On a reconstruction theorem for holonomic systems

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(Contributed by Masaki KASHIWARA, M.J.A., Nov. 12, 2012)

Abstract: Let X be a complex manifold. The classical Riemann-Hilbert correspondence associates to a regular holonomic system $\mathcal{M}$ the $\mathbf{C}$-constructible complex of its holomorphic solutions. Let t be the affine coordinate in the complex projective line. If $\mathcal{M}$ is not necessarily regular, we associate to it the ind-$\mathbf{R}$-constructible complex G of tempered holomorphic solutions to $\mathcal{M} \boxtimes \mathcal{D}e^t$. We conjecture that this provides a Riemann-Hilbert correspondence for holonomic systems. We discuss the functoriality of this correspondence, we prove that $\mathcal{M}$ can be reconstructed from G if $\dim X = 1$, and we show how the Stokes data are encoded in G.

Key words: Riemann-Hilbert problem; holonomic $\mathcal{D}$-modules; ind-sheaves; Stokes phenomenon.

Introduction. Let X be a complex manifold. The Riemann-Hilbert correspondence of [2] establishes an anti-equivalence

$$\mathbf{D}^{\mathrm{b}}_{\text{r-hol}}(\mathcal{D}_X) \underset{\Psi^0}{\overset{\Phi^0}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}_{\mathbf{C}\text{-c}}(\mathbf{C}_X)$$

between regular holonomic $\mathcal{D}$-modules and $\mathbf{C}$-constructible complexes. Here, $\Phi^0(\mathcal{L}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X)$ is the complex of holomorphic solutions to $\mathcal{L}$, and $\Psi^0(L) = T\mathcal{H}om(L, \mathcal{O}_X) = R\mathcal{H}om(L, \mathcal{O}^{\mathrm{t}}_X)$ is the complex of holomorphic functions tempered along L. Since $\mathcal{L} \simeq \Psi^0(\Phi^0(\mathcal{L}))$, this shows in particular that $\mathcal{L}$ can be reconstructed from $\Phi^0(\mathcal{L})$.

We are interested here in holonomic $\mathcal{D}$-modules which are not necessarily regular.

The theory of ind-sheaves from [6] allows one to consider the complex $\Phi^{\mathrm{t}}(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{\mathrm{t}}_X)$ of tempered holomorphic solutions to a holonomic module $\mathcal{M}$. The basic example $\Phi^{\mathrm{t}}(\mathcal{D}_{\mathbf{C}}e^{1/x})$ was computed in [7], and the functor Φ^{t} has been studied in [10,11]. However, since $\Phi^{\mathrm{t}}(\mathcal{D}_{\mathbf{C}}e^{1/x}) \simeq \Phi^{\mathrm{t}}(\mathcal{D}_{\mathbf{C}}e^{2/x})$, one cannot reconstruct $\mathcal{M}$ from $\Phi^{\mathrm{t}}(\mathcal{M})$.

Set $\Phi(\mathcal{M}) = \Phi^{\mathrm{t}}(\mathcal{M} \boxtimes \mathcal{D}_{\mathbf{P}}e^t)$, for t the affine variable in the complex projective line $\mathbf{P}$. This is an ind-$\mathbf{R}$-constructible complex in $X \times \mathbf{P}$. The arguments in [1] suggested us how $\mathcal{M}$ could be reconstructed from $\Phi(\mathcal{M})$ via a functor Ψ, described below (§3).

We conjecture that the contravariant functors

$$\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{C}_{X\times\mathbf{P}}),$$

between the derived categories of $\mathcal{D}_X$-modules and of ind-sheaves on $X \times \mathbf{P}$, provide a Riemann-Hilbert correspondence for holonomic systems.

To corroborate this statement, we discuss the functoriality of Φ and Ψ with respect to proper direct images and to tensor products with regular objects (§4). This allows us to reduce the problem to the case of holonomic modules with a good formal structure.

When X is a curve and $\mathcal{M}$ is holonomic, we prove that the natural morphism $\mathcal{M} \to \Psi(\Phi(\mathcal{M}))$ is an isomorphism (§6). Thus $\mathcal{M}$ can be reconstructed from $\Phi(\mathcal{M})$.

Recall that irregular holonomic modules are subjected to the Stokes phenomenon. We describe with an example how the Stokes data of $\mathcal{M}$ are encoded topologically in the ind-$\mathbf{R}$-constructible sheaf $\Phi(\mathcal{M})$ (§7).

