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Multivariate Christoffel functions and hyperinterpolation

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Abstract

We obtain upper bounds for Lebesgue constants (uniform norms) of hyperinterpolation operators via estimates for (the reciprocal of) Christoffel functions, with different measures on the disk and ball, and on the square and cube. As an application, we show that the Lebesgue constant of total-degree polynomial interpolation at the Morrow-Patterson minimal cubature points in the square has an $\mathcal{O}(\deg^3)$ upper bound, explicitly given by the square root of a sextic polynomial in the degree.

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1 Bounding uniform hyperinterpolation norms

Hyperinterpolation is a powerful tool for total-degree polynomial approximation of multivariate continuous functions, introduced by Sloan in the seminal paper [22]. In brief, it corresponds to a truncated Fourier expansion in a series of orthogonal polynomials for some measure on a given multidimensional domain, where the Fourier coefficients are discretized by means of a positive algebraic cubature formula. Since then, theoretical as well as computational aspects of hyperinterpolation as an alternative to interpolation have attracted much interest, due to the intrinsic difficulties in finding good interpolation nodes, with special attention to the case of the sphere; cf., e.g., [7, 8, 11, 13, 14, 24, 25].

One of the important features of hyperinterpolation as a polynomial projection consists in the growth of its uniform operator norm, which we call "Lebesgue constant" by analogy with interpolation (indeed, under certain conditions hyperinterpolation becomes an interpolation operator). In this note we give upper bounds of the Lebesgue constant of hyperinterpolation via estimates of a key function in the theory of orthogonal polynomials, the so-called Christoffel function.

Given a compact set $K \subset \mathbb{R}^d$ and a positive measure μ on K, we shall denote by $\mathbb{P}_n^d(K)$ the space of total-degree d-variate polynomials with degree not greater than n, restricted to K, and by $K_n(\boldsymbol{x},\boldsymbol{y})$ the reproducing kernel of $\mathbb{P}_n^d(K)$ in $L^2_{d\mu}(K)$, that has the representation (cf. [12, §3.5])

$$K_n(\boldsymbol{x},\boldsymbol{y}) = \sum_{j=1}^{N} p_j(\boldsymbol{x})p_j(\boldsymbol{y}) \,, \;\; \boldsymbol{x}=(x_1,\dots,x_d)\,, \;\boldsymbol{y}=(y_1,\dots,y_d)\,, \tag{1}$$

where $\{p_j\}$ is any orthonormal basis of $\mathbb{P}_n^d(K)$ in $L^2_{d\mu}(K)$, $N=\dim(\mathbb{P}_n^d(K))$. The function

$$K_n(\boldsymbol{x},\boldsymbol{x}) = \sum_{j=1}^{N} p_j^2(\boldsymbol{x}) \tag{2}$$

is known as the (reciprocal of) the n-th Christoffel function of μ on K.

We begin with the following observation

Proposition 1.1. *Let $K \subset \mathbb{R}^d$ be a compact set, μ a positive measure on K, and $\{a_n\}$ a sequence of positive real numbers such that*

$$a_n \ge \mathcal{C}_n(d\mu,K) = \sqrt{\max_{\boldsymbol{x}\in K} K_n(\boldsymbol{x},\boldsymbol{x})}\,. \tag{3}$$

Let

$$\mathcal{L}_n : \left(C(K), \|\cdot\|_{L^\infty(K)}\right) \to \left(\mathbb{P}_n^d, \|\cdot\|_{L^2_{d\mu}(K)}\right) \tag{4}$$

a sequence of uniformly bounded operators, that is, there is a constant $M>0$ such that

$$\|\mathcal{L}_n\| = \sup_{f\neq 0} \frac{\|\mathcal{L}_n f\|_{L^2_{d\mu}(K)}}{\|f\|_{L^\infty(K)}} \le M$$

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for every n.

