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On Lusztig's map for spherical unipotent conjugacy classes

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Abstract

We provide an alternative description of the restriction to spherical unipotent conjugacy classes, of Lusztig's map Ψ from the set of unipotent conjugacy classes in a connected reductive algebraic group to the set of conjugacy classes of its Weyl group. For irreducible root systems, we analyze the image of this restricted map and we prove that a conjugacy class in a finite Weyl group has a unique maximal length element if and only if it has a maximum.

1. *Introduction*

Springer [**24**] has shown how to associate to a unipotent conjugacy class of a connected reductive algebraic group G over an algebraically closed field k some irreducible representations of the associated Weyl group W. This led Kazhdan and Lusztig [**16**] to the definition of a conjecturally injective map from the set $\underline{G}$ of unipotent conjugacy classes of G to the set $\underline{W}$ of conjugacy classes of W, for $k = \mathbb{C}$. This map is not easily computable but Lusztig has very recently introduced in [**18**, **19**] a new, more computable, surjective map ϕ defined in all characteristics, from $\underline{W}$ to $\underline{G}$, and a right inverse Ψ which conjecturally coincides with the Kazhdan–Lusztig map over the complex numbers. The map ϕ is defined by assigning to a conjugacy class C in W a minimal unipotent conjugacy class in G, with respect to Zariski closure, having non-empty intersection with the Bruhat double coset corresponding to a minimal length element in C. It is a non-trivial result that this construction works. The proof of this important property is split into a proof for classical groups and one, based on a computer calculation, for exceptional ones. The right inverse Ψ is defined by taking, for a given unipotent class γ in G, the unique class C in W in the fiber of γ through ϕ for which the dimension of the fixed-point space of $w \in C$ in the geometric representation of W is minimal. Also in this case, the fact that this procedure actually works is a deep result.

The aim of this note is to give a different and direct combinatorial description of the restriction to spherical unipotent conjugacy classes of the map Ψ. We recall that a conjugacy class C in G is called spherical if a Borel subgroup B of G has a dense orbit in C. This new description is made possible by several recent results showing how the relation between spherical conjugacy classes and the Bruhat decomposition can be made very explicit. It has been shown in [**2**, **3**, **9**, **17**] that spherical (unipotent) classes may be characterized by means of a dimension formula involving the maximal Weyl group element w for which BwB meets a class. More precisely, let us define, for γ in $\underline{G}$, the element $w_\gamma \in W$ as the unique element in W for which $Bw_\gamma B \cap \gamma$ is Zariski dense in γ. Then γ is spherical if and only if $\dim \gamma = \ell(w_\gamma) + \mathrm{rk}(1 - w_\gamma)$, where ℓ is the length function on W and rk is the rank of the operator in the geometric representation of W. In addition, spherical conjugacy classes in good, odd characteristic are also characterized as those classes intersecting only Bruhat double cosets corresponding to

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involutions [3, 4]. Combining all these properties with the analysis of the elements w_γ in [7] leads us to the proof of our main result:

THEOREM. *Let γ be a spherical unipotent conjugacy class. Then $\Psi(\gamma) = W \cdot w_\gamma$.*

We also give some results on the map $\iota\colon \underline{G} \to \underline{W}$ defined by $\iota(\gamma) = W \cdot w_\gamma$.

This map can be defined on the set of all conjugacy classes in G. It was observed in [7, Remark 3] that the image of the set of all conjugacy classes and of the set of all spherical conjugacy classes through this map, in characteristic zero or good and odd characteristic, is the set $\underline{W}_m$ of classes in $\underline{W}$ having a unique maximal length element. We analyze the image of the restriction of ι to the set $\underline{G}_{\rm sph}$ of spherical unipotent conjugacy classes. A case-by-case analysis allows us to conclude the following proposition.

