LEVI'S FORMS OF HIGHER CODIMENSIONAL SUBMANIFOLDS

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Abstract. Let $X \cong \mathbf{C}^n$, let M be a C^2 hypersurface of X, S be a C^2 submanifold of M. Denote by L_M the Levi form of M at $z_0 \in S$. In [K-S 2] two numbers $s^\pm(S,p)$, $p \in (\dot{T}^*_S X)_{z_0}$ are defined; for $S = M$ they are the numbers of positive and negative eigenvalues for L_M. For $S \subset M$, $p \in S \times_M \dot{T}^*_S X$, we show here that $s^\pm(S,p)$ are still the numbers of positive and negative eigenvalues for L_M when restricted to $T^{\mathbf{C}}_{z_0} S$. Applications to the concentration in degree for microfunctions at the boundary are given.

Sunto. Sia $X \cong \mathbf{C}^n$, M una ipersuperficie di classe C^2 di X, S una sottovarietà C^2 di M. Sia L_M la forma di Levi di M al punto $z_0 \in S$. In [K-S 2] si definiscono dei numeri $s^\pm(S,p)$, $p \in (\dot{T}^*_S X)_{z_0}$ che per $S = M$ coincidono con i numeri di autovalori positivi e negativi di L_M. Per $S \subset M$, $p \in S \times_M \dot{T}^*_S X$, si prova che $s^\pm(S,p)$ sono ancora i numeri di autovalori positivi e negativi di L_M ristretta a $T^{\mathbf{C}}_{z_0} S$. Se ne dà applicazione alla concentrazione in grado di microfunzioni al bordo.

1. Let X be a complex analytic manifold of dimension n. We denote by $\tau : TX \to X$ the tangent bundle and by $\pi : T^*X \to X$ the cotangent bundle to X. If $X^{\mathbf{R}}$ denotes the underlying real analytic manifold structure on X, we recall that there is a natural identification $T^*(X^{\mathbf{R}}) \cong (T^*X)^{\mathbf{R}}$. We will denote by ∂ the holomorphic differential on X, and by d$=\partial + \overline{\partial}$ the differential on $X^{\mathbf{R}}$.

Let M be a C^2 hypersurface of X and S be a C^2 submanifold of M of real codimension $s-1$. We denote by $T^*_S X$ the conormal bundle to S in X, a closed submanifold of $T^*X^{\mathbf{R}}$.

Take a point $z_0 \in S$ and assume that, locally at z_0, one may express M as $\{z \in X; \phi(z) = 0\}$ and S as the set of zeros for the functions ϕ_i $(i = 1, \ldots, s)$. Here ϕ and the ϕ_i are real valued C^2 functions on X. Let $p = d\,\phi(z_0) \in S \times_M \dot{T}^*_M X \subset \dot{T}^*_S X$. Let L_M be the Levi form of M at z_0. Recall that, in a local system of coordinates $(z) \in X$ at z_0, one has:

$$L_M = \left(\frac{\partial^2}{\partial z_i \partial \overline{z}_j} \phi(z_0) \right)_{i,j}.$$

Appeared in: Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. (9) Mat. Appl. **2** (1991), no. 1, 29–33.

Let us set:

$$\begin{aligned}
T_{z_0}S &= \{z \in T_{z_0}X; Re\langle z, \partial\phi_i\rangle = 0\ \forall i\},\\
T^{\mathbf{C}}_{z_0}S &= \{z \in T_{z_0}X; \langle z, \partial\phi_i\rangle = 0\ \forall i\},\\
\lambda_S(p) &= T_pT^*_SX,\\
\lambda_0(p) &= T_p\pi^{-1}\pi(p),\\
\gamma(S,p) &= \dim_{\mathbf{C}}(\lambda_S(p)\cap i\lambda_S(p)\cap\lambda_0(p)),\\
s(S,p) &= \frac{1}{2}\tau(\lambda_S(p), i\lambda_S(p), \lambda_0(p)),\\
r(S,p) &= n - \operatorname{codim}_X S + 2\gamma(S,p) - \dim_{\mathbf{C}}(\lambda_S(p)\cap i\lambda_S(p)),
\end{aligned}$$

here $\tau(\cdot,\cdot,\cdot)$ denotes the Maslov index for three Lagrangian planes and $\operatorname{codim}_X S$ is the real codimension of S in X (cf. [K-S 2, §7]).

