

## A CLASSIFICATION OF UNIPOTENT SPHERICAL CONJUGACY CLASSES IN BAD CHARACTERISTIC

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ABSTRACT. Let  $G$  be a simple algebraic group over an algebraically closed field  $k$  of bad characteristic. We classify the spherical unipotent conjugacy classes of  $G$ . We also show that if the characteristic of  $k$  is 2, then the fixed point subgroup of every involutory automorphism (involution) of  $G$  is a spherical subgroup of  $G$ .

### 1. INTRODUCTION

Let  $G$  be a simple algebraic group over an algebraically closed field  $k$ , with Lie algebra  $\mathfrak{g}$ . In this paper we determine the unipotent spherical conjugacy classes of  $G$  (we recall that a conjugacy class  $\mathcal{O}$  in  $G$  is called *spherical* if a Borel subgroup of  $G$  has a dense orbit on  $\mathcal{O}$ ) when the characteristic of  $k$  is bad for  $G$ . There has been much work related to this field in the cases of good characteristic.

To fix the notation,  $B$  is a Borel subgroup of  $G$ ,  $T$  is a maximal torus of  $B$ ,  $B^-$  is the Borel subgroup opposite to  $B$ , and  $\{\alpha_1, \dots, \alpha_n\}$  is the set of simple roots with respect to the choice of  $(T, B)$ . Let  $W$  be the Weyl group of  $G$  and let us denote by  $s_i$  the reflection corresponding to the simple root  $\alpha_i$ ;  $\ell(w)$  is the length of the element  $w \in W$  and  $\text{rk}(1 - w)$  is the rank of  $1 - w$  in the geometric representation of  $W$ .

Initially, spherical  $G$ -orbits have been studied in the context of Lie algebras ([24], [25]) in characteristic zero. The classification of spherical nilpotent orbits has been obtained by Panyushev: in terms of height, a nilpotent orbit  $\mathcal{O} \subset \mathfrak{g}$  is spherical if and only if its height is at most 3, which means 2 or 3 if  $\mathcal{O}$  is not the zero orbit. Equivalently,  $\mathcal{O}$  is spherical if and only if it contains an element of the form  $e_{\gamma_1} + \dots + e_{\gamma_t}$ , where  $\gamma_1, \dots, \gamma_t$  are pairwise orthogonal simple roots (Panyushev [24], [26]).

More recently, in [5], we considered spherical conjugacy classes in  $G$  over  $\mathbb{C}$ . We classify all spherical conjugacy classes by means of the Bruhat decomposition: a conjugacy class  $\mathcal{O} \subset G$  is spherical if and only if  $\dim \mathcal{O} = \ell(w) + \text{rk}(1 - w)$ , where  $w = w(\mathcal{O})$  is the unique element of  $W$  such that  $\mathcal{O} \cap BwB$  is open dense in  $\mathcal{O}$  (we observe that the classification given in [5] over the complex numbers holds in general for characteristic zero).

In [15] the authors classify spherical nilpotent orbits in good characteristic using Kempf-Rousseau theory: the classification is the same as in the case of zero characteristic. In [8], the author obtains the classification of all spherical conjugacy classes in characteristic which is zero or is both good and odd by means of

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the Bruhat decomposition and by exploiting another characterization of spherical conjugacy classes available in such characteristic, namely a conjugacy class  $\mathcal{O}$  is spherical if and only if  $\{y \in W \mid \mathcal{O} \cap ByB \neq \emptyset\} \subseteq \{w \in W \mid w^2 = 1\}$  ([6, Theorem 2.7], [7, Theorem 5.7]).

In the present paper we complete the picture for unipotent spherical conjugacy classes by considering bad characteristic. Our strategy is to exhibit for each group  $G$  a set  $\mathcal{O}(G)$  of unipotent conjugacy classes, show that each element in  $\mathcal{O}(G)$  is spherical, and finally show that each conjugacy class not in  $\mathcal{O}(G)$  is not spherical. It turns out that in bad characteristic the classification of spherical unipotent conjugacy classes is the “same” as in zero characteristic, unless  $p = 2$  in type  $C_n$  and  $F_4$  or  $p = 3$  in type  $G_2$ . In these cases there are more classes than in characteristic zero. In particular, if  $p = 2$ , then the conjugacy class of a non-trivial unipotent element  $u$  is spherical if and only if  $u$  is an involution.

It is well known that in zero or odd characteristic the fixed point subgroup  $H$  of any involutory automorphism  $\sigma$  of  $G$  is a spherical subgroup (i.e.  $G/H$  is a spherical homogeneous space). This was proved by Vust in [32] in characteristic zero (see also [23] over  $\mathbb{C}$ ). Then Springer extended the result to odd characteristic in [29]. In [27], Seitz gives an alternative proof of Springer’s result. Here we prove that the result also holds in characteristic 2.

The paper is structured as follows. In Section 2 we introduce the notation. In Section 3 we recall some basic facts about the classification of unipotent conjugacy classes in bad characteristic and determine the spherical ones. We also determine which spherical unipotent conjugacy classes have a representative  $u$  of the form  $u = x_{\gamma_1}(1) \cdots x_{\gamma_t}(1)$ , where  $\gamma_1, \dots, \gamma_t$  are pairwise orthogonal simple roots. In Section 4 we prove that if  $G$  is a reductive connected algebraic group in characteristic 2 and  $\sigma$  is any involutory automorphism of  $G$ , then the fixed point subgroup  $H$  of  $\sigma$  is a spherical subgroup of  $G$ .

## 2. PRELIMINARIES

We denote by  $\mathbb{C}$  the complex numbers, by  $\mathbb{R}$  the reals, by  $\mathbb{Z}$  the integers and by  $\mathbb{N}$  the natural numbers.

Let  $A = (a_{ij})$  be a finite indecomposable Cartan matrix of rank  $n$  with associated root system  $\Phi$ , and let  $k$  be an algebraically closed field of characteristic  $\text{char } k = p$ . Let  $G$  be a simple algebraic group over  $k$  associated to  $A$ , with Lie algebra  $\mathfrak{g}$ . We fix a maximal torus  $T$  of  $G$  and a Borel subgroup  $B$  containing  $T$ :  $B^-$  is the Borel subgroup opposite to  $B$  and  $U$  (respectively  $U^-$ ) is the unipotent radical of  $B$  (respectively of  $B^-$ ). Then  $\Phi$  is the set of roots relative to  $T$  and  $B$  determines the set of positive roots  $\Phi^+$  and the simple roots  $\Delta = \{\alpha_1, \dots, \alpha_n\}$ . We fix a total ordering on  $\Phi^+$  compatible with the height function, and we shall write elements of  $U$  as products of root elements where the roots are taken in the order given by this ordering. We shall use the numbering and the description of the simple roots in terms of the canonical basis  $(e_1, \dots, e_k)$  of an appropriate  $\mathbb{R}^k$  as in [3, Planches I-IX]. For the exceptional groups, we shall write  $\beta = (m_1, \dots, m_n)$  for  $\beta = m_1\alpha_1 + \cdots + m_n\alpha_n$ . We denote by  $P$  the weight lattice, by  $P^+$  the monoid of dominant weights and by  $W$  the Weyl group. Also  $s_i$  is the simple reflection associated to  $\alpha_i$ ,  $\{\omega_1, \dots, \omega_n\}$  are the fundamental weights, and  $w_0$  is the longest element of  $W$ . The real space  $E = \mathbb{R}P$  is a Euclidean space, endowed with the scalar product  $(\alpha_i, \alpha_j) = d_i a_{ij}$ . Here  $\{d_1, \dots, d_n\}$  are relatively prime positive

integers such that if  $D$  is the diagonal matrix with entries  $d_1, \dots, d_n$ , then  $DA$  is symmetric.

We put  $\Pi = \{1, \dots, n\}$  and fix a Chevalley basis  $\{h_i, i \in \Pi; e_\alpha, \alpha \in \Phi\}$  of  $\mathfrak{g}$ .

We use the notation  $x_\alpha(\xi), h_\alpha(z)$ , for  $\alpha \in \Phi, \xi \in k, z \in k^*$  as in [31], [12]. For  $\alpha \in \Phi$  we put  $X_\alpha = \{x_\alpha(\xi) \mid \xi \in k\}$ , the root subgroup corresponding to  $\alpha$ , and  $H_\alpha = \{h_\alpha(z) \mid z \in k^*\}$ . We identify  $W$  with  $N/T$ , where  $N$  is the normalizer of  $T$ : given an element  $w \in W$  we shall denote a representative of  $w$  in  $N$  by  $\dot{w}$ . We choose the  $x_\alpha$ 's so that, for all  $\alpha \in \Phi, n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$  lies in  $N$  and has image the reflection  $s_\alpha$  in  $W$ . Then

$$(2.1) \quad x_\alpha(\xi)x_{-\alpha}(-\xi^{-1})x_\alpha(\xi) = h_\alpha(\xi)n_\alpha, \quad n_\alpha^2 = h_\alpha(-1)$$

for every  $\xi \in k^*, \alpha \in \Phi$  ([30, Proposition 11.2.1]).

For algebraic groups we use the notation in [17], [13]. In particular, for  $J \subseteq \Pi, \Delta_J = \{\alpha_j \mid j \in J\}, \Phi_J$  is the corresponding root system,  $W_J$  is the Weyl group,  $P_J$  is the standard parabolic subgroup of  $G$  and  $L_J = T\langle X_\alpha \mid \alpha \in \Phi_J \rangle$  is the standard Levi subgroup of  $P_J$ . For  $z \in W$  we put  $U_z = U \cap z^{-1}U^-z$ . Then the unipotent radical  $R_u P_J$  of  $P_J$  is  $U_{w_0 w_J}$ , where  $w_J$  is the longest element of  $W_J$ . Moreover,  $U \cap L_J = U_{w_J}$  is a maximal unipotent subgroup of  $L_J$ .

If  $\Psi$  is a subsystem of type  $X_r$  of  $\Phi$  and  $H$  is the subgroup generated by  $X_\alpha, \alpha \in \Psi$ , we say that  $H$  is an  $X_r$ -subgroup of  $G$ .

If  $X$  is a  $G$ -variety and  $x \in X$ , we denote by  $G.x$  the  $G$ -orbit of  $x$  and by  $G_x$  the isotropy subgroup of  $x$  in  $G$ . If the homogeneous space  $G/H$  is spherical, we say that  $H$  is a spherical subgroup of  $G$ .

If  $x$  is an element of a group  $K$  we denote by  $\mathcal{O}_x$  its conjugacy class and by  $C(x)$  its centralizer. If  $H \leq K$ , we shall denote by  $C_H(x)$  the centralizer of  $x$  in  $H$ . If  $x, y \in K$ , then  $x \sim y$  means that  $x, y$  are conjugate in  $K$ . For unipotent classes in exceptional groups we use the notation in [13]. We use the description of centralizers of involutions as in [18], [2].

We denote by  $\mathbb{Z}_r$  the cyclic group of order  $r$ .

For each conjugacy class  $\mathcal{O}$  in  $G, w = w(\mathcal{O})$  is the unique element of  $W$  such that  $BwB \cap \mathcal{O}$  is open dense in  $\mathcal{O}$ . We shall also say that  $\mathcal{O}$  lies over  $w$ .

In the remainder of the paper we shall denote by  $p$  the characteristic of  $k$  (hence  $p$  may be 0).

### 3. THE CLASSIFICATION

We recall that the bad primes for the individual types of simple groups are as follows:

- none when  $G$  has type  $A_n$ ;
- $p = 2$  when  $G$  has types  $B_n, C_n, D_n$ ;
- $p = 2$  or  $3$  when  $G$  has types  $G_2, F_4, E_6, E_7$ ;
- $p = 2, 3$  or  $5$  when  $G$  has type  $E_8$ .

One may find a detailed account of the classification of both unipotent classes and nilpotent orbits in bad characteristic in [13, §5.11].

*Strategy of the proof.* Let  $G_{\mathbb{C}}$  be the corresponding group over  $\mathbb{C}$ . We have shown in [5] that for every spherical conjugacy class  $\mathcal{C}$  of  $G_{\mathbb{C}}$  there exists an involution  $w = w(\mathcal{C})$  in  $W$  such that  $\dim \mathcal{C} = \ell(w) + \text{rk}(1 - w)$ , with  $\mathcal{C} \cap BwB \neq \emptyset$ . For each group  $G$  we introduce a certain set  $\mathcal{O}(G)$  of unipotent conjugacy classes which are

candidates for being spherical. For each  $\mathcal{O} \in \mathcal{O}(G)$  we show that there is a (not necessarily unipotent) spherical conjugacy class  $\mathcal{C}$  in  $G_{\mathbb{C}}$  such that

$$\dim \mathcal{O} = \dim \mathcal{C}.$$

Let  $w = w(\mathcal{C})$ . Our aim is to show that  $\mathcal{O} \cap BwB \neq \emptyset$ .