PROPOSITION. *For every irreducible root system there always exists a p such that in characteristic p, we have $\iota(\underline{G}_{\rm sph}) = \underline{W}_m$.*

It is worthwhile to mention that the element w_γ, for spherical classes, controls the G-module structure of the ring of regular functions $\mathbb{C}[\gamma]$. Indeed, Vinberg and Kimel'fel'd [27] proved that this module is multiplicity-free and it has been observed in [2] that the weights λ occurring in the decomposition of $\mathbb{C}[\gamma]$ all satisfy the equality $w_\gamma\lambda = -\lambda$ and that the rank of the lattice generated by these weights is $\mathrm{rk}(1 - w_\gamma)$. The precise G-module decomposition of $\mathbb{C}[\gamma]$ has been given in [8].

We conclude the paper by proving the following theorem:

THEOREM. *The set $\underline{W}_m$ coincides with the set of classes in $\underline{W}$ having maximum element with respect to the Bruhat order.*

This result holds for arbitrary finite Coxeter groups (see Remark 3).

2. *Notation and main result*

Throughout this paper, G is a semisimple algebraic group over an algebraically closed field k. Let T be a maximal torus of G, and let Φ be the associated root system. Let $B \supset T$ be a Borel subgroup, B^- its opposite Borel subgroup, and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the basis of Φ relative to (T, B). The Weyl group is denoted by $W = N(T)/T$, the symbol $\underline{W}$ will indicate the set of its conjugacy classes, and $\underline{W}_{\rm inv}$ will indicate the set of conjugacy classes of involutions in W, that is, the set of classes of those elements $w \in W$ such that $w^2 = 1$. The symbol $\underline{G}$ will stand for the set of unipotent conjugacy classes and $\underline{G}_{\rm sph}$ will denote the set of spherical unipotent ones. We recall that a conjugacy class γ in G is called spherical if B has a Zariski dense orbit in γ.

For any $C \in \underline{W}$, we define $C_{\min}$ to be the subset of C consisting of elements of minimal length. For $w \in W$, we define $\Sigma_w = \{\gamma \in \underline{G} \mid \gamma \cap BwB \neq \emptyset\}$.

For $\gamma \in \underline{G}$, we define $W_\gamma = \{w \in W \mid \gamma \cap BwB \neq \emptyset\}$. It is clear that W_γ is always not empty. It is also true that Σ_w is always not empty: indeed $BwB \cap B^- \neq \emptyset$ for every $w \in W$ [14, § A2], so $BwB \cap U^- \neq \emptyset$ for every $w \in W$.

As usual, w_0 denotes the longest element in W and, for $\Sigma \subseteq \Delta$, we shall denote by w_Σ the longest element in the parabolic subgroup W_Σ of W generated by simple reflections indexed by elements in Σ. The root subsystem of Φ generated by the roots in Σ will be denoted by Φ_Σ.

It follows from [12, 8.2.6(b); 18, 1.2(a)] that, for $w, \sigma \in C_{\min}$, then $\Sigma_w = \Sigma_\sigma$.

Let $\phi\colon \underline{W} \to \underline{G}$ be the map introduced in [18]. It is defined as follows: let $C \in \underline{W}$ and let $w \in C_{\min}$. The image of C through ϕ is the unique $\gamma \in \underline{G}$ such that $\gamma \in \Sigma_w$ and such that every $\gamma' \in \underline{G}$ lying in Σ_w contains γ in its closure. By Lusztig [18, Theorem 0.4], the map ϕ is surjective.

If $\gamma \in \underline{G}$ and $C \in \phi^{-1}(\gamma)$, then $\gamma \in \Sigma_w$ for some $w \in C_{\min}$. For γ a spherical unipotent conjugacy class the set W_γ has a particular structure. We recall the facts we will need.