Proposition 1.1. *One has:*

$$s(S,p) = \operatorname{sgn}(L_M|_{T^{\mathbf{C}}_{z_0}S}),$$

where sgn *denotes the signature.*

proof. The proof goes as in [K-S 1, Prop. 11.2.7] so we point out only the main lines.

Define the map:

$$\begin{aligned}
\psi : S &\longrightarrow T^*_SX\\
z &\mapsto \partial\phi(z)
\end{aligned}$$

Let:

$$\psi_* : \mathbf{C}\otimes_{\mathbf{R}} T_{z_0}S \longrightarrow \lambda_S(p) + i\lambda_S(p) \subset T_pT^*X,$$

be the map induced by ψ on the complexification $\mathbf{C}\otimes_{\mathbf{R}} T_{z_0}S$ of $T_{z_0}S$. If we identify $\mathbf{C}\otimes_{\mathbf{R}} T_{z_0}S$ to the subset of $T_{z_0}X \oplus \overline{T_{z_0}X}$:

$$\{(v,w); \langle v, \partial\phi_i\rangle + \langle w, \overline{\partial}\phi_i\rangle = 0,\ \forall i\},$$

we have:

$$\psi_*(v,w) = (v; \zeta = \partial\langle\partial\phi, v\rangle + \partial\langle\overline{\partial}\phi, w\rangle).$$

We have:

$$\begin{aligned}
\lambda_S(p) = \{(v,\zeta); &Re\langle v, \partial\phi_i\rangle = 0,\\
&\zeta = \sum t_j\partial\phi_j(z_0) + \partial\langle\partial\phi, v\rangle + \partial\langle\overline{\partial}\phi, \overline{v}\rangle)\}.
\end{aligned}$$

We need now a lemma.

Lemma 1.2. *One has:*

$$\psi_*\overline{T_{z_0}S} + (\lambda_0(p) \cap \lambda_S(p))^{\mathbf{C}} = \lambda_0(p) \cap (\lambda_S(p) + i\lambda_S(p)).$$

proof. Recall that one has the identification:

$$\psi_*\overline{T_{z_0}S} = \{(0,\zeta); \zeta = \partial\langle\overline{\partial}\phi, w\rangle,\ \langle w, \overline{\partial}\phi_i\rangle = 0,\ \forall i \le s\},$$

and also

$$(\lambda_0(p) \cap \lambda_S(p))^{\mathbf{C}} = \{(0,\zeta); \zeta = \sum_i \tau_i \partial\phi_i(z_0),\ \tau_i \in \mathbf{C}\}$$

In conclusion:

$$\begin{aligned}\psi_*\overline{T_{z_0}S} + (\lambda_0(p) \cap \lambda_S(p))^{\mathbf{C}} &= \{(0,\zeta); \zeta = \sum_i \tau_i\partial\phi_i(z_0) + \partial\langle\overline{\partial}\phi, w\rangle,\ \langle\overline{\partial}\phi_i, w\rangle = 0\ \forall i \le s\} \\ &= \lambda_0(p) \cap (\lambda_S(p) + i\lambda_S(p)).\end{aligned}$$

Q.E.D.

