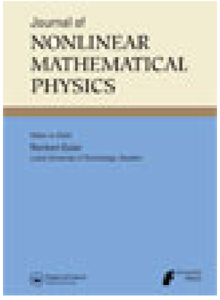

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Convergence to the Time Average by Stochastic Regularization

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In Ergodic Theory it is natural to consider the pointwise convergence of finite time averages of functions with respect to the flow of dynamical systems. Since the pointwise convergence is too weak for applications to Hamiltonian Perturbation Theory, requiring differentiability, we first introduce regularized averages obtained through a stochastic perturbation of an integrable Hamiltonian flow, and then we provide detailed estimates. In particular, for a special vanishing limit of the stochastic perturbation, we obtain convergence even in a Sobolev norm taking into account the derivatives.

Keywords: Stochastic regularization techniques; approximated first integrals; Hamiltonian Perturbation Theory; Ergodic Theory.

1. Introduction

In the last decades, complex non–linear dynamics turned out to be important in many fields of Physics. In particular, dynamics which exhibit an intermediate behaviour between integrability and ergodicity are still largely under investigation. Indeed, in Celestial Mechanics, Statistical Physics, Plasma Physics and Quantum Mechanics, one may typically find examples of orbits which visit, in a long time interval, several resonances displaying different transient chaos behaviors, temporary captures into resonances or stickiness phenomena (see, for example, [13], [15], [2], [7], [6]). Since in these cases random–phase approximations as well as the averaging principle are not efficient at all, these orbits are usually difficult to formally study with traditional tools. In this context, we find useful to utilize averaging techniques which are resonance independent, i.e. global in the phase–space, inspired by standard viscosity and stochastic regularizations of PDEs (see, for example [11], [5]). We remark that, in the last years, a completely new approach to Hamiltonian dynamics motivated by regularization techniques has been represented by the so–called weak KAM theories (see [10], [14], [9]), which focus on the existence of the Aubry–Mather invariant sets.

In this paper, we propose to use stochastic regularizations to generate global canonical transformations and approximated first integrals. Our starting point is a class of globally defined approximate first integrals previously introduced in [4], which generalize the usual time average of any phase–space function.

We recall that the time averages of functions with respect to the flow of Hamiltonian systems are extensively studied in Ergodic Theory and Hamiltonian Perturbation Theory. In particular, averages over integrable flows are commonly used as generating functions of averaging canonical transformations. In this setting it is well–known since Poincaré that resonances related to the so–called small divisors represent topological obstructions to the regularity of the time averages, which may be highly irregular in phase–space. The celebrated KAM and Nekhoroshev Theorems [12], [1], [16], [17] overcame this problem with a refined use of algebraic as well as geometric treatment of small divisors. More recently, the so–called weak KAM theories (see [10], [14], [9]) have studied the problem by new perspectives, based on variational and PDE regularizations by viscosity techniques.

In this paper, we obtain regularization by further averaging with respect to a stochastic perturbation (see [11], and also [4]) in order to deal with smooth functions. To use such regularized functions in any perturbation framework, we need estimates which include also the derivatives. We here provide such estimates with norms obtained by averaging over open domains of the phase–space. Therefore, our result has a probabilistic interpretation (probabilistic results in the weak KAM framework have been recently obtained by Evans, see [8], [9] and also [3]), whereas classical Hamiltonian Perturbation Theory provides uniform estimates valid for all initial conditions. The strength of the stochastic perturbation is a parameter of our construction. With evidence, it is interesting the limit of vanishing stochastic perturbations, which we study in detail by providing estimates based on Sobolev norms taking into account the first derivatives.

The paper is organized as follows. In Section 2 we define in detail the class of regularized approximated first integrals and we discuss their relation with resonances and small divisors. In Section 3 we introduce specific norms for phase–space functions and we state convergence results about the regularized averages. Section 4 is devoted to proofs.

2. Relations between time averages, resonances and stochastic regularization

Let us consider the integrable Hamiltonian system with Hamilton function $H(I,\varphi) := h(I)$, defined on the action–angle phase–space $A \times \mathbb{T}^n$, where $A \subseteq \mathbb{R}^n$ is open bounded and $g(I) := \nabla h(I)$ is a diffeomorphism over A such that

$$|g(I)| \leq C, \qquad \max_{i,j}\left|\frac{\partial g_i}{\partial I_j}(I)\right| \leq D, \qquad \left|\det\frac{\partial g}{\partial I}(I)\right| \geq m \tag{2.1}$$

$\forall I \in A$, for some constants $C, D, m > 0$. We also denote by $\lambda > 0$ a Lipschitz constant for g in the set A. For any smooth phase–space function $f(I,\varphi)$, we consider its finite time average

$$G^T(I,\varphi) := \frac{1}{T}\int_0^T f(\phi^t(I,\varphi))dt, \tag{2.2}$$

where $\phi^t(I,\varphi) = (I, \varphi + g(I)t)$ is the flow of the integrable Hamiltonian $h(I)$. By denoting with

$$f(I,\varphi) := \sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi}, \qquad G^T(I,\varphi) := \sum_{k\in\mathbb{Z}^n} G^T_k(I)e^{ik\cdot\varphi}$$

the Fourier expansions of f and G^T, we have

$$G^T_k(I) = \begin{cases} f_k(I) & \text{if } k\cdot g(I) = 0 \\ f_k(I)\dfrac{e^{ik\cdot g(I)T}-1}{ik\cdot g(I)T} & \text{if } k\cdot g(I) \neq 0 \end{cases}$$

With evidence, if $f_k \neq 0$ for a suitably large –that is generic– set of indices $k \in \mathbb{Z}^n$, the presence of small divisors $k \cdot g(I)$ represents an obstruction to the regularity both for G^T and for its limit

$$\bar{f}(I,\varphi) := \lim_{T\to+\infty} G^T(I,\varphi). \tag{2.3}$$

We remark that the Fourier coefficients $G_k^T(I)$ are similar to the Fourier coefficients of

$$\chi(I,\varphi) = -\sum_{k\in\mathbb{Z}^n\setminus 0} \frac{f_k(I)}{ik\cdot g(I)} e^{ik\cdot\varphi},$$

whose ε–time flow ϕ_χ^ε formally conjugates the quasi–integrable Hamiltonian system

$$H_\varepsilon(I,\varphi) = h(I) + \varepsilon f(I,\varphi)$$

to its first order average

$$(H_\varepsilon \circ \phi_\chi^\varepsilon)(I,\varphi) = h(I) + \varepsilon f_0(I) + \mathscr{O}(\varepsilon^2).$$

Of course, χ and G^T are affected by the same convergence problems.

We assume from now on that f is smooth and with generic Fourier expansion. Precisely, let us introduce for any $k \in \mathbb{Z}^n$ the resonant manifold

$$\mathscr{R}_k = \{I \in A : \ \ k\cdot g(I) = 0\}, \tag{2.4}$$

as well as

$$\mathscr{R}_k(f) = \{I \in A : \ \ k\cdot g(I) = 0 \ \text{ and } \ |f_k(I)| > 0\}. \tag{2.5}$$

Then, we assume that the set

$$\mathscr{R}(f) = \bigcup_{k\in\mathbb{Z}^n\setminus 0} \mathscr{R}_k(f) \tag{2.6}$$

is dense in A.

We now consider the regularization of G^T based on a vanishing stochastic perturbation, previously introduced in [4] by following a technique described in [11]. First, we consider the intermediate function

$$F^\mu(I,\varphi) := \mu \int_0^{+\infty} f(\phi^t(I,\varphi)) e^{-\mu t} dt, \tag{2.7}$$

with $\mu = 1/T$, which represents an unusual finite time average of f, in the sense that it is an exponentially damped average of f with respect to the integrable flow ϕ^t. Then, for any $\mu, \nu > 0$, we introduce the regularized function $F^{\mu,\nu}$ as follows. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $w_t : \Omega \to \mathbb{R}^n$ a n–dimensional Wiener process. Then, we obtain a stochastic differential equation by perturbing the Hamilton equations with a white noise

$$\begin{cases} \dot{I}_t = 0 \\ \dot{\varphi}_t = g(I) + 2\nu \dot{w}_t \end{cases} \tag{2.8}$$

whose flow is $\Phi^t_\nu(I,\varphi,\omega) = (I, \varphi + g(I)t + 2\nu w_t(\omega))$. As in [4], for $\mu,\nu > 0$ we introduce

$$F^{\mu,\nu}(I,\varphi) := \mu M_{(I,\varphi)} \left(\int_0^{+\infty} f(\Phi^t_\nu(I,\varphi,\omega)) e^{-\mu t} dt \right). \tag{2.9}$$

In the previous formula, $M_{(I,\varphi)}$ represents, for (I,φ) fixed, the average on all the trajectories of the Brownian motion (2.8), while the exponential damping $e^{-\mu t}$ allows us to interpret $F^{\mu,\nu}$ as an effective average over a time interval of some multiples of $1/\mu$ (see [4]).