**Definition 3.1.** Let  $\mathcal{O}$  be a conjugacy class of  $G$ . We say that  $\mathcal{O}$  satisfies  $(*)$  if there exists  $w \in W$  such that  $BwB \cap \mathcal{O} \neq \emptyset$  and  $\dim \mathcal{O} = \ell(w) + \text{rk}(1 - w)$ .

Let  $\mathcal{O}$  be a conjugacy class in  $G$ . There exists a unique element  $w = w(\mathcal{O}) \in W$  such that  $\mathcal{O} \cap BwB$  is open dense in  $\mathcal{O}$ . In particular,

$$(3.1) \quad \overline{\mathcal{O}} = \overline{\mathcal{O} \cap BwB} \subseteq \overline{BwB}.$$

It follows that if  $y$  is an element of  $\overline{\mathcal{O}}$  and  $y \in BzB$ , then  $z \leq w$  in the Chevalley-Bruhat order of  $W$ .

We recall the following result proved in [5, Theorem 5] over  $\mathbb{C}$ , but which is valid with the same proof over any algebraically closed field.

**Theorem 3.2.** *Suppose that  $\mathcal{O}$  contains an element  $x \in BwB$ . Then*

$$\dim B.x \geq \ell(w) + \text{rk}(1 - w).$$

*In particular,  $\dim \mathcal{O} \geq \ell(w) + \text{rk}(1 - w)$ . If, in addition,  $\dim \mathcal{O} \leq \ell(w) + \text{rk}(1 - w)$ , then  $\mathcal{O}$  is spherical,  $w = w(\mathcal{O})$  and  $B.x$  is the dense  $B$ -orbit in  $\mathcal{O}$ .  $\square$*

Let  $\mathcal{O}$  be a conjugacy class of  $G$  and let  $w = w(\mathcal{O})$ . If  $\mathcal{O}^{-1} = \mathcal{O}$  (i.e. if any element  $x \in \mathcal{O}$  is conjugate to its inverse), then  $w^2 = 1$ . It is well known that over any algebraically closed field any unipotent element is conjugate to its inverse ([9, Lemma 1.16], [10, Lemma 2.3]; see also [22, Proposition 2.5 (a)]) so that  $w$  is an involution for every non-trivial unipotent conjugacy class  $\mathcal{O}$ . However, it has recently been shown in [14] that  $w^2 = 1$  for every conjugacy class  $\mathcal{O}$  in  $G$ .

If  $g$  is in  $Z(G)$ , then  $g \in T$ , its conjugacy class  $\mathcal{O}$  is simply  $\{g\}$ , and  $w(\mathcal{O}) = 1$ . In the remainder of the paper we shall consider only non-central conjugacy classes.

We shall use the following result.

**Lemma 3.3.** *Assume the positive roots  $\beta_1, \dots, \beta_\ell$  are such that  $[X_{\pm\beta_i}, X_{\pm\beta_j}] = 1$  for every  $1 \leq i < j \leq \ell$ . Then, for  $\xi_1, \dots, \xi_\ell \in k^*$ ,  $g = x_{\beta_1}(-\xi_1^{-1}) \cdots x_{\beta_\ell}(-\xi_\ell^{-1})$ , and  $h = h_{\beta_1}(-\xi_1) \cdots h_{\beta_\ell}(-\xi_\ell)$  we have*

$$gx_{-\beta_1}(\xi_1) \cdots x_{-\beta_\ell}(\xi_\ell)g^{-1} = n_{\beta_1} \cdots n_{\beta_\ell} h x_{\beta_1}(2\xi_1^{-1}) \cdots x_{\beta_\ell}(2\xi_\ell^{-1}).$$

*Proof.* By (2.1) we have  $x_\alpha(-\xi^{-1})x_{-\alpha}(\xi)x_\alpha(\xi^{-1}) = n_\alpha h_\alpha(-\xi)x_\alpha(2\xi^{-1})$ . The result follows from  $[X_{\pm\beta_i}, X_{\pm\beta_j}] = 1$  for every  $1 \leq i < j \leq \ell$ .  $\square$

The hypothesis of the lemma is satisfied, for instance, if  $\beta_1, \dots, \beta_\ell$  are pairwise orthogonal and long, as in [11, Lemma 4.1]. In characteristic 2, we have  $[X_\gamma, X_\delta] = 1$  for every pair  $(\gamma, \delta)$  of orthogonal roots, so that for any set of pairwise orthogonal roots  $\beta_1, \dots, \beta_\ell$  and for  $g = x_{\beta_1}(1) \cdots x_{\beta_\ell}(1)$  we get

$$(3.2) \quad gx_{-\beta_1}(1) \cdots x_{-\beta_\ell}(1)g^{-1} = n_{\beta_1} \cdots n_{\beta_\ell}.$$

We shall prove the following theorem.

**Theorem 3.4.** *Let  $\mathcal{O}$  be a non-trivial unipotent conjugacy class of a simple algebraic group in characteristic 2. Then  $\mathcal{O}$  is spherical if and only if it consists of involutions.  $\square$*

From the classification it will follow that the unipotent conjugacy class  $\mathcal{O}$  is spherical if and only if it satisfies (\*). Moreover, we shall see that there is a unique maximal spherical conjugacy class  $\mathcal{O}_{\max}$  and that it has the following properties:  $w(\mathcal{O}_{\max}) = w_0$ , and the closure of  $\mathcal{O}_{\max}$  is the union of all spherical unipotent orbits.

**3.1. Classical groups in characteristic 2.** To deal with the classical groups with  $p = 2$ , we recall that the unipotent classes were determined by Wall in [33] (the nilpotent orbits were determined by Hesselink in [16]). For the convenience of the reader, here we recall the classification of unipotent classes in the classical groups following [28, §2]. Suppose  $G = GL(n)$  (any characteristic) and  $u$  is a unipotent element of  $G$ . Then one can associate to  $u$  a partition  $\lambda = (\lambda_1, \lambda_2, \dots) = 1^{c(1)} \oplus 2^{c(2)} \oplus \dots$  of  $n$  with  $\lambda_1 \geq \lambda_2 \geq \dots$ , where  $c(i)$  is the number of Jordan blocks of  $u$  of dimension  $i$ , for every  $i \geq 1$ . In this way the set  $\mathcal{CU}(G)$  of unipotent conjugacy classes of  $G$  is parametrized by the set of partitions of  $n$ . We denote by  $C_\lambda$  the unipotent class corresponding to the partition  $\lambda$ . The set  $\mathcal{CU}(G)$  has a natural partial order:  $\mathcal{O}_1 \leq \mathcal{O} \Leftrightarrow \mathcal{O}_1 \subseteq \overline{\mathcal{O}}$ . If we partially order the set of partitions of  $n$  by  $\lambda \leq \mu \Leftrightarrow \sum_{j=1}^i \lambda_j \leq \sum_{j=1}^i \mu_j$  for every  $i \geq 1$ , then the map  $\lambda \mapsto C_\lambda$  is an isomorphism of p.o. sets.

Now assume  $p = 2$ . In this case there exists a homomorphism (central isogeny) of  $SO(2n + 1)$  onto  $Sp(2n)$  which is an isomorphism of abstract groups. We shall therefore deal only with  $Sp(2n)$  and  $SO(2n)$ . Let  $\omega$  be an object distinct from 0 and 1, and consider the set  $\{\omega, 0, 1\}$  totally ordered by  $\omega < 0 < 1$ . Assume  $G = Sp(2n) \leq GL(2n)$  (resp.  $G = O(2n) \leq Sp(2n)$ ). The unipotent conjugacy classes of  $G$  are parametrized by pairs  $(\lambda, \varepsilon)$  such that

- a)  $\lambda = 1^{c(1)} \oplus 2^{c(2)} \oplus \dots$  is a partition of  $2n$  with  $c(i)$  even for every odd  $i$ .
- b)  $\varepsilon : \mathbb{N} \rightarrow \{\omega, 0, 1\}$  is such that
  - b<sub>1</sub>)  $\varepsilon(i) = \omega$  if  $i$  is odd or  $i \geq 1$  and  $c(i) = 0$ .
  - b<sub>2</sub>)  $\varepsilon(i) = 1$  if  $i$  is even and  $c(i)$  is odd ( $i \neq 0$ ).
  - b<sub>3</sub>)  $\varepsilon(i) \neq \omega$  if  $i$  is even and  $c(i) \neq 0$  ( $i \neq 0$ ).
  - b<sub>4</sub>)  $\varepsilon(0) = 1$  (resp.  $\varepsilon(0) = 0$ ).

The correspondence is obtained as follows. Let  $u$  be a unipotent element of  $G$ . Then  $u$  determines a class in  $GL(2n)$ , hence the partition  $\lambda$  of  $2n$ . This partition satisfies a). Now, if  $i$  is even,  $i \neq 0$  and  $c(i) \neq 0$ , we put  $\varepsilon(i) = 0$  if

$$f((u - 1)^{i-1}(x), x) = 0$$

for every  $x \in \ker(u - 1)^i$ , and  $\varepsilon(i) = 1$  otherwise (here  $f$  is the bilinear form used to define  $Sp(2n)$ ). In view of condition b), this uniquely defines  $\varepsilon$ . For  $i \in \mathbb{N}$  we shall write  $c_i$  and  $\varepsilon_i$  for  $c(i)$  and  $\varepsilon(i)$ .

We denote by  $C_{\lambda, \varepsilon}$  the unipotent class of  $G$  corresponding to  $(\lambda, \varepsilon)$ . We observe that every unipotent class of  $Sp(2n)$  intersects  $O(2n)$  in a unique class of  $O(2n)$ . Moreover, the unipotent classes of  $O(2n)$  contained in  $SO(2n)$  (the connected component of  $O(2n)$ ) are those for which  $\lambda_1^*$  is even ( $\lambda^*$  is the dual partition of  $\lambda$ ). If all  $\lambda_i$ 's and  $c(i)$ 's are even and if  $\varepsilon(i) \neq 1$  for every  $i$ , then the unipotent class  $C_{\lambda, \varepsilon}$  of  $O(2n)$  splits into two classes of  $SO(2n)$ . All the other unipotent classes in  $SO(2n)$  are unipotent classes in  $O(2n)$ .

We shall add the subscript 0 in the case when  $i$  is even,  $c(i)$  is even and  $\varepsilon(i) = 0$ .

In [28, §2.8, 2.9, 2.10], there are formulas for the dimensions of centralizers of unipotent elements in  $Sp(2n)$ ,  $O(2n)$  (hence also in  $SO(2n)$ ), the determination of

TABLE 1. Spherical unipotent classes in  $A_n, p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$X_\ell = 2^\ell \oplus 1^{n+1-2\ell}$ $\ell = 1, \dots, \lfloor \frac{n+1}{2} \rfloor$	$J_\ell$	$s_{\beta_1} \cdots s_{\beta_\ell}$	$n_{\beta_1} \cdots n_{\beta_\ell}$	$2\ell(n+1-\ell)$

the component groups  $C(u)/C(u)^\circ$  in the various cases, and an explicit definition of a partial order on pairs  $(\lambda, \varepsilon)$  such that  $C_{\lambda, \varepsilon} \leq C_{\mu, \phi} \Leftrightarrow (\lambda, \varepsilon) \leq (\mu, \phi)$ .

We shall use the notation as in [28]. As mentioned above, we need only treat  $Sp(2n)$  and  $SO(2n)$ . However, for completeness, we shall also deal with the case when  $G$  has type  $A_n$ , since this is dealt with in [15] but not in [8].

3.1.1. *Type  $A_n, n \geq 1$ .* We show that every spherical unipotent conjugacy class satisfies (\*). The spherical nilpotent orbits (and therefore the spherical unipotent classes) have been classified in [15], and it follows that a unipotent conjugacy class  $\mathcal{O}$  is spherical if and only if  $\mathcal{O} = X_\ell$ , the unipotent class  $2^\ell \oplus 1^{n+1-2\ell}$  for  $\ell = 1, \dots, \lfloor \frac{n+1}{2} \rfloor$ . For every  $i = 1, \dots, \lfloor \frac{n+1}{2} \rfloor$ , let  $\beta_i = e_i - e_{n+2-i}$ . Then, as for  $\mathbb{C}$  (and for any algebraically closed field of odd or zero characteristic) the element  $u = x_{-\beta_1}(1) \cdots x_{-\beta_\ell}(1)$  lies in  $X_\ell \cap BwB$ , where  $w = s_{\beta_1} \cdots s_{\beta_\ell}$  is such that  $\dim X_\ell = \ell(w) + \text{rk}(1-w)$ . In fact one may take  $n_{\beta_1} \cdots n_{\beta_\ell} \in s_{\beta_1} \cdots s_{\beta_\ell}B \cap X_\ell$ . For  $\ell = 1, \dots, \lfloor \frac{n+1}{2} \rfloor - 1$  we put  $J_\ell = \{\ell + 1, \dots, n - \ell\}$ ,  $J_{\lfloor \frac{n+1}{2} \rfloor} = \emptyset$ . We have established the information presented in Table 1, where  $w(\mathcal{O}) = w_0 w_j$ .

