EXTENSION OF CR FUNCTIONS TO "WEDGE TYPE" DOMAINS

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Abstract. Let X be a complex manifold, S a generic submanifold of $X^{\mathbf{R}}$, the real underlying manifold to X. Let Ω be an open subset of S with $\partial\Omega$ analytic, Y a complexification of S.

We first recall the notion of Ω-tuboid of X and of Y and then give a relation between; we then give the corresponding result in terms of microfunctions at the boundary.

We relate the regularity at the boundary for $\overline{\partial}_b$ to the extendability of CR functions on Ω to Ω-tuboids of X.

Next, if X has complex dimension 2, we give results on extension for some classes of hypersurfaces (which correspond to some $\overline{\partial}_b$ whose Poisson bracket between real and imaginary part is ≥ 0.

The main tools of the proof are the complex $\mathcal{C}_{\Omega|Y}$ by Schapira and the theorem of Ω-regularity of [S-Z] and [U-Z].

1. The system $\overline{\partial}_b$. Let X be a complex manifold of complex dimension n, S a real analytic submanifold of $X^{\mathbf{R}}$ of dimension m ($X^{\mathbf{R}}$ being the real underlying manifold to X), Y a complexification of S. Due to the complex structure of X we get a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & X \\ \downarrow & \nearrow_{\widetilde{\phi}} & \\ Y & & \end{array}$$

In this article we will assume S to be a generic submanifold of X, i.e. $S \times_X TX = TS +_S \sqrt{-1}\,TS$. In particular a hypersurface is always generic.

Remark 1.1. The genericity of S implies that $\widetilde{\phi}$ is smooth. In fact one has: $\widetilde{\phi}'(S \times_Y TY) = \widetilde{\phi}'(TS \oplus_S \sqrt{-1}\,TS) = \widetilde{\phi}'(TS) +_S \sqrt{-1}\,\widetilde{\phi}'(TS) = TS +_S \sqrt{-1}\,TS = S \times_X TX$. Where the third equality follows from $\widetilde{\phi}\Big|_S = \phi$.

Due to Remark 1.1, ${}^t\widetilde{\phi}'(T^*X) = Y \times_X T^*X$ is a sub-bundle of T^*Y.

One defines $\overline{\partial}_b$ as the system of complex vector fields on Y which annihilate $Y \times_X T^*X$.

Remark 1.2. One has

(1) $\widetilde{\phi}^{-1}(\mathcal{O}_X) = \mathcal{O}_Y^{\overline{\partial}_b}$,
(2) $char(\overline{\partial}_b) = Y \times_X T^*X$.

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(Here $\mathcal{O}_Y^{\overline{\partial}_b}$ is the sheaf of germs of holomorphic functions annihilated by $\overline{\partial}_b$.) In fact, according to Remark 1.1 one can take as a system of coordinates in Y $(z_i)_{i=1,\dots,m}$ with $z_i = \widetilde{\phi}_i$, $i = 1, \dots, n$. Then clearly $\overline{\partial}_b = (\partial/\partial z_{n+1}, \dots, \partial/\partial z_m)$ and the claim follows. In particular, since TS is preserved by $\widetilde{\phi}'$, one has

$$(char(\overline{\partial}_b)) \cap T^*_S Y \cong T^*_S X. \tag{1.1}$$

2. A brief review on the language of tuboids. Let $S \subset WX$ be C^2-manifolds, $\Omega \subset X$ an open set with $N(\Omega) \neq \emptyset$ (here $N(\Omega)$ denotes the normal cone to Ω in S of [K-S,§1.2.3]).

Definition 2.1. Let γ be an open convex cone of $\overline{\Omega} \times_S T_S X$. A set $U \subset X$ is said to be an Ω-tuboid of X with profile γ iff

(1) $\rho(TX \setminus C(X \setminus U, \overline{\Omega})) \supset \gamma$.

(Where $\rho : TX \to T_S X$.)

Remark 2.2. If one chooses a local coordinate system $(x, y) \in X$, $S = \{(x, y) : y = 0\}$ then U is an Ω-tuboid with profile γ iff for every $\gamma' \subset\subset \gamma$ there exists $\varepsilon = \varepsilon_{\gamma'}$, so that

$$U \supset \{(x, y) \in \Omega \times_V \gamma' : |y| < \varepsilon\, dist(x, \partial\Omega) \wedge 1\}.$$

(Here we identify $T_S X \cong X$ in local coordinates.)