THEOREM 2.1 [3, 9]. *Let γ be a spherical conjugacy class, and let $\gamma \cap BwB$ be non-empty. Assume in addition that γ is unipotent if $\mathrm{char}(k) = 2$. Then w is an involution.*

Proof. If the characteristic of k is zero or if it is good and odd, the statement is [3, Theorem 2.7]. The same proof holds as long as $\mathrm{char}(k) \neq 2$. For $\mathrm{char}(k) = 2$, let u be an element of $\gamma \cap BwB$. From the classification of spherical unipotent conjugacy classes it follows that u is an involution, see [9, Theorem 3.18]. Thus, $u = u^{-1} \in Bw^{-1}B \cap BwB$, forcing $w = w^{-1}$. □

So, $\phi^{-1}(\underline{G}_{\mathrm{sph}}) \subseteq \underline{W}_{\mathrm{inv}}$. One may wish to see whether $\underline{G}_{\mathrm{sph}}$ can be characterized as the image of a suitable subset of $\underline{W}_{\mathrm{inv}}$.

The statement of the lemma below was communicated to the first named author by Kei–Yuen Chan.

LEMMA 2.2. *Let $\mathrm{char}(k) \neq 2$. Let γ be a (not necessarily unipotent) spherical conjugacy class and let $\gamma \cap BwB \neq \emptyset$ for some $w \in C$ and $C \in \underline{W}$. Then $\gamma \cap B\sigma B \neq \emptyset$ for every $\sigma \in C$. The same conclusion holds for $\mathrm{char}(k) = 2$ if γ is a spherical unipotent conjugacy class.*

Proof. Let $\sigma = s_{i_l} \ldots s_{i_1} w s_{i_1} \ldots s_{i_l}$ with $\tau = s_{i_l} \ldots s_{i_1}$ of minimal length l such that $\sigma = \tau w \tau^{-1}$. Let us put $\sigma_j = s_{i_j} \ldots s_{i_1} w s_{i_1} \ldots s_{i_j}$ for $j = 0, \ldots l$, so that $\sigma_0 = w$ and $\sigma_l = \sigma$. We shall prove by induction on j that $\gamma \cap B\sigma_j B \neq \emptyset$ for every $j \in \{0, \ldots, l\}$. The basis of the induction is our assumption. Assume $\gamma \cap B\sigma_j B \neq \emptyset$ for a given j. Then there is also $x \in B\sigma_j \cap \gamma$. Let $\dot{s}_{i_{j+1}}$ be a lift of $s_{i_{j+1}}$ in $N(T)$. We have

$$\dot{s}_{i_{j+1}} x \dot{s}_{i_{j+1}}^{-1} \in s_{i_{j+1}} B\sigma_j s_{i_{j+1}} \subseteq B\sigma_{j+1} B \cup B\sigma_j s_{i_{j+1}} B.$$

By Theorem 2.1, the class γ intersects only cells corresponding to involutions. Hence, w and σ_j are involutions. On the other hand, $\sigma_j s_{i_{j+1}}$ is an involution if and only if σ_j and $s_{i_{j+1}}$ commute, but this would contradict minimality of the length of τ. Thus, $\gamma \cap B\sigma_j s_{i_{j+1}} B = \emptyset$, and we necessarily have $\dot{s}_{i_{j+1}} x \dot{s}_{i_{j+1}}^{-1} \in B\sigma_{j+1} B \cap \gamma$, yielding the statement. □

Let γ be any conjugacy class in G. We shall denote by w_γ the unique element in W for which $Bw_\gamma B \cap \gamma$ is dense in γ, and by $C^\gamma = W \cdot w_\gamma$, the conjugacy class of w_γ in W. Let us denote by $\underline{W}_m$ the set of classes in $\underline{W}$ containing a unique maximal length element. We recall some basic facts.

THEOREM 2.3 [7]. *Let γ be a conjugacy class in G and let w_γ and C^γ be as above. Then*

(i) *C^γ lies in $\underline{W}_m$ and w_γ is its maximal length element;*
(ii) *$\underline{W}_m \subseteq \underline{W}_{\mathrm{inv}}$;*
(iii) *if $\mathrm{char}(k)$ is either 0 or good and odd, then for every $C \in \underline{W}_m$ there is a spherical conjugacy class γ such that $C = C^\gamma$.*

Proof. Statement (i) is Corollary 2.11 in [7], the proof of which is characteristic-free. Statement (ii) follows from the fact that any w is conjugate to w^{-1} [6, Theorem C]. Statement (iii) is observed in Remark 3 in [6]. □

We will also make use of the following result.