3. Convergence results

Inspired by Birkhoff–Kinchin Theorem, which considers the (pointwise) convergence of the finite time averages G^T, it is natural to study also the convergence of the regularized functions $F^{\mu,\nu}$ for vanishing μ,ν. Since the pointwise convergence is too weak for applications requiring at least a $\mathscr{C}^1$ smoothness, we introduce specific norms on $A \times \mathbb{T}^n$. In more detail, for any function $u(I,\varphi) = \sum_{k\in\mathbb{Z}^n} u_k(I) e^{ik\cdot\varphi}$ on $A \times \mathbb{T}^n$, the uniform Fourier norm

$$|u|^\infty := \sum_{k\in\mathbb{Z}^n} \sup_{I\in A} |u_k(I)| \tag{3.1}$$

as well as the norms obtained with averages over the action space

$$|u|^0 := \sum_{k\in\mathbb{Z}^n} \int_A |u_k(I)| dI \tag{3.2}$$

and[a]

$$|u|^1 := |u|^0 + \sum_{k\in\mathbb{Z}^n} \sum_{j=1}^n \int_A \left(\left| \frac{\partial u_k}{\partial I_j}(I) \right| + |k_j u_k(I)| \right) dI. \tag{3.3}$$

Let us remark that, by considering the usual L^1 and Sobolev $W^{1,1}$ norms on $A \times \mathbb{T}^n$, in particular

$$\|u\|_{W^{1,1}} = \|u\|_{L^1} + \sum_{j=1}^n \left(\left\| \frac{\partial u}{\partial I_j} \right\|_{L^1} + \left\| \frac{\partial u}{\partial \varphi_j} \right\|_{L^1} \right)$$

and we have

$$\frac{1}{(2\pi)^n} \|u\|_{W^{1,1}} \le |u|^1 \le \frac{1}{(2\pi)^n} \sum_{k\in\mathbb{Z}^n} \|u_k(I) e^{ik\cdot\varphi}\|_{W^{1,1}}$$

For any phase–space function f, we discuss the convergence of the approximated first integrals G^T, F^μ and $F^{\mu,\nu}$ to the time average $\bar{f}$ both in the uniform Fourier norm $|\cdot|^\infty$ –see (3.1)– and in the action–averages based norm $|\cdot|^0$ given in (3.2). In particular, we prove the next

[a]The notation $|u|^0, |u|^1, |u|^\infty$ has been here chosen in order to avoid any confusion with the standard uniform and Sobolev norms.

Proposition 3.1. *Let us consider a smooth phase–space function $f(I,\varphi)$. The functions G^T, F^μ and $F^{\mu,\nu}$ converge to $\bar{f}$ in the $|\cdot|^0$ norm on $A\times\mathbb{T}^n$. Precisely, we have*

$$|G^T-\bar{f}|^0 \leq \frac{4C^{n-1}}{m}\sum_{k\in\mathbb{Z}^n\setminus 0}|f_k|^\infty\,\frac{3+\log(\|k\|TC)}{T} \tag{3.4}$$

$$|F^\mu-\bar{f}|^0 \leq \frac{2C^{n-1}}{m}\sum_{k\in\mathbb{Z}^n\setminus 0}|f_k|^\infty\,\frac{\mu}{\|k\|}\left(1+\log\frac{\|k\|C}{\mu}\right) \tag{3.5}$$

$$|F^{\mu,\nu}-\bar{f}|^0 \leq \frac{2C^{n-1}}{m}\sum_{k\in\mathbb{Z}^n\setminus 0}|f_k|^\infty\,\frac{\mu}{\|k\|}\left(1+\log\frac{\|k\|C}{\mu+\nu\,\|k\|^2}\right). \tag{3.6}$$

Instead, if the set $\mathscr{R}(f)$ defined in (2.6) is dense in A, the functions G^T, F^μ and $F^{\mu,\nu}$ do not uniformly Fourier converge to $\bar{f}$ in any set $B\times\mathbb{T}^n$ with $B\subseteq A$ open.

As it arises from the previous proposition, the three different finite time approximations G^T, F^μ and $F^{\mu,\nu}$ behave in the same way with respect to the $|\cdot|^\infty$ and $|\cdot|^0$ norms. Indeed, the difference consists in the convergence in the $|\cdot|^1$ norm given in (3.3). In such a case, the G^T, F^μ do not converge to $\bar{f}$, and it is remarkable that the convergence of $F^{\mu,\nu}$ is obtained only in a special limit of vanishing stochastic perturbation, as stated in the proposition below.

Proposition 3.2. *Let us consider a smooth phase–space function $f(I,\varphi)$. For any $\mu,\nu>0$ the function $F^{\mu,\nu}$ satisfies*

$$\left|F^{\mu,\nu}-\bar{f}\right|^1 \leq \frac{2C^{n-1}}{m}\sum_{k\in\mathbb{Z}^n\setminus 0}\left(\mu\left[1+\log\frac{\|k\|C}{\mu+\nu\,\|k\|^2}\right]\left((1+n)\,|f_k|^\infty+\right.\right.$$

$$\left.\left.+\sum_{j=1}^n\left|\frac{\partial f_k}{\partial I_j}\right|^\infty\right)+\frac{1}{2}n^2\pi D\frac{\mu}{\mu+\nu\,\|k\|^2}\,|f_k|^\infty\right) \tag{3.7}$$

on $A\times\mathbb{T}^n$. In particular, for any sequence μ_i, $\nu_i>0$ converging to zero and such that

$$\lim_{i\to+\infty}\frac{\mu_i}{\nu_i}=0,$$

we have

$$\lim_{i\to+\infty}|F^{\mu_i,\nu_i}-\bar{f}|^1=0.$$

Differently, if $\mathscr{R}(f)$ defined in (2.6) is dense in A, the functions G^T and F^μ do not converge to $\bar{f}$ in the $|\cdot|^1$ norm on any set $B\times\mathbb{T}^n$ with $B\subseteq A$ open.

Let us remark that the convergence of $F^{\mu,\nu}$ to $\bar{f}$ requires a restriction of the sub-sequences μ_i,ν_i because in (3.7) we find contributions proportional to $\mu/\mu+\nu\,\|k\|^2$, while the contributions $\mu\log\left(\|k\|C/\mu+\nu\,\|k\|^2\right)$ which are dominant in (3.6) converge for $(\mu,\nu)\to(0,0)$.

The proofs of Propositions 3.1, 3.2 are reported in Section 4.

4. Proofs

The different time averages (2.2), (2.7) and (2.9) can be alternatively expressed in terms of their Fourier coefficients, as discussed in the following technical

Lemma 4.1. *Let us consider*

$$f(I,\varphi) = \sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi}. \tag{4.1}$$

The Fourier coefficients of

$$G^T(I,\varphi) = \sum_{k\in\mathbb{Z}^n} G_k^T(I)e^{ik\cdot\varphi}, \quad F^\mu(I,\varphi) = \sum_{k\in\mathbb{Z}^n} F_k^\mu(I)e^{ik\cdot\varphi}, \quad F^{\mu,\nu}(I,\varphi) = \sum_{k\in\mathbb{Z}^n} F_k^{\mu,\nu}(I)e^{ik\cdot\varphi}$$

are respectively

$$G_k^T(I) = \begin{cases} f_k(I) & \textit{if } k\cdot g(I) = 0 \\ f_k(I)\dfrac{e^{ik\cdot g(I)T}-1}{ik\cdot g(I)T} & \textit{if } k\cdot g(I) \neq 0 \end{cases} \tag{4.2}$$

$$F_k^\mu(I) = -\mu\frac{f_k(I)}{ik\cdot g(I)-\mu} \tag{4.3}$$

and

$$F_k^{\mu,\nu}(I) = -\mu\frac{f_k(I)}{ik\cdot g(I)-\mu-\nu\|k\|^2} \tag{4.4}$$

Proof. The first equality easily follows from (2.2) and (4.1) by direct calculations. Indeed

$$\begin{aligned} G^T(I,\phi) &= \frac{1}{T}\int_0^T f(\phi^t(I,\varphi))dt = \frac{1}{T}\int_0^T \sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi}e^{ik\cdot g(I)t}dt \\ &= \frac{1}{T}\sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi}\int_0^T e^{ik\cdot g(I)t}dt. \end{aligned}$$