In this Note, the proofs are only sketched. Details will appear in a forthcoming paper. There, we will also describe some of the properties of the essential image of holonomic systems by the functor Φ. Such a category is related to a construction of [13].

1. Notations. We refer to [3–6].

Let X be a real analytic manifold.

Denote by $\mathbf{D}^{\mathrm{b}}(\mathbf{C}_X)$ the bounded derived category of sheaves of $\mathbf{C}$-vector spaces, and by

2010 Mathematics Subject Classification. Primary 32C38, 35A20, 32S60, 34M40.

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doi: 10.3792/pjaa.88.178

$\mathbf{D}^{\mathrm{b}}_{\mathbf{R}\text{-c}}(\mathbf{C}_X)$ the full subcategory of objects with $\mathbf{R}$-constructible cohomologies. Denote by $\otimes$, $R\mathcal{H}om$, f^{-1}, Rf_*, $Rf_!$, $f^!$ the six Grothendieck operations for sheaves. (Here $f\colon X \to Y$ is a morphism of real analytic manifolds.)

For $S \subset X$ a locally closed subset, we denote by $\mathbf{C}_S$ the zero extension to X of the constant sheaf on S.

Recall that an ind-sheaf is an ind-object in the category of sheaves with compact support. Denote by $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{C}_X)$ the bounded derived category of ind-sheaves, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbf{R}\text{-c}}(\mathrm{I}\mathbf{C}_X)$ the full subcategory of objects with ind-$\mathbf{R}$-constructible cohomologies. Denote by $\otimes$, $R\mathcal{I}\mathcal{H}om$, f^{-1}, Rf_*, $Rf_{!!}$, $f^!$ the six Grothendieck operations for ind-sheaves.

Denote by α the left adjoint of the embedding of sheaves into ind-sheaves. One has $\alpha(\text{“}\varinjlim\text{”}\, F_i) = \varinjlim F_i$. Denote by β the left adjoint of α.

Denote by $\mathcal{D}b^{\mathrm{t}}_X$ the ind-$\mathbf{R}$-constructible sheaf of tempered distributions.

Let X be a complex manifold. We set for short $d_X = \dim X$.

Denote by $\mathcal{O}_X$ and $\mathcal{D}_X$ the rings of holomorphic functions and of differential operators. Denote by Ω_X the invertible sheaf of differential forms of top degree.

Denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ the bounded derived category of left $\mathcal{D}_X$-modules, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ and $\mathbf{D}^{\mathrm{b}}_{\text{r-hol}}(\mathcal{D}_X)$ the full subcategories of objects with holonomic and regular holonomic cohomologies, respectively. Denote by $\otimes^{\mathsf{D}}$, $\mathsf{D}f^{-1}$, $\mathsf{D}f_*$ the operations for $\mathcal{D}$-modules. (Here $f\colon X \to Y$ is a morphism of complex manifolds.)

Denote by $\mathbf{D}\mathcal{M}$ the dual of $\mathcal{M}$ (with shift such that $\mathbf{D}\mathcal{O}_X \simeq \mathcal{O}_X$).

For $Z \subset X$ a closed analytic subset, we denote by $\mathrm{R}\Gamma_{[Z]}\mathcal{M}$ and $\mathcal{M}(*Z)$ the relative algebraic cohomologies of a $\mathcal{D}_X$-module $\mathcal{M}$.

Denote by $\mathrm{ss}(\mathcal{M}) \subset X$ the singular support of $\mathcal{M}$, that is the set of points where the characteristic variety is not reduced to the zero-section.

Denote by $\mathcal{O}^{\mathrm{t}}_X \in \mathbf{D}^{\mathrm{b}}_{\mathrm{I}\mathbf{R}\text{-c}}(\mathrm{I}\mathbf{C}_X)$ the complex of tempered holomorphic functions. Recall that $\mathcal{O}^{\mathrm{t}}_X$ is the Dolbeault complex of $\mathcal{D}b^{\mathrm{t}}_X$ and that it has a structure of $\beta\mathcal{D}_X$-module. We will write for short $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{\mathrm{t}}_X)$ instead of $R\mathcal{I}\mathcal{H}om_{\beta\mathcal{D}_X}(\beta\mathcal{M}, \mathcal{O}^{\mathrm{t}}_X)$.

