A PROPAGATION THEOREM FOR A CLASS OF SHEAVES OF MICROFUNCTIONS

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Abstract. Let A be a closed set of $M \cong \mathbf{R}^n$, whose conormal cones $x + \gamma_x^*(A)$, $x \in A$, have locally empty intersection. We first show in §1 that $\mathrm{dist}(x, A)$, $x \in M \setminus A$ is a C^1 function. We then represent the microfunctions of $\mathcal{C}_{A|X}$, $X \cong \mathbf{C}^n$, (cf [S]), using cohomology groups of $\mathcal{O}_X$ of degree 1. By the results of §1–3, we are able to prove in §4 that the sections of $\mathcal{C}_{A|X}\big|_{(T^*_M X \oplus \gamma^*(A))_{x_0}}$, $x_0 \in \partial A$, satisfy the principle of analytic continuation in the complex integral manifolds of $\{H(\phi_i^{\mathbf{C}})\}_{i=1,\dots,m}$, $\{\phi_i\}$ being a base for the linear hull of $\gamma^*_{x_0}(A)$ in $T^*_{x_0}M$; in particular we get $\Gamma_{A\times_M T^*_M X}(\mathcal{C}_{A|X})\big|_{\partial A\times_M \dot{T}^*_M X} = 0$. In the proof we deeply use a variant of Bochner's theorem due to [Kan]. When A is a half space with C^ω boundary, all above results were already proved by Kataoka in [Kat 1]. Finally for a $\mathcal{E}_X$-module $\mathcal{M}$ we show that $\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{A|X})_p = 0$, $p \in \dot{\pi}^{-1}(x_0)$, when at least one conormal $\theta \in \dot{\gamma}^*_{x_0}(A)$ is non-characteristic for $\mathcal{M}$. We also show that for an open domain Ω such that the set $A = M \setminus \Omega$ verifies the above conditions, we have $(\mathcal{C}_{\Omega|X})_{T^*_M X} = H^0(\mathcal{C}_{\Omega|X})$ (cf [S] for the case of A convex).

Un teorema di propagazione per certi fasci di microfunzioni

Sunto. Sia A un insieme chiuso di $M \cong \mathbf{R}^n$, i cui coni conormali $x + \gamma_x^*(A)$, $x \in A$, hanno localmente intersezione vuota. Si prova nel §1 che $\mathrm{dist}(x, A)$, $x \in M \setminus A$ è una funzione C^1. Si rappresentano poi le microfunzioni di $\mathcal{C}_{A|X}$, $X \cong \mathbf{C}^n$, (cf [S]), mediante gruppi di coomologia di $\mathcal{O}_X$ in grado 1. Se ne deduce nel §4 un principio di prolungamento analitico per sezioni di $\mathcal{C}_{A|X}\big|_{(T^*_M X \oplus \gamma^*(A))_{x_0}}$, $x_0 \in \partial A$, che generalizza i risultati di [Kat 1]. (Per la dimostrazione si usa essenzialmente un'idea di [Kan].) Se ne dà infine applicazione ai problemi ai limiti.

§1. Let X be a C^∞-manifold, A a closed set of X. We denote by $\gamma(A) \subset TX$ the set

$$\gamma_x(A) = C(A, \{x\}), \qquad x \in A,$$

where $C(A, \{x\})$ is the normal cone to A along $\{x\}$ in the sense of [K-S]; we also denote by $\gamma^*(A)$ the polar cone to $\gamma(A)$. We assume that in some coordinates in a neighborhood of a point $x_0 \in \partial A$:

$$(1.1) \qquad \begin{aligned} &\text{(i)} \quad (x - \gamma_x^*(A)) \cap (y - \gamma_y^*(A)) \cap S = \emptyset \quad \forall x \neq y \in \partial A \cap S, \\ &\text{(ii)} \quad x \mapsto \gamma_x^*(A) \quad \text{is upper semicontinuous.} \end{aligned}$$

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Remark 1.1.