Then, the following estimate holds for the uniform norm

$$\|\mathcal{L}_n\|_\infty = \sup_{f\neq 0} \frac{\|\mathcal{L}_n f\|_{L^\infty(K)}}{\|f\|_{L^\infty(K)}} \leq a_n M \,. \tag{5}$$

Proof. First, we recall the well-known fact that

$$\mathcal{C}_n(d\mu, K) = \max\left\{ \frac{\|p\|_{L^\infty(K)}}{\|p\|_{L^2_{d\mu}(K)}} \,,\, p \in \mathbb{P}^d_n(K) \,,\, p \neq 0 \right\} , \tag{6}$$

or in other words $\mathcal{C}_n(d\mu,K)$ is the norm of the identity $Id : (\mathbb{P}^d_n(K), \|\cdot\|_{L^2_{d\mu}(K)}) \to (\mathbb{P}^d_n(K), \|\cdot\|_{L^\infty(K)})$. Indeed, by the Cauchy-Schwarz inequality in $\mathbb{R}^N$

$$|p(\boldsymbol{x})| = \left| \sum_{j=1}^N \langle p, p_j\rangle_{L^2_{d\mu}(K)} \, p_j(\boldsymbol{x}) \right| \leq \sqrt{\sum_{j=1}^N \langle p, p_j\rangle^2_{L^2_{d\mu}(K)}} \sqrt{K_n(\boldsymbol{x},\boldsymbol{x})}$$

$$= \|p\|_{L^2_{d\mu}(K)} \sqrt{K_n(\boldsymbol{x},\boldsymbol{x})} \leq \|p\|_{L^2_{d\mu}(K)} \sqrt{\max_{\boldsymbol{x}\in K} K_n(\boldsymbol{x},\boldsymbol{x})} \,,$$

for every $p \in \mathbb{P}^d_n(K)$ and for every $\boldsymbol{x} \in K$. On the other hand, the maximum on the right-hand side of (6) is attained at the polynomial $p(\boldsymbol{x}) = \sum_{j=1}^N p_j(\overline{\boldsymbol{x}}) p_j(\boldsymbol{x})$, where $\overline{\boldsymbol{x}}$ is a maximum point for $K_n(\boldsymbol{x},\boldsymbol{x})$ in K, since $p(\overline{\boldsymbol{x}}) = K_n(\overline{\boldsymbol{x}},\overline{\boldsymbol{x}})$ and $\|p\|_{L^2_{d\mu}(K)} = \sqrt{K_n(\overline{\boldsymbol{x}},\overline{\boldsymbol{x}})}$.

Then estimate (5) is a consequence of the chain of inequalities

$$\|\mathcal{L}_n f\|_{L^\infty(K)} \leq \mathcal{C}_n(d\mu, K) \, \|\mathcal{L}_n f\|_{L^2_{d\mu}(K)}$$

$$\leq a_n \|\mathcal{L}_n\| \, \|f\|_{L^\infty(K)} \leq a_n M \, \|f\|_{L^\infty(K)} \,. \quad \square$$

Let a cubature formula $(\boldsymbol{w}, X)$ for μ be given, exact in $\mathbb{P}^d_{2n}(K)$, with nodes $X = X_n = \{\boldsymbol{\xi}_i(n) \,,\ i = 1, \ldots, \mathcal{V}\} \subset K$ and positive weights $\boldsymbol{w} = \boldsymbol{w}_n = \{w_i(n) \,,\ i = 1, \ldots, \mathcal{V}\}$, $\mathcal{V} \geq N = \dim(\mathbb{P}^d_n(K))$, and let $\{p_j \,,\ j = 1, \ldots, N\}$ be any orthonormal basis of $\mathbb{P}^d_n(K)$ in $L^2_{d\mu}(K)$. We recall that the hyperinterpolation operator corresponding to the cubature formula, is the discretized orthogonal projection $\mathcal{L}_n : C(K) \to \mathbb{P}^d_n(K)$ defined as

$$\mathcal{L}_n f(\boldsymbol{x}) = \sum_{j=1}^N \langle f, p_j\rangle_{\ell^2_w(X)} \, p_j(\boldsymbol{x}) \,,$$

where $\ell^2_w(X)$ denotes the Euclidean space of functions defined on X equipped with the scalar product

$$\langle f, g\rangle = \sum_{i=1}^{\mathcal{V}} w_i f(\boldsymbol{\xi}_i) g(\boldsymbol{\xi}_i) \,.$$