2. Exponential $\mathcal{D}$-modules. Let X be a complex analytic manifold. Let $D \subset X$ be a hypersurface, and set $U = X \setminus D$. For $\varphi \in \mathcal{O}_X(*D)$, we set

$$\mathcal{D}_X e^{\varphi} = \mathcal{D}_X / \{P\colon Pe^{\varphi} = 0 \text{ on } U\},$$
$$\mathcal{E}^{\varphi}_{D|X} = (\mathcal{D}_X\, e^{\varphi})(*D).$$

As an $\mathcal{O}_X(*D)$-module, $\mathcal{E}^{\varphi}_{D|X}$ is generated by e^{φ}. Note that $\mathrm{ss}(\mathcal{E}^{\varphi}_{D|X}) = D$, and $\mathcal{E}^{\varphi}_{D|X}$ is holonomic. It is regular if $\varphi \in \mathcal{O}_X$, since then $\mathcal{E}^{\varphi}_{D|X} \simeq \mathcal{O}_X(*D)$.

One easily checks that $(\mathbf{D}\mathcal{E}^{\varphi}_{D|X})(*D) \simeq \mathcal{E}^{-\varphi}_{D|X}$.

Proposition 2.1. *If $\dim X = 1$, and φ has an effective pole at every point of D, then $\mathbf{D}\mathcal{E}^{\varphi}_{D|X} \simeq \mathcal{E}^{-\varphi}_{D|X}$.*

Let $\mathbf{P}$ be the complex projective line and denote by t the coordinate on $\mathbf{C} = \mathbf{P} \setminus \{\infty\}$.

For $c \in \mathbf{R}$, we set for short

$$\{\operatorname{Re}\varphi < c\} = \{x \in U\colon \operatorname{Re}\varphi(x) < c\},$$
$$\{\operatorname{Re}(t+\varphi) < c\} = \{(x,t)\colon x \in U,\ t \in \mathbf{C},\ \operatorname{Re}(t+\varphi(x)) < c\}.$$

Consider the ind-$\mathbf{R}$-constructible sheaves on X and on $X \times \mathbf{P}$, respectively,

$$\mathbf{C}_{\{\operatorname{Re}\varphi<?\}} = \text{“}\varinjlim_{c\to+\infty}\text{”}\, \mathbf{C}_{\{\operatorname{Re}\varphi<c\}},$$
$$\mathbf{C}_{\{\operatorname{Re}(t+\varphi)<?\}} = \text{“}\varinjlim_{c\to+\infty}\text{”}\, \mathbf{C}_{\{\operatorname{Re}(t+\varphi)<c\}}.$$

The following result is analogous to [1, Proposition 7.1]. Its proof is simpler than loc. cit., since φ is differentiable.

Proposition 2.2. *One has an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$*

$$\mathcal{E}^{\varphi}_{D|X} \xrightarrow{\sim} Rq_* R\mathcal{H}om_{p^{-1}\mathcal{D}_{\mathbf{P}}}(p^{-1}\mathcal{E}^{t}_{\infty|\mathbf{P}}, R\mathcal{H}om(\mathbf{C}_{\{\operatorname{Re}(t+\varphi)<?\}}, \mathcal{O}^{\mathrm{t}}_{X\times\mathbf{P}})),$$

for q and p the projections from $X \times \mathbf{P}$.

The following result is analogous to [7, Proposition 7.3].

Lemma 2.3. *Denote by (u,v) the coordinates in $\mathbf{C}^2$. There is an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{C}_{\mathbf{C}^2})$*

$$R\mathcal{H}om_{\mathcal{D}_{\mathbf{C}^2}}(\mathcal{E}^{u/v}_{\{v=0\}|\mathbf{C}^2}, \mathcal{O}^{\mathrm{t}}_{\mathbf{C}^2}) \simeq R\mathcal{I}\mathcal{H}om(\mathbf{C}_{\{v\neq0\}}, \mathbf{C}_{\{\operatorname{Re} u/v<?\}}).$$

Proposition 2.4. *There is an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{C}_X)$*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathbf{D}\mathcal{E}^{-\varphi}_{D|X}, \mathcal{O}^{\mathrm{t}}_X) \simeq R\mathcal{I}\mathcal{H}om(\mathbf{C}_U, \mathbf{C}_{\{\operatorname{Re}\varphi<?\}}).$$