One sees that $\psi_*(0,v) + \psi_*(\overline{v},0) \in \lambda_S(p)$ and $\psi_*(0,v) - \psi_*(\overline{v},0) \in i\lambda_S(p)$ and thus $\psi_*(\overline{v},0)$ is the "conjugate" to $\psi_*(0,v)$ in $T_pT^*X/(\lambda_S(p) \cap \lambda_S(p))$ with respect to the space $\lambda_S(p)/(\lambda_S(p) \cap \lambda_S(p))$. The conclusion is then as in [K-S 1]. Q.E.D.

Proposition 1.3. *One has:*

$$r(S,p) = \operatorname{rank}(L_M|_{T^{\mathbf{C}}_{z_0}S}).$$

proof. One has:

$$\begin{aligned}\lambda_S(p) \cap i\lambda_S(p) =& \{(v,\zeta); \langle v, \partial\phi_i\rangle = 0\ \forall i \le s, \zeta = \sum_i t_i\partial\phi_i(z_0) + \partial\langle\partial\phi, v\rangle + \partial\langle\overline{\partial}\phi, v\rangle, \\ & \zeta = \sum_i is_i\partial\phi_i(z_0) + \partial\langle\partial\phi, v\rangle - \partial\langle\overline{\partial}\phi, v\rangle\} \\ =& \{(v,\zeta); \langle v, \partial\phi_i\rangle = 0\ \forall i \le s, \langle\partial\langle\overline{\partial}\phi, \overline{v}\rangle, w\rangle = 0,\ \forall w \in T^{\mathbf{C}}_{z_0}S, \\ & \zeta = \sum_i t_i\partial\phi_i(z_0) + \partial\langle\partial\phi, v\rangle + \partial\langle\overline{\partial}\phi, v\rangle, \\ & \sum_i(-t_i + is_i)\partial\phi_i(z_0) = 2\partial\langle\overline{\partial}\phi, \overline{v}\rangle\}.\end{aligned}$$

Thus $\zeta - \partial\langle\partial\phi, v\rangle - \partial\langle\overline{\partial}\phi, v\rangle$ is the first term in a decomposition:

$$-2\partial\langle\overline{\partial}\phi, \overline{v}\rangle = \sum_i t_i\partial\phi_i(z_0) - i\sum_i s_i\partial\phi_i(z_0),$$

$t_i, s_i \in \mathbf{R}$. This last decomposition being unique modulo $\lambda_S(p) \cap i\lambda_S(p) \cap \lambda_0(p)$, we get:

$$\begin{aligned}\dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p)) = \dim_{\mathbf{C}}\{v \in T^{\mathbf{C}}_{z_0}S; \langle\partial\langle\overline{\partial}\phi, \overline{v}\rangle, w\rangle = 0,\ \forall w \in T^{\mathbf{C}}_{z_0}S\} + \\ + \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p) \cap \lambda_0(p)).\end{aligned}$$

We have, recalling that $T^{\mathbf{C}}_{z_0}S = T_{z_0}S \cap iT_{z_0}S$:

$$\dim_{\mathbf{C}} T^{\mathbf{C}}_{z_0}S = n - \operatorname{codim}{}_X S + \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p) \cap \lambda_0(p)),$$

and hence:

$$\begin{aligned}\operatorname{rank} L_M|_{T^{\mathbf{C}}_{z_0}S} =& \dim_{\mathbf{C}} T^{\mathbf{C}}_{z_0}S - \dim_{\mathbf{C}}\{v \in T^{\mathbf{C}}_{z_0}S; \langle \partial\langle\overline{\partial}\phi, \overline{v}\rangle, w\rangle = 0,\ \forall w \in T^{\mathbf{C}}_{z_0}S\}\\ =&(n - \operatorname{codim}{}_X S + \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p) \cap \lambda_0(p))) -\\ & (\dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p)) - \dim_{\mathbf{C}}(\lambda_S(p) \cap i\lambda_S(p) \cap \lambda_0(p)))\\ =& r(S,p).\end{aligned}$$

Q.E.D.