(a) If A is convex in $X \cong \mathbf{R}^n$ then (1.1) holds. Moreover in this case $\gamma(A) = \overline{N(A)}$ where $N(A)$ is the normal cone to A in the linear hull of A.

(b) All sets A with C^2-boundary satisfy (1.1).

(c) The set $A = \{(x_1, x_2) \in \mathbf{R}^2; x_1 \leq -\sqrt{|x_2|}\}$ verifies (1.1). (Here $\gamma_0^*(A) = \mathbf{R}^-_{x_2}$ but $N_0^*(A) = \mathbf{R}^2$.)

(d) The set $A = \{(x_1, x_2) \in \mathbf{R}^2; x_1 \leq |x_2|^{3/2}\}$ does not verify (1.1); in particular (1.1) is not C^1-coordinate invariant.

Lemma 1.2. *Fix coordinates in X at x_0 and assume that (1.1) holds. Then for every $x \in (X \setminus A) \cap S_\varepsilon$ ($S_\varepsilon = \{y; |y - x_0| < \varepsilon\}$, ε small) there exists an unique point $a = a(x) \in \partial A \cap S_{2\varepsilon}$ such that*

$$x \in a - \gamma_a^*(A). \tag{1.2}$$

Proof. One takes a point $a = a(x)$ verifying

$$|x - a| = \operatorname{dist}(x, \partial A), \tag{1.3}$$

and verifies easily that a also verifies (1.2). The uniqueness is assured by (1.1). □

From the uniqueness it easily follows that $a(x)$ is a continuous function. (One should even easily prove that it is Lipschitz-continuous.)

We set $d(x) = \operatorname{dist}(x, A)$ and, for $t \geq 0$, $A_t = \{x; d(x) \leq t\}$; we also set $\delta_x = \gamma_x(A_{d(x)})$.

Lemma 1.3. *Let (1.1) hold in some coordinate system; then δ_x, $x \in \partial A$ are half-spaces and the mapping $x \mapsto \delta_x$ is continuous.*

Proof. We shall show that

$$\delta_x = \{y; \langle y - x, a(x) - x\rangle \geq 0\}. \tag{1.4}$$

(The function $x \mapsto a(x)$ being continuous, the lemma will follow at once.) In fact since

$$\{y; |y - a(x)| \leq d(x)\} \subset A_{d(x)},$$

then "$\supseteq$" holds in (1.4). On the other hand we reason by absurd and find a sequence $\{x_\nu\}$ such that

$$\begin{cases} x_\nu \to x, \\ d(x_\nu) \leq d(x), \\ \langle a(x) - x, x_\nu - x\rangle \leq -\delta|x_\nu - x|, \quad \delta > 0. \end{cases} \tag{1.5}$$

By continuity we can replace $a(x) - x$ by $a(x_\nu) - x_\nu$ in (1.5) and conclude that, for large ν,

$$|x - a(x_\nu)| < |x_\nu - a(x_\nu)| = d(x_\nu) \leq d(x),$$

a contradiction. □

Let $N(A)$ be the normal cone to A in the sense of [K-S].

Lemma 1.4. *Let B be closed and assume that:*
(1.6) *$\gamma_x(B)$ is a half space for every $x \in \partial B$,*
(1.7) *$x \mapsto \gamma_x(B)$ is continuous.*
Then $N_x(B)$, $x \in \partial B$ are also half spaces.

Proof. Suppose by absurd that there exist $\Gamma' \subset\subset \gamma_{x_0}(B)$ and two sequences $\{z_\nu\}$, $\{y_\nu\}$ with

$$\begin{cases} z_\nu, y_\nu \to x_0, \\ z_\nu, y_\nu \in \partial B, \\ \theta_\nu = y_\nu - z_\nu \in \mathrm{int}\Gamma', \\ [z_\nu, y_\nu] \subset B \end{cases}$$

(Here $[z_\nu, y_\nu]$ denotes the segment from z_ν to y_ν.) But then $\gamma_{y_\nu} \supset \Gamma' \cup \{-\theta_\nu\}$, a contradiction. □