Proof. As $\mathbf{D}\mathcal{E}^{u/v}_{\{v=0\}|\mathbf{C}^2} \simeq \mathcal{E}^{-u/v}_{\{v=0\}|\mathbf{C}^2}$, Lemma 2.3 gives

$$\Omega^{\mathrm{t}}_{\mathbf{C}^2} \otimes^{L}_{\mathcal{D}_{\mathbf{C}^2}} \mathcal{E}^{-u/v}_{\{v=0\}|\mathbf{C}^2}[-2] \simeq R\mathcal{I}\mathcal{H}om(\mathbf{C}_{\{v\neq0\}}, \mathbf{C}_{\{\operatorname{Re} u/v<?\}}).$$

Write $\varphi = a/b$ for $a, b \in \mathcal{O}_X$ such that $b^{-1}(0) \subset D$, and consider the map

$$f = (a, b): X \to \mathbf{C}^2.$$

As $\mathsf{D}f^{-1}\mathcal{E}^{-u/v}_{\{v=0\}|\mathbf{C}^2} \simeq \mathcal{E}^{-\varphi}_{D|X}$, [6, Theorem 7.4.1] implies

$$\Omega^{\mathsf{t}}_X \otimes^L_{\mathcal{D}_X} \mathcal{E}^{-\varphi}_{D|X}[-d_X] \simeq R\mathcal{IH}om(\mathbf{C}_U, \mathbf{C}_{\{\mathrm{Re}\,\varphi<?\}}).$$

Finally, one has

$$\Omega^{\mathsf{t}}_X \otimes^L_{\mathcal{D}_X} \mathcal{E}^{-\varphi}_{D|X}[-d_X] \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathbf{D}\mathcal{E}^{-\varphi}_{D|X}, \mathcal{O}^{\mathsf{t}}_X).$$

□

3. A correspondence. Let X be a complex analytic manifold. Recall that $\mathbf{P}$ denotes the complex projective line. Consider the contravariant functors

$$\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{C}_{X\times\mathbf{P}})$$

defined by

$$\Phi(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_{X\times\mathbf{P}}}(\mathcal{M} \boxtimes^{\mathsf{D}} \mathcal{E}^t_{\infty|\mathbf{P}}, \mathcal{O}^{\mathsf{t}}_{X\times\mathbf{P}}),$$
$$\Psi(F) = Rq_* R\mathcal{H}om_{p^{-1}\mathcal{D}_{\mathbf{P}}}(p^{-1}\mathcal{E}^t_{\infty|\mathbf{P}}, R\mathcal{H}om(F, \mathcal{O}^{\mathsf{t}}_{X\times\mathbf{P}})),$$

for q and p the projections from $X \times \mathbf{P}$.

We conjecture that this provides a Riemann-Hilbert correspondence for holonomic systems:

Conjecture 3.1.
(i) *The natural morphism of endofunctors of* $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$

$$\mathrm{id} \to \Psi \circ \Phi \tag{3.1}$$

is an isomorphism on $\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$.
(ii) *The restriction of* Φ

$$\Phi|_{\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)}: \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathbf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{C}_{X\times\mathbf{P}})$$

is fully faithful.

Let us prove some results in this direction.

4. Functorial properties. The next two Propositions are easily deduced from the results in [6].

Proposition 4.1. *Let $f: X \to Y$ be a proper map, and set $f_{\mathbf{P}} = f \times \mathrm{id}_{\mathbf{P}}$. Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ and $F \in \mathbf{D}^{\mathrm{b}}_{\mathrm{IR\text{-}c}}(\mathrm{I}\mathbf{C}_{X\times\mathbf{P}})$. Then*

$$\Phi(\mathsf{D}f_*\mathcal{M}) \simeq Rf_{\mathbf{P}!!}\Phi(\mathcal{M})[d_X - d_Y],$$
$$\Psi(Rf_{\mathbf{P}!!}F) \simeq \mathsf{D}f_*\Psi(F)[d_X - d_Y].$$

For $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}_{\text{r-hol}}(\mathcal{D}_X)$, set

$$\Phi^0(\mathcal{L}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X).$$

Recall that $\Phi^0(\mathcal{L})$ is a $\mathbf{C}$-constructible complex of sheaves on X.