We can now prove the following result

Corollary 1.2. *Assume that (3) holds. Then, the "Lebesgue constant" of any hyperinterpolation operator (i.e., its uniform operator norm) has the upper bound*

$$\|\mathcal{L}_n\|_\infty \leq a_n \sqrt{\mu(K)} \,. \tag{7}$$

Proof. Following [22], we can write by exactness in $\mathbb{P}^d_{2n}(K)$ and the Pythagorean theorem in $\ell^2_w(X)$

$$\|\mathcal{L}_n f\|_{L^2_{d\mu}(K)} = \|\mathcal{L}_n f\|_{\ell^2_w(X)} \leq \|f\|_{\ell^2_w(X)} = \sqrt{\sum_{i=1}^{\mathcal{V}} w_i f^2(\boldsymbol{\xi}_i)}$$

$$\leq \sqrt{\sum_{i=1}^{\mathcal{V}} w_i} \, \|f\|_{\ell^\infty(X)} = \sqrt{\mu(K)} \, \|f\|_{\ell^\infty(X)} \leq \sqrt{\mu(K)} \, \|f\|_{L^\infty(K)} \,,$$

so that we can take $M = \sqrt{\mu(K)}$ in Proposition 1. $\square$

2 Estimates for Christoffel functions

If an orthonormal basis $\{p_j\}$ is known analytically, $\mathcal{C}_n(d\mu,K)$ can be estimated via the representation (2). Such estimates are scattered in the literature on multivariate approximation theory. There are several asymptotic results (cf., e.g., [2, 4, 15, 27, 28] and references therein), and few explicit bounds. Below, we recall or derive some bounds related to standard compact sets and standard measures.

2.1 Disk and ball

In [2] (cf. also [1]) Bos proved that for the measure with density $W_0(\boldsymbol{x})$ on the *d-dimensional unit euclidean ball* $K = B_d$, where

$$W_\lambda(\boldsymbol{x}) = (1-|\boldsymbol{x}|^2)^{\lambda-1/2}\,,\ \ \lambda > -\frac{1}{2}\,, \tag{8}$$

the following explicit estimate holds

$$\mathcal{C}_n(W_0(\boldsymbol{x})\,d\boldsymbol{x}, B_d) \le \sqrt{\frac{2}{\omega_d}\left(\binom{n+d}{d}+\binom{n+d-1}{d}\right)} = \mathcal{O}(n^{d/2})\,, \tag{9}$$

ω_d being the surface area of the unit sphere $S^d \subset \mathbb{R}^{d+1}$. Observe that (9) gives the exact order of growth, since for any measure $\int_K K_n(\boldsymbol{x},\boldsymbol{x})\,d\mu = N$ where $N = \dim(\mathbb{P}_n^d(K))$ and thus

$$\mathcal{C}_n(d\mu,K) \ge \sqrt{\frac{N}{\mu(K)}}\,. \tag{10}$$

Here $K = B_d$ is polynomial determining and hence $N = \binom{n+d}{d} = \mathcal{O}(n^d)$.

In [1] it is also recalled that by the method of [2] it can be proved that $\mathcal{C}_n$ has polynomial growth on the ball for all the measures with density $d\mu = W_\lambda(\boldsymbol{x})\,d\boldsymbol{x}$, $\lambda \ge 0$, for example for the Lebesgue measure $d\mu = d\boldsymbol{x}$, but neither bounds nor the order of growth are explicitly provided.

We work out in detail the case of the disk with the Lebesgue measure, $d = 2$ and $d\mu = d\boldsymbol{x}$, where a more direct approach can be conveniently used. Indeed, consider the Zernike polynomials, an orthogonal basis on the disk with respect to the Lebesgue measure, which is widely used in optics. We recall that the Zernike *orthonormal* basis is defined in polar coordinates as

$$\hat{Z}_h^m(r,\theta) = \begin{cases} \sqrt{\frac{2(h+1)}{\alpha_m}}\,R_h^m(r)\cos(m\theta)\,, & m \ge 0 \\ \sqrt{\frac{2(h+1)}{\alpha_m}}\,R_h^m(r)\sin(m\theta)\,, & m < 0 \end{cases} \tag{11}$$

