z : |y| \le \delta'/t\,\rho(x)\} \subset\subset U_x$.

If $\widetilde{\rho}(x) = t'(1-\alpha)$ then $\overline{(\widetilde{U}_\alpha)_x} = \sqrt{-1}\,\overline{\Gamma'} \cap \{z : |y| \le \delta'/t\,\rho(x)\} \subset\subset (\widetilde{U}_1)_x \cup U_x$.

If $\widetilde{\rho}(x) > t'(1-\alpha)$, since $\widetilde{\Phi}_\alpha(x) \le \Phi_\alpha(x)$ we have $\widetilde{U}_\alpha \subset U_\alpha$ and (3.5) holds with U_α replaced by $\widetilde{U}_\alpha$, hence

$$\emptyset \ne \overline{(\widetilde{U}_\alpha)_x} \cap \{z : |y| \ge k^{-1}\Phi_\alpha(x)\} \subset\subset U_x,$$

and moreover it is easily seen that

$$\overline{(\widetilde{U}_\alpha)_x} \cap \{z : |y| < k^{-1}\Phi_\alpha(x)\} \subset\subset (U_1)_x.$$

Last, for $\rho(x)$ near t we have

$$\overline{(\widetilde{U}_\alpha)_x} \subset -cr\,\eta + \sqrt{-1}\,\overline{\Gamma'} \cap \{z : |y| \le \delta'\} \subset\subset U_x,$$

since $c < 1$ and $\Gamma' \subset\subset \Gamma$ (in the sense of Remark 2.7).

Concerning (iv), first note that for every $z \in \partial\widetilde{U}_\alpha \cap \overline{\widetilde{U}_1 \setminus U}$ we have

$$\begin{cases} |y| < k^{-1}\widetilde{\Phi}_\alpha(x) \le k^{-1}\Phi_1(x) \\ \text{-the solution of } \quad \widetilde{\Phi}_\alpha(u) = 0 \quad \text{for} \quad u' = x' \quad \text{verifies} \quad \widetilde{\rho}(u) < t'\text{-}. \end{cases}$$

If one follows the lines of the proof of Lemma 3.1 it is easy to check that for such z and for $\zeta \in N^*_z(\widetilde{U}_\alpha)$ we have

$$\frac{|\xi|}{|\eta|} < c_1, \qquad \frac{|\xi|}{|\eta||y|} > c_2.$$

It is clear that $\eta \in \Gamma'^{*a}$ and $\xi \in (N^*_{x_0}(\Omega))_\varepsilon$ due to (3.8)-(iii). Since $\sigma(P)$ verifies (2.3) (even replacing $\dot{N}^*_{x_0}(\Omega)$ by $(\dot{N}^*_{x_0}(\Omega))_\varepsilon$ according to Remark 2.5), (i)-(iv) imply $\sigma(P)(z,\zeta) \ne 0$. □

Proof of Theorem 2.15. Let be given $f \in \mathcal{O}_X(U), Pf \in \mathcal{O}_X(V)$ as in the statement. The family $\{\widetilde{U}_\alpha\}_\alpha$ of (3.9) has been so defined that one can find S', depending on T' of (3.8)-(i), with

$$\widetilde{U}_1 \cap S' = V \cap S'.$$

Using Lemma 3.10, the proof of the theorem follows immediately from Lemma 1.1. □

Proof of Theorem 2.21. The proof is the the same as the one of Theorem 2.15. One only needs to replace in the definition of $\widetilde{U}_\alpha$ the functions $\widetilde{\phi}_\alpha$ by $k_\alpha \widetilde{\phi}_\alpha$ with k_α so chosen that $k_\alpha \widetilde{\phi}_\alpha < g_1(x)$. Note that it is not restrictive to assume the map $\alpha \to k_\alpha$ to be a continuous one. Thus the family $V_\alpha = \bigcup_{\beta<\alpha} \widetilde{U}_\beta$ satisfies the conditions of Lemma 1.1 and hence f extends to $\bigcup_\alpha V_\alpha$. Note that, on a small $S' \subset S$, the function $\sup_\alpha k_\alpha \widetilde{\phi}_\alpha$ is in the form $h(s)$ $(s = \rho(x))$ for a $\mathcal{C}^1$-function h satisfying all requirements in the statement. □

Proof of Theorem 2.8. Let $f \in \mathcal{O}_X(U \cap S)$ and $Pf \in \mathcal{O}_X(W \cap S)$ where U (resp W) is a tuboid whose fiber verifies (2.9) (resp (2.9)'), (2.10). Then for every t, t' and for suitable δ and $r = r_{t,t'}$, we can write $U \cap S$ as in (2.16) (possibly with a new S). Moreover for a suitable c, $(W \cup U) \cap S$ contains a set V as in (2.19). Applying Theorem 2.15, we get $f \in \mathcal{O}_X(V \cap S')$; then the conclusion follows from Remark 2.20. □