**Proposition 3.5.** *Let  $\mathcal{O}$  be a non-trivial unipotent conjugacy class of  $SL(n+1)$  in characteristic 2. Then  $\mathcal{O}$  is spherical if and only if it consists of involutions if and only if  $\mathcal{O}$  has a representative of the form  $x_{\gamma_1}(1) \cdots x_{\gamma_t}(1)$ , where  $\gamma_1, \dots, \gamma_t$  are pairwise orthogonal simple roots. All non-trivial spherical unipotent classes are listed in Table 1.  $\square$*

3.1.2. *Type  $C_n$  (and  $B_n$ ),  $n \geq 2$ .* We first show that if  $u$  is an involution of  $G$ , then  $\mathcal{O}_u$  is spherical, by showing that  $\mathcal{O}_u$  satisfies (\*). Therefore, let  $u$  be an involution of  $G = Sp(2n)$ . Then the partition corresponding to  $u$  is of the form  $\lambda = 1^{c_1} \oplus 2^{c_2}$ , with  $c_2 = \ell, c_1 = 2n - 2\ell, \ell = 1, \dots, n$ .

Using the description of unipotent conjugacy classes recalled above, let  $\lambda = 2^\ell \oplus 1^{2n-2\ell}$ , for  $\ell = 1, \dots, n$ . Then we have  $\varepsilon_0 = 1, \varepsilon_1 = \omega, \varepsilon_i = \omega$ , for  $i \geq 3$ . As for  $\varepsilon_2$ , we have  $\varepsilon_2 = 1$  if  $\ell$  is odd. On the other hand, if  $\ell$  is even, we have both possibilities  $\varepsilon_2 = 0$  or  $1$ . We denote by  $X_\ell$  the class corresponding to  $\varepsilon_2 = 1$  and by  $Y_\ell$  the class corresponding to  $\varepsilon_2 = 0$  (when  $\ell$  is even).

We denote by  $X_{\ell, \mathbb{C}}$  the unipotent class in  $Sp(2n, \mathbb{C})$  corresponding to  $\lambda = 2^\ell \oplus 1^{2n-2\ell}$ . Then we get

$$\begin{aligned} \dim X_\ell &= \dim X_{\ell, \mathbb{C}} &= \ell(2n - \ell + 1) & \ell \in \{1, \dots, n\}, \\ \dim Y_\ell &= \dim X_{\ell, \mathbb{C}} - \ell &= \ell(2n - \ell) & \ell \in \{1, \dots, n\}, \ell \text{ even.} \end{aligned}$$

Note that if  $\ell$  is even and we write  $\ell = 2\ell'$ , then  $\dim Y_\ell = \dim \mathcal{O}_{\sigma_{\ell'}, \mathbb{C}}$ , where  $\mathcal{O}_{\sigma_{\ell'}, \mathbb{C}}$  is the conjugacy class in  $Sp(2n, \mathbb{C})$  of the involution  $\sigma_{\ell'}$  ([5, Table 1]). In  $Sp(2n, \mathbb{C})$ , the spherical semisimple conjugacy class  $\mathcal{O}_{\sigma_{\ell'}, \mathbb{C}}$  lies over  $w = s_{\gamma_1} \cdots s_{\gamma_{\ell'}}$  ([5, Table 5], [11, §4.2.2]).

We observe that if the partition associated to the involution  $u$  is  $\lambda = 2^{2\ell} \oplus 1^{2n-4\ell}$ , then  $\mathcal{O}_u = 2_0^{2\ell} \oplus 1^{2n-4\ell}$  if and only if  $f((u-1)v, v) = 0$  for every  $v \in V$  (here  $f$  is the bilinear form on  $V$  used to define  $Sp_{2n}(k)$ ). Let  $w$  be an involution of  $W$ ,  $L(w) = \{\beta \in \Phi^+ \mid w(\beta) = -\beta, \beta \text{ long}\}$ , and  $L(w)_\perp = \{\gamma \in \Phi^+ \mid w(\gamma) = -\gamma, (\gamma, L(w)) = 0, \gamma \text{ short}\}$ . Then  $w = \prod_{\beta \in L(w)} s_\beta \prod_{\gamma \in L(w)_\perp} s_\gamma$ . Let  $x = \prod_{\beta \in L(w)} n_\beta \prod_{\gamma \in L(w)_\perp} n_\gamma$ . Then  $x$  is an involution in  $BwB$  and the number of blocks of dimension 2 in the Jordan canonical form of  $x$  is  $|L(w)+2|L(w)_\perp|$ . If this number is even, then  $f((x-1)v, v) = 0$  for every  $v \in V$  if and only if  $L(w) = \emptyset$ .

We put  $\beta_i = 2e_i$  for  $i = 1, \dots, n$  and  $\gamma_i = e_{2i-1} + e_{2i}$  for  $i = 1, \dots, [\frac{n}{2}]$ .

Then it is straightforward to show that

$$\begin{aligned} x_{-\beta_1}(1) \cdots x_{-\beta_\ell}(1) &\in X_\ell \cap Bs_{\beta_1} \cdots s_{\beta_\ell} B \cap U^- \quad \text{for } \ell = 1, \dots, n, \\ x_{-\gamma_1}(1) \cdots x_{-\gamma_\ell}(1) &\in Y_{2\ell} \cap Bs_{\gamma_1} \cdots s_{\gamma_\ell} B \cap U^- \quad \text{for } \ell = 1, \dots, [\frac{n}{2}]. \end{aligned}$$

By (3.2), we can choose

$$\begin{aligned} n_{\beta_1} \cdots n_{\beta_\ell} &\in X_\ell \cap wB \quad \text{for } \ell = 1, \dots, n, \\ n_{\gamma_1} \cdots n_{\gamma_\ell} &\in Y_{2\ell} \cap wB \quad \text{for } \ell = 1, \dots, [\frac{n}{2}]. \end{aligned}$$

One can easily deduce which classes of involutions have a representative of the form  $u = \prod_{i \in K} x_{\alpha_i}(1)$  for a certain subset  $K$  of  $\Pi$ . Note that since  $u$  is an involution,  $(\alpha_i, \alpha_j) = 0$  if  $i, j \in K$  with  $i \neq j$ . Up to the  $W$  action, we have only the following representatives  $u$ :

$$\begin{aligned} \prod_{i=1}^\ell x_{\alpha_{n-2(i-1)}}(1) &\in X_{2\ell-1} \quad \text{for } \ell = 1, \dots, [\frac{n+1}{2}], \\ \prod_{i=1}^\ell x_{\alpha_{2i-1}}(1) &\in Y_{2\ell} \quad \text{for } \ell = 1, \dots, [\frac{n}{2}]. \end{aligned}$$

These exhaust the conjugacy classes of involutions with representative of the form  $\prod_{i \in K} x_{\alpha_i}(1)$ . In particular, all  $X_{2\ell}$  have no representative of the form  $\prod_{i \in K} x_{\alpha_i}(1)$ . The point is that in good characteristic, for  $\ell = 1, \dots, [\frac{n}{2}]$  the element  $\prod_{i=1}^\ell x_{\alpha_{2i-1}}(1)$  is conjugate to  $\prod_{i=1}^{2\ell} x_{\beta_i}(1)$  (which lies in  $X_{2\ell}$ ).

By Theorem 3.2, we have proved

**Proposition 3.6.** *Let  $\mathcal{O}$  be the conjugacy class of an involution of  $Sp(2n)$  in characteristic 2. Then  $\mathcal{O}$  is spherical.  $\square$*

Our aim is to show that a (non-trivial) unipotent conjugacy class  $\mathcal{O}_u$  is spherical if and only if  $u$  is an involution. By [15, Remark 2.14], the orbit  $\mathcal{O}$  is spherical if and only if  $G/H$  is spherical, where  $H$  is the isotropy subgroup of an element in  $\mathcal{O}$ . Moreover,  $G/H$  is spherical if and only if  $G/H^\circ$  is spherical, where  $H^\circ$  denotes the connected component of  $H$ . We shall therefore use the following.

**Lemma 3.7.** *Let  $\mathcal{O}$  be a  $G$ -orbit with isotropy subgroup  $H$ . Then  $\mathcal{O}$  is spherical if and only if  $G/H^\circ$  is spherical.  $\square$*

By Proposition 3.6, we are left to show that if the (non-trivial) unipotent class  $\mathcal{O}$  does not consist of involutions, then  $\mathcal{O}$  is not spherical. Let  $u$  be a unipotent element of order greater than 4, and let  $v$  be an element of order 4 in the subgroup generated by  $u$ . Since  $C(u) \leq C(v)$ , if  $\mathcal{O}_v$  is non-spherical, then  $\mathcal{O}_u$  is also non-spherical. We

are therefore left to consider the set  $X$  of conjugacy classes of elements of order 4. By [19, Theorem 2.2], it is enough to show that the minimal elements in  $X$  are not spherical.

From the explicit definition of a partial order on pairs  $(\lambda, \varepsilon)$  such that  $C_{\lambda, \varepsilon} \leq C_{\mu, \phi} \Leftrightarrow (\lambda, \varepsilon) \leq (\mu, \phi)$  given in [28, §2.10], it follows that the minimal elements in  $X$  are the classes  $3^2 \oplus 1^{2n-6}$  (if  $n \geq 3$ ) and  $4 \oplus 1^{2n-4}$ .

In the following lemmas we deal with these cases. We also consider a case in  $D_n$ .

**Lemma 3.8.** *Let  $\mathcal{O}$  be the unipotent conjugacy class of type  $3^2 \oplus 1^{2n-6}$  in  $C_n$  or  $D_n$ . Then  $\mathcal{O}$  is not spherical ( $p = 2$ ).*

*Proof.* Let  $u$  be an element in  $\mathcal{O} = 3^2 \oplus 1^{2n-6}$  (this exists in  $C_n$  if  $n \geq 3$ ). In  $Sp(2n)$ , we may take  $u = x_{\alpha_2}(1)x_{\gamma-\alpha_2}(1)$ , where  $\gamma$  is the highest short root ( $\gamma = e_1 + e_2$ ), and get  $C(u)^\circ \leq P$ , where  $P$  is the maximal parabolic subgroup  $P_{I \setminus \{\alpha_2\}}$ . Then  $C(u)^\circ = CR$ ,  $C = H_{\alpha_1} \times K$ , where  $K$  is the  $C_{n-3}$ -subgroup of  $G$  corresponding to  $\{\alpha_4, \dots, \alpha_n\}$ , and  $R$  is the subgroup

$$\{u \in R_uP \mid u = \prod_{\alpha \in \Phi^+} x_\alpha(z_\alpha) \mid z_{\alpha_2} = z_{\gamma-\alpha_2}\}$$

of codimension 1 in  $R_uP$ . It follows that  $C(u)^\circ$  fixes the element  $e_{\alpha_2} + e_{\gamma-\alpha_2}$  of  $\mathfrak{g}$ . However, we clearly have  $C(u) \leq C(u^2)$ , and  $u^2 = x_\gamma(1)$ . But  $C(x_\gamma(1))$  fixes the element  $e_\gamma$  of  $\mathfrak{g}$ , since  $x_\gamma(1) = 1 + e_\gamma$  in  $M_{2n}(k)$ . It follows that  $C(u)^\circ$  has 2 linearly independent invariants in  $\mathfrak{g}$  so that we can find linearly independent functions  $f, g$  in  $k[Sp(2n)/C(u)^\circ]$  which are  $B$ -weight vectors of the same weight. Then  $f/g$  is a non-constant  $B$ -invariant rational function on  $Sp(2n)/C(u)^\circ$ , which is therefore not spherical by Rosenlicht's Theorem (see [4, Lemma 1.1]).

Since  $u$  lies in  $SO(2n)$  and both  $e_{\alpha_2} + e_{\gamma-\alpha_2}$  and  $e_\gamma$  are in the Lie algebra of  $SO(2n)$ , the  $SO(2n)$ -orbit of  $u$  is not spherical as well.  $\square$

**Lemma 3.9.** *Let  $\mathcal{O}$  be the unipotent conjugacy class of type  $4 \oplus 1^{2n-4}$  in  $C_n$ . Then  $\mathcal{O}$  is not spherical ( $p = 2$ ).*

*Proof.* Let  $u$  be an element in  $\mathcal{O} = 4 \oplus 1^{2n-4}$ . In  $Sp(2n)$ , we may take  $u = x_{\alpha_1}(1)x_\delta(1)$ , where  $\delta = 2e_2$ , and get  $C(u)^\circ \leq P$ , where  $P$  is the parabolic subgroup  $P_{I \setminus \{\alpha_1, \alpha_2\}}$ . Then  $C(u)^\circ = CR$ , where  $C$  is the  $C_{n-2}$ -subgroup of  $G$  corresponding to  $\{\alpha_3, \dots, \alpha_n\}$ , and  $R$  is a subgroup of  $U$ . In fact  $\dim R = 2n - 2$ , and  $R$  is the product of  $X_\alpha$ 's, where  $\alpha = e_1 \pm e_i, i = 3, \dots, n$ , or  $\alpha = e_1 + e_2$  or  $\alpha = 2e_1$ .