Proposition 4.2. *Let $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}_{\text{r-hol}}(\mathcal{D}_X)$, $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ and $F \in \mathbf{D}^{\mathrm{b}}_{\mathrm{IR\text{-}c}}(\mathrm{I}\mathbf{C}_{X\times\mathbf{P}})$. Then*

$$\Phi(\mathbf{D}(\mathcal{L} \otimes^{\mathsf{D}} \mathbf{D}\mathcal{M})) \simeq R\mathcal{IH}om(q^{-1}\Phi^0(\mathcal{L}), \Phi(\mathcal{M})),$$
$$\Psi(F \otimes q^{-1}\Phi^0(\mathcal{L})) \simeq \Psi(F) \otimes^{\mathsf{D}} \mathcal{L}.$$

Noticing that

$$\Phi(\mathcal{O}_X) \simeq \mathbf{C}_X \boxtimes R\mathcal{IH}om(\mathbf{C}_{\{t\neq\infty\}}, \mathbf{C}_{\{\mathrm{Re}\,t<?\}}),$$

one checks easily that $\Psi(\Phi(\mathcal{O}_X)) \simeq \mathcal{O}_X$. Hence, Proposition 4.2 shows:

Theorem 4.3.
(i) *For $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}_{\text{r-hol}}(\mathcal{D}_X)$, we have*

$$\Phi(\mathcal{L}) \simeq q^{-1}\Phi^0(\mathcal{L}) \otimes \Phi(\mathcal{O}_X)$$
$$\simeq \Phi^0(\mathcal{L}) \boxtimes R\mathcal{IH}om(\mathbf{C}_{\{t\neq\infty\}}, \mathbf{C}_{\{\mathrm{Re}\,t<?\}}).$$

(ii) *The morphism* (3.1) *is an isomorphism on* $\mathbf{D}^{\mathrm{b}}_{\text{r-hol}}(\mathcal{D}_X)$.
(iii) *For any $\mathcal{L}, \mathcal{L}' \in \mathbf{D}^{\mathrm{b}}_{\text{r-hol}}(\mathcal{D}_X)$, the natural morphism*

$$\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{L}, \mathcal{L}') \to \mathrm{Hom}(\Phi(\mathcal{L}'), \Phi(\mathcal{L}))$$

is an isomorphism.

Therefore, Conjecture 3.1 holds true for regular holonomic $\mathcal{D}$-modules.

5. Review on good formal structures. Let $D \subset X$ be a hypersurface. A flat meromorphic connection with poles at D is a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ such that $\mathrm{ss}(\mathcal{M}) = D$ and $\mathcal{M} \simeq \mathcal{M}(*D)$.

We recall here the classical results on the formal structure of flat meromorphic connections on curves. (Analogous results in higher dimension have been obtained in [8,9,12].)

Let X be an open disc in $\mathbf{C}$ centered at 0.

For $\mathcal{F}$ an $\mathcal{O}_X$-module, we set

$$\mathcal{F}\,\widehat{|}_0 = \widehat{\mathcal{O}}_{X,0} \otimes_{\mathcal{O}_{X,0}} \mathcal{F}_0,$$

where $\widehat{\mathcal{O}}_{X,0}$ is the completion of $\mathcal{O}_{X,0}$.

One says that a flat meromorphic connection $\mathcal{M}$ with poles at 0 has a good formal structure if

$$\mathcal{M}\,\widehat{|}_0 \simeq \bigoplus_{i\in I}\Big(\mathcal{L}_i \otimes^{\mathsf{D}} \mathcal{E}^{\varphi_i}_{0|X}\Big)\widehat{|}_0 \tag{5.1}$$

as $(\widehat{\mathcal{O}}_{X,0} \otimes_{\mathcal{O}_{X,0}} \mathcal{D}_{X,0})$-modules, where I is a finite set, $\mathcal{L}_i$ are regular holonomic $\mathcal{D}_X$-modules, and $\varphi_i \in \mathcal{O}_X(*0)$.

A ramification at 0 is a map $X \to X$ of the form $x \mapsto x^m$ for some $m \in \mathbf{N}$.

The Levelt-Turrittin theorem asserts:

Theorem 5.1. *Let $\mathcal{M}$ be a meromorphic connection with poles at* 0*. Then there is a ramification $f\colon X \to X$ such that $\mathsf{D}f^{-1}\mathcal{M}$ has a good formal structure at* 0*.*

Assume that $\mathcal{M}$ satisfies (5.1). If $\mathcal{M}$ is regular, then $\varphi_i \in \mathcal{O}_X$ for all $i \in I$, and (5.1) is induced by an isomorphism

$$\mathcal{M}_0 \simeq \bigoplus_{i\in I}\Big(\mathcal{L}_i \otimes^{\mathsf{D}} \mathcal{E}^{\varphi_i}_{0|X}\Big)_0.$$

However, such an isomorphism does not hold in general.