Proof of Theorem 2.12. As in the proof of Theorem 2.8 we can assume that f is analytic in $U \cap S$ and Pf in $V \cap S$, where U, V are defined by (2.16) and (2.19) respectively. On account of (2.3), $z_1 = 0$ is non characteristic for $\sigma(P)$ at x_0 and then there exist C so that:

$$\sigma(P) \neq 0 \qquad \text{if} \qquad |\zeta_1| > C|\zeta'|. \tag{3.11}$$

We then set

$$\tilde{\widetilde{U}}_\alpha = \{z \in X : |z' - \widetilde{z}'| < C|z_1 - \widetilde{z}_1|, x_1 = \widetilde{x}_1 \Rightarrow \widetilde{z} \in \widetilde{U}_\alpha \cap \{z : y_1 = 0\} \cap S\}.$$

According to (3.11) we get

$$f \in \mathcal{O}_X(\widetilde{U}_\alpha \cap \{z : y_1 = 0\} \cap S), Pf \in \mathcal{O}_X(\tilde{\widetilde{U}}_\alpha) \quad \text{implies} \quad f \in \mathcal{O}_X(\tilde{\widetilde{U}}_\alpha). \tag{3.12}$$

On the other hand we have

$$\sigma(P)(z,\zeta) \neq 0 \quad \text{for} \quad \begin{cases} z \in \partial\widetilde{U}_\alpha \cap \overline{\widetilde{U}_1 \setminus U} \cap S \cap \{z : y_1 = 0\} \\ \zeta \in N^*_z(\tilde{\widetilde{U}}_\alpha) \end{cases}$$

and then

$$f \in \mathcal{O}_X(\tilde{\widetilde{U}}_\alpha \cap S), Pf \in (\mathcal{O}_X)_z \qquad \text{implies} \qquad f \in (\mathcal{O}_X)_z.$$

The conclusion then follows from (3.11),(3.12), via Lemma 1.1, in the same way as it was for Theorem 2.8. □

References

[B-S 1] J.-M. Bony, P. Schapira, *Existence et prolongement des solutions holomorphes des équations aux dérivées partielles*, Inventiones Math. **17** (1972), 95–105.

[B-S 2] J.-M. Bony, P. Schapira, *Solutions hyperfonctions du problème de Cauchy*, Lecture Notes in Math., Springer-Verlag **287** (1973), 82–98.

[B-S 3] J.-M. Bony, P. Schapira, *Propagation des singularités analytiques pour les solutions des équations aux dérivées partielles*, Ann. Inst. Fourier **26,1** (1976), 81–140.

[D'A-Z] A. D'Agnolo, G. Zampieri, *Analysis of the action of a pseudodifferential operator over* $(\mathcal{C}_{\Omega|X})_{T^*_M X}$, Tsukuba J. Math., No. 1 **15** (1991), 175–184.

[Kan] A. Kaneko, *Singular spectrum of boundary values of solutions of partial differential equations with real analytic coefficients*, Sci. Papers College Gen. Ed. Univ. Tokyo **25** (1975), 59–68.

[K-K] M. Kashiwara, T. Kawai, *Micro-hyperbolic pseudo-differential operators*, I. J. Math. Soc. Japan **27** (1975), 359–404.

[K-S 1] M. Kashiwara, P. Schapira, *Microlocal study of sheaves*, Asterisque **128** (1985).

[K-S 2] M. Kashiwara, P. Schapira, *Micro-hyperbolic systems*, Acta Math. **142** (1979), 1–55.

[Kat] K. Kataoka, *Microlocal theory of boundary value problems I*, J. Fac. Sci. Univ. Tokyo Sect. 1A **27** (1980), 355-399; *II* **28** (1981), 31–56.

[S] P. Schapira, *Front d'onde analytique au bord I*, C. R. Acad. Sci. Paris Sér. I Math. **302** **10** (1986), 383–386; *II*, Sém. E.D.P. École Polytechnique Exp. 13 (1986).

[S-K-K] M. Sato, M. Kashiwara, T. Kawai, *Hyperfunctions and pseudo-differential equations*, Lecture Notes in Math., Springer-Verlag **287** (1973), 265–529.

[S-Z] P. Schapira, G. Zampieri, *Regularity at the boundary for systems of microdifferential equations*, Pitman Res. Notes in Math. **158** (1987), 186–201.

[Z] G. Zampieri, *Tuboids of* $\mathbf{C}^n$ *with cone property and domains of holomorphy*, Proc. Japan Acad. Ser. A **67** (1991).

Dipartimento di matematica, via Belzoni 7, 35131 Padova, Italy