Let us define $s^{\pm}(S,p)$ as the solutions of the system:

$$\begin{cases} s(S,p) = s^+(S,p) - s^-(S,p) \\ r(S,p) = s^+(S,p) + s^-(S,p), \end{cases}$$

and $s^0(S,p)$ as:

$$s^0(S,p) = \dim_{\mathbf{C}}\{v \in T^{\mathbf{C}}_{z_0}S; \langle \partial\langle\overline{\partial}\phi, \overline{v}\rangle, w\rangle = 0,\ \forall w \in T^{\mathbf{C}}_{z_0}S\}$$

Due to Propositions 1.1, 1.3, we get at once the following theorem.

Theorem 1.4. *$s^+(S,p)$, $s^-(S,p)$ and $s^0(S,p)$ are the number of the positive, negative and null eigenvalues for the form $L_M|_{T^{\mathbf{C}}_{z_0}S}$.*

2. We give now application of the preceding results to the concentration in degree for microfunctions at the boundary.

We shall be working with the derived category $D^b(X)$ of the category of bounded complexes of sheaves of abelian groups on X. In [K-S 1] a bifunctor μhom : $D^b(X)^\circ \times D^b(X) \to D^b(T^*X)$ is defined. For a subset $Z \subset X$ one sets $\mu_Z(\mathcal{O}_X) = \mu hom(\mathbf{C}_Z, \mathcal{O}_X)$, where $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X and $\mathbf{C}_Z$ is the sheaf which is 0 on $X \setminus Z$ and the constant sheaf with fiber $\mathbf{C}$ on Z.

In what follows π_M will denote the projection $\pi_M : T^*_MX \to M$, ω will denote the complex canonical 1-form on X, H the hamiltonian isomorphism $T^*T^*X \cong TT^*X$ induced by the symplectic 2-form $d\,\omega$ and, in the case of T^*_MX being $d\,\omega^I$-symplectic, H^I will denote the hamiltonian isomorphism $T^*T^*_MX \cong TT^*_MX$ induced by the symplectic 2-form $d\,\omega^I$ (ω^I being the imaginary part of ω)

Proposition 2.1. *Let $S \subset M \subset X$ be C^2 subvarieties of X, M being an hypersurface. Then, for $p \in S \times_M \dot{T}^*_MX$:*

(i) $0 \le s^{\pm}(M,p) - s^{\pm}(S,p) \le cod_M S - [\gamma(S,p) - \gamma(M,p)]$,
(ii) $-cod_M S + [\gamma(S,p) - \gamma(M,p)] \le s^0(M,p) - s^0(S,p) \le cod_M S - [\gamma(S,p) - \gamma(M,p)]$.

Proof. Recall that $T^{\mathbf{C}}_{z_0}M = T_{z_0}M \cap iT_{z_0}M$. We have:

$$\dim_{\mathbf{C}} T^{\mathbf{C}}_{z_0}S = \dim_{\mathbf{C}} T^{\mathbf{C}}_{z_0}M - [\operatorname{codim}{}_M S - (\gamma(S,p) - \gamma(M,p))].$$

Let us define (for $* = +, -, 0$):

$$V_M^* = \{v \in T_{z_0}^{\mathbf{C}} M; v \text{ and } v^\perp \text{ generate } T_{z_0}^{\mathbf{C}} M, {}^t\!v L_M v \begin{cases} > 0 & \text{for } * = + \\ < 0 & \text{for } * = - \\ = 0 & \text{for } * = 0 \end{cases} \}$$

(where $v^\perp = \{w \in T_{z_0}^{\mathbf{C}} M; {}^t\!v L_M w = 0\}$), and similarly for V_S^*. In order to prove that the right hand side estimates on (i) and (ii) hold, it is enough to observe that $V_S^* \supset V_M^* \cap T_{z_0}^{\mathbf{C}} S$.