It follows that  $C(u)^\circ$  fixes the first 2 basis vectors  $v_1$  and  $v_2$  of the natural module of  $G$  so that  $Sp(2n)/C(u)^\circ$  is not spherical, as in the previous case.  $\square$

If  $J_\ell = \{\ell + 1, \dots, n\}$  ( $J_n = \emptyset$ ) for  $\ell = 1, \dots, n$ ,  $K_\ell = \{1, 3, \dots, 2\ell - 1, 2\ell + 1, 2\ell + 2, \dots, n\}$  for  $\ell = 1, \dots, \lfloor \frac{n}{2} \rfloor$ , we have established the information presented in Table 2, where  $w(\mathcal{O}) = w_0w_j$ .

We have proved

**Proposition 3.10.** *Let  $\mathcal{O}$  be a non-trivial unipotent conjugacy class of  $Sp(2n)$  in characteristic 2. Then  $\mathcal{O}$  is spherical if and only if it consists of involutions. All non-trivial spherical unipotent classes are listed in Table 2.  $\square$*

3.1.3. *Type  $D_n, n \geq 4$ .* Let  $m = \lfloor \frac{n}{2} \rfloor$ . We put  $\beta_i = e_{2i-1} + e_{2i}, \delta_i = e_{2i-1} - e_{2i}$  for  $i = 1, \dots, m$ . For  $\ell = 1, \dots, m - 1$  we put  $J_\ell = \{2\ell + 1, \dots, n\}, J_m = \emptyset; K_\ell = J_\ell \cup \{1, 3, \dots, 2\ell - 1\}$  for  $\ell = 1, \dots, m$ . Also, if  $n$  is even, we put  $K'_m = \{1, 3, \dots, n - 3, n\}$ .



TABLE 2. Spherical unipotent classes in  $C_n$ ,  $n \geq 2$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$X_\ell = 2^\ell \oplus 1^{2n-2\ell}$ $\ell = 1, \dots, n$	$J_\ell$	$s_{\beta_1} \cdots s_{\beta_\ell}$	$n_{\beta_1} \cdots n_{\beta_\ell}$	$\ell(2n - \ell + 1)$
$Y_{2\ell} = 2_0^{2\ell} \oplus 1^{2n-4\ell}$ $\ell = 1, \dots, \lfloor \frac{n}{2} \rfloor$	$K_\ell$	$s_{\gamma_1} \cdots s_{\gamma_\ell}$	$n_{\gamma_1} \cdots n_{\gamma_\ell}$	$4\ell(n - \ell)$

Let  $u$  be an involution of  $G = SO(2n)$ . Then the partition corresponding to  $u$  is of the form  $\lambda = 2^{c_2} \oplus 1^{c_1}$ , with  $c_2 = 2\ell$ ,  $c_1 = 2n - 4\ell$ ,  $\ell = 1, \dots, m$ .

For each  $\ell = 1, \dots, m - 1$  there are 2 conjugacy classes corresponding to  $\lambda = 2^{2\ell} \oplus 1^{2n-4\ell}$ : we denote by  $X_\ell$  the class  $2_0^{2\ell} \oplus 1^{2n-4\ell}$  and by  $Z_\ell$  the class  $2^{2\ell} \oplus 1^{2n-4\ell}$ . If  $\ell = m$ , then we denote by  $Z_m$  the class  $2^{2m} \oplus 1^{2n-4m}$ . The conjugacy class in  $O(2n)$  corresponding to  $2_0^{2m} \oplus 1^{2n-4m}$  is a single class  $X_m$  in  $SO(2n)$  if  $n$  is odd, while it splits into 2 conjugacy classes  $X_m$  and  $X'_m$  in  $SO(2n)$  if  $n$  is even.

We have

$$\dim Z_\ell = 4\ell(n - \ell), \quad \dim X_\ell = 2\ell(2n - 2\ell - 1) \quad \text{for } \ell = 1, \dots, m,$$

with  $\dim X'_m = \dim X_m$  if  $n$  is even. We have chosen the notation so that for each conjugacy class of involutions  $\mathcal{O}$  in  $G$ , the conjugacy class  $\mathcal{C}$  in  $G_{\mathbb{C}}$  denoted by the same symbol in [11, §4.3] has the same dimension. For the corresponding  $w$ , we write  $w$  as a product of commuting reflections,  $w = s_{\gamma_1} \cdots s_{\gamma_t}$ . It is straightforward to prove that in each case the element  $x = n_{\gamma_1} \cdots n_{\gamma_t}$  lies in  $\mathcal{O}$ . By Theorem 3.2, we have proved

**Proposition 3.11.** *Let  $\mathcal{O}$  be the conjugacy class of an involution of  $SO(2n)$  in characteristic 2. Then  $\mathcal{O}$  is spherical.  $\square$*

Our aim is to show that a (non-trivial) unipotent conjugacy class  $\mathcal{O}_u$  is spherical if and only if  $u$  is an involution. Using the same arguments as in case  $C_n$ , we are left to consider the set  $X$  of conjugacy classes of elements of order 4, and then show that the minimal elements in  $X$  are not spherical. From the explicit definition of a partial order on pairs  $(\lambda, \varepsilon)$  such that  $C_{\lambda, \varepsilon} \leq C_{\mu, \phi} \Leftrightarrow (\lambda, \varepsilon) \leq (\mu, \phi)$  given in [28, §2.10], it follows that the minimal element in  $X$  is the class  $3^2 \oplus 1^{2n-6}$ . By Lemma 3.8, this class is not spherical.

We have established the information presented in Tables 3 and 4, where  $w(\mathcal{O}) = w_0 w_j$ .

TABLE 3. Spherical unipotent classes in  $D_n$ ,  $n \geq 4$ ,  $n = 2m$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$Z_\ell = 2^{2\ell} \oplus 1^{2n-4\ell}$ $\ell = 1, \dots, m$	$J_\ell$	$s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell}$	$n_{\beta_1} n_{\delta_1} \cdots n_{\beta_\ell} n_{\delta_\ell}$	$4\ell(n - \ell)$
$X_\ell = 2_0^{2\ell} \oplus 1^{2n-4\ell}$ $\ell = 1, \dots, m$	$K_\ell$	$s_{\beta_1} \cdots s_{\beta_\ell}$	$n_{\beta_1} \cdots n_{\beta_\ell}$	$2\ell(2n - 2\ell - 1)$
$X'_m = (2_0^{2m})'$	$K'_m$	$s_{\beta_1} \cdots s_{\beta_{m-1}} s_{\alpha_{n-1}}$	$n_{\beta_1} \cdots n_{\beta_{m-1}} n_{\alpha_{n-1}}$	$n(n - 1)$

TABLE 4. Spherical unipotent classes in  $D_n$ ,  $n \geq 4$ ,  $n = 2m + 1$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$Z_\ell = 2^{2\ell} \oplus 1^{2n-4\ell}$ $\ell = 1, \dots, m$	$J_\ell$	$s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell}$	$n_{\beta_1} n_{\delta_1} \cdots n_{\beta_\ell} n_{\delta_\ell}$	$4\ell(n - \ell)$
$X_\ell = 2_0^{2\ell} \oplus 1^{2n-4\ell}$ $\ell = 1, \dots, m$	$K_\ell$	$s_{\beta_1} \cdots s_{\beta_\ell}$	$n_{\beta_1} \cdots n_{\beta_\ell}$	$2\ell(2n - 2\ell - 1)$

We have therefore proved

**Proposition 3.12.** *Let  $\mathcal{O}$  be a non-trivial unipotent conjugacy class of  $SO(2n)$  in characteristic 2. Then  $\mathcal{O}$  is spherical if and only if it consists of involutions. All non-trivial spherical unipotent classes are listed in Table 3 if  $n$  is even and in Table 4 if  $n$  is odd.  $\square$*

*Remark 3.13.* From our discussion, it follows that for  $D_n$  the map  $\pi_G : X^{G'} \rightarrow X^G$  defined in [28, Theorem III.5.2] induces an isomorphism of p.o. sets between  $X_{\text{sph}}^{G'}$  and  $X_{\text{sph}}^G$ , where  $X_{\text{sph}}^{G'}$ ,  $X_{\text{sph}}^G$  are the corresponding sets of spherical unipotent classes. In particular, every spherical unipotent conjugacy class has a representative of the form  $\prod_{\alpha \in K} x_\alpha(1)$  for a certain set of pairwise orthogonal simple roots  $K$ .

**3.2. Exceptional groups.** For the exceptional groups, we use [21, Table 2]. In this table, for each group  $G$ , there are all unipotent conjugacy classes  $\mathcal{O}_u$ , in every characteristic, for which the dimension of  $C(u)$  is greater than a certain number  $l_G$ . From this we deduce Table 5 where the value of  $\phi C(u)/C(u)^\circ$  for  $E_7$  refers to the adjoint group (see [1]).

TABLE 5. Unipotent classes of small dimension in exceptional groups.

$G$	$\dim B$	$u$ with $\dim \mathcal{O}_u \leq \dim B$	$\dim \mathcal{O}_u$	$\phi C(u)/C(u)^\circ$
$E_6$	42	$A_1, 2A_1, 3A_1, A_2$	22, 32, 40, 42	1, 1, 1, 2
$E_7$	70	$A_1, 2A_1, 3A_1', 3A_1'', A_2, 4A_1$	34, 52, 54, 64, 66, 70	1, 1, 1, 1, 2, 1
$E_8$	128	$A_1, 2A_1, 3A_1, A_2, 4A_1$	58, 92, 112, 114, 128	1, 1, 1, 2, 1
$F_4$	28	$A_1, \hat{A}_1(p=2), \hat{A}_1(p \neq 2), \hat{A}_1^{(2)}(p=2), A_1 \hat{A}_1$	16, 16, 22, 22, 28	1, 1, 2, 1, 1
$G_2$	8	$A_1, \hat{A}_1(p=3), \hat{A}_1(p \neq 3), \hat{A}_1^{(3)}(p=3)$	6, 6, 8, 8	1, 1, 1, 1

The unipotent conjugacy classes appearing in Table 5 are the only candidates for being spherical. We shall show that they are all spherical, except for the classes of type  $A_2$  in  $E_6$ ,  $E_7$  and  $E_8$ . Note that when  $p = 2$ , then all classes of involutions appear in Table 5 by the results in [2].

**Lemma 3.14.** *Let  $\mathcal{O}$  be the unipotent conjugacy class of type  $A_2$  in  $E_6$ ,  $E_7$  or  $E_8$ . Then  $\mathcal{O}$  is not spherical (in any characteristic).*

*Proof.* Let  $u$  be an element in  $\mathcal{O}$ . From [21, Table 2], it follows that the type of  $C(u)^\circ$  is independent of the characteristic. For completeness, we determine  $C(u)^\circ$  in all cases.