Remark 1.5. Let B verify $N_{x_0}(B) \neq \{0\}$; then if one takes coordinates with $(0, \ldots, 0, 1) \in N_{x_0}(B)$ and sets $x = (x', x_n)$, one can represent $\partial B = \{x; x_n - \varphi(x') = 0\}$ for a Lipschitz-continuous function φ. Moreover if $N_{x_0}(B)$ is a half-space and if we let $\mathbf{R}^+(0, \ldots, 0, 1) = N^*_{x_0}(B)$ then φ is differentiable at x_0 due to

$$|\varphi(x') - \varphi(x'_0)| = o(|x' - x'_0|).$$

Proposition 1.6. *Let (1.1) hold in some coordinates; then $d(x)$, $x \notin A$ is a C^1 function.*

Proof. By Lemmas 1.3, 1.4, $N_x(A_{d(x)})$ are half-spaces; set $\tau_x = \partial N_x(A_{d(x)})$ and denote by n_x the normal to τ_x. Let $y \in \tau_x$; according to Remark 1.5 there exists $\tilde{y} \in \partial A_{d(x)}$ with $|\tilde{y} - y| = o(|y - x|)$. It follows:

$$|d(y) - d(x)| = |d(y) - d(\tilde{y})| \leq k|y - \tilde{y}| = o(|y - x|). \tag{1.8}$$

By (1.8) we obtain $\partial/\partial\tau_x d(x) = 0$. On the other hand one has $\partial/\partial n_x d(x) = 1$. Finally $\partial d(x) = n_x$, $x \notin A$, and hence d is C^1. □

§**2.** Let X be a C^∞-manifold, $Y \subset X$ a C^1-submanifold, $M^\cdot$ a complex of $\mathbf{Z}$-modules of finite rank, and set $M^{\cdot *} = \mathbf{R}\mathcal{H}om_{\mathbf{Z}}(M^\cdot, \mathbf{Z})$. Let $\mu\mathrm{hom}(\cdot, \cdot)$ be the bifunctor of [K-S, §5]; one easily proves that

$$\mu\mathrm{hom}(\mathbf{Z}_Y, M^\cdot_Y) \cong M^\cdot_{T^*_Y X}, \tag{2.1}$$

$$\mu\mathrm{hom}(M^\cdot_Y, \mathbf{Z}_Y) \cong (M^\cdot_{T^*_Y X})^*. \tag{2.2}$$

Lemma 2.1. *Let $M^\cdot_Y \cong \mathbf{Z}_Y$ in $D^+(X; p)$, $p \in \dot{T}^*_Y X$* ([K-S, §6]). *Then $M^\cdot \cong \mathbf{Z}$.*

Proof. The proof is a straightforward consequence of the formula

$$Hom_{D^+(X;p)}(\cdot, \cdot) \xrightarrow{\sim} H^0\mu\mathrm{hom}(\cdot, \cdot)_p,$$

and of (2.1), (2.2). □

§**3.** Let M be a C^ω-manifold of dimension n, X a complexification of M, $A \subset M$ a closed set. According to [S] we define

$$\mathcal{C}_{A|X} = \mu\mathrm{hom}(\mathbf{Z}_A, \mathcal{O}_X) \otimes \mathrm{or}_{M|X}[n].$$

We assume here that

$$\text{(3.1)} \qquad \begin{aligned} &\text{(i)} \quad A \text{ satisfies (1.1) in some coordinates at } x_0 = 0,\\ &\text{(ii)} \quad A = \overline{\mathrm{int} A} \text{ in the linear hull of } A,\\ &\text{(iii)} \quad SS(\mathbf{Z}_A) \subset \gamma^*(A). \end{aligned}$$

We take coordinates $(x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} \cong M$, $(z', z_n) \in \mathbf{C}^n \cong X$, and suppose that $A = A' \times \mathbf{R}$. We define

$$G_A = \{z; y_n \geq \inf_{a' \in A'} a'^2/4 + \langle y', a' \rangle /2\}. \tag{3.2}$$

Lemma 3.1. *∂G_A is C^1.*

Proof. One defines the set

$$\text{(3.3)} \qquad \begin{aligned} &\{z; y_n \geq -y'^2/4 \text{ for } y' \in -A',\\ &\quad y_n \geq a^2(-y')/4 + \langle y', a(-y')\rangle/2 \text{ for } y' \notin -A'\}, \end{aligned}$$

(where $a(-y')$ is the point of $\partial A'$ such that $-y' \in a(-y') - \gamma^*_{a(-y')}(A')$).