Moreover, from

$$\frac{1}{T}\int_0^T e^{ik\cdot g(I)t}dt = \begin{cases} 1 & \text{if } k\cdot g(I) = 0 \\ \dfrac{e^{ik\cdot g(I)T}-1}{ik\cdot g(I)T} & \text{if } k\cdot g(I) \neq 0 \end{cases}$$

we immediately obtain formula (4.2). Similarly for (4.3)

$$\begin{aligned} F^\mu(I,\varphi) &= \mu\int_0^{+\infty} f(\phi^t(I,\varphi))e^{-\mu t}dt = \mu\int_0^{+\infty}\sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi}e^{ik\cdot g(I)t-\mu t}dt \\ &= \mu\sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi}\int_0^{+\infty} e^{(ik\cdot g(I)-\mu)t}dt = -\mu\sum_{k\in\mathbb{Z}^n}\frac{f_k(I)}{ik\cdot g(I)-\mu}e^{ik\cdot\varphi}. \end{aligned}$$

We conclude by proving the equality (4.4). We first take into account (2.9), so that

$$\int_0^{+\infty} f(\Phi_\nu^t(I,\varphi,\omega))e^{-\mu t}dt = \sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi}\int_0^{+\infty} e^{(ik\cdot g(I)-\mu)t}e^{2i\nu k\cdot w_t(\omega)}dt.$$

As a consequence –see (2.9)– we obtain

$$\begin{aligned} F^{\mu,\nu}(I,\varphi) &= \mu M_{(I,\varphi)}\left(\int_0^{+\infty} f(\Phi_\nu^t(I,\varphi,\omega))e^{-\mu t}dt\right) \\ &= \mu\sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi}\int_\Omega\left[\int_0^{+\infty} e^{(ik\cdot g(I)-\mu)t}e^{2i\nu k\cdot w_t(\omega)}dt\right]P(d\omega) \\ &= \mu\sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi}\int_0^{+\infty}\left[e^{(ik\cdot g(I)-\mu)t}\int_\Omega e^{2i\nu k\cdot w_t(\omega)}P(d\omega)\right]dt. \end{aligned} \tag{4.5}$$

Since $w_t:\Omega\to\mathbb{R}^n$ is a n–dimensional Wiener process, the corresponding covariance matrix $R(t) = R_{ij}(t) = t\delta_{ij}$ and therefore

$$\int_\Omega e^{2i\nu k\cdot w_t(\omega)}P(d\omega) = e^{-\nu\|k\|^2 t}.$$

Therefore, from equation (4.5) we have

$$F^{\mu,\nu}(I,\varphi) = \mu\sum_{k\in\mathbb{Z}^n} f_k(I)e^{ik\cdot\varphi}\int_0^{+\infty} e^{(ik\cdot g(I)-\mu-\nu\|k\|^2)t}dt = -\mu\sum_{k\in\mathbb{Z}^n}\frac{f_k(I)}{ik\cdot g(I)-\mu-\nu\|k\|^2}e^{ik\cdot\varphi}.$$

□

The next sections are devoted to the convergence results, in three different norms, of G^T, F^μ and $F^{\mu,\nu}$ to the time average $\bar{f}$ defined in (2.3). From (4.2), we immediately obtain $\bar{f} = \sum_{k\in\mathbb{Z}^n}\bar{f}_k(I)e^{ik\cdot\varphi}$, with

$$\bar{f}_k(I) = \begin{cases} f_k(I) & \text{if } k\cdot g(I) = 0 \\ 0 & \text{if } k\cdot g(I)\neq 0 \end{cases} \tag{4.6}$$

4.1. *Proof of Proposition 3.1*

We start by proving that G^T does not converge to $\bar{f}$ in the uniform Fourier norm. Let us consider

$$(G^T-\bar{f})(I,\varphi) := \sum_{k\in\mathbb{Z}^n}(G^T-\bar{f})_k(I)e^{ik\cdot\varphi}.$$

From (4.2) and (4.6) we immediately obtain

$$(G^T-\bar{f})_k(I) = \begin{cases} 0 & \text{if } k\cdot g(I) = 0 \\ f_k(I)\dfrac{e^{ik\cdot g(I)T}-1}{ik\cdot g(I)T} & \text{if } k\cdot g(I)\neq 0 \end{cases} \tag{4.7}$$

Since the set $\mathscr{R}(f)$ defined in (2.6) is dense, there exists a dense set of points $\bar{I}\in A$ such that $\bar{k}\cdot g(\bar{I}) = 0$ and $|f_{\bar{k}}(\bar{I})| > 0$ for some $\bar{k}\in\mathbb{Z}^n\backslash 0$. Since g is a diffeomorphism, we have

$$\lim_{I\notin\mathscr{R}_{\bar{k}},\ I\to\bar{I}}\left|\frac{e^{i\bar{k}\cdot g(I)T}-1}{i\bar{k}\cdot g(I)T}\right| = \lim_{J\to 0}\left|\frac{e^{iJT}-1}{iJT}\right| = \lim_{J\to 0}\frac{\sqrt{2[1-\cos(JT)]}}{|JT|} = 1,$$

and also

$$\sup_{I\in A\setminus\mathscr{R}_{\bar k}}\left|f_{\bar k}(I)\frac{e^{i\bar k\cdot g(I)T}-1}{i\bar k\cdot g(I)T}\right|\geq \lim_{I\notin\mathscr{R}_{\bar k},\, I\to\bar I}\left|f_{\bar k}(I)\frac{e^{i\bar k\cdot g(I)T}-1}{i\bar k\cdot g(I)T}\right| = |f_{\bar k}(\bar I)|\,.$$

As a consequence,

$$|G^T-\bar f|^\infty=\sum_{k\in\mathbb{Z}^n}\sup_{I\in A}\left|(G^T-\bar f)_k(I)\right|\geq |f_{\bar k}(\bar I)|>0$$

that is, G^T does not uniformly Fourier converge to $\bar f$ in any set $B\times\mathbb{T}^n$ with $B\subseteq A$ open.

We proceed with the same discussion for F^μ. By denoting

$$(F^\mu-\bar f)(I,\varphi):=\sum_{k\in\mathbb{Z}^n}(F^\mu-\bar f)_k(I)e^{ik\cdot\varphi},$$

from (4.3) and (4.6) we have

$$(F^\mu-\bar f)_k(I)=\begin{cases}0 & \text{if } k\cdot g(I)=0\\ -\mu\dfrac{f_k(I)}{ik\cdot g(I)-\mu} & \text{if } k\cdot g(I)\neq 0\end{cases}\tag{4.8}$$

By considering as before $\bar I\in\mathscr{R}(f)$ such that $\bar k\cdot g(\bar I)=0$ and $|f_{\bar k}(\bar I)|>0$, for some $\bar k\neq 0$, from

$$\lim_{I\notin\mathscr{R}_{\bar k},\, I\to\bar I}\left|-\mu\frac{f_{\bar k}(I)}{i\bar k\cdot g(I)-\mu}\right|=|f_{\bar k}(\bar I)|$$

we have

$$|F^\mu-\bar f|^\infty=\sum_{k\in\mathbb{Z}^n}\sup_{I\in A}\left|(F^\mu-\bar f)_k(I)\right|\geq |f_{\bar k}(\bar I)|>0$$

that is, F^μ does not uniformly Fourier converge to $\bar f$ in any set $B\times\mathbb{T}^n$ with $B\subseteq A$ open.