3. A link between tuboids in Y and in X. Let S,X,Y be as in §1, let $\Omega \subset S$ be an open set with analytic boundary.

Our aim is to give a relation between Ω-tuboids in Y and in X.

Let $U \subset X$ be an open set, $\gamma \subset T_S X$, $U' = \widetilde{\phi}^{-1}(U) \subset Y$, $\gamma' = \widetilde{\phi}'^{-1}(\gamma) \subset T_S Y$ (we still denote by $\widetilde{\phi}'$ the induced map $\widetilde{\phi}' : T_S Y \to T_S X$).

Lemma 3.1. *U is an Ω-tuboid of X with profile γ iff U' is an Ω-tuboid of Y with profile γ'.*

Proof. Since $\Omega \subset S$, we have $\overline{\Omega} = \widetilde{\phi}(\overline{\Omega})$.

If $\rho(TY \setminus C(Y \setminus U', \overline{\Omega})) \supset \gamma'$ then $\rho(TX \setminus C(X \setminus U, \overline{\Omega})) = \rho(TX \setminus C(X \setminus \widetilde{\phi}(U'), \overline{\Omega})) = \rho(\widetilde{\phi}'(TY \setminus C(Y \setminus U', \overline{\Omega}))) = \widetilde{\phi}'(\rho(TY \setminus C(Y \setminus U', \overline{\Omega}))) \supset \widetilde{\phi}'(\widetilde{\phi}'^{-1}(\gamma)) = \gamma$.

If $\rho(TX \setminus C(X \setminus U, \overline{\Omega})) \supset \gamma$ then $\rho(TY \setminus C(Y \setminus U', \overline{\Omega})) = \rho(TY \setminus C(Y \setminus \widetilde{\phi}^{-1}(U), \overline{\Omega})) = \rho(\widetilde{\phi}'^{-1}(TX \setminus C(X \setminus U, \overline{\Omega}))) = \widetilde{\phi}'^{-1}(\rho(TX \setminus C(X \setminus U, \overline{\Omega}))) = \widetilde{\phi}'^{-1}(\gamma) = \gamma'$. □

Using this lemma and 1, 2 of Remark 1.2 we can then claim

Proposition 3.2. *Let U be an Ω-tuboid of X with profile γ, $U' = \widetilde{\phi}^{-1}(U)$, $\gamma' = \widetilde{\phi}'^{-1}(\gamma)$. We have $f \in \mathcal{O}_X(U)$ iff $f \circ \widetilde{\phi} \in \mathcal{O}_Y^{\overline{\partial}_b}(U')$.*

4. A microlocal approach. Let S, X, Y as before, $\Omega \subset S$ an open set with analytic boundary (Ω locally on one side of $\partial\Omega$).

The framework of this paragraph is the microlocal study of sheaves by Kashiwara and Schapira (cf [K-S]).

We will still denote by $\overline{\partial}_b$ the coherent $\mathcal{D}_Y$-module associated to the system of complex vector fields, i.e. $\overline{\partial}_b = \widetilde{\phi}^*(\mathcal{D}_X)$.

In [S] Schapira defined the complex of microfunctions at the boundary

$$\mathcal{C}_{\Omega|Y} = \mu hom(\mathbf{Z}_\Omega, \mathcal{O}_Y) \otimes or_{S|Y}[m],$$

similarly we set

$$\mathcal{C}_{\Omega|X} = \mu hom(\mathbf{Z}_\Omega, \mathcal{O}_X) \otimes or_{S|X}[2n-m].$$

To give a relation between $\mathcal{C}_{\Omega|X}$ and $\mathcal{C}_{\Omega|Y}$ we first need to translate in the language of derived categories the results of section 1.