Theorem 2.4 [2, 3, 9, 17]. *Let γ be a unipotent conjugacy class, let w_γ be as above, and let $w \in W$.*

(i) *If $\gamma \in \Sigma_w$, then $\dim \gamma \geqslant \ell(w) + \mathrm{rk}(1-w)$.*
(ii) *We have $\dim \gamma \geqslant \ell(w_\gamma) + \mathrm{rk}(1-w_\gamma)$.*
(iii) *The class γ is spherical if and only if $\dim \gamma = \ell(w_\gamma) + \mathrm{rk}(1-w_\gamma)$.*

Proposition 2.5. *Let γ be a spherical unipotent conjugacy class and let C^γ be as above. Then $\phi(C^\gamma) = \gamma$.*

Proof. Let $w \in (C^\gamma)_{\min}$. We need to show that $\gamma \in \Sigma_w$ and that it is the unique minimal element therein.

By construction γ lies in Σ_{w_γ} so by Lemma 2.2, it also lies in Σ_w. It follows from [7, Propositions 2.8, 2.9], which in turn uses [10, Proposition 3.4; 11, § 2.9], that if $\sigma \in C^\gamma$ and y is a maximal length element in C^γ, then $\Sigma_\sigma \subseteq \Sigma_y$. In particular, this holds for $\sigma = w$ and $y = w_\gamma$ by Theorem 2.3(i).

Let $\gamma' \in \Sigma_w$. Then $\gamma' \in \Sigma_{w_\gamma}$ and by part (i) of Theorem 2.4 we have $\dim \gamma' \geqslant \ell(w_\gamma) + \mathrm{rk}(1 - w_\gamma)$. However, by Theorem 2.4, we have $\dim \gamma = \ell(w_\gamma) + \mathrm{rk}(1-w_\gamma)$ so γ is minimal in Σ_{w_γ}, and, *a fortiori*, in Σ_w. The assertion follows from uniqueness of the minimal element in Σ_w (see [18]). □

The above result can be rephrased by saying that the restriction to $\underline{G}_{\mathrm{sph}}$ of the map

$$\iota \colon \underline{G} \longrightarrow \underline{W}_{\mathrm{inv}}, \quad \gamma \longmapsto C^\gamma$$

is a right inverse for ϕ on $\underline{G}_{\mathrm{sph}}$.

In [19, Theorem 0.2], a right inverse Ψ to ϕ has been constructed. It is defined as follows. For any $\gamma \in \underline{G}$ one considers $\phi^{-1}(\gamma)$. This set contains a unique element $C_0 \in \underline{W}$ for which the dimension d_C of the fixed-point space of an (thus any) element in C is minimal. Then $\Psi(\gamma) = C_0$. We want to compare the maps ι and Ψ on $\underline{G}_{\mathrm{sph}}$.

It is shown in [7, Lemma 3.2] that $w_\gamma = w_0 w_\Sigma$ for some $\Sigma \subseteq \Delta$ such that w_Σ coincides with w_0 on Σ. Using the same arguments, one can prove the following result, that we report here for completeness.