In  $E_8$ , we may take  $u = x_{\alpha_8}(1)x_{\beta-\alpha_8}(1)$ , where  $\beta$  is the highest root, and get  $C(u)^\circ \leq P$ , where  $P$  is the maximal parabolic subgroup  $P_{I \setminus \{\alpha_8\}}$ . Then  $C(u)^\circ = CR$ , where  $C$  is the  $E_6$ -subgroup of  $G$  corresponding to  $\{\alpha_1, \dots, \alpha_6\}$ , and

$$R = \{g = \prod_{\alpha \in \Phi^+} x_\alpha(k_\alpha) \in R_u P \mid k_{\alpha_8} = k_{\beta-\alpha_8}\}.$$

In  $E_7$ , we may take  $u = x_{\alpha_1}(1)x_{\beta-\alpha_1}(1)$ , where  $\beta$  is the highest root, and get  $C(u)^\circ \leq P$ , where  $P$  is the maximal parabolic subgroup  $P_{I \setminus \{\alpha_1\}}$ . Then  $C(u)^\circ = CR$ , where  $C$  is the  $A_5$ -subgroup of  $G$  corresponding to  $\{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ , and

$$R = \{g = \prod_{\alpha \in \Phi^+} x_\alpha(k_\alpha) \in R_u P \mid k_{\alpha_1} = k_{\beta-\alpha_1}\}.$$

In  $E_6$ , we may take  $u = x_{\alpha_2}(1)x_{\beta-\alpha_2}(1)$ , where  $\beta$  is the highest root, and get  $C(u)^\circ \leq P$ , where  $P$  is the maximal parabolic subgroup  $P_{I \setminus \{\alpha_2\}}$ . Then  $C(u)^\circ = CR$ , where  $C$  is the  $A_2 \times A_2$ -subgroup of  $G$  corresponding to  $\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$ , and

$$R = \{g = \prod_{\alpha \in \Phi^+} x_\alpha(k_\alpha) \in R_u P \mid k_{\alpha_2} = k_{\beta-\alpha_2}\}.$$

It is well known that the class  $A_2$  is not spherical in  $E_6, E_7$  or  $E_8$  over any algebraically closed field of characteristic zero ([24, Theorem (3.1)]). We may now apply [4, Theorem 2.2 (i)]. Note that the groups  $C(u)^\circ$  involved are all defined over  $\mathbb{Z}$ , and the argument in the proof of [4, Theorem 2.2 (i)] is valid in our situation. Therefore  $G/C(u)^\circ$  is not spherical in any positive characteristic. It follows that  $\mathcal{O}_u$  is not spherical by Lemma 3.7.  $\square$

3.2.1. *Type  $E_6$ .* We put

$$\begin{aligned} \beta_1 &= (1, 2, 2, 3, 2, 1), & \beta_2 &= (1, 0, 1, 1, 1, 1), \\ \beta_3 &= (0, 0, 1, 1, 1, 0), & \beta_4 &= (0, 0, 0, 1, 0, 0). \end{aligned}$$

For groups of type  $E_6$  we have to consider  $p = 2, 3$ . If  $p = 3$ , then we may apply the arguments in [5, Theorem 13] to prove that the orbits of type  $A_1, 2A_1$  and  $3A_1$  satisfy (\*). Hence they are spherical, since to handle  $3A_1$  we need results for  $D_4$  which hold due to [8, Theorem 3.4] and its proof (in fact what we need is that the maximal spherical unipotent conjugacy class  $\mathcal{O}'$  of  $D_4$  satisfies (\*) when  $p = 3$ ). Therefore, now assume  $p = 2$ . Then again we may use the proof of [5, Theorem 3.4] to deal with  $A_1$  and  $2A_1$ . Note that in these cases

$$x_{-\beta_1}(1) \in A_1 \cap Bs_{\beta_1}B \cap U^-, \quad x_{-\beta_1}(1)x_{-\beta_2}(1) \in 2A_1 \cap Bs_{\beta_1}s_{\beta_2}B \cap U^-$$

with  $x_{-\beta_1}(1) \sim n_{\beta_1}, x_{-\beta_1}(1)x_{-\beta_2}(1) \sim n_{\beta_1}n_{\beta_2}$ . To deal with  $3A_1$ , we may still use the arguments in the proof of [5, Theorem 3.4] since we have shown in §3.1.3 that the maximal spherical unipotent conjugacy class  $\mathcal{O}'$  of  $D_4$  satisfies (\*) when  $p = 2$ , or directly observe that  $x = n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\beta_4}$  is an involution in  $Bn_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\beta_4}B = Bw_0B$ . Then  $\dim \mathcal{O}_x \geq \ell(w_0) + \text{rk}(1 - w_0) = 40$  by Theorem 3.2, so that  $x \in 3A_1$  by Table 5, since elements in  $A_2$  are not involutions.

We have established the information presented in Tables 6 and 7, where  $w(\mathcal{O}) = w_0w_j$ .

TABLE 6. Spherical unipotent classes in  $E_6, p = 3$  (or any  $p \neq 2$ ).

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$\dim \mathcal{O}$
$A_1$	$\{1, 3, 4, 5, 6\}$	$s_{\beta_1}$	22
$2A_1$	$\{3, 4, 5\}$	$s_{\beta_1}s_{\beta_2}$	32
$3A_1$	$\emptyset$	$w_0$	40

TABLE 7. Spherical unipotent classes in  $E_6$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$A_1$	$\{1, 3, 4, 5, 6\}$	$s_{\beta_1}$	$n_{\beta_1}$	22
$2A_1$	$\{3, 4, 5\}$	$s_{\beta_1}s_{\beta_2}$	$n_{\beta_1}n_{\beta_2}$	32
$3A_1$	$\emptyset$	$w_0$	$n_{\beta_1} \cdots n_{\beta_4}$	40

We have proved

**Proposition 3.15.** *Let  $\mathcal{O}$  be a non-trivial unipotent conjugacy class of  $E_6$ . Then  $\mathcal{O}$  is spherical if and only if it is listed in Tables 6 or 7. If  $p = 2$ , these are precisely the classes consisting of involutions.*  $\square$

We observe that, independent of the characteristic,  $x_{\alpha_1}(1) \in A_1$ ,  $x_{\alpha_1}(1)x_{\alpha_4}(1) \in 2A_1$  and  $x_{\alpha_1}(1)x_{\alpha_4}(1)x_{\alpha_6}(1) \in 3A_1$ . Hence every spherical unipotent conjugacy class has a representative of the form  $\prod_{\alpha \in K} x_{\alpha}(1)$  for a certain set of pairwise orthogonal simple roots  $K$ .

3.2.2. *Type  $E_7$ .* We put

$$\beta_1 = (2, 2, 3, 4, 3, 2, 1), \beta_2 = (0, 1, 1, 2, 2, 2, 1), \beta_3 = (0, 1, 1, 2, 1, 0, 0),$$

$$\beta_4 = \alpha_7, \beta_5 = \alpha_5, \beta_6 = \alpha_3, \beta_7 = \alpha_2.$$

For groups of type  $E_7$  we have to consider  $p = 2, 3$ . If  $p = 3$ , then we may apply the arguments in [5, Theorem 13] to prove that the orbits of type  $A_1, 2A_1, (3A_1)', (3A_1)''$  and  $4A_1$  are spherical, since we need results for  $D_n$  which hold due to [8, Theorem 3.4] and its proof. So now assume  $p = 2$ . Then again we may use the proof of [5, Theorem 3.4] to deal with  $A_1, 2A_1$  and  $(3A_1)''$ . Note that in these cases

$$x_{-\beta_1}(1) \in A_1 \cap Bs_{\beta_1}B \cap U^-, \quad x_{-\beta_1}(1)x_{-\beta_2}(1) \in 2A_1 \cap Bs_{\beta_1}s_{\beta_2}B \cap U^-,$$

$$x_{-\beta_1}(1)x_{-\beta_2}(1)x_{-\beta_4}(1) \in (3A_1)'' \cap Bs_{\beta_1}s_{\beta_2}s_{\beta_4}B \cap U^-.$$

To deal with  $(3A_1)'$  and  $4A_1$ , again we may apply the arguments in [5, Theorem 13], since we need results for  $D_n$  when  $p = 2$ , which we proved in §3.1.3. However, it is also possible to show directly that  $n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\beta_6} \in s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_6}B \cap (3A_1)'$ . To deal with  $4A_1$  one can observe that  $x = n_{\beta_1} \cdots n_{\beta_7}$  is an involution in  $Bw_0B$ . Then  $\dim \mathcal{O}_x \geq \ell(w_0) + \text{rk}(1 - w_0) = 70$  by Theorem 3.2, so that  $x \in 4A_1$  by Table 5.

We have established the information presented in Tables 8 and 9, where  $w(\mathcal{O}) = w_0w_j$ .

TABLE 8. Spherical unipotent classes in  $E_7$ ,  $p = 3$  (or any  $p \neq 2$ ).

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$\dim \mathcal{O}$
$A_1$	$\{2, 3, 4, 5, 6, 7\}$	$s_{\beta_1}$	34
$2A_1$	$\{2, 3, 4, 5, 7\}$	$s_{\beta_1}s_{\beta_2}$	52
$(3A_1)''$	$\{2, 3, 4, 5\}$	$s_{\beta_1}s_{\beta_2}s_{\beta_4}$	54
$(3A_1)'$	$\{2, 5, 7\}$	$s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_6}$	64
$4A_1$	$\emptyset$	$w_0$	70

TABLE 9. Spherical unipotent classes in  $E_7$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$A_1$	$\{2, 3, 4, 5, 6, 7\}$	$s_{\beta_1}$	$n_{\beta_1}$	34
$2A_1$	$\{2, 3, 4, 5, 7\}$	$s_{\beta_1}s_{\beta_2}$	$n_{\beta_1}n_{\beta_2}$	52
$(3A_1)''$	$\{2, 3, 4, 5\}$	$s_{\beta_1}s_{\beta_2}s_{\beta_4}$	$n_{\beta_1}n_{\beta_2}n_{\beta_4}$	54
$(3A_1)'$	$\{2, 5, 7\}$	$s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_6}$	$n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\beta_6}$	64
$4A_1$	$\emptyset$	$w_0$	$n_{\beta_1} \cdots n_{\beta_7}$	70

We have proved

**Proposition 3.16.** *Let  $\mathcal{O}$  be a non-trivial unipotent conjugacy class of  $E_7$ . Then  $\mathcal{O}$  is spherical if and only if it is listed in Table 8 or 9. If  $p = 2$ , these are precisely the classes consisting of involutions.  $\square$*

We observe that, independent of the characteristic,  $x_{\alpha_1}(1) \in A_1$ ,  $x_{\alpha_1}(1)x_{\alpha_4}(1) \in 2A_1$ ,  $x_{\alpha_3}(1)x_{\alpha_5}(1)x_{\alpha_7}(1) \in (3A_1)''$ ,  $x_{\alpha_1}(1)x_{\alpha_4}(1)x_{\alpha_6}(1) \in (3A_1)'$  and  $x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_5}(1)x_{\alpha_7}(1) \in 4A_1$ . Hence every spherical unipotent conjugacy class has a representative of the form  $\prod_{\alpha \in K} x_{\alpha}(1)$  for a certain set of pairwise orthogonal simple roots  $K$ .

3.2.3. *Type  $E_8$ .* We put

$$\beta_1 = (2, 3, 4, 6, 5, 4, 3, 2), \beta_2 = (2, 2, 3, 4, 3, 2, 1, 0), \beta_3 = (0, 1, 1, 2, 2, 2, 1, 0),$$

$$\beta_4 = (0, 1, 1, 2, 1, 0, 0, 0), \beta_5 = \alpha_7, \beta_6 = \alpha_5, \beta_7 = \alpha_3, \beta_8 = \alpha_2.$$

For groups of type  $E_8$  we have to consider  $p = 2, 3, 5$ . If  $p = 3$  or  $5$ , then we may apply the arguments in [5, Theorem 13] to prove that the orbits of types,  $A_1, 2A_1, 3A_1$  and  $4A_1$  are spherical, since to handle  $3A_1$  and  $4A_1$  we need results for  $D_4$  and  $D_6$  which hold due to [8, Theorem 3.4] and its proof. Therefore, now assume  $p = 2$ . Then again we may use the proof of [5, Theorem 3.4] to deal with  $A_1$  and  $2A_1$ . Note that in these cases

$$x_{-\beta_1}(1) \in A_1 \cap Bs_{\beta_1}B \cap U^-,$$

$$x_{-\beta_1}(1)x_{-\beta_2}(1) \in 2A_1 \cap Bs_{\beta_1}s_{\beta_2}B \cap U^-.$$

To deal with  $3A_1$  and  $4A_1$ , again we may apply the arguments in [5, Theorem 13], since we need results for  $D_n$  when  $p = 2$ , which we proved in §3.1.3. However, it is also possible to show directly that  $n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\beta_5} \in s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5}B \cap 3A_1$ . To deal with  $4A_1$  one can observe that  $x = n_{\beta_1} \cdots n_{\beta_8}$  is an involution in  $Bw_0B$ . Then  $\dim \mathcal{O}_x \geq \ell(w_0) + \text{rk}(1 - w_0) = 128$  by Theorem 3.2, so that  $x \in 4A_1$  by Table 5.

We have established the information presented in Tables 10 and 11, where  $w(\mathcal{O}) = w_0w_j$ .

We have proved

**Proposition 3.17.** *Let  $\mathcal{O}$  be a non-trivial unipotent conjugacy class of  $E_8$ . Then  $\mathcal{O}$  is spherical if and only if it is listed in Table 10 or 11. If  $p = 2$ , these are precisely the classes consisting of involutions.  $\square$*

We observe that, independent of the characteristic,  $x_{\alpha_1}(1) \in A_1$ ,  $x_{\alpha_1}(1)x_{\alpha_4}(1) \in 2A_1$ ,  $x_{\alpha_1}(1)x_{\alpha_4}(1)x_{\alpha_6}(1) \in 3A_1$ , and  $x_{\alpha_1}(1)x_{\alpha_4}(1)x_{\alpha_6}(1)x_{\alpha_8}(1) \in 4A_1$ . Hence every spherical unipotent conjugacy class has a representative of the form  $\prod_{\alpha \in K} x_{\alpha}(1)$  for a certain set of pairwise orthogonal simple roots  $K$ .

TABLE 10. Spherical unipotent classes in  $E_8$ ,  $p = 3$  or  $5$  (or any  $p \neq 2$ ).