Proposition 4.1. *One has*

$$\widetilde{\phi}^{-1}(\mathcal{O}_X) = \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\overline{\partial}_b, \mathcal{O}_Y).$$

Proof. $\widetilde{\phi}^{-1}(\mathcal{O}_X) = \widetilde{\phi}^{-1}\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{O}_X) = \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\overline{\partial}_b, \mathcal{O}_Y)$, where the second equality is the Cauchy-Kowalevsky-Kashiwara's theorem which holds since $\widetilde{\phi}$ is non-characteristic for $\mathcal{D}_X$. □

We then have

Theorem 4.2.

$$\mathcal{C}_{\Omega|X} \cong \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\overline{\partial}_b, \mathcal{C}_{\Omega|Y}). \tag{4.1}$$

Proof. One has $\mu hom(\mathbf{Z}_\Omega, \mathcal{O}_X) \cong \mu hom(\mathbf{Z}_\Omega, \widetilde{\phi}^!\mathcal{O}_X)$ due to [K-S, Corollary 5.5.6]. Here one notices that both complexes are supported by $Y \times_X T^*X$.

On the other hand by [K-S, Proposition 1.3.1] $\widetilde{\phi}^!\mathcal{O}_X = \widetilde{\phi}^{-1}\mathcal{O}_X \otimes or_{Y|X}[2m-2n] = \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\overline{\partial}_b, \mathcal{O}_Y) \otimes or_{Y|X}[2m-2n]$, and the claim follows. □

Next, similarly to the sheaf of Sato's hyperfunctions

$$\mathcal{B}_S = H^0(\mathbf{R}\Gamma_S(\mathcal{O}_Y) \otimes or_{S|Y}[m]),$$

one sets (e.g. cf [S-T])

$$\mathcal{B}_{S|X} = H^0(\mathbf{R}\Gamma_S(\mathcal{O}_X) \otimes or_{S|X}[2n-m]).$$

Recall that, S being generic, $H^j(\mathbf{R}\Gamma_S\mathcal{O}_X) = 0\,\forall j < 2n-m$, then by applying $\mathbf{R}^0\pi_*$ in Theorem 4.2 we get

$$\mathcal{H}om_{\mathcal{D}_Y}(\overline{\partial}_b, \Gamma_\Omega(\mathcal{B}_S)) \cong \Gamma_\Omega(\mathcal{B}_{S|X}). \tag{4.2}$$

Let

$$\begin{aligned}\alpha &: \pi^{-1}\mathcal{H}om_{\mathcal{D}_Y}(\overline{\partial}_b, \Gamma_\Omega(\mathcal{B}_S)) \to H^0(\mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\overline{\partial}_b, \mathcal{C}_{\Omega|Y}))\\ \beta &: \pi^{-1}\Gamma_\Omega(\mathcal{B}_{S|X}) \to H^0(\mathcal{C}_{\Omega|X}),\end{aligned}$$

be the canonical maps and define

$$\begin{aligned}\mathrm{SS}^{\overline{\partial}_b,0}_{\Omega|Y}(f) &= supp(\alpha(f)), \qquad f \in \mathcal{H}om_{\mathcal{D}_Y}(\overline{\partial}_b, \Gamma_\Omega(\mathcal{B}_S)),\\ \mathrm{SS}_{\Omega|X}(g) &= supp(\beta(g)), \qquad g \in \Gamma_\Omega(\mathcal{B}_{S|X}).\end{aligned}$$

Corollary 4.3. *Let* $u \in \Gamma_\Omega(\mathcal{B}_{S|X})$ *then*

$$SS_{\Omega|X}(u) = SS^{\overline{\partial}_b,0}_{\Omega|Y}(u \circ \phi).$$

Note that, after [Z], there is a tight relation between this corollary and Proposition 3.2.

Remark 4.4. Note that $\mathcal{H}om_{\mathcal{D}_Y}(\overline{\partial}_b, \Gamma_\Omega(\mathcal{B}_S))$ are nothing but the CR functions in Ω (i.e. hyperfunction solutions of the system $\overline{\partial}_b$).

5. The case of a hypersurface. Let X, S, Y, Ω as before; from now on assume moreover S being a hypersurface of $X^{\mathbf{R}}$.

In this case $\dot{T}_S X$ is the union of two half rays, say $\pm\gamma$; set $\pm\gamma' = \widetilde{\phi}'^{-1}(\pm\gamma)$.

Fix a point $x_0 \in \partial\Omega$ and call $X^\pm$ the two connected components of $X \setminus S$ near x_0.