We conclude the first part of the proof by showing that also $F^{\mu,\nu}$ does not uniformly Fourier converge to $\bar f$. Indeed, in such a case, formulas (4.4) and (4.6) give

$$(F^{\mu,\nu}-\bar f)_k(I)=\begin{cases}f_k(I)\left[\dfrac{\mu}{\mu+\nu\|k\|^2}-1\right] & \text{if } k\cdot g(I)=0\\[2ex] -\mu\dfrac{f_k(I)}{ik\cdot g(I)-\mu-\nu\|k\|^2} & \text{if } k\cdot g(I)\neq 0\end{cases}\tag{4.9}$$

By considering again $\bar k\cdot g(\bar I)=0$ and $|f_{\bar k}(\bar I)|>0$ with $\bar k\neq 0$, and sequences $(\mu_i,\nu_i)\to 0$, we discuss the following two cases.

(i) If $\lim_{i\to+\infty}\nu_i/\mu_i=0$, we have

$$\lim_{i\to+\infty}\lim_{I\notin\mathscr{R}_{\bar k},\, I\to\bar I}|(F^{\mu_i,\nu_i}-\bar f)_{\bar k}(I)|=\lim_{i\to+\infty}\left|\mu_i\frac{f_{\bar k}(\bar I)}{\mu_i+\nu_i\|\bar k\|}\right|=|f_{\bar k}(\bar I)|.$$

(ii) On the contrary, if the sequence ν_i/μ_i does not converge to zero, we consider

$$\left|(F^{\mu_i,\nu_i}-\bar{f})_{\bar{k}}(\bar{I})\right| = |f_{\bar{k}}(\bar{I})| \left|\left[\frac{\mu_i}{\mu_i+\nu_i\|\bar{k}\|^2}-1\right]\right| = |f_{\bar{k}}(\bar{I})| \frac{\nu_i\left\|\bar{k}\right\|^2}{\mu_i+\nu_i\left\|\bar{k}\right\|^2}$$

which does not converge to zero as i tends to infinity.

As a consequence of all previous cases, we conclude that $F^{\mu,\nu}$ does not uniformly Fourier converge to $\bar{f}$ in any set $B\times\mathbb{T}^n$ with $B\subseteq A$ open.

We proceed by discussing the convergence to $\bar{f}$ in the $|\cdot|^0$ norm. Since $g: A\to\mathbb{R}^n$ is a diffeomorphism, the set of all resonances $\mathscr{R} := \bigcup_{k\in\mathbb{Z}^n\setminus 0}\mathscr{R}_k$ has measure zero. Consequently, the norm $|\cdot|^0$ can be rewritten as

$$|u|^0 = \sum_{k\in\mathbb{Z}^n}\int_{\tilde{A}}|u_k(I)|dI$$

where

$$\tilde{A} := A\setminus\mathscr{R} = \{I\in A:\ k\cdot g(I)\neq 0 \text{ for all } k\in\mathbb{Z}^n\setminus 0\}. \tag{4.10}$$

We first prove $\lim_{T\to+\infty}|G^T-\bar{f}|^0=0$. From (4.2) and (4.6) we immediately obtain

$$(G^T-\bar{f})_k(I) = f_k(I)\frac{e^{ik\cdot g(I)}-1}{ik\cdot g(I)T} \qquad \forall I\in\tilde{A}$$

so that

$$\int_{\tilde{A}}|(G^T-\bar{f})_k(I)|dI = \int_{\tilde{A}}\left|f_k(I)\frac{e^{ik\cdot g(I)T}-1}{ik\cdot g(I)T}\right|dI$$

$$\leq |f_k|^\infty\int_{\tilde{A}}\frac{\sqrt{\sin^2(k\cdot g(I)T)+[\cos(k\cdot g(I)T)-1]^2}}{|k\cdot g(I)|T}dI = |f_k|^\infty\int_{\tilde{A}}\frac{\sqrt{2[1-\cos(k\cdot g(I)T)]}}{|k\cdot g(I)|T}dI.$$

Using the change of variables

$$I\mapsto J := g(I) \tag{4.11}$$

and assumption (2.1), we obtain

$$\int_{\tilde{A}}|(G^T-\bar{f})_k(I)|dI \leq |f_k|^\infty\int_{\tilde{A}}\frac{\sqrt{2[1-\cos(k\cdot g(I)T)]}}{|k\cdot g(I)|T}dI \leq \frac{|f_k|^\infty}{m}\int_{g(\tilde{A})}\frac{\sqrt{2[1-\cos(k\cdot JT)]}}{|k\cdot J|T}dJ. \tag{4.12}$$

Let now $\tilde{e}_1,\ldots,\tilde{e}_n$ be an orthonormal basis of $\mathbb{R}^n$ with $k\in\langle\tilde{e}_2,\ldots,\tilde{e}_n\rangle^\perp$ and R a rotation matrix such that $Rk=\|k\|\tilde{e}_1$ (the dependence of the basis and the rotation matrix on $k\in\mathbb{Z}^n$ is here omitted). By the further change of variables

$$J \mapsto x := RJ \tag{4.13}$$

the quantity $k \cdot J$ in (4.12) becomes $k \cdot J = \|k\| x_1$, and for any x in the integration domain $Rg(\tilde{A})$ we have $x = Rg(I)$ with $I \in \tilde{A}$ and $\|x\| \leq \|g(I)\| \leq C$. As a consequence, we obtain

$$\int_{\tilde{A}} |(G^T - \bar{f})_k(I)| dI \leq \frac{|f_k|^\infty}{m} \int_{g(\tilde{A})} \frac{\sqrt{2[1-\cos(k\cdot JT)]}}{|k\cdot J|T} dJ$$

$$= \frac{|f_k|^\infty}{m} \int_{Rg(\tilde{A})} \frac{\sqrt{2[1-\cos(\|k\| x_1 T)]}}{\|k\||x_1|T} dx_1 \dots dx_n$$

$$\leq \frac{|f_k|^\infty C^{n-1}}{m} \int_{-C}^{C} \frac{\sqrt{2[1-\cos(\|k\|x_1 T)]}}{\|k\||x_1|T} dx_1 = \frac{|f_k|^\infty C^{n-1}}{m\|k\|T} \int_{-\|k\|CT}^{\|k\|CT} \frac{\sqrt{2(1-\cos y)}}{|y|} dy$$

$$= \frac{2|f_k|^\infty C^{n-1}}{m\|k\|T} \int_{-\|k\|CT/2}^{\|k\|CT/2} \left|\frac{\sin y}{y}\right| dy = \frac{4|f_k|^\infty C^{n-1}}{m\|k\|T} \int_0^{\|k\|CT/2} \left|\frac{\sin y}{y}\right| dy$$

$$\leq \frac{4|f_k|^\infty C^{n-1}}{m\|k\|T} \int_0^{2\pi} \left|\frac{\sin y}{y}\right| dy + \frac{4|f_k|^\infty C^{n-1}}{m\|k\|T} \int_{2\pi}^{\|k\|TC/2} \frac{1}{y} dy \leq \frac{4|f_k|^\infty C^{n-1}}{m\|k\|T} [l_0 + \log(\|k\|TC)] \tag{4.14}$$

with $l_0 := \int_0^{2\pi} \left|\frac{\sin y}{y}\right| dy \leq 3$. Consequently,

$$|G^T - \bar{f}|^0 \leq \sum_{k\in\mathbb{Z}^n\setminus 0} \frac{4|f_k|^\infty C^{n-1}}{m\|k\|T} [3 + \log(\|k\|TC)]$$

proving that G^T converges to $\bar{f}$ in the $|\cdot|^0$ norm.