for $0 \le h \le n$, $|m| \le h$, $h-m \in 2\mathbb{Z}$, where

$$\alpha_m = \begin{cases} 2\,, & m = 0 \\ 1\,, & m \ne 0 \end{cases} \tag{12}$$

$$R_h^m(r) = (-1)^{(h-m)/2} r^m P_{(h-m)/2}^{m,0}(1-2r^2) \tag{13}$$

and $P_j^{m,0}$ is the corresponding Jacobi polynomial of degree j. We refer the reader, e.g., to [9, 20] for the properties of Zernike polynomials. In particular, the relevant property to our purposes is that (cf., e.g., [9, Prop. 3.1])

$$|\hat{Z}_h^m(r,\theta)| \le \sqrt{\frac{2h+2}{\pi}}\,,\ \ \boldsymbol{x} = (r\cos(\theta), r\sin(\theta)) \in B_2\,,$$

for $0 \le h \le n$, $|m| \le h$, $h-m \in 2\mathbb{Z}$. Then,

$$K_n(\boldsymbol{x},\boldsymbol{x}) = \sum_{h=0}^{n} \sum_{|m|\le h, h-m\in 2\mathbb{Z}} (\hat{Z}_h^m(r,\theta))^2 \le \frac{1}{\pi}\sum_{h=0}^{n} \sum_{|m|\le h, h-m\in 2\mathbb{Z}} (2h+2)$$

$$= \frac{1}{\pi}\sum_{h=0}^{n}(2h+2)(n-h+1) = \frac{1}{3\pi}(n+1)(n+2)(n+3)\,,$$

and hence

$$\mathcal{C}_n(d\boldsymbol{x}, B_2) \le \frac{1}{\sqrt{3\pi}}\sqrt{(n+1)(n+2)(n+3)} = \mathcal{O}(n^{3/2})\,. \tag{14}$$

2.2 Square and cube

Consider now the case of the *d-dimensional cube*, $K = [-1,1]^d$, with a product Jacobi measure

$$d\mu = W_{\alpha,\beta}(\boldsymbol{x})\,d\boldsymbol{x}\ ,\ \ W_{\alpha,\beta}(\boldsymbol{x}) = \prod_{i=1}^{d}(1-x_i)^{\alpha}(1+x_i)^{\beta}\ ,\ \ \alpha,\beta > -1\ , \tag{15}$$

and the corresponding total-degree *orthonormal* product basis

$$\Pi_{\boldsymbol{k}}^{\alpha,\beta}(\boldsymbol{x}) = \prod_{i=1}^{d}\hat{P}_{k_i}^{\alpha,\beta}(x_i)\ ,\ \ 0 \le |\boldsymbol{k}| \le n\ , \tag{16}$$

where $\boldsymbol{k} = (k_1,\dots,k_d)$ with $k_i \ge 0$ and $|\boldsymbol{k}| = \sum_{i=1}^{d} k_i$, and $\hat{P}_m^{\alpha,\beta}$ denotes the m-th degree polynomial of the univariate orthonormal Jacobi basis with parameters α and β.

It is known that, for $\max\{\alpha,\beta\} \ge -1/2$, the maximum modulus of the orthonormal polynomials is attained at one of the endpoints ± 1, in particular

$$|\hat{P}_m^{\alpha,\beta}(t)| \le |\hat{P}_m^{\alpha,\beta}(\text{sign}(\alpha-\beta))| = \sqrt{\frac{(2m+\alpha+\beta+1)\Gamma(m+\alpha+\beta+1)\Gamma(m+q+1)}{2^{\alpha+\beta+1}\,m!\,\Gamma(m+\min\{\alpha,\beta\}+1)}}$$

$$\le c(\alpha,\beta)\,m^{q+1/2}\ ,\ \ t \in [-1,1]\ ,\ \ q = \max\{\alpha,\beta\} \ge -\frac{1}{2}\ , \tag{17}$$

with a suitable positive constant $c(\alpha,\beta)$; cf., e.g., [23, Ch. X] and references therein.