Consider the real oriented blow-up

$$\pi\colon B = \mathbf{R}\times S^1 \to X, \quad (\rho,\theta)\mapsto \rho e^{i\theta}. \tag{5.2}$$

Set $V = \{\rho > 0\}$ and let $Y = \{\rho \geq 0\}$ be its closure. If W is an open neighborhood of $(0,\theta)\in\partial Y$, then $\pi(W\cap V)$ contains a germ of open sector around the direction θ centered at 0.

Consider the commutative ring

$$\mathcal{A}_Y = R\mathcal{H}om_{\pi^{-1}\mathcal{D}_{\overline{X}}}(\pi^{-1}\mathcal{O}_{\overline{X}}, R\mathcal{H}om(\mathbf{C}_V, \mathcal{D}b^{\mathrm{t}}_B)),$$

where $\overline{X}$ is the complex conjugate of X.

To a $\mathcal{D}_X$-module $\mathcal{M}$, one associates the $\mathcal{A}_Y$-module

$$\pi^*\mathcal{M} = \mathcal{A}_Y \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{M}.$$

The Hukuara-Turrittin theorem states that (5.1) can be extended to germs of open sectors:

Theorem 5.2. *Let $\mathcal{M}$ be a flat meromorphic connection with poles at 0. Assume that $\mathcal{M}$ admits the good formal structure (5.1). Then for any $(0,\theta)\in\partial Y$ one has*

$$(\pi^*\mathcal{M})_{(0,\theta)} \simeq \left(\bigoplus_{i\in I}\pi^*(\mathcal{E}^{\varphi_i}_{0|X})^{m_i}\right)_{(0,\theta)}, \tag{5.3}$$

where m_i is the rank of $\mathcal{L}_i$.

(Note that only the ranks of the $\mathcal{L}_i$'s appear here, since $x^\lambda(\log x)^m$ belongs to $\mathcal{A}_Y$ for any $\lambda\in\mathbf{C}$ and $m\in\mathbf{Z}_{\geq 0}$.)

One should be careful that the above isomorphism depends on θ, giving rise to the Stokes phenomenon.

We will need the following result:

Lemma 5.3. *If $\mathcal{M}$ is a flat meromorphic connection with poles at 0, then*

$$R\pi_*(\pi^*\mathcal{M}) \simeq \mathcal{M}.$$

6. Reconstruction theorem on curves. Let X be a complex curve. Then Conjecture 3.1 (i) holds true:

Theorem 6.1. *For $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ there is a functorial isomorphism*

$$\mathcal{M} \xrightarrow{\sim} \Psi(\Phi(\mathcal{M})). \tag{6.1}$$

Sketch of proof. Since the statement is local, we can assume that X is an open disc in $\mathbf{C}$ centered at 0, and that $\mathrm{ss}(\mathcal{M}) = \{0\}$.

By devissage, we can assume from the beginning that $\mathcal{M}$ is a flat meromorphic connection with poles at 0.

Let $f\colon X\to X$ be a ramification as in Theorem 5.1, so that $\mathsf{D}f^{-1}\mathcal{M}$ admits a good formal structure at 0.

Note that $\mathsf{D}f_*\mathsf{D}f^{-1}\mathcal{M}\simeq\mathcal{M}\oplus\mathcal{N}$ for some $\mathcal{N}$. If (6.1) holds for $\mathsf{D}f^{-1}\mathcal{M}$, then it holds for $\mathcal{M}\oplus\mathcal{N}$ by Proposition 4.1, and hence it also holds for $\mathcal{M}$.

We can thus assume that $\mathcal{M}$ admits a good formal structure at 0.

Consider the real oriented blow-up (5.2).

By Lemma 5.3, one has $\mathcal{M}\simeq R\pi_*\pi^*\mathcal{M}$. Hence Proposition 4.1 (or better, its analogue for π) implies that we can replace $\mathcal{M}$ with $\pi^*\mathcal{M}$.