We conclude the proof with the convergence of F^μ and $F^{\mu,\nu}$ to $\bar{f}$. By using formulas (4.4) and (4.6), we have

$$(F^{\mu,\nu} - \bar{f})_k(I) = -\mu \frac{f_k(I)}{ik\cdot g(I) - \mu - \nu\|k\|^2} \qquad \forall I \in \tilde{A}.$$

Hence

$$\int_{\tilde{A}} |(F^{\mu,\nu} - \bar{f})_k(I)| dI = \mu \int_{\tilde{A}} \frac{|f_k(I)|}{\sqrt{(\mu+\nu\|k\|^2)^2 + (k\cdot g(I))^2}} dI$$

$$\leq \mu |f_k|^\infty \int_{\tilde{A}} \frac{1}{\sqrt{(\mu+\nu\|k\|^2)^2 + (k\cdot g(I))^2}} dI$$

The same changes of variables of the previous case, see (4.11) and (4.13), provide

$$\int_{\tilde{A}} |(F^{\mu,\nu} - \bar{f})_k(I)| dI \leq \frac{\mu |f_k|^\infty C^{n-1}}{m} \int_{-C}^{C} \frac{1}{\sqrt{(\mu+\nu\|k\|^2)^2 + \|k\|^2 x_1^2}} dx_1$$

$$= \frac{\mu|f_k|^\infty C^{n-1}}{m\|k\|} \int_{-\frac{\|k\|C}{\mu+\nu\|k\|^2}}^{\frac{\|k\|C}{\mu+\nu\|k\|^2}} \frac{1}{\sqrt{1+x^2}} dx = \frac{2\mu|f_k|^\infty C^{n-1}}{m\|k\|} \int_0^{\frac{\|k\|C}{\mu+\nu\|k\|^2}} \frac{1}{\sqrt{1+x^2}} dx$$

$$= \frac{2\mu|f_k|^\infty C^{n-1}}{m\|k\|} \left[\int_0^1 \frac{1}{\sqrt{1+x^2}} dx + \int_1^{\frac{\|k\|C}{\mu+\nu\|k\|^2}} \frac{1}{\sqrt{1+x^2}} dx \right] \leq \frac{2\mu|f_k|^\infty C^{n-1}}{m\|k\|} \left[l_1 + \int_1^{\frac{\|k\|C}{\mu+\nu\|k\|^2}} \frac{1}{x} dx \right]$$

$$= \frac{2\mu|f_k|^\infty C^{n-1}}{m\|k\|} \left[l_1 + \log \frac{\|k\|C}{\mu+\nu\|k\|^2} \right] \tag{4.15}$$

with $l_1 := \operatorname{arcsinh} 1 \leq 1$. Consequently

$$|F^{\mu,\nu} - \bar{f}|^0 \leq \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} |f_k|^\infty \frac{\mu}{\|k\|} \left[1 + \log \frac{\|k\|C}{\mu+\nu\|k\|^2} \right]$$

and for $\nu = 0$ we obtain also (3.5).

Inequalities (3.6) and (3.5) respectively prove that $F^{\mu,\nu}$ converges to $\bar{f}$ for $(\mu,\nu) \to (0,0)$ and F^μ converges to $\bar{f}$ for $\mu \to 0$ in the $|\cdot|^0$ norm. □

4.2. *Proof of Proposition 3.2*

Let us consider any open set $B \subseteq A$. Since $g : A \to \mathbb{R}^n$ is a diffeomorphism, the $|\cdot|^1$ norm in $B \times \mathbb{T}^n$ –see (3.3)– can be rewritten as

$$|u|^1 = \sum_{k \in \mathbb{Z}^n} \left\{ \int_{\tilde{B}} |u_k(I)| dI + \sum_{j=1}^n \int_{\tilde{B}} \left(\left| \frac{\partial u_k}{\partial I_j}(I) \right| + \left| k_j u_k(I) \right| \right) dI \right\}$$

with $\tilde{B} = B \cap \tilde{A}$, see (4.10).

We first prove that G^T does not converge to $\bar{f}$ in the set $B \times \mathbb{T}^n$. It is sufficient to prove that there exists $\varepsilon > 0$ such that for any large T we have

$$\sum_{j=1}^n \sum_{k \in \mathbb{Z}^n} \int_{\tilde{B}} \left| \left(\frac{\partial G_k^T}{\partial I_j} - \frac{\partial \bar{f}_k}{\partial I_j} \right)(I) \right| dI > \varepsilon. \tag{4.16}$$

From (4.2) and (4.6), for any $I \in \tilde{B}$ we have

$$(G^T - \bar{f})_k(I) = \begin{cases} f_k(I) \dfrac{e^{ik\cdot g(I)T} - 1}{ik \cdot g(I)T} & \text{if } 0 \neq k \in \mathbb{Z}^n \\ 0 & \text{if } k = 0 \end{cases}$$

so that

$$\left(\frac{\partial G_k^T}{\partial I_j} - \frac{\partial \bar{f}_k}{\partial I_j} \right)(I) = \begin{cases} \dfrac{\partial f_k}{\partial I_j}(I) \dfrac{e^{ik\cdot g(I)T} - 1}{ik \cdot g(I)T} + f_k(I) \dfrac{\partial}{\partial I_j} \left(\dfrac{e^{ik\cdot g(I)T} - 1}{ik \cdot g(I)T} \right) & \text{if } 0 \neq k \in \mathbb{Z}^n \\ 0 & \text{if } k = 0 \end{cases} \tag{4.17}$$

We notice that the first addendum in (4.17) tends to 0, that is

$$\lim_{T\to+\infty}\int_{\tilde{B}}\left|\frac{\partial f_k}{\partial I_j}(I)\frac{e^{ik\cdot g(I)T}-1}{ik\cdot g(I)T}\right|dI=0.$$

Indeed, by using the changes of variables (4.11) and (4.13) as in the proof of Proposition 3.1 –see also (4.14)– we obtain

$$\int_{\tilde{B}}\left|\frac{\partial f_k}{\partial I_j}(I)\frac{e^{ik\cdot g(I)T}-1}{ik\cdot g(I)T}\right|dI\le\left|\frac{\partial f_k}{\partial I_j}\right|^{\infty}\frac{4C^{n-1}}{m\|k\|T}[l_0+\log(\|k\|TC)].$$

As a consequence, it remains to study the other term of the equality (4.17), precisely

$$\sum_{j=1}^{n}\sum_{k\in\mathbb{Z}^n}\int_{\tilde{B}}|f_k(I)|\left|\frac{\partial}{\partial I_j}\left(\frac{e^{ik\cdot g(I)T}-1}{ik\cdot g(I)T}\right)\right|dI$$

$$=\sum_{j=1}^{n}\sum_{k\in\mathbb{Z}^n}\int_{\tilde{B}}|f_k(I)|\left|\frac{\partial}{\partial I_j}(ik\cdot g(I))\right|\frac{\sqrt{2+(k\cdot g)^2T^2-2k\cdot gT\sin(k\cdot gT)-2\cos(k\cdot gT)}}{(k\cdot g)^2T}dI$$

$$=\sum_{k\in\mathbb{Z}^n}\int_{\tilde{B}}|f_k(I)|\left\|\frac{\partial g}{\partial I}^T k\right\|_1\frac{\sqrt{2+(k\cdot g)^2T^2-2k\cdot gT\sin(k\cdot gT)-2\cos(k\cdot gT)}}{(k\cdot g)^2T}dI$$

where

$$\left\|\frac{\partial g}{\partial I}^T k\right\|_1:=\sum_{j=1}^{n}\left|\left(\frac{\partial g}{\partial I}^T k\right)_j\right|=\sum_{j=1}^{n}\left|\sum_{i=1}^{n}\frac{\partial g_i}{\partial I_j}k_i\right|=\sum_{j=1}^{n}\left|\frac{\partial}{\partial I_j}(ik\cdot g(I))\right|.$$

Since $\mathscr{R}(f)$ is dense in A, there exists $\bar{I}\in B\cap\mathscr{R}(f)$ such that $k\cdot g(\bar{I})=0$ and $|f_k(\bar{I})|>0$ for some $k\in\mathbb{Z}^n\backslash 0$. In particular, there exist $\delta,\lambda_1,\lambda_2>0$ (independent of T) such that the closed ball

$$B_\delta(\bar{I})=\{I:\|I-\bar{I}\|\le\delta\}$$

is contained in B, and also for any $I\in B_\delta(\bar{I})$ we have

$$|f_k(I)|\ge\lambda_1$$

and

$$\min_{\|u\|=1}\left\|\frac{\partial g}{\partial I}^T u\right\|_1\ge\lambda_2.$$

Let us remark that the constant λ_1 satisfies $0<\lambda_1\le|f_k|^{\infty}$. The constant λ_2 is indeed strictly positive, since otherwise there would exist $u\neq 0$ with $\frac{\partial g}{\partial I}^T u=0$, which is in contradiction with (2.1). From

(2.1), there exists also a constant $M > 0$ such that

$$\left|\det \frac{\partial g}{\partial I}(I)\right| \le M \tag{4.18}$$

for any $I \in A$. As a consequence, we have

$$\sum_{j=1}^{n} \sum_{\tilde{k}\in\mathbb{Z}^n} \int_{\tilde{B}} |f_{\tilde{k}}(I)| \left| \frac{\partial}{\partial I_j} \left(\frac{e^{i\tilde{k}\cdot g(I)T} - 1}{i\tilde{k}\cdot g(I)T} \right) \right| dI$$

$$\ge \lambda_1 \lambda_2 \|k\| \int_{B_\delta(\bar{I})} \frac{\sqrt{2 + (k\cdot g)^2 T^2 - 2k\cdot gT \sin(k\cdot gT) - 2\cos(k\cdot gT)}}{(k\cdot g)^2 T} dI.$$