By (17) we can estimate the function $K_n(\boldsymbol{x},\boldsymbol{x})$,

$$\max_{\boldsymbol{x}\in[-1,1]^d} K_n(\boldsymbol{x},\boldsymbol{x}) = \max_{\boldsymbol{x}\in[-1,1]^d}\sum_{0\le|\boldsymbol{k}|\le n}\left(\Pi_{\boldsymbol{k}}^{\alpha,\beta}(\boldsymbol{x})\right)^2$$

$$= \sum_{0\le|\boldsymbol{k}|\le n}\prod_{i=1}^{d}\left(\hat{P}_{k_i}^{\alpha,\beta}(\text{sign}(\alpha-\beta))\right)^2 \le (c(\alpha,\beta))^{2d}\sum_{0\le|\boldsymbol{k}|\le n}\prod_{i=1}^{d}k_i^{2q+1}$$

$$= (c(\alpha,\beta))^{2d}\sum_{k_1=0}^{n}k_1^{2q+1}\sum_{k_2=0}^{n-k_1}k_2^{2q+1}\cdots\sum_{k_d=0}^{n-\sum_{j=1}^{d-1}k_j}k_d^{2q+1} = \mathcal{O}(n^{(2q+2)d})\ ,$$

which gives the qualitative bound

$$\mathcal{C}_n(W_{\alpha,\beta}(\boldsymbol{x})\,d\boldsymbol{x},[-1,1]^d) = \mathcal{O}(n^{(q+1)d})\ . \tag{18}$$

In the case of the Lebesgue measure ($\alpha = \beta = 0$), it is worth recalling that (18) is also a consequence of a general result by Bos and Milman [5, Thm. 7.1], concerning generalized Markov type inequalities. Such a result, specialized to the present context, says that there is a universal constant $c > 0$ such that for every $p \in \mathbb{P}_n^d([-1,1]^d)$

$$\left\|\frac{\partial^{k_1}}{\partial x_1^{k_1}}\cdots\frac{\partial^{k_d}}{\partial x_d^{k_d}}\,p\right\|_{L^{p_2}([-1,1]^d)} \le 8^d(cn^2)^{|\boldsymbol{k}|+d\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|p\|_{L^{p_1}([-1,1]^d)} \tag{19}$$

for $1 \le p_1 < p_2 \le \infty$, so that for $\boldsymbol{k} = \boldsymbol{0}$, $p_1 = 2$ and $p_2 = \infty$ we get the bound

$$\mathcal{C}_n(d\boldsymbol{x},[-1,1]^d) \le (8\sqrt{c}\,n)^d\ . \tag{20}$$

In specific instances, it is possible to compute exactly $\mathcal{C}_n$. We work out in detail three relevant cases: $W_{0,0} \equiv 1$ (the Lebesgue measure), $W_{-\frac{1}{2},-\frac{1}{2}}$ (product Chebyshev measure of the first kind), and $W_{\frac{1}{2},\frac{1}{2}}$ (product Chebyshev measure of the second kind), in dimensions $d = 1,2,3$.

Concerning the Lebesgue measure, we start from the following bound for the univariate Legendre *orthonormal* polynomials (cf., e.g., [19, Ch. 18])

$$|\hat{P}_m^{0,0}(t)| \le \hat{P}_m^{0,0}(1) = \sqrt{\frac{2m+1}{2}}\ ,\ \ t \in [-1,1]\ ,$$

from which we have

$$\max_{\boldsymbol{x}\in[-1,1]^d} K_n(\boldsymbol{x},\boldsymbol{x}) = \sum_{0\le|\boldsymbol{k}|\le n}\prod_{i=1}^{d}\left(\hat{P}_{k_i}^{0,0}(1)\right)^2$$

$$= \frac{1}{2^d}\sum_{k_1=0}^{n}(2k_1+1)\sum_{k_2=0}^{n-k_1}(2k_2+1)\cdots\sum_{k_d=0}^{n-\sum_{j=1}^{d-1}k_j}(2k_d+1)\ , \tag{21}$$

and then by easy calculations

$$\mathcal{C}_n(dx,[-1,1]) = \frac{1}{\sqrt{2}}(n+1)\,, \tag{22}$$

$$\mathcal{C}_n(d\boldsymbol{x},[-1,1]^2) = \frac{1}{2\sqrt{6}}\sqrt{(n+1)(n+2)(n^2+3n+3)}\,, \tag{23}$$

$$\mathcal{C}_n(d\boldsymbol{x},[-1,1]^3) = \frac{1}{12\sqrt{10}}\sqrt{(n+1)(n+2)^2(n+3)(2n^2+8n+15)} \tag{24}$$

(here and below, the one-dimensional instances are well-known and reported only for completeness).