One easily proves that the above set coincides with G_A. Then one observes that the boundary of (3.3) is defined by

$$y_n = \begin{cases} -y'/4, & \text{for } y' \in -A' \\ a^2(-y')/4 + \langle y', a(-y')\rangle/2 & \text{for } y' \notin -A' \\ \qquad = |y' + a(-y')|^2/4 - y'^2/4 & \\ \qquad = \mathrm{dist}^2(y', -A')/4 - y'^2/4, & . \end{cases} \tag{3.4}$$

Since $\mathrm{dist}(y', -A')$, $y' \in M \setminus -A'$, is C^1 due to Proposition 1.4, then the function defined in (3.4) is also C^1.

Proposition 3.2. *(cf* [S]*)*

(i) *We can find a complex homogeneous symplectic transformation ϕ such that*

$$\phi(A \times_M T^*_M X \oplus \gamma^*(A)) = N^*(G_A). \tag{3.5}$$

(ii) *ϕ can be quantized to Φ such that*

$$\Phi(\mathbf{Z}_A) = \mathbf{Z}_{G_A}[n-1]; \tag{3.6}$$

in particular

$$(\mathcal{C}_{A|X})_p \cong \mathcal{H}^1_{G_A}(\mathcal{O}_X)_{\pi(\phi(p))}, \quad p \in \dot{\pi}^{-1}(x_0). \tag{3.7}$$

Proof. One takes coordinates $(z,\zeta) \in T^*X$, and defines ϕ for Im $\zeta_n > 0$ by:

$$\begin{cases} z' \mapsto \zeta'/\zeta_n - \sqrt{-1}\, z', \\ z_n \mapsto \langle z, \zeta/\zeta_n\rangle/2 - \sqrt{-1}\, z'^2/4, \\ \zeta' \mapsto -z'\zeta_n/2, \\ \zeta_n \mapsto \zeta_n. \end{cases}$$

Then by recalling that G_A coincides with the set (3.3), one gets (i). As for (ii) one sets $\mathcal{F} = \Phi(\mathbf{Z}_A)$, $Y = \partial G_A$, and denotes by $j : Y \hookrightarrow X$ the embedding. One gets $SS(\mathcal{F}) \subset N^*(G_A) \subset \pi^{-1}(Y)$ at p; hence $\mathcal{F} \cong \mathbf{R}j_*\mathcal{G}$ at p for some $\mathcal{G}$ in $D^+(Y)$ (cf [K-S, §6]). On the other hand one has $SS(\mathcal{G}) \subset T^*_X X$ at $\pi(p)$ ([K-S, Prop. 4.1.1]); hence $\mathcal{G} \cong M^{\cdot}_Y$ at $\pi(p)$ for a complex of $\mathbf{Z}$-modules.

Due to (3.1)(ii) there exists $q \sim p$ such that A is a manifold at $\pi(q)$ and hence by [K-S, §11] we get $\mathcal{F} \cong \mathbf{Z}_Y[n-1]$ at $\phi(q)$. Thus (3.6) follows from Lemma 2.1, and (3.7) from the fact that $X \setminus G_A$ is pseudoconvex. $\square$

For convex A, the above proposition is stated in [S].

§**4.** Let M be a C^ω-manifold, X a complexification of M, $A \subset M$ a closed set satisfying conditions (3.1).

Proposition 4.1. *Let* $\{\phi_i\}_{i=1,\dots,m}$ *be a base for the space spanned by* $\gamma^*_{x_0}(A)$ *in* $T^*_{x_0}X$. *Then the sections of* $\mathcal{C}_{A|X}\big|_{(T^*_M X \oplus \gamma^*(A))_{x_0}}$ *satisfy the principle of the analytic continuation on the complex integral manifolds of* $\{H(\phi_i^{\mathbf{C}})\}_{i=1,\dots,m}$.