By performing the change of variables $J := g(I)$ and using (4.18), the above term has the lower bound

$$\frac{\lambda_1\lambda_2}{M} \|k\| \int_{g(B_\delta(\bar{I}))} \frac{\sqrt{2 + (k\cdot J)^2 T^2 - 2k\cdot JT \sin(k\cdot JT) - 2\cos(k\cdot JT)}}{(k\cdot J)^2 T} dJ,$$

which, using the additional change of variables $x := RJ$ as in (4.13), equals to

$$\frac{\lambda_1\lambda_2}{M} \|k\| \int_{Rg(B_\delta(\bar{I}))} \frac{\sqrt{2 + \|k\|^2 x_1^2 T^2 - 2\|k\| x_1 T \sin(\|k\| x_1 T) - 2\cos(\|k\| x_1 T)}}{\|k\|^2 x_1^2 T} dx.$$

We consider $\tilde{\delta} > 0$ possibly depending on $k, \bar{I}$ (but independent of T) such that

$$\left\{ x : \max_{j=1,\dots,n} \left| x_j - Rg(\bar{I})_j \right| \le \tilde{\delta} \right\} \subseteq Rg(B_\delta(\bar{I})),$$

so that we have

$$\frac{\lambda_1\lambda_2}{M} \|k\| \int_{Rg(B_\delta(\bar{I}))} \frac{\sqrt{2 + \|k\|^2 x_1^2 T^2 - 2\|k\| x_1 T \sin(\|k\| x_1 T) - 2\cos(\|k\| x_1 T)}}{\|k\|^2 x_1^2 T} dx$$

$$\ge \frac{\lambda_1\lambda_2}{M} \|k\| \tilde{\delta}^{n-1} \int_{Rg(\bar{I})_1 - \tilde{\delta}}^{Rg(\bar{I})_1 + \tilde{\delta}} \frac{\sqrt{2 + \|k\|^2 x_1^2 T^2 - 2\|k\| x_1 T \sin(\|k\| x_1 T) - 2\cos(\|k\| x_1 T)}}{\|k\|^2 x_1^2 T} dx_1$$

$$= \frac{\lambda_1\lambda_2}{M} \tilde{\delta}^{n-1} \int_{\|k\|T(Rg(\bar{I})_1 - \tilde{\delta})}^{\|k\|T(Rg(\bar{I})_1 + \tilde{\delta})} \frac{\sqrt{2 + y^2 - 2y\sin y - 2\cos y}}{y^2} dy.$$

We remark that, since the change of variables (4.13) is performed by a matrix R such that $Rk = \|k\|\tilde{e}_1$, so that

$$Rg(\bar{I})_1 = \tilde{e}_1 \cdot Rg(\bar{I}) = \frac{1}{\|k\|} Rk \cdot Rg(\bar{I}) = \frac{1}{\|k\|} k \cdot g(\bar{I}) = 0,$$

we have

$$\frac{\lambda_1\lambda_2}{M} \tilde{\delta}^{n-1} \int_{\|k\|T(Rg(\bar{I})_1 - \tilde{\delta})}^{\|k\|T(Rg(\bar{I})_1 + \tilde{\delta})} \frac{\sqrt{2 + y^2 - 2y\sin y - 2\cos y}}{y^2} dy$$

$$= \frac{\lambda_1 \lambda_2}{M} \tilde{\delta}^{n-1} \int_{-\|k\| T \tilde{\delta}}^{\|k\| T \tilde{\delta}} \frac{\sqrt{2 + y^2 - 2y \sin y - 2\cos y}}{y^2} dy.$$

Since for any $y \in \mathbb{R}$ we have

$$2 + y^2 - 2y \sin y - 2\cos y \geq \frac{y^4}{4(1+y^2)},$$

we conclude

$$\frac{\lambda_1 \lambda_2}{M} \tilde{\delta}^{n-1} \int_{-\|k\| T \tilde{\delta}}^{\|k\| T \tilde{\delta}} \frac{\sqrt{2 + y^2 - 2y \sin y - 2\cos y}}{y^2} dy \geq \frac{\lambda_1 \lambda_2}{2M} \tilde{\delta}^{n-1} \int_{-\|k\| T \tilde{\delta}}^{\|k\| T \tilde{\delta}} \frac{1}{\sqrt{1+y^2}} dy$$

$$= \frac{\lambda_1 \lambda_2}{M} \tilde{\delta}^{n-1} \int_{0}^{\|k\| T \tilde{\delta}} \frac{1}{\sqrt{1+y^2}} dy = \frac{\lambda_1 \lambda_2}{M} \tilde{\delta}^{n-1} \text{arcsinh}(\|k\| T \tilde{\delta}).$$

Since

$$\lim_{T \to +\infty} \text{arcsinh}(\|k\| T \tilde{\delta}) = +\infty,$$

with a suitable definition of ε, one immediately obtains (4.16).

We proceed by proving that F^μ does not converge to $\bar{f}$ in the set $B \times \mathbb{T}^n$. It is sufficient to prove that there exists $\varepsilon > 0$ such that for any small μ we have

$$\sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n} \int_{\tilde{B}} \left| \left(\frac{\partial F_k^\mu}{\partial I_j} - \frac{\partial \bar{f}_k}{\partial I_j} \right)(I) \right| dI > \varepsilon. \tag{4.19}$$

From (4.8), for any $I \in \tilde{B}$ we have

$$(F^\mu - \bar{f})_k(I) = \begin{cases} F_k^\mu(I) = -\dfrac{\mu f_k(I)}{ik \cdot g(I) - \mu} & \text{if } 0 \neq k \in \mathbb{Z}^n \\ 0 & \text{if } k = 0 \end{cases}$$

so that we have to estimate

$$\sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n \setminus 0} \int_{\tilde{B}} \left| \frac{\partial F_k^\mu}{\partial I_j}(I) \right| dI.$$

By direct computations we obtain

$$\begin{aligned} &\sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n \setminus 0} \int_{\tilde{B}} \left| \frac{\partial F_k^\mu}{\partial I_j}(I) \right| dI \\ &= \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n \setminus 0} \mu \int_{\tilde{B}} \frac{1}{|ik \cdot g(I) - \mu|^2} \left| \frac{\partial f_k}{\partial I_j}(I)(ik \cdot g(I) - \mu) - f_k(I) \frac{\partial}{\partial I_j}(ik \cdot g(I)) \right| dI \\ &= \sum_{j=1}^{n} \sum_{k \in \mathbb{Z}^n \setminus 0} \mu \int_{\tilde{B}} \frac{1}{|ik \cdot g(I) - \mu|^2} \sqrt{\mu^2 \left(\frac{\partial f_k}{\partial I_j} \right)^2 + \left(k \cdot g(I) \frac{\partial f_k}{\partial I_j} - f_k k \cdot \frac{\partial g}{\partial I_j} \right)^2} dI \end{aligned}$$

$$\geq \sum_{j=1}^{n} \sum_{k\in\mathbb{Z}^n\setminus 0} \mu \int_{\tilde{B}} \frac{1}{|ik\cdot g(I)-\mu|^2} \left| k\cdot g(I)\frac{\partial f_k}{\partial I_j} - f_k k\cdot \frac{\partial g}{\partial I_j}\right| dI. \tag{4.20}$$

As before, we consider $\bar{I}\in\tilde{B}\cap\mathscr{R}(f)$, so that there exists $k\in\mathbb{Z}^n$ such that $k\cdot g(\bar{I})=0$ and $|f_k(\bar{I})|>0$. In particular, there exist $\delta,\lambda_1,\lambda_2>0$ (independent of T) such that the closed ball

$$B_\delta(\bar{I}) = \{I : \|I-\bar{I}\|\leq\delta\}$$

is contained in B, and also for any $I\in B_\delta(\bar{I})$ we have

$$|f_k(I)|\geq\lambda_1,$$

and

$$\min_{\|u\|=1} \left\| \frac{\partial g}{\partial I}^T u \right\|_1 \geq \lambda_2.$$