Let now $v_1, \dots, v_s$ be a base for V_S^+, completed in a basis $v_1, \dots, v_s, \tilde{v}_{s+1}, \dots, \tilde{v}_r$ of $T_{z_0}^{\mathbf{C}} M$, and let $(a_{ij})_{i,j \le r}$ be the matrix of L_M in such a base. If we write $v_{s+1} = \tilde{v}_{s+1} - (a_{1s+1}/a_{11} v_1 + \cdots + a_{ss+1}/a_{ss} v_s)$, and similarly for v_j, $s+1 < j \le r$, we get a new base for $T_{z_0}^{\mathbf{C}} M$ such that ${}^t\!v_i L_M v_j = 0$, $\forall i \le s$, $\forall j \le m$. This proves that the left hand side estimate on (i) holds.

In order to prove that the left hand side estimates on (ii) holds, assume at first that $\dim_{\mathbf{C}} T_{z_0}^{\mathbf{C}} M - \dim_{\mathbf{C}} T_{z_0}^{\mathbf{C}} S = 1$ and let $u \in T_{z_0}^{\mathbf{C}} M \setminus T_{z_0}^{\mathbf{C}} S$. Let $v_1, v_2 \in T_{z_0}^{\mathbf{C}} S$, with

$${}^t\!v_i (L_M|_{T_{z_0}^{\mathbf{C}} S}) w = 0,$$

$\forall i = 1, 2$, $\forall w \in T_{z_0}^{\mathbf{C}} S$. Assume that ${}^t\!v_1 L_M u \neq 0$ and hence there exists an $\alpha \in \mathbf{C}$ with ${}^t(\alpha v_1 + v_2) L_M u = 0$. This means $(\alpha v_1 + v_2)^\perp = T_{z_0}^{\mathbf{C}} M$. The case $\dim_{\mathbf{C}} T_{z_0}^{\mathbf{C}} M - \dim_{\mathbf{C}} T_{z_0}^{\mathbf{C}} S > 1$ is similarly proven. Q.E.D.

Proposition 2.2. *Let $S \subset M \subset X$ be C^2 subvarieties of X, M an hypersurface, and take $p_0 \in S \times_M \dot{T}_M^* X$. Let $\Omega = M \setminus S$. Assume the following conditions for p in a neighborhood of p_0:*

1. $\dim_{\mathbf{R}} (\lambda_S(p) \cap \nu(p)) = 1$ *(here $\nu(p) = \mathbf{C}\, H(\omega(p))$ is the radial direction),*
2. $s^-(S,p) - \gamma(S,p)$ *is constant for $p \in T_S^* X$,*
3. $s^-(M,p) - \gamma(M,p)$ *is constant for $p \in T_M^* X$.*

Then $\mu_\Omega(\mathcal{O}_X)$ is concentrated in degrees $[\alpha - 1, \alpha' \vee \alpha]$, where we set $\alpha = \operatorname{codim}_X M + s^-(M,p) - \gamma(M,p)$, $\alpha' = \alpha + \operatorname{codim}_M S - [\gamma(S,p) - \gamma(M,p)] - 1$.

proof. By [K-S 1, Prop. 2.3] we have that $\mu_M(\mathcal{O}_X)$ is concentrated in degree α and $\mu_S(\mathcal{O}_X)$ is concentrated in degree $\alpha'' = cod_X S + s^-(S,p) - \gamma(S,p)$. Due to Corollary 2.1 we then have $\alpha \le \alpha'' \le \alpha' + 1$ and we conclude. Q.E.D.