Let U be a neighborhood of Ω at x_0 and let $f \in \mathcal{O}_X(U \cap X^+)$. In this case, using Proposition 3.2, we then get an equivalent of (4.1), (4.2) without using the results of §4:

Proposition 5.1. *f extends to an Ω-tuboid of X with profile $\overline{\Omega} \times_S \gamma$ iff $f \circ \widetilde{\phi}$ extends, as a solution of $\overline{\partial}_b$, to an Ω-tuboid of Y with profile $\overline{\Omega} \times_S \gamma'$.*

To prove this statement, recall that, by using [Z] we get that f (resp $f \circ \widetilde{\phi}$) extends to a tuboid with profile γ (resp $\gamma' = \widetilde{\phi}'^{-1}\gamma$) iff $\gamma^* \notin \mathrm{SS}_{\Omega|X}(b(f))$ (resp $\gamma'^* \notin \mathrm{SS}^{\overline{\partial}_b,0}_{\Omega|Y}(b(f \circ \widetilde{\phi})))$.

In fact the latter is equivalent to $b(f) \in \pi_*\Gamma_{\gamma^{*a}}((\mathcal{C}_{\Omega|X})_{T^*_S X})$ (resp. is equivalent to $b(f \circ \widetilde{\phi}) \in \pi_*\Gamma_{\gamma'^{*a}}((\mathcal{C}_{\Omega|Y})_{T^*_S Y}))$. (We recall that $H^j(\mathcal{C}_{\Omega|X})_{T^*_S X} = 0 \, \forall j < 0$.)

This last remark, together with Proposition 5.1, gives the following:

$$\mathrm{SS}_{\Omega|X}(b(f)) = \mathrm{SS}^{\overline{\partial}_b,0}_{\Omega|Y}(b(f \circ \widetilde{\phi})). \tag{5.1}$$

We will make use of the following mixed version of (5.1) and Proposition 5.1:

Proposition 5.2. *f extends to a tuboid of X with profile $\overline{\Omega} \times_S \gamma$ iff $\gamma'^* \cap SS^{\overline{\partial}_b,0}_{\Omega|Y}(b(f \circ \widetilde{\phi})) = \emptyset$.*

6. Ω-regularity. Let S be a real analytic manifold, Y a complexification of S, $\Omega \subset S$ an open set with analytic boundary (Ω locally on one side of $\partial\Omega$). Let ω be the canonical 1-form.

We shall endow T^*Y of a real symplectic structure by $\mathrm{Re}\, d\omega$ and $T^*_S Y$ by $\mathrm{Im}\, d\omega$. We shall denote by $H^{\mathbf{R}}$ and $H^{\mathbf{I}}$ the corresponding hamiltonian isomorphisms.

Choose coordinates $(x; \partial/\partial x) \in TS$, and the dual coordinates $(x; \sqrt{-1}\,\eta) \in T^*_S Y$; assume $\Omega = \{x : \varphi > 0\}$.

Let $P(x; \partial/\partial x) \in (\mathcal{E}_Y)_\lambda$, $\lambda \in \partial\Omega \times_S \dot{T}^*_S Y$. Set $p = \mathrm{Re}\, \sigma(P)|_{T^*_S Y}$, $q = \mathrm{Im}\, \sigma(P)|_{T^*_S Y}$. We assume that $\{p, \varphi\} \equiv 1$ (and $p(\lambda) = q(\lambda) = \varphi(\lambda) = 0$).

It follows that $dp \wedge d\varphi \wedge \mathrm{Im}\, \omega \neq 0$ and thus one can divide $q = a + \varphi b$ with $\{p, a\} \equiv 0$.

Proposition 6.1. *Assume that in a neighborhood of λ:*

$$\begin{cases} \{p, \varphi\} \equiv 1, \\ \{\varphi, q\}|_{\{\varphi=0\}} \equiv 0, \\ da \neq 0 \quad or \quad da \equiv 0, \\ \{b, a\} \equiv 0. \end{cases} \tag{6.1}$$

Assume also

$$b \geq 0 \qquad for \quad \varphi \geq 0. \tag{6.2}$$

Then P is Ω-regular at λ (i.e.

$$\mathcal{H}om(P, \Gamma_{\dot{\pi}^{-1}(\overline{S \setminus \Omega})} \mathcal{C}_{\Omega|Y})_\lambda = 0 \ ). \tag{6.3}$$

(Here we still denote by P the module $\mathcal{M} = \mathcal{D}_Y/\mathcal{D}_Y P$.)