Since $\lambda>0$ is a Lipschitz constant for g in the set A, for any $I\in B_\delta(\bar{I})$ we also have

$$|k\cdot g(I)| \leq \|k\|\,\lambda\,\delta.$$

The series in (4.20) has therefore the lower bound

$$\begin{aligned}
&\mu \sum_{j=1}^{n}\sum_{\tilde{k}\in\mathbb{Z}^n\setminus 0} \int_{\tilde{B}} \frac{1}{\left|i\tilde{k}\cdot g(I)-\mu\right|^2} \left|\tilde{k}\cdot g(I)\frac{\partial f_{\tilde{k}}}{\partial I_j} - f_{\tilde{k}}\tilde{k}\cdot\frac{\partial g}{\partial I_j}\right| dI \\
&\geq \mu \sum_{j=1}^{n} \int_{B_\delta(\bar{I})} \frac{1}{|ik\cdot g(I)-\mu|^2} \left|k\cdot g(I)\frac{\partial f_k}{\partial I_j} - f_k k\cdot\frac{\partial g}{\partial I_j}\right| dI \\
&\geq \mu \int_{B_\delta(\bar{I})} \frac{1}{|ik\cdot g(I)-\mu|^2}\,|f_k| \left\|\frac{\partial g}{\partial I}^T k\right\|_1 dI - \mu \sum_{j=1}^{n}\int_{B_\delta(\bar{I})} \frac{1}{|ik\cdot g(I)-\mu|^2}\,|k\cdot g| \left|\frac{\partial f_k}{\partial I_j}\right| dI \\
&\geq \lambda_1\lambda_2\,\|k\|\,\mu \int_{B_\delta(\bar{I})} \frac{1}{|ik\cdot g(I)-\mu|^2} dI - \mu\,\|k\|\,\delta\lambda\Big(\sum_{j=1}^{n}\left|\frac{\partial f_k}{\partial I_j}\right|^\infty\Big) \int_{B_\delta(\bar{I})} \frac{1}{|ik\cdot g(I)-\mu|^2} dI \\
&= \left(\lambda_1\lambda_2 - \delta\lambda\sum_{j=1}^{n}\left|\frac{\partial f_k}{\partial I_j}\right|^\infty\right)\|k\|\,\mu\int_{B_\delta(\bar{I})} \frac{1}{|ik\cdot g(I)-\mu|^2} dI.
\end{aligned}$$

First, we remark that in the case $\sum_{j=1}^{n}\left|\frac{\partial f_k}{\partial I_j}\right|^\infty>0$, it is not restrictive to choose δ satisfying

$$\delta \leq \frac{\lambda_1\lambda_2}{2\lambda\sum_{j=1}^{n}\left|\frac{\partial f_k}{\partial I_j}\right|^\infty},$$

so that we have

$$\sum_{j=1}^{n}\sum_{\tilde{k}\in\mathbb{Z}^n\setminus 0} \mu\int_{\tilde{B}} \frac{1}{\left|i\tilde{k}\cdot g(I)-\mu\right|^2} \left|\tilde{k}\cdot g(I)\frac{\partial f_{\tilde{k}}}{\partial I_j} - f_{\tilde{k}}\tilde{k}\cdot\frac{\partial g}{\partial I_j}\right| dI \geq \frac{\lambda_1\lambda_2}{2}\,\|k\|\,\mu\int_{B_\delta(\bar{I})} \frac{1}{|ik\cdot g(I)-\mu|^2} dI.$$

Then by performing the change of variables $J:=g(I)$ and using (4.18) we obtain the lower bound

$$\frac{\lambda_1\lambda_2}{2}\,\|k\|\,\mu\int_{B_\delta(\bar{I})} \frac{1}{|ik\cdot g(I)-\mu|^2} dI \geq \frac{\lambda_1\lambda_2}{2M}\,\|k\|\,\mu\int_{g(B_\delta(\bar{I}))} \frac{1}{|ik\cdot J-\mu|^2} dJ$$

$$= \frac{\lambda_1 \lambda_2}{2M} \|k\| \mu \int_{g(B_\delta(\bar{I}))} \frac{1}{\sqrt{(k \cdot J)^2 + \mu^2}} dJ$$

which, by the additional change of variables $x := RJ$ as in (4.13), can be written as

$$\frac{\lambda_1 \lambda_2}{2M} \|k\| \mu \int_{Rg(B_\delta(\bar{I}))} \frac{1}{\sqrt{\|k\|^2 x_1^2 + \mu^2}} dx.$$

Since there exists $\tilde{\delta} > 0$ possibly depending on $k, \bar{I}$ (but independent of μ) such that

$$\left\{ x : \max_{j=1,\dots,n} \left| x_j - Rg(\bar{I})_j \right| \leq \tilde{\delta} \right\} \subseteq Rg(B_\delta(\bar{I})),$$

we obtain the lower bound

$$\frac{\lambda_1 \lambda_2}{2M} \|k\| \mu \int_{Rg(B_\delta(\bar{I}))} \frac{1}{\sqrt{\|k\|^2 x_1^2 + \mu^2}} dx \geq \frac{\lambda_1 \lambda_2}{2M} \|k\| \tilde{\delta}^{n-1} \mu \int_{-\tilde{\delta}}^{\tilde{\delta}} \frac{1}{\sqrt{\|k\|^2 x_1^2 + \mu^2}} dx_1$$

$$= \frac{\lambda_1 \lambda_2}{2M} \int_{-\frac{\|k\|}{\mu}\tilde{\delta}}^{\frac{\|k\|}{\mu}\tilde{\delta}} \frac{1}{1+y^2} dy = \frac{\lambda_1 \lambda_2}{M} \arctan \frac{\|k\|}{\mu} \tilde{\delta}.$$

Since we have

$$\lim_{\mu \to 0^+} \arctan \frac{\|k\|}{\mu} \tilde{\delta} = \frac{\pi}{2},$$

with a suitable definition of ε one immediately obtains (4.19).

We conclude our proof by showing the convergence of $F^{\mu,\nu}$ to $\bar{f}$ in $A \times \mathbb{T}^n$ on sequences $(\mu_i, \nu_i) \to (0,0)$ such that

$$\lim_{i \to 0} \frac{\mu_i}{\nu_i} = 0. \tag{4.21}$$

We first provide an estimate on the different contributions to

$$\left| F^{\mu,\nu} - \bar{f} \right|^1 = \left| F^{\mu,\nu} - \bar{f} \right|^0 + \sum_{k \in \mathbb{Z}^n} \sum_{j=1}^n \int_A \left| \frac{\partial}{\partial I_j} (F^{\mu,\nu} - \bar{f})_k \right| dI + \sum_{k \in \mathbb{Z}^n} \sum_{j=1}^n \int_A \left| k_j \right| \left| (F^{\mu,\nu} - \bar{f})_k \right| dI.$$

The first term $\left| F^{\mu,\nu} - \bar{f} \right|^0$ has been already estimated (see (3.6))

$$|F^{\mu,\nu} - \bar{f}|^0 \leq \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \setminus 0} |f_k|^\infty \frac{\mu}{\|k\|} \left(1 + \log \frac{\|k\| C}{\mu + \nu \|k\|^2} \right). \tag{4.22}$$

Then, for any $I \in \tilde{A}$, from (4.9) we have

$$(F^{\mu,\nu} - \bar{f})_k(I) = \begin{cases} F_k^{\mu,\nu}(I) = -\mu \dfrac{f_k(I)}{ik \cdot g(I) - \mu - \nu \|k\|^2} & \text{if } 0 \neq k \in \mathbb{Z}^n \\ 0 & \text{if } k = 0 \end{cases}$$

so that we need to estimate

$$\int_{\tilde{A}} \left| \frac{\partial F_k^{\mu,\nu}}{\partial I_j}(I) \right| dI = \mu \int_{\tilde{A}} \left| \frac{\partial f_k}{\partial I_j}(I) \frac{1}{ik \cdot g(I) - \mu - \nu \|k\|^2} + f_k(I) \frac{\partial}{\partial I_j} \left(\frac{1}{ik \cdot g(I) - \mu - \nu \|k\|^2} \right) \right| dI$$

for any $k \in \mathbb{Z}^n \backslash 0$. By using the changes of variables (4.11) and (4.13) as in the proof of Proposition 3.1 –and proceeding as in estimate (4.15)– we obtain

$$\mu \sum_{k \in \mathbb{Z}^n \backslash 0} \sum_{j=1}^{n} \int_{\tilde{A}} \left| \frac{\partial f_k}{\partial I_j}(I) \frac{1}{ik \cdot g(I) - \mu - \nu_i \|k\|^2} \right| dI$$

$$\leq \frac{2C^{n-1}}{m} \sum_{k \in \mathbb{Z}^n \backslash 0} \left(\sum_{j=1}^{n} \left| \frac{\partial f_k}{\partial I_j} \right|^{\infty} \right) \frac{\mu}{\|k\|} \left[1 + \log \frac{\|k\| C}{\mu + \nu \|k\|^2} \right]. \tag{4.23}$$