Lemma 2.6. *Let γ be a spherical unipotent conjugacy class or any spherical conjugacy class if $\mathrm{char}(k)$ is either 0 or good and odd, and let $\sigma \in W_\gamma$ be a maximal length element in its conjugacy class C. Then $\sigma = w_0 w_\Sigma$ for some $\Sigma \subseteq \Delta$ such that w_Σ coincides with w_0 on Σ.*

Proof. Since W_γ consists of involutions, we may apply [21, Theorem 1.1(ii)], so $\sigma = w_0 w_\Sigma$ for some $\Sigma \subseteq \Delta$. In addition, w_0 and w_Σ necessarily commute so $(-w_0)\Sigma = \Sigma$. Let $\alpha \in \Sigma$. We have $\beta = w_0 w_\Sigma \alpha \in \Sigma \subseteq \Phi^+$ so $\ell(w_0 w_\Sigma s_\alpha) = \ell(w_0 w_\Sigma) + 1$. Then, by maximality of the length of σ in C, we have $\ell(s_\alpha w_0 w_\Sigma s_\alpha) = \ell(w_0 w_\Sigma)$. By Springer [25, Lemma 3.2], we get $\alpha = \beta$. □

Lemma 2.7. *Let $\Pi \subseteq \Delta$ and let $w = w_0 w_\Pi$ be an involution with the property that w_0 restricted to Φ_Π is w_Π. Then $(-w_0)(\Pi) = \Pi$ and*

$$\mathrm{rk}(1-w_0) = \mathrm{rk}(1-w_\Pi) + \mathrm{rk}(1-w).$$

Proof. The first statement follows from $w_0 w_\Pi(\alpha) = \alpha$ for every $\alpha \in \Pi$.

Let us denote by $E_m(x)$ the m-eigenspace of an operator x. Clearly, if x is an involution, then $\dim E_{-1}(x) = \mathrm{rk}(1-x)$. It is an immediate exercise in linear algebra that if x and y are commuting involutions, then $\dim E_{-1}(x) + \dim E_{-1}(y) = \dim E_{-1}(xy)$ if and only if $E_{-1}(x) \cap E_{-1}(y) = \{0\}$.

We have $\Pi \subseteq E_1(w_0 w_\Pi) = E_{-1}(w_0 w_\Pi)^\perp$ so, since w_Π can be written as a product of reflections with respect to roots in Π, for every $v \in E_{-1}(w_0 w_\Pi)$ we have $w_\Pi(v) = v$. In other words,

$$E_{-1}(w_0 w_\Pi) \cap E_{-1}(w_\Pi) \subseteq E_1(w_\Pi) \cap E_{-1}(w_\Pi) = \{0\},$$

whence the second statement. □

We are ready to state the main result of this paper.

THEOREM 2.8. *Lusztig's map Ψ coincides with ι on $\underline{G}_{\mathrm{sph}}$.*

Proof. Let $\gamma \in \underline{G}_{\mathrm{sph}}$. By Proposition 2.5, we have $C^\gamma \in \phi^{-1}(\gamma)$, so we need to show only that the dimension d_C of the fixed-point space $E_1(w)$ of an element $w \in C$ for $C \in \phi^{-1}(\gamma)$ is minimal for $w \in C^\gamma$.

Let C be a class in $\phi^{-1}(\gamma)$. Then every σ in C lies in W_γ by Lemma 2.2. By Theorem 2.1, the set W_γ is a union of classes in $\underline{W}_{\mathrm{inv}}$. Moreover, all elements in W_γ are less than or equal to w_γ in the Bruhat ordering, in particular, this holds for all elements in C. Let z be a maximal length element in C. By Lemma 2.6, $z = w_0 w_\Sigma$ and $w_\gamma = w_0 w_\Pi$, where Σ and Π are subsets of Δ on which z and w_γ, respectively, act as the identity, and $z \leqslant w_\gamma$, or, equivalently, $w_\Pi \leqslant w_\Sigma$. Since w_Σ has a reduced expression as a product of reflections with respect to roots in Σ, the simple reflections occurring in some reduced expression of w_Π correspond to some simple roots in Σ by [**1**, Corollary 2.2.3]. By Björner and Brenti [**1**, Corollary 1.4.8(ii)], the set of simple roots occurring in any reduced expression of w_Π is precisely Π. Hence, $\Pi \subseteq \Sigma$. Moreover, the restriction of w_Σ to Π coincides with w_Π so by Lemma 2.7 applied to Φ_Σ we have $\mathrm{rk}(1-w_\Sigma) = \mathrm{rk}(1 - w_\Pi w_\Sigma) + \mathrm{rk}(1-w_\Pi)$, so $\mathrm{rk}(1-w_\Pi) \leqslant \mathrm{rk}(1-w_\Sigma)$. Applying Lemma 2.7 once more we see that