By Theorem 5.2, we finally reduce to prove

$$\mathcal{E}^{\varphi}_{0|X} \xrightarrow{\sim} \Psi(\Phi(\mathcal{E}^{\varphi}_{0|X})).$$

Set $D' = \{x=0\}\cup\{t=\infty\}$ and $U' = (X\times\mathbf{P})\setminus D'$. By Proposition 2.1,

$$\mathbf{D}\mathcal{E}^{t+\varphi}_{D'|X\times\mathbf{P}} \simeq \mathbf{D}(\mathcal{E}^{\varphi}_{0|X}\boxtimes^{\mathsf{D}}\mathcal{E}^{t}_{\infty|\mathbf{P}}) \simeq \mathcal{E}^{-t-\varphi}_{D'|X\times\mathbf{P}}.$$

By Proposition 2.4, we thus have

$$\Phi(\mathcal{E}^{\varphi}_{0|X}) \simeq R\mathcal{IH}om(\mathbf{C}_{U'}, \mathbf{C}_{\{\mathrm{Re}(t+\varphi)<?\}}).$$

Noticing that $\Phi(\mathcal{E}^{\varphi}_{0|X})\otimes\mathbf{C}_{D'}\in\mathbf{D}^{\mathrm{b}}_{\mathbf{C}\text{-c}}(\mathbf{C}_{X\times\mathbf{P}})$, one checks that $\Psi(\Phi(\mathcal{E}^{\varphi}_{0|X})\otimes\mathbf{C}_{D'})\simeq 0$.

Hence, Proposition 2.2 implies

$$\Psi(\Phi(\mathcal{E}^{\varphi}_{0|X})) \simeq \Psi(\mathbf{C}_{\{\mathrm{Re}(t+\varphi)<?\}}) \simeq \mathcal{E}^{\varphi}_{0|X}.$$

□

Example 6.2. Let $X=\mathbf{C}$, $\varphi(x)=1/x$ and $\mathcal{M}=\mathcal{E}^{\varphi}_{0|X}$. Then we have

$$H^k\Phi(\mathcal{M}) = \begin{cases} \mathbf{C}_{\{\mathrm{Re}(t+\varphi)<?\}}, & \text{for } k=0,\\ \mathbf{C}_{\{x=0,\, t\neq\infty\}}\oplus \\ \qquad\mathbf{C}_{\{x\neq 0,\, t=\infty\}}, & \text{for } k=1,\\ 0, & \text{otherwise.}\end{cases}$$

7. Stokes phenomenon. We discuss here an example which shows how, in our setting, the Stokes phenomenon arises in a purely topological fashion.

Let X be an open disc in $\mathbf{C}$ centered at 0. (We will shrink X if necessary.) Set $U = X \setminus \{0\}$.

Let $\mathcal{M}$ be a flat meromorphic connection with poles at 0 such that

$$\mathcal{M}\widehat{|}_0 \simeq (\mathcal{E}^{\varphi}_{0|X} \oplus \mathcal{E}^{\psi}_{0|X})\widehat{|}_0, \quad \varphi, \psi \in \mathcal{O}_X(*0).$$

Assume that $\psi - \varphi$ has an effective pole at 0.

The Stokes curves of $\mathcal{E}^{\varphi}_{0|X} \oplus \mathcal{E}^{\psi}_{0|X}$ are the real analytic arcs ℓ_i, $i \in I$, defined by

$$\{\mathrm{Re}(\psi - \phi) = 0\} = \bigsqcup\nolimits_{i \in I} \ell_i.$$

(Here we possibly shrink X to avoid crossings of the ℓ_i's and to ensure that they admit the polar coordinate $\rho > 0$ as parameter.)

Since $\mathcal{E}^{\varphi}_{0|X} \simeq \mathcal{E}^{\varphi+\varphi_0}_{0|X}$ for $\varphi_0 \in \mathcal{O}_X$, the Stokes curves are not invariant by isomorphism.

The Stokes lines L_i, defined as the limit tangent half-lines to ℓ_i at 0, are invariant by isomorphism.

The Stokes matrices of $\mathcal{M}$ describe how the isomorphism (5.3) changes when θ crosses a Stokes line.

Let us show how these data are topologically encoded in $\Phi(\mathcal{M})$.

Set $D' = \{x = 0\} \cup \{t = \infty\}$ and $U' = (X \times \mathbf{P}) \setminus D'$. Set

$$F_c = \mathbf{C}_{\{\mathrm{Re}(t+\varphi)<c\}}, \quad G_c = \mathbf{C}_{\{\mathrm{Re}(t+\psi)<c\}},$$
$$F = \mathbf{C}_{\{\mathrm{Re}(t+\varphi)<?\}}, \quad G = \mathbf{C}_{\{\mathrm{Re}(t+\psi)<?\}}.$$

By Proposition 2.4 and Theorem 5.2,

$$\Phi(\mathcal{M}) \simeq R\mathcal{I}\mathcal{H}om(\mathbf{C}_{U'}, H),$$

where H is an ind-sheaf such that

$$H \otimes \mathbf{C}_{q^{-1}S} \simeq (F \oplus G) \otimes \mathbf{C}_{q^{-1}S}$$

for any sufficiently small open sector S.