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$\dim \mathcal{O}$
$A_1$	$\{1, 2, 3, 4, 5, 6, 7\}$	$s_{\beta_1}$	58
$2A_1$	$\{2, 3, 4, 5, 6, 7\}$	$s_{\beta_1}s_{\beta_2}$	92
$3A_1$	$\{2, 3, 4, 5\}$	$s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5}$	112
$4A_1$	$\emptyset$	$w_0$	128

TABLE 11. Spherical unipotent classes in  $E_8$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$A_1$	$\{1, 2, 3, 4, 5, 6, 7\}$	$s_{\beta_1}$	$n_{\beta_1}$	58
$2A_1$	$\{2, 3, 4, 5, 6, 7\}$	$s_{\beta_1}s_{\beta_2}$	$n_{\beta_1}n_{\beta_2}$	92
$3A_1$	$\{2, 3, 4, 5\}$	$s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5}$	$n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\beta_5}$	112
$4A_1$	$\emptyset$	$w_0$	$n_{\beta_1} \cdots n_{\beta_8}$	128

3.2.4. *Type  $F_4$ .* We put

$$\begin{aligned} \beta_1 &= (2, 3, 4, 2), & \beta_2 &= (0, 1, 2, 2), \\ \beta_3 &= (0, 1, 2, 0), & \beta_4 &= (0, 1, 0, 0); \end{aligned}$$

also  $\gamma_1$  is the highest short root  $(1, 2, 3, 2)$ .

For groups of type  $F_4$  we have to consider  $p = 2, 3$ . If  $p = 3$ , then we may apply the arguments in [5, Theorem 13] to prove that the orbits of types  $A_1, \tilde{A}_1$  and  $A_1\tilde{A}_1$  are spherical, since to handle  $A_1\tilde{A}_1$  we need results for  $D_4$  which hold due to [8, Theorem 3.4] and its proof. Therefore, now assume  $p = 2$ . Here there are more conjugacy classes  $\mathcal{O}_u$  (due to the presence of the graph automorphism of  $G$ ) for which  $\dim \mathcal{O}_u \leq \dim B$  (see Table 5). Each class consists of involutions. We may take the following representatives:

$$\begin{aligned} x_{-\beta_1}(1) &\in A_1 \cap Bs_{\beta_1}B \cap U^-, \\ x_{-\gamma_1}(1) &\in \tilde{A}_1 \cap Bs_{\gamma_1}B \cap U^-. \end{aligned}$$

To deal with  $\tilde{A}_1^{(2)}$ , we observe that  $u = x_{\beta_1}(1)x_{\gamma_1}(1) \in \tilde{A}_1^{(2)}$  by [2, (13.1)]. Let  $K$  be the  $C_2$ -subgroup of  $G$  with basis  $\{(1, 1, 1, 0), \beta_2\}$ . Then  $\beta_1$  and  $\gamma_1$  are the highest long and short roots in  $K$ , respectively. A direct calculation in  $C_2$  shows that  $u$  is conjugate to  $v = x_{\beta_1}(1)x_{\beta_2}(1)$ ; hence

$$x_{-\beta_1}(1)x_{-\beta_2}(1) \in \tilde{A}_1^{(2)} \cap Bs_{\beta_1}s_{\beta_2}B \cap U^-.$$

Finally, to deal with  $A_1\tilde{A}_1$  we observe that  $x = n_{\beta_1} \cdots n_{\beta_4}$  is an involution in  $Bw_0B$ . Then  $\dim \mathcal{O}_x \geq \ell(w_0) + \text{rk}(1 - w_0) = 28$  by Theorem 3.2, so that  $x \in A_1\tilde{A}_1$  by Table 5. We have established the information presented in Tables 12 and 13, where  $w(\mathcal{O}) = w_0w_j$ .

**Proposition 3.18.** *Let  $\mathcal{O}$  be a non-trivial unipotent conjugacy class in  $F_4$ . Then  $\mathcal{O}$  is spherical if and only if it is listed in Table 12 or 13. If  $p = 2$ , these are precisely the classes consisting of involutions.*

We note that in  $G_{\mathbb{C}}$  there is an involution  $\sigma$  such that  $C(\sigma)$  is of type  $B_4$  and such that  $\mathcal{O}_{\sigma}$  lies over  $s_{\gamma_1}$ . We also observe that if  $p = 2$ , then  $x_{\alpha_4}(1) \in A_1$ ,

TABLE 12. Spherical unipotent classes in  $F_4$ ,  $p = 3$  (or any  $p \neq 2$ ).

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$\dim \mathcal{O}$
$A_1$	$\{2, 3, 4\}$	$s_{\beta_1}$	16
$\tilde{A}_1$	$\{2, 3\}$	$s_{\beta_1} s_{\beta_2}$	22
$A_1 \tilde{A}_1$	$\emptyset$	$w_0$	28

TABLE 13. Spherical unipotent classes in  $F_4$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$A_1$	$\{2, 3, 4\}$	$s_{\beta_1}$	$n_{\beta_1}$	16
$\tilde{A}_1$	$\{1, 2, 3\}$	$s_{\gamma_1}$	$n_{\gamma_1}$	16
$\tilde{A}_1^{(2)}$	$\{2, 3\}$	$s_{\beta_1} s_{\beta_2}$	$n_{\beta_1} n_{\beta_2}$	22
$A_1 \tilde{A}_1$	$\emptyset$	$w_0$	$n_{\beta_1} n_{\beta_2} n_{\beta_3} n_{\beta_4}$	28

$x_{\alpha_1}(1) \in \tilde{A}_1$ ,  $x_{\alpha_1}(1)x_{\alpha_3}(1) \in A_1 \tilde{A}_1$  and these exhaust the conjugacy classes of involutions with a representative of the form  $\prod_{i \in K} x_{\alpha_i}(1)$ ,  $K \subseteq \Pi$ . In particular,  $\tilde{A}_1^{(2)}$  has no representative of the form  $\prod_{i \in K} x_{\alpha_i}(1)$ .

3.2.5. *Type  $G_2$ .* We put  $\beta_1 = (3, 2)$ ,  $\beta_2 = \alpha_1$ ,  $\gamma_1 = (2, 1)$  (the highest short root).

For groups of type  $G_2$  we have to consider  $p = 2, 3$ . The p.o. set of unipotent conjugacy classes is described in the tables in [28, Proposition II 10.4].

If  $p = 2$ , the classification of unipotent conjugacy classes  $\mathcal{O}$  for which  $\dim \mathcal{O} \leq \dim B$  is the same as over  $\mathbb{C}$  and each class consists of involutions. We may take

$$x_{-\beta_1}(1) \in A_1 \cap Bs_{\beta_1}B \cap U^-.$$

To deal with  $\tilde{A}_1$  we observe that  $x = n_{\beta_1} n_{\beta_2}$  is an involution in  $Bw_0B$ . Then  $\dim \mathcal{O}_x \geq \ell(w_0) + \text{rk}(1 - w_0) = 8$  by Theorem 3.2, so that  $x \in \tilde{A}_1$  by Table 5.

Therefore, now assume  $p = 3$ . Here there are more conjugacy classes  $\mathcal{O}$  for which  $\dim \mathcal{O}_u \leq \dim B$  (see Table 5), due to the presence of the graph automorphism of  $G$ . We may take the following representatives:

$$\begin{aligned} x_{-\beta_1}(1) &\in A_1 \cap Bs_{\beta_1}B \cap U^-, \\ x_{-\gamma_1}(1) &\in \tilde{A}_1 \cap Bs_{\gamma_1}B \cap U^-. \end{aligned}$$

To deal with  $\tilde{A}_1^{(3)}$ , we observe that since  $A_1 \leq \tilde{A}_1^{(3)}$  and  $\tilde{A}_1 \leq \tilde{A}_1^{(3)}$ , we get  $s_{\beta_1} \leq w(\tilde{A}_1^{(3)})$  and  $s_{\gamma_1} \leq w(\tilde{A}_1^{(3)})$  so that  $w(\tilde{A}_1^{(3)}) = w_0$ , and we are done since  $\dim \tilde{A}_1^{(3)} = 8$ .

We have established the information presented in Tables 14 and 15, where  $w(\mathcal{O}) = w_0 w_j$ .

TABLE 14. Spherical unipotent classes in  $G_2$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$A_1$	$\{1\}$	$s_{\beta_1}$	$n_{\beta_1}$	6
$\tilde{A}_1$	$\emptyset$	$s_{\beta_1} s_{\beta_2}$	$n_{\beta_1} n_{\beta_2}$	8

TABLE 15. Spherical unipotent classes in  $G_2$ ,  $p = 3$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$\dim \mathcal{O}$
$A_1$	$\{1\}$	$s_{\beta_1}$	6
$\tilde{A}_1$	$\{2\}$	$s_{\gamma_1}$	6
$\tilde{A}_1^{(3)}$	$\emptyset$	$s_{\beta_1} s_{\beta_2}$	8

**Proposition 3.19.** *Let  $\mathcal{O}$  be a non-trivial unipotent conjugacy class in  $G_2$ . Then  $\mathcal{O}$  is spherical if and only if it is listed in Table 14 or 15. If  $p = 2$ , these are precisely the classes consisting of involutions.  $\square$*

Note that if  $p = 3$ , then  $x_{\alpha_2}(1) \in A_1$ ,  $x_{\alpha_1}(1) \in \tilde{A}_1$ , while  $\tilde{A}_1^{(3)}$  has no representative of the form  $\prod_{i \in K} x_{\alpha_i}(1)$ ,  $K \subseteq \Pi$ .

This completes the classification of unipotent spherical conjugacy classes in bad characteristic and the proof of Theorem 3.4, which clearly holds for every connected reductive algebraic group in characteristic 2.

#### 4. SYMMETRIC HOMOGENEOUS SPACES

In this section we shall prove that if  $G$  is a connected reductive algebraic group over an algebraically closed field  $k$  of characteristic 2, then  $H = C(\sigma)$  is a spherical subgroup of  $G$  for every involutory automorphism  $\sigma$  of  $G$ . This was proved by Vust in [32] in characteristic zero (see also [23] over  $\mathbb{C}$ ). Then Springer extended the result to odd characteristic in [29]. In [27], Seitz gives an alternative proof of Springer’s result.

We start by observing that it is enough to assume  $G$  simple, for if  $G$  is connected reductive, then  $G = S G_1 \cdots G_t$ , where  $S$  is the connected component of  $Z(G)$  and  $G_1, \dots, G_t$  are the simple components of  $G$ . Let  $\sigma$  be an involutory automorphism of  $G$ . Then  $\sigma$  fixes  $S$  and induces a permutation  $\rho$  of the set  $\{1, \dots, t\}$ . If  $\rho$  is non-trivial, then it is the product of disjoint cycles of length 2. Suppose one of these cycles is  $(1, 2)$ . Then  $\sigma$  induces an isomorphism  $\varphi : G_1 \rightarrow G_2$ . Let  $B_1 = T_1 U_1$  be a Borel subgroup of  $G_1$ , where  $U_1$  is the unipotent radical of  $B_1$  and  $T_1$  is a maximal torus. Let  $V_1$  be the maximal unipotent subgroup of the Borel subgroup of  $G_1$  opposite to  $B_1$ , and let  $R = B_1 V_1^\varphi \leq G_1 G_2$ . Then  $C_{G_1 G_2}(\sigma) \cap R$  is finite so that  $C_{G_1 G_2}(\sigma)$  is a spherical subgroup of  $G_1 G_2$ , since  $\dim C_{G_1 G_2}(\sigma) + \dim B_1 V_1^\varphi = \dim G_1 G_2$ . Of course  $C_S(\sigma)$  is a spherical subgroup of  $S$ , so to conclude it is enough to assume that  $G$  is a (connected) simple algebraic group.

In the following we assume that  $G$  is a simple algebraic group over  $k$ , of any characteristic. We shall use a generalization of Theorem 3.2.

Let  $\tau$  be an automorphism of  $G$  fixing  $B$  and  $T$ , and consider  $G : \langle \tau \rangle$ . Assume  $\tau$  has order  $r$ . Then we have the Bruhat decomposition

$$G : \langle \tau \rangle = \bigcup_{w \in W, i \in \mathbb{Z}_r} B \tau^i w B.$$

Let  $\mathcal{O}$  be a  $G$ -orbit in  $G : \langle \tau \rangle$ . Then there exists a unique  $i \in \mathbb{Z}_r$  such that  $\mathcal{O} \subseteq \bigcup_{w \in W} B \tau^i w B$ , and there is a unique  $z = z(\mathcal{O})$  such that  $\mathcal{O} \cap B \tau^i z B$  is open dense in  $\mathcal{O}$ . In particular,

$$(4.1) \quad \overline{\mathcal{O}} = \overline{\mathcal{O} \cap B \tau^i z B} \subseteq \overline{B \tau^i z B} = \tau^i \overline{B z B}.$$



It follows that if  $y$  is an element of  $\overline{\mathcal{O}}$  and  $y \in B\tau^i wB$ , then  $w \leq z$  in the Chevalley-Bruhat order of  $W$ . Let us observe that if  $\mathcal{O}$  is spherical and  $B.x$  is the dense  $B$ -orbit in  $\mathcal{O}$ , then  $B.x \subseteq B\tau^i zB$ .