proof. We first choose coordinates $x = (x_1, x')$, $x' = (x_2, x'')$ in S, $(x; \sqrt{-1}\,\eta) \in T^*_S Y$ so that

$$p = \eta_1, \qquad \varphi = x_1.$$

We observe that (6.2) implies $\{\varphi, a\} \equiv 0$. Thus:

$$q(x; \sqrt{-1}\,\eta) = a(x'; \sqrt{-1}\,\eta') + x_1 b(x; \sqrt{-1}\,\eta).$$

Assume $da \neq 0$; by the trick of the dummy variable (that do not affect the conclusion of the theorem) it is not restrictive to assume $da \wedge \omega \neq 0$.

One can then change the coordinates $(x'; \sqrt{-1}\,\eta')$ so that

$$a = \eta_2, \quad b = b(x_1, x''; \sqrt{-1}\,\eta),$$

$$\lambda = (0; \sqrt{-1}\,\eta_0), \quad \eta_0 = (0, \ldots, 0, 1).$$

Let

$$\begin{aligned} N &= \{x : \varphi = 0\}, \\ V &= \{(x; \sqrt{-1}\,\eta) : \eta_2 = 0\}. \end{aligned}$$

We note that $N \times_S V$ is regular involutive. We also recall that $b \geq 0$ when $x_1 \geq 0$.

We claim that then

$$-H^{\mathbf{R}}(-d\varphi) \notin C_\lambda(char(\mathcal{M}), \widetilde{V_{\overline{\Omega}}}), \tag{6.4}$$

$\widetilde{V_{\overline{\Omega}}}$ being the union of the leaves of $V^{\mathbf{C}}$ issued from $\Omega \times_S V$ and $C(\cdot, \cdot)$ the normal cone in the sense of [K-S]. In fact let $(z; \zeta)$, $z = x + \sqrt{-1}\,y$, $\zeta = \xi + \sqrt{-1}\,\eta$ be coordinates on T^*Y. If $\operatorname{Im} \sigma(p + \sqrt{-1}\,q) = 0$ then

$$\xi_1 = \eta_2 + x_1 b^{\mathbf{R}} - y_1 b^{\mathbf{I}}.$$

We have

$$b^{\mathbf{R}} = b|_{T^*_S Y} + O((|y_1| + |y''|)|\eta| + |\xi|),$$

thus we have for some c:

$$x_1 b^{\mathbf{R}} + c((|y_1| + |y''|)|\eta| + |\xi|) \geq \begin{cases} 0, & x_1 \geq 0 \\ -c|x_1||\eta|, & x_1 \leq 0. \end{cases}$$

It follows for a new c:

$$\xi_1 \geq -c[|\zeta_2| + |\xi''| + (|y_1| + |y''| + Y(-x_1)|x_1|)|\eta|],$$

and hence (6.4).

Finally (6.4) implies (6.3) by [S-Z], [U-Z].

As for the case $a \equiv 0$ it can be handled by using the results on $\overline{\Omega}$-hyperbolicity instead of $\overline{\Omega} - V-$hyperbolicity (i.e. for $V = T^*_S Y$).(cf [S-Z,§3].) □

7. An application. Let $X \cong \mathbf{C}^2 \ni (u_1, u_2)$, $S \ni (x_1, x_2, x_3)$ a real hypersurface of X, Y a complexification of S, $\Omega = \{x : \varphi > 0\} \subset S$ an open set with analytic boundary. Let $x_0 \in \partial\Omega$, U a neighborhood of Ω at x_0, $X^\pm$ the two components of $X \setminus S$ near x_0.

In this case $\overline{\partial}_b$ is a vector field $p(x; \partial/\partial x) + \sqrt{-1}\, q(x; \partial/\partial x)$. We still denote by $p = \sqrt{-1}\, q$ the symbol $\sigma(\overline{\partial}_b)\big|_{T^*_S Y}$.

Let γ be the half space $N(X^+)$ and γ'^* the half ray $\gamma'^* = {}^t\widetilde{\phi}'(\gamma^*)$. Let U be a neighborhood of Ω at x_0.