Let $\mathfrak{b}^{\pm}$ be the vector space of upper/lower triangular matrices in $\mathrm{M}_2(\mathbf{C})$, and let $\mathfrak{t} = \mathfrak{b}^+ \cap \mathfrak{b}^-$ be the vector space of diagonal matrices.

Lemma 7.1. *Let S be an open sector, and $\mathfrak{v}$ a vector space, which satisfy one of the following conditions:*

(i) $\mathfrak{v} = \mathfrak{b}^{\pm}$ *and* $S \subset \{\pm\mathrm{Re}(\psi - \varphi) > 0\}$,

(ii) $\mathfrak{v} = \mathfrak{t}$, $S \supset L_i$ *for some* $i \in I$ *and* $S \cap L_j = \emptyset$ *for* $i \neq j$.

Then, for $c' \gg c$, one has

$$\mathrm{Hom}((F_c \oplus G_c)|_{q^{-1}S}, (F_{c'} \oplus G_{c'})|_{q^{-1}S}) \simeq \mathfrak{v}.$$

In particular,

$$\mathrm{End}((F \oplus G) \otimes \mathbf{C}_{q^{-1}S}) \simeq \mathfrak{v}.$$

This proves that the Stokes lines are encoded in H. Let us show how to recover the Stokes matrices of $\mathcal{M}$ as glueing data for H.

Let S_i be an open sector which contains L_i and is disjoint from L_j for $i \neq j$. We choose S_i so that $\bigcup_{i \in I} S_i = U$.

Then for each $i \in I$, there is an isomorphism

$$\alpha_i \colon H \otimes \mathbf{C}_{q^{-1}S_i} \simeq (F \oplus G) \otimes \mathbf{C}_{q^{-1}S_i}.$$

Take a cyclic ordering of I such that the Stokes lines get ordered counterclockwise.

Since $\{S_i\}_{i \in I}$ is an open cover of U, the ind-sheaf H is reconstructed from $F \oplus G$ via the glueing data given by the Stokes matrices

$$A_i = \alpha_{i+1}^{-1}\alpha_i|_{q^{-1}(S_i \cap S_{i+1})} \in \mathfrak{b}^{\pm}.$$

Acknowledgment. The first author expresses his gratitude to the RIMS of Kyoto University for hospitality during the preparation of this paper.

References

[1] A. D'Agnolo, On the Laplace transform for temperate holomorphic functions, arXiv:1207.5278.

[2] M. Kashiwara, The Riemann-Hilbert problem for holonomic systems, Publ. Res. Inst. Math. Sci. **20** (1984), no. 2, 319–365.

[3] M. Kashiwara, *D-modules and microlocal calculus*, translated from the 2000 Japanese original by Mutsumi Saito, Translations of Mathematical Monographs, 217, Amer. Math. Soc., Providence, RI, 2003.

[4] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften, 292, Springer, Berlin, 1990.

[5] M. Kashiwara and P. Schapira, Moderate and formal cohomology associated with constructible sheaves, Mém. Soc. Math. France (N.S.) No. 64 (1996), iv+76 pp.

[6] M. Kashiwara and P. Schapira, Ind-sheaves, Astérisque **271** (2001), 136 pp.

[7] M. Kashiwara and P. Schapira, Microlocal study of ind-sheaves. I. Micro-support and regularity, Astérisque **284** (2003), 143–164.

[8] K. S. Kedlaya, Good formal structures for flat meromorphic connections, II: excellent schemes, J. Amer. Math. Soc. **24** (2011), no. 1, 183–229.

[9] T. Mochizuki, Wild harmonic bundles and wild pure twistor D-modules, Astérisque **340** (2011), x+607 pp.

[10] G. Morando, An existence theorem for tempered solutions of $\mathcal{D}$-modules on complex curves, Publ. Res. Inst. Math. Sci. **43** (2007), no. 3,

625–659.
[11] G. Morando, Preconstructibility of tempered solutions of holonomic $\mathcal{D}$-modules, arXiv:1007.4158.
[12] C. Sabbah, Équations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension 2, Astérisque **263** (2000), viii+190 pp.
[13] D. Tamarkin, Microlocal condition for non-displaceablility, arXiv:0809.1584.