$$\begin{aligned}\mathrm{rk}(1-w_\gamma) = \mathrm{rk}(1-w_0 w_\Pi) &= \mathrm{rk}(1-w_0) - \mathrm{rk}(1-w_\Pi)\\ &\geqslant \mathrm{rk}(1-w_0) - \mathrm{rk}(1-w_\Sigma) = \mathrm{rk}(1-z),\end{aligned}$$

so $\mathrm{rk}(1-x)$ reaches its maximum over $\phi^{-1}(\gamma)$ at $x = w_\gamma$. Since all elements in $\phi^{-1}(\gamma)$ are involutions, this gives precisely minimality of $d_{C^\gamma} = \dim E_1(w_\gamma)$. Thus, $\Psi(\gamma) = C^\gamma$. □

COROLLARY 2.9. *The map ι is injective on spherical unipotent conjugacy classes.*

REMARK 1. Except for type A_1, the maps ι and Ψ do not coincide on the full set $\underline{G}$ because Ψ is necessarily injective whereas ι is not. Indeed, the regular unipotent class γ_{reg} intersects every BwB (see [**15**] or the result of Springer in the Appendix of [**10**]), so $\iota(\gamma_{\mathrm{reg}}) = W \cdot w_0$. On the other hand, there is always a spherical unipotent conjugacy class intersecting Bw_0B.

An important feature of the maps ϕ and Ψ is that they are defined in all characteristic and they satisfy compatibility conditions as follows. For a fixed irreducible root system Φ, let G^p denote a corresponding group in characteristic p and let ϕ_p, Ψ_p and ι_p denote the corresponding maps ϕ, Ψ and ι. If in the sequel reference to p is omitted, we shall mean that the statement holds for every $p \geqslant 0$. Let us recall that there is a dimension-preserving and order-preserving injective map $\pi\colon \underline{G}^0 \to \underline{G}^p$, where the order is given by inclusion of Zariski closures [**19**, § 3.1; **22**, III, 5.2]. It is shown in [**19**, Theorem 0.4(b)] that $\Psi_0 = \Psi_p\pi$ and $\pi = \phi_p\Psi_0$. The compatibility behaves well when we restrict ourselves to spherical conjugacy classes.

Proposition 2.10. *The map π maps $\underline{G}^0{}_{\rm sph}$ into $\underline{G}^p{}_{\rm sph}$, and if γ lies in $\underline{G}^0{}_{\rm sph}$, then $w_{\pi(\gamma)} = w_\gamma$.*

Proof. Let $\gamma \in \underline{G}^0{}_{\rm sph}$. Then

$$\pi(\gamma) = \phi_p\Psi_0(\gamma) = \phi_p\iota_0(\gamma) = \phi_p(C^\gamma).$$

Let σ be a minimal length element in C^γ. Then $\pi(\gamma) \in \Sigma_\sigma$ and, arguing as in the proof of Lemma 2.5, since w_γ is the maximal length element, $\pi(\gamma) \in \Sigma_{w_\gamma}$. Thus, $w_\gamma \leqslant w_{\pi(\gamma)}$. It is not hard to show, by induction on the length of a word in W, that if $w \leqslant \tau$ in the Bruhat order, then $\ell(w) + {\rm rk}(1-w) \leqslant \ell(\tau) + {\rm rk}(1-\tau)$ (see the proof of [**2**, Proposition 6]). Therefore, invoking part (ii) of Theorem 2.4 for γ we have $\dim(\pi(\gamma)) = \dim(\gamma) = \ell(w_\gamma) + {\rm rk}(1-w_\gamma) \leqslant \ell(w_{\pi(\gamma)}) + {\rm rk}(1-w_{\pi(\gamma)})$. Applying Theorem 2.4 to $\pi(\gamma)$, we have the first statement. The second one is immediate from $\Psi_0 = \Psi_p\pi$ and Theorem 2.8. □

In the remainder of the paper, we analyze the image of the restriction of Ψ to spherical unipotent conjugacy classes.