Proposition 7.1. *Assume that the functions p, q, φ satisfy (6.1), (6.2) at $\lambda = \gamma'^*_{x_0}$ and let $f \in \mathcal{O}_X(X^+ \cap U)$. Then f extends to a tuboid of X with profile $\overline{\Omega} \times_S \gamma$.*

Proof. Clearly $b(f \circ \widetilde{\phi}) \in \mathcal{H}om(\overline{\partial}_b, \Gamma_{\dot{\pi}^{-1}(\overline{S \setminus \Omega})}(\mathcal{C}_{\Omega|Y}))_\lambda$. By Theorem 6.1, $\lambda \notin \mathrm{SS}^{\overline{\partial}_b, 0}_{\Omega|Y}(b(f \circ \widetilde{\phi}))$. Then f extends to U verifying (2.5) on account of Corollary 4.3. □

Example 7.2. Assume that

(i) $S = \{(u_1, u_2) \in X : u_j = \chi_j(x) + \sqrt{-1}\, \psi_j(x),\ j = 1, 2,\ x \in S\}$,
(ii) $\varphi = \psi_1$,
(iii) $d\,\chi_1 \wedge d\,\chi_2 \wedge d\,\varphi \neq 0$.
(iv) $\partial_{x_2}\psi_2 + \partial_{x_1}\psi_2 \partial_{x_3}\psi_2$.

By (ii), (iii), $\|\partial\chi_j/\partial x_i\|_{j=1,2; i=2,3}$ is non singular; one can then set $\chi_1 = x_2$, $\chi_2 = x_3$, $\psi_1 = x_1$.

In such a case we have:

$$\overline{\partial}_b = \partial_{x_1} - \sqrt{-1}\,[\partial_{x_2} + \beta(x_1, x_2, x_3)\partial_{x_3}],$$

for β solving:

$$\sqrt{-1}\,\partial_{x_1}\psi_2 + \partial_{x_2}\psi_2 - \sqrt{-1}\,\beta + \beta\partial_{x_3}\psi_2 = 0.$$

Setting $\beta = \partial_{x_1}\psi_2$, we get:

$$\overline{\partial}_b = \partial_{x_1} - \sqrt{-1}\,[\partial_{x_2} + \partial_{x_1}\psi_2\partial_{x_3}]. \tag{7.1}$$

Write $\psi_2 = x_1 a(x_2, x_3) + x_1^2 c(x_1, x_2, x_3)$ and set $b = 2c + x_1\partial_{x_1} c$. Assume $\{\xi_2 + a\xi_3, b\xi_3\} \equiv 0$ (for instance take $a(x_2, x_3) = a$ and $c(x_1, x_2, x_3) = c(x_1)$, or take any $a(x_2, x_3)$ and let $c(x_1, x_2, x_3) = 0$).

Under such hypotheses (6.1) is satisfied. If we then assume $b \leq 0$ for $x_1 \geq 0$ and $(x_0; \sqrt{-1}\,\eta \sim (x_0; \sqrt{-1}\,\eta_0)$, we get Ω-regularity at $(x_0; \sqrt{-1}\,\eta_0)$.

Remark 7.3. Note that if $b \leq 0$ for $x_1 \leq 0$, we get $S\backslash\overline{\Omega}$-regularity at $(x_0; \sqrt{-1}\,\eta_0)$.

Thus for instance for $S = \{u : u_1 = x_2 + \sqrt{-1}\,x_1, u_2 = x_3 + \sqrt{-1}\,x_1^2\}$, $\Omega = \{x : x_1 > 0\}$ and $\gamma^+ = N(\{u : \operatorname{Im} u_2 > \operatorname{Im} u_1^2\})$ then any f^+ (resp g^+) defined in $X^+ \cap W^+$ (resp $X^+ \cap W^-$) for $W^\pm$ a neighborhood of $S \cap \{\pm\mathrm{Im}u_1 > 0\}$, extends to a domain of type $\{u : \operatorname{Im} u_1 > 0, (\text{resp}\operatorname{Im} u_1 < 0)\operatorname{Im} u_1^2 < \operatorname{Im} u_2 < \varepsilon \operatorname{Im} u_1\}$ (for $\gamma^{+*} = -\sqrt{-1}\,d\operatorname{Re} u_2$ in the duality $T_M X \times T_M^* X \to \mathbf{R}$ associated to $-\operatorname{Im}\omega$).

This is of course classical by Bochner's theorem.