We still denote by  $\tau$  the automorphism of  $E = X(T) \otimes \mathbb{R}$  induced by  $\tau$  (i.e.  $\tau(\chi)(t) = \chi(\tau^{-1}t\tau)$  for every  $\chi \in X = X(T)$ ,  $t \in T$ ). For every  $w \in W$  we put

$$T^{\tau w} = \{t \in T \mid w^{-1}\tau^{-1}t\tau w = t\}.$$

We have  $\dim T^{\tau w} = n - \text{rk}(1 - \tau w)$ .

**Theorem 4.1.** *Let  $\sigma \in G : \langle \tau \rangle$ ,  $\sigma = \tau^i g$ , for a certain  $g \in G$ ,  $i \in \mathbb{Z}_r$ , and let  $\mathcal{O} = G.\sigma$ . Suppose that  $\mathcal{O}$  contains an element  $x \in B\tau^i wB$ , for a certain  $w \in W$ , where  $U_{w^{-1}}$  is  $\tau$ -invariant. Then*

$$\dim B.x \geq \ell(w) + \text{rk}(1 - \tau^i w).$$

*In particular,  $\dim \mathcal{O} \geq \ell(w) + \text{rk}(1 - \tau^i w)$ . If, in addition,*

$$\dim \mathcal{O} \leq \ell(w) + \text{rk}(1 - \tau^i w),$$

*then  $\mathcal{O}$  is spherical,  $w = z(\mathcal{O})$  and  $B.x$  is the dense  $B$ -orbit in  $\mathcal{O}$ . □*

*Proof.* Without loss of generality, we may assume  $x = \tau^i \dot{w}u$ , for a certain representative  $\dot{w}$  of  $w$  in  $N$  and  $u \in U$ . Let us estimate the dimension of the orbit  $B_{w^{-1}}.x$ , where  $B_{w^{-1}} = TU_{w^{-1}}$ .

Let  $vt \in C_{B_{w^{-1}}}(x)$ , with  $v \in U_{w^{-1}}$ ,  $t \in T$ . Then

$$\tau^i \dot{w}vvt = vt\tau^i \dot{w}u = \tau^i \tau^{-i} vt\tau^i \dot{w}u = \tau^i \tau^{-i} v\tau^i \tau^{-i} t\tau^i \dot{w}u = \tau^i \tau^{-i} v\tau^i \dot{w} \dot{w}^{-1} \tau^{-i} t\tau^i \dot{w}u$$

so that, by the uniqueness of the decomposition,  $v = 1$  since  $\tau^{-i} v\tau^i \in U_{w^{-1}}$ . Moreover, from  $ut = \dot{w}^{-1} \tau^{-i} t\tau^i \dot{w}u$  it follows that  $t = \dot{w}^{-1} \tau^{-i} t\tau^i \dot{w}$ . Therefore  $C_{B_{w^{-1}}}(x) \leq T^{\tau^i w}$ . Thus  $\dim C_{B_{w^{-1}}}(x) \leq \dim T^{\tau^i w} = n - \text{rk}(1 - \tau^i w)$  and

$$\begin{aligned} \dim B_{w^{-1}}.x &= \dim B_{w^{-1}} - \dim C_{B_{w^{-1}}}(x) \geq \ell(w) + n - n + \text{rk}(1 - \tau^i w) \\ &= \ell(w) + \text{rk}(1 - \tau^i w). \end{aligned}$$

If, in addition,  $\ell(w) + \text{rk}(1 - \tau^i w) \geq \dim \mathcal{O}$ , then  $\dim \mathcal{O} = \ell(w) + \text{rk}(1 - \tau^i w)$ . In particular  $B.x$  is the dense  $B$ -orbit in  $\mathcal{O}$ . □

We observe that the condition  $\tau(U_{w^{-1}}) = U_{w^{-1}}$  is clearly satisfied if  $w = s_{r_1} \cdots s_{r_k}$ , where  $r_1, \dots, r_k$  are roots fixed by  $\tau$ , or if  $\{r_1, \dots, r_k\}$  is a  $\tau$ -invariant set of pairwise orthogonal roots.

In the remainder of this section we assume that the characteristic of  $k$  is 2.

In the previous section we have already shown that  $C(\sigma)$  is a spherical subgroup of  $G$  when  $\sigma$  is an inner involution of  $G$ . We are therefore left to deal with outer involutions, which exist only in the following cases:  $A_\ell$ ,  $\ell \geq 2$ ,  $D_\ell$ ,  $\ell \geq 4$  and  $E_6$ .

To prove that the fixed point subgroup of any outer involution of  $G$  is spherical, we shall use the classification of outer involutions of  $G$  as in [28] and [2]. We fix the graph automorphism  $\tau$  (of order 2) of  $G$ , and for each  $G$ -orbit  $\mathcal{O}$  of outer involutions of  $G$  we show that there exists an involution  $w = s_{\delta_1} \cdots s_{\delta_\ell} \in W$  such that  $\mathcal{O} \cap B\tau wB$  is not empty,  $\dim \mathcal{O} = \ell(w) + \text{rk}(1 - \tau w)$ , the  $\delta_i$ 's are pairwise orthogonal positive roots and  $\{\delta_1, \dots, \delta_\ell\}$  is  $\tau$ -invariant. By Theorem 4.1,  $\mathcal{O}$  is spherical (with  $z(\mathcal{O}) = w$ ). We consider the various cases: if  $\sigma = \tau g \in G : \langle \tau \rangle$ ,  $C(\sigma)$  stands for  $C_G(\sigma)$ . In each case we use the notation introduced in Section 3.

4.1. **Type  $A_n$ ,  $n = 2m$ ,  $m \geq 1$ .** We take  $G = SL(2m + 1)$ . In this case there is only one (class of) outer involution  $\tau$ , the graph automorphism of  $SL(2m + 1)$ , and  $C(\tau) = SO(2m + 1)$ . Then

$$\dim SL(2m + 1)/SO(2m + 1) = 2m^2 + 3m,$$

which is the dimension of a Borel subgroup of  $SL(2m + 1)$ . We may take

$$x = \tau n_{\beta_1} \cdots n_{\beta_m} \in \tau w_0 B.$$

Then  $x^2 = 1$  since  $\tau(\beta_k) = \beta_k$  for each  $k$ , so that  $x$  lies in  $\mathcal{O}_\tau$ . Since  $\tau w_0 = -1$ , we get

$$\ell(w_0) + \text{rk}(1 - \tau w_0) = \dim B,$$

and we are done.

We have established the information presented in Table 16, where  $w(\mathcal{O}) = w_0 w_J$ .

TABLE 16. Outer involutions of  $SL(2m + 1)$ ,  $m \geq 1$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau B w(\mathcal{O}) B$	$\dim \mathcal{O}$
$\tau$	$\emptyset$	$w_0 = s_{\beta_1} \cdots s_{\beta_m}$	$\tau n_{\beta_1} \cdots n_{\beta_m}$	$2m^2 + 3m$

4.2. **Type  $A_n$ ,  $n = 2m - 1$ ,  $m \geq 2$ .** We take  $G = SL(2m)$ . In this case there are two (classes of) outer involutions:  $\tau$  and  $\tau x_{\beta_1}(1)$ , with  $C(\tau) = Sp(2m)$ ,  $C(\tau x_{\beta_1}(1)) = C_{Sp(2m)}(x_{\beta_1}(1))$ . We have

$$\dim SL(2m)/Sp(2m) = 2m^2 - m - 1.$$

We put  $J = \{1, 3, \dots, n\}$ ,  $w = w_0 w_J$ . We have

$$\ell(w) = \ell(w_0) - \ell(w_J) = m(2m - 1) - m = 2m^2 - 2m$$

and

$$\text{rk}(1 - \tau w) = \text{rk}(w_J + 1) = m - 1$$

since  $\tau w_0 = -1$  and the (simple) roots in  $J$  are pairwise orthogonal. Hence

$$\ell(w) + \text{rk}(1 - \tau w) = 2m^2 - m - 1 = \dim SL(2m)/Sp(2m).$$

We are left to exhibit a conjugate  $x \in \tau B w B$  of  $\tau$ . For this purpose we distinguish 2 cases.

Assume  $m$  is even,  $m = 2r$ . Then there are precisely  $m$  positive roots  $\gamma_1, \dots, \gamma_m$  for which  $w(\gamma) = -\gamma$ , namely

$$\gamma_{2i-1} = e_{2i-1} - e_{2m-2i+1}, \quad \gamma_{2i} = e_{2i} - e_{2m+2-2i}$$

for  $i = 1, \dots, r$  and

$$w = s_{\gamma_1} \cdots s_{\gamma_m}.$$

We also note that  $\tau$  exchanges  $\gamma_{2i-1}$  and  $\gamma_{2i}$  for each  $i = 1, \dots, r$ . In this case we take

$$x = g \tau g^{-1},$$

where  $g$  is the involution  $g = x_{-\gamma_1}(1) x_{-\gamma_3}(1) \cdots x_{-\gamma_{m-1}}(1)$ . Then

$$x = \tau x_{-\gamma_1}(1) x_{-\gamma_2}(1) x_{-\gamma_3}(1) \cdots x_{-\gamma_{m-1}}(1) x_{-\gamma_m}(1) \in \tau B w B,$$

and we are done.

If  $m$  is odd,  $m - 1 = 2r$ , then there are precisely  $m - 1$  positive roots  $\gamma_1, \dots, \gamma_{m-1}$  for which  $w(\gamma) = -\gamma$ , namely

$$\gamma_{2i-1} = e_{2i-1} - e_{2m-2i+1}, \quad \gamma_{2i} = e_{2i} - e_{2m+2-2i}$$

for  $i = 1, \dots, r$  and

$$w = s_{\gamma_1} \cdots s_{\gamma_{m-1}}.$$

We also note that  $\tau$  exchanges  $\gamma_{2i-1}$  and  $\gamma_{2i}$  for each  $i = 1, \dots, r$ . In this case we take

$$x = g\tau g^{-1},$$

where  $g$  is the involution  $g = x_{-\gamma_1}(1)x_{-\gamma_3}(1) \cdots x_{-\gamma_{m-2}}(1)$ . Then

$$x = \tau x_{-\gamma_1}(1)x_{-\gamma_2}(1)x_{-\gamma_3}(1) \cdots x_{-\gamma_{m-2}}(1)x_{-\gamma_{m-1}}(1) \in \tau BwB,$$

and we are done.

We finally deal with  $\tau x_{\beta_1}(1)$ ,  $H = C(\tau x_{\beta_1}(1)) = C_{Sp(2m)}(x_{\beta_1}(1))$ . We have  $\dim SL(2m)/H = \dim SL(2m)/Sp(2m) + \dim \mathcal{O}' = 2m^2 - m - 1 + 2m = 2m^2 + m - 1$ , where  $\mathcal{O}'$  is the  $Sp(2m)$ -orbit of  $x_{\beta_1}(1)$ , which has dimension  $2m$  (note that  $\dim SL(2m)/H = \dim SL(2m)/SO(2m)$ , and  $SO(2m)$  is the centralizer of an outer involution of  $SL(2m)$  if the characteristic is not 2). Therefore  $\dim SL(2m)/H$  is the dimension of a Borel subgroup of  $SL(2m)$ . As in the case when  $n$  is even, we take  $w = w_0$ ,

$$x = \tau n_{\beta_1} \cdots n_{\beta_m} \in \tau w_0 B.$$

Then  $x^2 = 1$  since  $\tau(\beta_k) = \beta_k$  for each  $k$ , and for dimensional reasons,  $x$  is conjugate to  $\tau x_{\beta}(1)$ .

We have established the information presented in Tables 17 and 18, where  $w(\mathcal{O}) = w_0 w_J$ .

TABLE 17. Outer involutions of  $SL(2m)$ ,  $m \geq 2$ ,  $m$  even,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$\tau$	$\{1, 3, \dots, 2m - 1\}$	$s_{\gamma_1} \cdots s_{\gamma_m}$	$\tau x_{-\gamma_1}(1) \cdots x_{-\gamma_m}(1)$	$2m^2 - m - 1$
$\tau x_{\beta_1}(1)$	$\emptyset$	$w_0$	$\tau n_0$	$2m^2 + m - 1$

TABLE 18. Outer involutions of  $SL(2m)$ ,  $m \geq 3$ ,  $m$  odd,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$\tau$	$\{1, 3, \dots, 2m - 1\}$	$s_{\gamma_1} \cdots s_{\gamma_{m-1}}$	$\tau x_{-\gamma_1}(1) \cdots x_{-\gamma_{m-1}}(1)$	$2m^2 - m - 1$
$\tau x_{\beta_1}(1)$	$\emptyset$	$w_0$	$\tau n_0$	$2m^2 + m - 1$

4.3. **Type  $D_n$ ,  $n \geq 4$ .** To deal with  $G$  of type  $D_n$  we shall, as usual, consider  $G = SO(2n)$ . Then the outer involutions of  $G$  are obtained by conjugation with involutions of  $O(2n)$ . Note that if  $n = 4$  and if  $G$  is adjoint or simply-connected, then there are other outer involutions in  $\text{Aut } G$ . However, they are conjugate in  $\text{Aut } G$  to involutions in  $O(2n)$ .