In the case of the Chebyshev measure of the first kind, $\alpha=\beta=-\frac{1}{2}$, consider the following bound for the univariate orthonormal Chebyshev polynomials (cf., e.g., [16])

$$|\hat{P}_m^{-\frac{1}{2},-\frac{1}{2}}(t)| = |\hat{T}_m(t)| \le \hat{T}_m(1) = \sqrt{\frac{2-\delta_{0,m}}{\pi}}\,,\quad t\in[-1,1]\,,$$

which entails by a little algebra

$$\pi^d \max_{\boldsymbol{x}\in[-1,1]^d} K_n(\boldsymbol{x},\boldsymbol{x}) = \pi^d \sum_{0\le|\boldsymbol{k}|\le n}\prod_{i=1}^d \big(\hat{T}_{k_i}(1)\big)^2$$

$$= \sum_{k_1=0}^{n}(2-\delta_{0,k_1})\sum_{k_2=0}^{n-k_1}(2-\delta_{0,k_2})\cdots\sum_{k_2=0}^{n-\sum_{j=1}^{d-1}k_j}(2-\delta_{0,k_d})$$

$$= c_{d,0} + c_{d,1}\sum_{k_1=1}^{n}1 + c_{d,2}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n-k_1}1 + \cdots + c_{d,d}\sum_{k_1=1}^{n}\cdots\sum_{k_d=1}^{n-\sum_{j=1}^{d-1}k_j}1$$

$$= c_{d,0} + c_{d,1}\sum_{h_1=1}^{n}1 + c_{d,2}\sum_{h_1=1}^{n}\sum_{h_2=1}^{h_1-1}1 + \cdots + c_{d,d}\sum_{h_1=1}^{n}\cdots\sum_{h_d=1}^{h_{d-1}-1}1\,, \tag{25}$$

by the change of variables $h_i = n+1-\sum_{j=1}^{i}k_j$, $1\le i\le d$, where $c_{d,0}=1$, $c_{d,d}=2^d$ and $\{c_{d,i}\}$ satisfies the recurrence relation

$$c_{d+1,i} = c_{d,i} + 2c_{d,i-1}\,,\quad i=1,\dots,d\,,\quad d\ge 1\,. \tag{26}$$

The last row in (25) shows that the i-th nested summation counts the number of subsets of $\{1,\dots,n\}$ with cardinality i, that is $\binom{n}{i}$, so finally we get

$$\mathcal{C}_n(W_{-\frac{1}{2},-\frac{1}{2}}(\boldsymbol{x})\,d\boldsymbol{x},[-1,1]^d) = \frac{1}{\pi^{d/2}}\sqrt{\sum_{i=0}^{d}c_{d,i}\binom{n}{i}} \tag{27}$$

and in particular

$$\mathcal{C}_n(W_{-\frac{1}{2},-\frac{1}{2}}(x)\,dx,[-1,1]) = \frac{1}{\sqrt{\pi}}\sqrt{2n+1} \tag{28}$$

(observe that (28) coincides with the bound in (9) for $d=1$),

$$\mathcal{C}_n(W_{-\frac{1}{2},-\frac{1}{2}}(\boldsymbol{x})\,d\boldsymbol{x},[-1,1]^2) = \frac{1}{\pi}\sqrt{2n^2+2n+1}\,, \tag{29}$$

$$\mathcal{C}_n(W_{-\frac{1}{2},-\frac{1}{2}}(\boldsymbol{x})\,d\boldsymbol{x},[-1,1]^3) = \frac{1}{\sqrt{3\pi^3}}\sqrt{4n^3+6n^2+8n+3}\,. \tag{30}$$