On the contrary for $S = \{u : u_1 = x_2 + \sqrt{-1}\,x_1, u_2 = x_3 + \sqrt{-1}\,x_1^3\}$ and for $W^\pm$ a neighborhood of $S \cap \{\sqrt{-1}\,u_1 > 0\}$, one has extension for f^+ (resp g^-) from $X^+ \cap W^+$ (resp $X^- \cap W^-$) to a domain of type $\{u; \mathrm{Im}u_1 > 0, \mathrm{Im}u_1^2 < \mathrm{Im}u_2 < \varepsilon\mathrm{Im}u_1\}$ (resp. $\{u; \mathrm{Im}u_1 < 0, -\varepsilon\mathrm{Im}u_1 < \mathrm{Im}u_2 < -\mathrm{Im}u_1^2\}$)

Remark 7.4. Let $S = \{u : u_1 = x_2 + \sqrt{-1}\,x_1, u_2 = x_3 + \sqrt{-1}\,a(x_2, x_3)x_1\}$, with $\partial a/\partial x_2 + a\partial a/\partial x_3 = 0$ and $\Omega = \{x : x_1 > 0\}$.

We have

$$\overline{\partial}_b = \frac{\partial}{\partial x_1} - \sqrt{-1}\,[\frac{\partial}{\partial x_2} + a(x_2, x_3)\frac{\partial}{\partial x_3}],$$

(which corresponds to the case $b \equiv 0$ in Proposition 6.1). Then one gets Ω and $S \setminus \overline{\Omega}$-regularity at both points in $T_S^* Y \cap char\overline{\partial}_b$.

8. Removable singularities. Let $S \subset X \cong \mathbf{C}^2$ be a generic hypersurface, Y a complexification of S. Let $N \subset S$ be an hypersurface, generic on X, given by $N = \{x; \varphi(x) = 0\}$. Let $N^{\mathbf{C}}$ be a complexification of N. Assume that, for $\overline{\partial}_b = p + \sqrt{-1}\,q$, one has $\{p, \varphi\} \equiv 1$. For $q = a + \varphi b$ ($\{p, a\} \equiv 0$), set $V = \{x; a(x) = 0\}$. Assume (6.1) to hold and moreover:

(6.2)' $b \geq 0$ on $T_S^* X$ (for any φ).

Let $\Sigma \subset N$ be such that $\sqrt{-1}\,N^*(\Sigma) \subset \rho\varpi(V)$ (here we denoted by ρ and ϖ the maps: $T^*N^{\mathbf{C}} \overset{\rho}{\leftarrow} N^{\mathbf{C}} \times_Y T_S^* Y \overset{\varpi}{T}{}^* Y$).

Take $u \in \Gamma_{S\setminus\Sigma}(\mathcal{B}_{S|Y})_{x_0}$, $x_0 \in \partial\Sigma$.

Proposition 8.1. *If $\pm\lambda \notin SS(u|_{S\setminus\Sigma})$ then u extends to S at x_0 to a function $\tilde{u}$ with $\pm\lambda \notin SS(\tilde{u})$.*

Sketch of the proof. We can look at u as being a section of $\mathcal{H}om_{\mathcal{D}_Y}(\overline{\partial}_b, \Gamma_{S\setminus\Sigma}\mathcal{B}_{S|Y})_{x_0}$. Let $\varphi = x_1$, let $\Omega^\pm = \{\pm x_1 > 0\}$ and denote by $\gamma^\pm(u)$ be the traces of u on N. We have $SS(\gamma^\pm(u) \subset \rho\varpi^{-1}SS_\Omega^{\overline{\partial}_b,0}(u)$ and so, by Proposition 6.1, $\rho(\lambda^\pm) \notin SS(\gamma^\pm(u))$. Hence also $\rho\varpi^{-1}(V) \cap SS(\gamma^\pm(u)) = \emptyset$.

Since $\mathrm{char}(\overline{\partial}_b) \cap \rho^{-1}\rho\varpi^{-1}v^{\mathbf{C}} \subset T_S^* Y$, then $SS(\gamma^\pm) \cap \rho\varpi^{-1}(V) = \emptyset$. Since $\gamma^+ - \gamma^- = 0$ on $S \setminus \Sigma$, we can propagate by the classical sweeping-out theorem. □

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