By part (i) of Theorem 2.3, the image of the restriction of ι to $\underline{G}_{\rm sph}$ lies in $\underline{W}_m$. We observe that the map ι can be defined in the same way for any conjugacy class.

Identifying a class in $\underline{W}_m$ with its unique maximal length element, we may endow $\underline{W}_m$ with a poset structure from the Bruhat order of W. Inclusion of Zariski closures induces a poset structure on the set of conjugacy classes in G and on $\underline{G}$.

We observe that if for some conjugacy classes γ, γ' we have $\bar\gamma \subseteq \overline{\gamma'}$, then

$$\emptyset \neq Bm_\gamma B \cap \gamma \subseteq \overline{Bm_\gamma B \cap \gamma} = \bar\gamma \subseteq \overline{\gamma'} = \overline{Bm_{\gamma'}B \cap \gamma'} \subseteq \overline{Bm_{\gamma'}B},$$

so $m_\gamma \leqslant m_{\gamma'}$ in the Bruhat order and ι is order-preserving.

By Theorem 2.3, in zero or good and odd characteristic the image of the set of all spherical classes through ι is exactly $\underline{W}_m$. Let us analyze the situation for spherical unipotent conjugacy classes.

Proposition 2.11. *For every Φ there is some p such that $\iota_p(\underline{G}^p{}_{\rm sph})$ is $\underline{W}_m$.*

Proof. The list of the maximal length representatives for all elements in $\underline{W}_m$ is given in [**7**] in terms of subdiagrams of the Dynkin diagram, and it can be deduced from [**20**]. In zero or good and odd characteristic, we have $\iota_p(\underline{G}^p{}_{\rm sph}) = \underline{W}_m$ precisely in type A_n, $n \geqslant 1$; D_n, $n \geqslant 4$; E_6; E_7; E_8 (see [**2**, Table 3; **5**, **7**, Lemma 3.5]). From Proposition 2.10, it follows that in these cases, we have $\iota_p(\underline{G}^p{}_{\rm sph}) = \underline{W}_m$ also when p is a bad prime or $p = 2$.

In type C_n (and B_n), $n \geqslant 2$, in characteristic 2 there are $n + [n/2]$ non-trivial spherical unipotent conjugacy classes (see [**9**, 3.1.2]) and therefore we have $\iota_2(\underline{G}^2{}_{\rm sph}) = \underline{W}_m$.

In type F_4, for $p = 3$ the poset of spherical unipotent conjugacy classes is the same as the corresponding poset in good characteristic, while for $p = 2$ we have $\iota_2(\underline{G}^2{}_{\rm sph}) = \underline{W}_m$ (see [**9**, Table 6, 7]).

In type G_2, for $p = 2$ the poset of spherical unipotent conjugacy classes is the same as the corresponding poset in good characteristic, while for $p = 3$ we have $\iota_3(\underline{G}^3{}_{\rm sph}) = \underline{W}_m$ (see [**9**, Table 8, 9]). □

Corollary 2.12. *The following are equivalent*:

(i) $\iota_p(\underline{G^p}_{\mathrm{sph}}) = \underline{W}_m$ *for every* $p \geqslant 0$;
(ii) $\iota_0(\underline{G^0}_{\mathrm{sph}}) = \underline{W}_m$;
(iii) *the restriction of* π *to* $\underline{G^0}_{\mathrm{sph}}$ *is an isomorphism onto* $\underline{G^p}_{\mathrm{sph}}$ *for every* $p \geqslant 0$.