Concerning the product Chebyshev measure of the second kind, $\alpha=\beta=\frac{1}{2}$, we start from the following bound for the corresponding univariate orthonormal polynomials (cf., e.g., [16])

$$|\hat{P}_m^{\frac{1}{2},\frac{1}{2}}(t)| = |\hat{U}_m(t)| \le \hat{U}_m(1) = \sqrt{\frac{2}{\pi}}(m+1)\,,\quad t\in[-1,1]\,,$$

which leads to

$$\max_{\boldsymbol{x}\in[-1,1]^d} K_n(\boldsymbol{x},\boldsymbol{x}) = \sum_{0\le|\boldsymbol{k}|\le n}\prod_{i=1}^d \big(\hat{U}_{k_i}(1)\big)^2$$

$$= \left(\frac{2}{\pi}\right)^d \sum_{k_1=0}^{n}(k_1+1)^2\sum_{k_2=0}^{n-k_1}(k_2+1)^2\cdots\sum_{k_d=0}^{n-\sum_{j=1}^{d-1}k_j}(k_d+1)^2\,, \tag{31}$$

and by easy calculations

$$\mathcal{C}_n(W_{\frac{1}{2},\frac{1}{2}}(x)\,dx,[-1,1]) = \frac{1}{\sqrt{3\pi}}\sqrt{(n+1)(n+2)(2n+3)}\ , \tag{32}$$

$$\mathcal{C}_n(W_{\frac{1}{2},\frac{1}{2}}(\boldsymbol{x})\,d\boldsymbol{x},[-1,1]^2) = \frac{1}{3\pi\sqrt{10}}\sqrt{P_6(n)}\ ,$$

$$P_6(n) = (n+1)(n+2)(n+3)(n+4)(2n^2+10n+15)\,, \tag{33}$$

$$\mathcal{C}_n(W_{\frac{1}{2},\frac{1}{2}}(\boldsymbol{x})\,d\boldsymbol{x},[-1,1]^3) = \frac{1}{18\sqrt{35\pi^3}}\sqrt{P_9(n)}\ ,$$

$$P_9(n) = (n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(2n+7)(n^2+7n+18)\,. \tag{34}$$

2.3 Upper bounds for Lebesgue constants

Corollary 1 allows to provide upper bounds for the uniform norm of *any* hyperinterpolation operator, i.e. for its “Lebesgue constant”, independent of the underlying cubature formula, whenever we are able to estimate the maximum of $K_n(\boldsymbol{x},\boldsymbol{x})$. Such bounds are typically overestimates of the actual order of growth.

The fact that Lebesgue constants of hyperinterpolation can be estimated independently of the sampling nodes is not surprising. Indeed, recently Wade in [24] has proved that for hyperinterpolation operators with respect to Gegenbauer measures (8) on the *d-dimensional ball*, the following bilateral bound holds

$$a_{d,\lambda}n^{(d-1)/2+\lambda} \le \|\mathcal{L}_n\|_\infty \le b_{d,\lambda}n^{(d-1)/2+\lambda}\ ,\ \ n\ even\ ,\ \ d>1\,, \tag{35}$$

where $a_{d,\lambda}$ and $b_{d,\lambda}$ are positive constants. Such a result improves a previous upper bound $\mathcal{O}(n\log(n))$ for hyperinterpolation with respect to the Lebesgue measure on the disk ($\lambda=1/2$, $d=2$), cf. [13]. Incidentally, we observe that differently from what is stated in [24] the upper bound cannot hold for $\lambda=0$ and $d=1$, since it would imply that the Lebesgue constant is bounded, which is certainly false since any polynomial projection operator has a uniform norm that increases at least logarithmically in the degree (cf. [21]).

If $\lambda=0$, Corollary 1 gives via (9) $\|\mathcal{L}_n\|_\infty=\mathcal{O}(n^{d/2})$, that is in view of [24] an overestimate of the actual order of growth by a factor $\sqrt{n}$ (for any d). On the other hand, in the case of the Lebesgue measure on the disk ($\lambda=1/2$, $d=2$), by (7) and (14) we get $\|\mathcal{L}_n\|_\infty=\mathcal{O}(n^{3/2})$, again an overestimate by a factor $\sqrt{n}$.