Remark 2.3. If we assume in addition that:

$$\begin{cases} T_M^* X \text{ is } d\,\omega^I \text{ symplectic } (\operatorname{cod}_X M = 1), \\ S \times_M T_M^* X \text{ is } d\,\omega^I \text{ involutive}, \\ ip_0 \notin T_S^* M \text{ in the identification } T_S^* M \cong i H^I(\pi_M^*(T_S^* M)) \end{cases}$$

then $\mu_\Omega(\mathcal{O}_X)$ is concentrated in degrees $[\alpha, \alpha + \operatorname{codim}_M S - 1]$. In fact $\mu_M(\mathcal{O}_X)$ is concentrated in degree $\alpha = \operatorname{codim}_X M + s^-(M,p)$. By a complex quantized contact transformation it is not restrictive to assume $s^-(M,p) = 0$, and hence $s^-(S,p) = 0$ due to Proposition 2.1. Moreover, since $(T_M^* X, d\,\omega^I)$ is symplectic and $S \times_M T_M^* X$ is regular involutive, then $\gamma(S,p) = 0$ (cf [D'A-Z]). This implies (cf [K-S 2, Prop. 11.2.8]) that $\mu_S(\mathcal{O}_X)$ is concentrated in degree $\operatorname{codim}_X S + s^-(S,p)$. Note that if $\operatorname{codim}_X M + s^-(M,p) = \operatorname{codim}_X S + s^-(S,p)$, the concentration in degree α for $\mu_\Omega(\mathcal{O}_X)$ follows by applying a quantized contact transformation that interchanges $T_M^* X$ with $T_{M'}^* X'$ (for $M' \cong \mathbf{R}^n$) and $S \times_M T_M^* X$ with $S' \times_{M'} T_{M'}^* X'$. The concentration for $\mu_{\Omega'}(\mathcal{O}_X)$ ($\Omega' = M' \setminus S'$) is then a classical argument due to P. Schapira.

Remark 2.4. If we assume that:

$$\begin{cases} T^*_M X \text{ is } d\,\omega^I \text{ symplectic } (\mathrm{cod}_X M = 1), \\ \mathrm{codim}\,_M S = 1, \\ ip_0 \notin T^*_S M \text{ in the identification } T^*_S M \cong iH^I(\pi^*_M(T^*_S M)) \end{cases}$$

then $\mu_\Omega(\mathcal{O}_X)$ is concentrated in degree $\alpha = \mathrm{codim}\,_X M + s^-(M,p)$. In fact, by a quantized contact transformation, we can reduce to the case $s^-(S,p) \equiv 0$, $s^-(M,p) \equiv 0$, M being the boundary of a strictly pseudo-convex set, moreover, in this case of $\mathrm{codim}\,_S X = 2$, we have $\gamma(S,p) = 0$.

Proposition 2.5. *Let $S \subset M \subset X$ be C^2 subvarieties of X, set $\Omega = M \setminus S$ and take $p_0 \in S \times_M \dot{T}^*_M X$. Then $\mu_\Omega(\mathcal{O}_X)$ is concentrated in degrees $[\alpha - 1, \beta + \mathrm{codim}\,_M S - 1]$, where we set $\alpha = \mathrm{codim}\,_X M + s^-(M,p) - \gamma(M,p)$ and $\beta = n - s^+(M,p) + \gamma(M,p)$.*

proof. Due to [K-S 2, Th. 2.2] one has that $\mu_M(\mathcal{O}_X)$ is concentrated in degree $[\alpha, \beta]$ and $\mu_S(\mathcal{O}_X)$ is concentrated in degree $[\alpha', \beta']$, for: $\alpha' = \mathrm{codim}\,_X S + s^-(S,p) - \gamma(S,p)$, $\beta' = n - s^+(S,p) + \gamma(S,p)$. Due to Corollary 2.1 we have:

$$s^\pm(S,p) \geq s^\pm(M,p) - \mathrm{codim}\,_M S + [\gamma(S,p) - \gamma(M,p)],$$

and hence:

$$\begin{aligned} \alpha &\leq \alpha', \\ \beta' &\leq n - s^+(M,p) + cod_M S - (\gamma(S,p) - \gamma(M,p)) + \gamma(S,p) \\ &= \beta + cod_M S. \end{aligned}$$

Q.E.D.

References

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