Proof. Using the trick of the dummy variable we can assume A being of the form $A' \times \mathbf{R}$ and hence use the transformation ϕ of §3. Let $p, q \in \pi\phi((T^*_M X \oplus \gamma^*(A))_{x_0})$ belong to the same integral leaf of $\{H(\phi_i^{\mathbf{C}})\}_{i=1,\dots,m}$. We then have to show that if f is holomorphic in $X \setminus G_A$ and extends holomorphically at p , then it also extends at q.

We observe that $\pi((T^*_M X \oplus \gamma^*(A))_{x_0}) = \pi(\partial G_A \cap \phi(\dot{\pi}^{-1}(x_0))$ is plane. Thus the claim follows from the Bochner's tube theorem at least when $\rho_M(p)$ belongs to the interior of $\gamma^*_{x_0}(A)$ in the plane of $\{\phi_i\}$ ($\rho_M : T^*X \to T^*M$).

Otherwise one has to remember that ∂G_A is C^1 , and use the following result whose proof is straightforward.

Lemma 4.2. *(cf* [Kan]*) Let* $(z_1, z_2) \in \mathbf{C}^2$, $z_i = x_i + \sqrt{-1}\, y_i$, $i = 1, 2$, *and let* ψ *be a* C^1 *function on* $\mathbf{R}^2_{y_1,y_2}$ *at* 0 *such that* $\psi \geq 0$ *and* $\psi = 0$ *for* $y_1 \geq 0$. *If* f *is analytic in the set*

$$\{|x_i| < \varepsilon\} \times (\{|y_i| < \varepsilon, y_2 > \psi(y_1)\} \cup \{y_1 = \varepsilon, -\delta < y_2 \leq 0\}),$$

then f *is analytic at* 0.

Remark 4.3.

(a) When A is a half-space with C^ω-boundary, Proposition 4.1 was already stated in [Kat 1].

(b) In the situation of Proposition 4.1, one has (cf [Kat 1]):

$$\Gamma_{A\times_M T^*_M X}(\mathcal{C}_{A|X})\big|_{\partial A\times_M \dot{T}^*_M X} = 0.$$

(c) Let $\mathcal{M}$ be an $\mathcal{E}_X$-module at $p \in \dot{\pi}^{-1}(x_0)$. Suppose that there exists $\theta \in \dot{\gamma}^*_{x_0}(A)$ non-characteristic for $\mathcal{M}$. Then:

$$\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_{A|X})_p = 0.$$

(This was announced by Uchida when A is convex and all $\theta \in \dot{N}^*_{x_0}(A)$ (or $\partial N^*_{x_0}(A)$) are non-characteristic.)

Let now Ω be an open set of M and assume that $A = M \setminus \Omega$ satisfies the hypotheses (3.1). By the distinguished triangle

$$\mathcal{C}_{A|X} \to \mathcal{C}_{M|X} \to \mathcal{C}_{\Omega|X} \overset{+1}{\to},$$

by (3.7), and by the corresponding formula for $\mathcal{C}_{M|X}$, one gets (cf [S]):

Proposition 4.4.

$$H^0(\mathcal{C}_{\Omega|X}) = (\mathcal{C}_{\Omega|X})_{T^*_M X}.$$

By (4.1) and by Remark 4.3 (c), one also gets, for a $\mathcal{D}_X$-module $\mathcal{M}$:

$$\begin{aligned}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_{\overline{\Omega}})_{x_0} =&\{f \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_\Omega(\mathcal{B}_M))_{x_0};\\ & SS^{\mathcal{M},0}_\Omega(f) \cap \dot{\pi}^{-1}(x_0) = \emptyset\},\end{aligned}$$

$SS^{\mathcal{M},0}_\Omega(f)$ being the micro-support in the sense of [S]. (One needs perhaps to assume here $\mathbf{Z}_\Omega$ cohomologically constructible; but this follows probably from (1.1).)

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