Using (2.1) we first obtain

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}^n \backslash 0} \sum_{j=1}^{n} \mu \int_{\tilde{A}} \left| f_k(I) \frac{\partial}{\partial I_j} \left(\frac{1}{ik \cdot g(I) - \mu - \nu \|k\|^2} \right) \right| dI \\
&\leq \sum_{k \in \mathbb{Z}^n \backslash 0} \sum_{j=1}^{n} \mu |f_k|^{\infty} \int_{\tilde{A}} \frac{1}{|ik \cdot g(I) - \mu - \nu \|k\|^2|^2} \left| \frac{\partial}{\partial I_j}(ik \cdot g(I)) \right| dI \\
&\leq \sum_{k \in \mathbb{Z}^n \backslash 0} \mu |f_k|^{\infty} n^2 \|k\| D \int_{\tilde{A}} \frac{1}{(k \cdot g(I))^2 + (\mu + \nu \|k\|^2)^2} dI,
\end{aligned}$$

then using the change of variables (4.11) and (4.13) we have

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}^n \backslash 0} \mu |f_k|^{\infty} n^2 \|k\| D \int_{\tilde{A}} \frac{1}{(k \cdot g(I))^2 + (\mu + \nu \|k\|^2)^2} dI \\
&\leq \sum_{k \in \mathbb{Z}^n \backslash 0} \frac{n^2 D}{m} |f_k|^{\infty} \|k\| \mu \int_{g(\tilde{A})} \frac{1}{(k \cdot J)^2 + (\mu + \nu \|k\|^2)^2} dJ \\
&\leq \sum_{k \in \mathbb{Z}^n \backslash 0} \frac{n^2 C^{n-1} D}{m} |f_k|^{\infty} \|k\| \mu \int_{-C}^{C} \frac{1}{\|k\|^2 x_1^2 + (\mu + \nu \|k\|^2)^2} dx_1 \\
&\leq \sum_{k \in \mathbb{Z}^n \backslash 0} \frac{2n^2 C^{n-1} D}{m} |f_k|^{\infty} \frac{\|k\|}{(\mu + \nu \|k\|^2)^2} \mu \int_{0}^{C} \frac{1}{\frac{\|k\|^2}{(\mu + \nu \|k\|^2)^2} x_1^2 + 1} dx_1 \\
&= \sum_{k \in \mathbb{Z}^n \backslash 0} \frac{2n^2 C^{n-1} D}{m} |f_k|^{\infty} \frac{\mu}{(\mu + \nu \|k\|^2)} \int_{0}^{\frac{\|k\| C}{\mu + \nu \|k\|^2}} \frac{1}{1 + y^2} dy \\
&= \sum_{k \in \mathbb{Z}^n \backslash 0} \frac{2n^2 C^{n-1} D}{m} |f_k|^{\infty} \frac{\mu}{(\mu + \nu \|k\|^2)} \arctan \left(\frac{\|k\| C}{\mu + \nu \|k\|^2} \right).
\end{aligned}$$

From the previous inequality, we obtain

$$\sum_{k \in \mathbb{Z}^n \backslash 0} \sum_{j=1}^{n} \mu \int_{\tilde{A}} \left| f_k(I) \frac{\partial}{\partial I_j} \left(\frac{1}{ik \cdot g(I) - \mu - \nu \|k\|^2} \right) \right| dI \leq \frac{n^2 \pi C^{n-1} D}{m} \sum_{k \in \mathbb{Z}^n \backslash 0} |f_k|^{\infty} \frac{\mu}{\mu + \nu \|k\|^2}. \tag{4.24}$$

In order to conclude the estimate of $\left|F^{\mu,\nu}-\bar{f}\right|^1$ it remains to consider

$$\sum_{k\in\mathbb{Z}^n\setminus 0}\sum_{j=1}^{n}\int_{\tilde{A}}\left|k_j\frac{\mu_i f_k(I)}{ik\cdot g(I)-\mu_i-\nu_i\|k\|^2}\right|dI.$$

This term is estimated by using the changes of variables (4.11) and (4.13), so that

$$\sum_{k\in\mathbb{Z}^n\setminus 0}\sum_{j=1}^{n}\mu|k_j|\int_{\tilde{A}}\left|\frac{f_k(I)}{ik\cdot g(I)-\mu-\nu\|k\|^2}\right|dI\leq\sum_{k\in\mathbb{Z}^n\setminus 0}\sum_{j=1}^{n}\frac{2\mu|k_j|\,|f_k|^\infty C^{n-1}}{m\|k\|}\left[l_1+\log\frac{\|k\|C}{\mu+\nu\|k\|^2}\right]$$

$$\leq\frac{2nC^{n-1}}{m}\sum_{k\in\mathbb{Z}^n\setminus 0}|f_k|^\infty\mu\left[1+\log\frac{\|k\|C}{\mu+\nu\|k\|^2}\right].\tag{4.25}$$

By collecting inequalities (4.22), (4.23), (4.24) and (4.25), we obtain

$$\left|F^{\mu,\nu}-\bar{f}\right|^1\leq\frac{2C^{n-1}}{m}\sum_{k\in\mathbb{Z}^n\setminus 0}\left(\frac{\mu}{\|k\|}\left[1+\log\frac{\|k\|C}{\mu+\nu\,\|k\|^2}\right]|f_k|^\infty+\frac{\mu}{\|k\|}\left[1+\log\frac{\|k\|C}{\mu+\nu\|k\|^2}\right]\left(\sum_{j=1}^{n}\left|\frac{\partial f_k}{\partial I_j}\right|^\infty\right)\right.$$

$$\left.+\frac{1}{2}n^2\pi D\frac{\mu}{\mu+\nu\,\|k\|^2}|f_k|^\infty+n\mu\left[1+\log\frac{\|k\|C}{\mu+\nu\|k\|^2}\right]|f_k|^\infty\right)$$

$$\leq\frac{2C^{n-1}}{m}\sum_{k\in\mathbb{Z}^n\setminus 0}\left(\mu\left[1+\log\frac{\|k\|C}{\mu+\nu\,\|k\|^2}\right]\left((1+n)\,|f_k|^\infty+\sum_{j=1}^{n}\left|\frac{\partial f_k}{\partial I_j}\right|^\infty\right)+\frac{1}{2}n^2\pi D\frac{\mu}{\mu+\nu\,\|k\|^2}|f_k|^\infty\right)$$

so that (3.7) is proved. Since for $\mu,\nu>0$ and $\|k\|\geq 1$, we have

$$\mu\log\frac{\|k\|C}{\mu+\nu\,\|k\|^2}\leq\mu\log\frac{C}{\nu}\leq\frac{\mu}{\nu}\left(\nu\log\frac{C}{\nu}\right)$$

and

$$\frac{\mu}{\mu+\nu\,\|k\|^2}\leq\frac{\mu}{\nu},$$

from (3.7) we obtain

$$\left|F^{\mu,\nu}-\bar{f}\right|^1\leq\left(\frac{\mu}{\nu}\right)\frac{2C^{n-1}}{m}\sum_{k\in\mathbb{Z}^n\setminus 0}\left(\left(\nu+\nu\log\frac{C}{\nu}\right)\left((1+n)\,|f_k|^\infty+\sum_{j=1}^{n}\left|\frac{\partial f_k}{\partial I_j}\right|^\infty\right)+\frac{1}{2}n^2\pi D\,|f_k|^\infty\right).$$

Therefore, for any sequence $\mu_i,\nu_i>0$ converging to zero with μ_i/ν_i converging to zero, we have

$$\lim_{i\to+\infty}\left|F^{\mu_i,\nu_i}-\bar{f}\right|^1=0.$$

The proof of Proposition 3.2 is concluded. □

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