Proof. The equivalence of (i) and (ii) is immediate from $\iota_0 = \iota_p \pi$. Let us assume (i). By bijectivity of ι_0 and injectivity of ι_p, we have

$$|\underline{G^p}_{\mathrm{sph}}| \leqslant |\underline{W}_m| = |\underline{G^0}_{\mathrm{sph}}|$$

so injectivity of π implies (iii). Finally, Proposition 2.11 shows that (iii) implies (i). □

Remark 2. Let $\mathcal{J}$ be the set of subsets of Δ such that

$$\underline{W}_m = \{W \cdot w_0 w_J \mid J \in \mathcal{J}\}.$$

We can identify $\underline{W}_m$ with $\mathcal{J}$ and the partial order on $\underline{W}_m$ becomes reverse inclusion of subsets in $\mathcal{J}$. We observe that for $J, K \in \mathcal{J}$ both $J \cap K$ and $J \cup K$ are in $\mathcal{J}$ and therefore $\underline{W}_m$ is a lattice. It can be proved by inspection that for every p the order-preserving map ι_p restricted to $\underline{G^p}_{\mathrm{sph}}$ is a poset isomorphism onto its image and that $\underline{G^p}_{\mathrm{sph}}$ is always a lattice.

Theorem 2.13. *The set* $\underline{W}_m$ *is the set of conjugacy classes in* W *having maximum with respect to the Bruhat order.*

Proof. By Theorem 2.3(iii) or by Proposition 2.11 if C lies in $\underline{W}_m$, then $C = C^\gamma$ for some spherical conjugacy class in G^p for some p. By Lemma 2.2, every $w \in C$ lies in W_γ so it must be less than or equal to w_γ in the Bruhat ordering. Thus, the maximal length element in C is the sought maximum in C. Conversely, if C has maximum σ with respect to the Bruhat ordering, then σ has maximal length in C. Hence, σ is the unique maximal length element in C because for any $\tau \in C$ different from σ we have $\ell(\tau) < \ell(\sigma)$. □

Remark 3. It was kindly suggested to us by A. Hultman that the statement of Theorem 2.13 for arbitrary finite Coxeter groups follows from the observation in [**11**, p. 577]. Indeed, it is shown therein that for $C \in \underline{W}$ and any $w \in C$ there exists some σ of maximal length in C and a chain of simple reflections $s_{i_1}, \ldots, s_{i_r}$ satisfying

$$\sigma_0 = \sigma; \quad \sigma_j = s_{i_j} \sigma_{j-1} s_{i_j}; \quad \sigma_r = w$$

and $\ell(\sigma_j) \geqslant \ell(\sigma_{j+1})$ for $j = 0, \ldots, r$. Now, if $C \in \underline{W}_m$, then $C \in \underline{W}_{\mathrm{inv}}$ (see [**23**, Theorem 8.7] for H_3 and H_4 or [**12**, Corollary 3.2.14] for arbitrary finite Coxeter groups). By Springer [**25**, Lemma 3.2], we have $\ell(\sigma_j) = \ell(\sigma_{j+1})$ if and only if $\sigma_j = \sigma_{j+1}$, and if $\ell(\sigma_j) > \ell(\sigma_{j+1})$, we necessarily have $\ell(\sigma_j) = \ell(\sigma_{j+1}) + 2$. This forces

$$\sigma_{j-1} \geqslant \sigma_{j-1} s_{i_j} \geqslant s_{i_j} \sigma_{j-1} s_{i_j} = \sigma_j$$

in the Bruhat order, so the unique maximal length element σ is the sought maximum in C.

The main result in [**11**] is based on a case-by-case analysis, but a new case-free proof is available in [**13**]. On the other hand, surjectivity of ι on $\underline{W}_m$ relies on the case-by-case analysis in [**2**]. This could be shortened by looking at the image through ι of the classes of involutions (in the adjoint group) in [**26**, Table 1] and using [**2**] only for the few missing cases.

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