Let  $\tau$  be the involution of  $O(2n)$  inducing the graph automorphism of  $SO(2n)$ , i.e. the graph automorphism acting trivially on  $\langle X_{\pm\alpha_i} \mid i \in \{1, \dots, n - 2\} \rangle$  and such that  $x_{\alpha_{n-1}}(\xi) \leftrightarrow x_{\alpha_n}(\xi)$ ,  $x_{-\alpha_{n-1}}(\xi) \leftrightarrow x_{-\alpha_n}(\xi)$  for  $\xi \in k$ .

The classes of involutions in  $O(2n) \setminus SO(2n)$  correspond to partitions  $2^\ell \oplus 1^{2n-2\ell}$  for  $\ell \leq n$  with  $\ell$  odd, with  $\tau$  corresponding to  $2 \oplus 1^{2n-2}$ . Let  $\mathcal{O}_\ell$  be the class corresponding to  $2^\ell \oplus 1^{2n-2\ell}$ . From [28, 2.9 b)] we get

$$\dim \mathcal{O}_\ell = \dim \mathcal{O}_{\ell, Sp(2n, \mathbb{C})} - 2n + \lambda_1^*,$$

where  $\lambda_1^* = c_1 + c_2 = (2n - 2\ell) + \ell = 2n - \ell$ . Hence

$$\dim \mathcal{O}_\ell = k(2n - \ell + 1) - 2n + 2n - \ell = \ell(2n - \ell)$$

for  $\ell \leq n$  with  $\ell$  odd.

Let

$$\mu_1 = e_1 - e_n, \quad \nu_1 = e_1 + e_n, \quad w = s_{\mu_1} s_{\nu_1}.$$

Then

$$\ell(w) + \text{rk}(1 - \tau w) = 2(n - 1) + 1 = 2n - 1 = \dim \mathcal{O}_1$$

and  $\tau(\mu_1) = \nu_1$ . We have

$$n_{\nu_1} \tau n_{\nu_1} = \tau n_{\mu_1} n_{\nu_1}$$

so that

$$\tau n_{\mu_1} n_{\nu_1} \in \mathcal{O}_1 \cap wB.$$

To deal with the remaining classes, we put  $m = \lfloor \frac{n}{2} \rfloor$  and

$$\mu_i = e_{2i-2} - e_{2i-1}, \quad \nu_i = e_{2i-2} + e_{2i-1}, \quad w_i = s_{\mu_1} s_{\nu_1} \cdots s_{\mu_i} s_{\nu_i}$$

for  $i = 2, \dots, m$ .

Arguing as above, we can prove that

$$\ell(w_\ell) + \text{rk}(1 - \tau w_\ell) = \dim \mathcal{O}_{2\ell-1}$$

and

$$\tau n_{\mu_1} n_{\nu_1} \cdots n_{\mu_\ell} n_{\nu_\ell} \in \mathcal{O}_{2\ell-1} \cap w_\ell B$$

for  $\ell = 2, \dots, m$ . In fact it is enough to count the number of Jordan blocks of dimension 2 in  $\tau n_{\mu_1} n_{\nu_1} \cdots n_{\mu_\ell} n_{\nu_\ell}$ : in  $\tau n_{\mu_1} n_{\nu_1}$  there is 1, and in  $n_{\mu_i} n_{\nu_i}$  there are 2 for each  $i = 2, \dots, \ell$ .

If  $n$  is even, then there are  $\frac{1}{2}n$  conjugacy classes of outer involutions, and we are done. In particular, the maximal one is  $2^{n-1} \oplus 1^2$  and corresponds to  $w = s_{\mu_1} s_{\nu_1} \cdots s_{\mu_m} s_{\nu_m} = w_0 = -1$ .

If  $n$  is odd, then there are  $\frac{1}{2}(n + 1)$  conjugacy classes of outer involutions: the maximal one is  $\mathcal{O}_n = 2^n$ , which is the only one not in the previous list. We have

$$\dim \mathcal{O}_n = n^2.$$

Let  $n_0$  be any representative in  $N$  of  $w_0$  with  $n_0$  of order 2 and commuting with  $\tau$ . Then  $x = \tau n_0$  is an involution in  $\tau w_0 B$ . Since

$$\ell(w_0) + \text{rk}(1 - \tau w_0) = n^2 - n + n = n^2$$

by Theorem 4.1, we have  $\dim B.x \geq n^2$ . However,  $x$  is an involution in  $O(2n) \setminus SO(2n)$ , so that  $\dim G.x \leq \dim \mathcal{O}_n = n^2$ . Therefore  $x$  lies in  $\mathcal{O}_n \cap \tau w_0 B$ , and we are done.

Let  $J_\ell = \{2\ell, 2\ell + 1, \dots, n\}$  for  $\ell = 1 \dots, m$ . We have established the information presented in Tables 19 and 20, where  $w(\mathcal{O}) = w_0 w_j$ .

TABLE 19. Outer involutions of  $D_n$ ,  $n \geq 4$ ,  $n = 2m$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$2^{2\ell-1} \oplus 1^{2n-4\ell+2}$ $\ell = 1, \dots, m-1$	$J_\ell$	$s_{\mu_1} s_{\nu_1} \cdots s_{\mu_\ell} s_{\nu_\ell}$	$\tau n_{\mu_1} n_{\nu_1} \cdots n_{\mu_\ell} n_{\nu_\ell}$	$(2\ell-1)(2n-2\ell+1)$
$2^{n-1} \oplus 1^2$	$\emptyset$	$w_0 = s_{\mu_1} s_{\nu_1} \cdots s_{\mu_m} s_{\nu_m}$	$\tau n_{\mu_1} n_{\nu_1} \cdots n_{\mu_m} n_{\nu_m}$	$n^2 - 1$

TABLE 20. Outer involutions of  $D_n$ ,  $n \geq 4$ ,  $n = 2m + 1$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$2^{2\ell-1} \oplus 1^{2n-4\ell+2}$ $\ell = 1, \dots, m$	$J_\ell$	$s_{\mu_1} s_{\nu_1} \cdots s_{\mu_\ell} s_{\nu_\ell}$	$\tau n_{\mu_1} n_{\nu_1} \cdots n_{\mu_\ell} n_{\nu_\ell}$	$(2\ell-1)(2n-2\ell+1)$
$2^n$	$\emptyset$	$w_0$	$\tau n_0$	$n^2$

4.4. **Type  $E_6$ .** There are two (classes of) outer involutions,  $\tau$  and  $\tau x_{\beta_1}(1)$ , where  $\tau$  is the graph automorphism of  $G$ . We recall from §3.2.1 that

$$\begin{aligned} \beta_1 &= (1, 2, 2, 3, 2, 1), & \beta_2 &= (1, 0, 1, 1, 1, 1), \\ \beta_3 &= (0, 0, 1, 1, 1, 0), & \beta_4 &= (0, 0, 0, 1, 0, 0). \end{aligned}$$

Note that each  $\beta_i$  is fixed by  $\tau$ .

Let us start with  $\tau x_{\beta_1}(1)$ . We have  $K = C(\tau x_{\beta_1}(1)) \cong C_{F_4}(x_{\beta_1}(1))$ ,  $\dim K = 36$ . Let  $x = \tau n_{\beta_1} n_{\beta_2} n_{\beta_3} n_{\beta_4}$ . Since  $x$  is an involution in  $\tau w_0 B$ , with  $\ell(w_0) + \text{rk}(1 - \tau w_0) = 36 + 6 = \dim E_6/K$ , it follows that  $\tau x_{\beta_1}(1) \sim x$ .

To deal with  $\tau$ , we put  $\delta_1 = (1, 1, 2, 2, 1, 1)$ ,  $\delta_2 = (1, 1, 1, 2, 2, 1)$ . We have

$$\dim E_6/F_4 = 26.$$

Note that  $\tau(\delta_1) = \delta_2$  and

$$\ell(s_{\delta_1} s_{\delta_2}) + \text{rk}(1 - \tau s_{\delta_1} s_{\delta_2}) = 24 + 2 = 26.$$

In fact, here  $J = \{2, 3, 4, 5\}$ ,  $w = s_{\delta_1} s_{\delta_2} = w_0 w_J$ .

We show that  $\tau \sim \tau n_{\delta_1} n_{\delta_2}$ . Let  $g = x_{-\delta_1}(1)$ . Then

$$g \tau g^{-1} = \tau x_{-\delta_1}(1) x_{-\delta_2}(1) \in \tau BwB.$$

Moreover, since  $[\tau, x_{\delta_1}(1) x_{\delta_2}(1)] = 1$ , we get

$$x_{\delta_1}(1) x_{\delta_2}(1) \tau x_{-\delta_1}(1) x_{-\delta_2}(1) x_{\delta_1}(1) x_{\delta_2}(1) = \tau n_{\delta_1} n_{\delta_2}$$

and we are done. We have established the information presented in Table 21, where  $w(\mathcal{O}) = w_0 w_J$ .

TABLE 21. Outer involutions of  $E_6$ ,  $p = 2$ .

$\mathcal{O}$	$J$	$w(\mathcal{O})$	$x \in \mathcal{O} \cap \tau Bw(\mathcal{O})B$	$\dim \mathcal{O}$
$\tau$	$\{2, 3, 4, 5\}$	$s_{\delta_1} s_{\delta_2}$	$\tau n_{\delta_1} n_{\delta_2}$	26
$\tau x_{\beta_1}(1)$	$\emptyset$	$w_0$	$\tau n_{\beta_1} n_{\beta_2} n_{\beta_3} n_{\beta_4}$	42

This completes the list of outer involutions of simple algebraic groups in characteristic 2. We have proved

**Theorem 4.2.** *Let  $G$  be a reductive connected algebraic group in characteristic 2, and let  $\sigma$  be any involutory automorphism of  $G$ . Then the fixed point subgroup  $H$  of  $\sigma$  is a spherical subgroup of  $G$ .  $\square$*

We conclude with another application of Theorem 4.1.

**4.5. Type  $G_2$  in  $D_4$ .** We show briefly how one can prove that the subgroup of type  $G_2$  in  $D_4$  is spherical (in all characteristics). Let us assume  $G$  of type  $D_4$ . Without loss of generality, we may assume  $G$  is adjoint. Hence if we denote by  $\varphi$  the graph automorphism of  $G$  fixing  $\alpha_2$  and mapping  $\alpha_1 \mapsto \alpha_3 \mapsto \alpha_4 \mapsto \alpha_1$ , then the fixed point subgroup  $K$  of  $\varphi$  is of type  $G_2$ . Let  $\delta_1 = \alpha_1 + \alpha_2 + \alpha_3$ ,  $\delta_2 = \alpha_1 + \alpha_2 + \alpha_4$ ,  $\delta_3 = \alpha_2 + \alpha_3 + \alpha_4$ , and let  $w = w_0 s_2 = s_{\delta_1} s_{\delta_2} s_{\delta_3}$ . We have

$$\ell(w) + \text{rk}(1 - \varphi w) = 14 = \dim D_4/G_2.$$

It remains to show that a  $G$ -conjugate of  $\varphi$  lies in  $\varphi B w_0 s_2 B \subseteq G : \langle \varphi \rangle$ .

Let  $g = x_{-\delta_1}(\xi_1)x_{-\delta_2}(\xi_2)x_{-\delta_3}(\xi_3)$ . Then

$$g\varphi g^{-1} = \varphi x_{-\delta_1}(-\xi_1 - \xi_3)x_{-\delta_2}(-\xi_2 - \xi_1)x_{-\delta_3}(-\xi_3 + \xi_2).$$

If we choose  $\xi_1, \xi_2, \xi_3$  such that  $\xi_1 + \xi_3, \xi_2 + \xi_1$  and  $-\xi_3 + \xi_2$  are non-zero, then  $g\varphi g^{-1} \in \varphi B w_0 s_2 B$ , and we are done.

*Remark 4.3.* If the characteristic is zero, we may apply the arguments in [11]. Since  $T^{\varphi w} = H_{\alpha_2}$  is connected, it follows that, in the simply-connected case, the monoid  $\lambda(D_4/G_2)$  of  $B$ -weights in  $k[D_4/G_2]$  is generated by  $\omega_1, \omega_3, \omega_4$ , since  $G_2$  is connected and has no non-trivial characters, so that the monoid  $\lambda(D_4/G_2)$  is free (and it contains  $(1 - \varphi w)P^+$ , which is the monoid generated by  $\omega_1 + \omega_3, \omega_1 + \omega_4, \omega_3 + \omega_4, \omega_1 + \omega_3 + \omega_4$ ) (see also [20]).

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