Concerning the *d-dimensional cube*, a quite recent result has given an affirmative answer to a conjecture stated in [8] for $d=3$, namely that for *any* hyperinterpolation operator with respect to the Chebyshev measure of the first kind $d\mu=W_{-1/2,-1/2}(\boldsymbol{x})\,d\boldsymbol{x}$ (cf. (15)), the following estimate holds

$$\|\mathcal{L}_n\|_\infty=\mathcal{O}(\log^d(n))\,. \tag{36}$$

An estimate of this kind was previously obtained in the case of hyperinterpolation at the Morrow-Patterson-Xu points of the square [7]. Observe that such a growth order of the Lebesgue constant is optimal, in view of the general result for polynomial projection operators in [21]. In the context of the present note, Corollary 1 gives via (27) $\|\mathcal{L}_n\|_\infty=\mathcal{O}(n^{d/2})$, that is an ovestimate of the actual order of growth by a factor $(\sqrt{n}/\log(n))^d$.

For other Jacobi measures, there are apparently no results in the literature for $d>1$. Again, Corollary 1 and (27) provide the estimate

$$\|\mathcal{L}_n\|_\infty=\mathcal{O}(n^{(q+1)d})\,, \tag{37}$$

for *any* hyperinterpolation operator with respect to *any* Jacobi measure with $q=\max\{\alpha,\beta\}\ge -1/2$. The fact that (37) is an overestimate of the Lebesgue constant is manifest in dimension one, where it is well-known that the Lebesgue constant of interpolation at the zeros of $P_{n+1}^{\alpha,\beta}$ increases asymptotically like $\log(n)$ for $q\le -1/2$, and like $n^{q+1/2}$ for $q>-1/2$, in view of a classical result by Szëgo; cf. [17] and references therein. Indeed, in such cases the hyperinterpolation operator based on the Gauss-Jacobi formula is just an interpolation operator at the Gauss-Jacobi nodes, cf. [22, Lemma 3].

We specialize now (37) to the case of the product Chebyshev measure of the second kind, and in particular to *interpolation at the Morrow-Patterson points* of the square, originally studied in [18]. We recall that, for *even* degree n, such points are the set $\{(x_m,y_k)\}\subset(-1,1)^2$ defined as

$$x_m=\cos\left(\frac{m\pi}{n+2}\right)\ ,\ \ y_k=\begin{cases}\cos\left(\frac{2k\pi}{n+3}\right) & m\ odd\\[2ex] \cos\left(\frac{(2k-1)\pi}{n+3}\right) & m\ even\end{cases} \tag{38}$$

$1\le m\le n+1$, $1\le k\le \frac{n}{2}+1$. This set consists of $N=\binom{n+2}{2}=\dim(\mathbb{P}_n^2)$ points, and is unisolvent for polynomial interpolation on the square. Indeed, by the bivariate Christoffel-Darboux formula [26], the corresponding fundamental Lagrange polynomials have an explicit expression in terms of second kind Chebyshev polynomials, i.e., the interpolation problem has a constructive solution.

On the other hand, the Morrow-Patterson points support one of the few known minimal positive cubature formulas, namely a formula that has degree of exactness $2n$ for the product Chebyshev measure of the second kind, $d\mu=W_{\frac{1}{2},\frac{1}{2}}(x_1,x_2)\,dx_1dx_2$, cf. [18]. Hence, we can construct an hyperinterpolation polynomial of degree not greater than n at these points, that in view of minimality of the formula turns out to be the interpolation polynomial, by [22, Lemma 3].

Concerning the growth of the Lebesgue constant, say Λ_n^{MP}, Bos [3] proved that $\Lambda_n^{MP} = \mathcal{O}(n^6)$, by means of the bivariate Christoffel-Darboux formula. Now, interpreting interpolation at the Morrow-Patterson points as hyperinterpolation, we can state the following

Proposition 2.1. *The Lebesgue constant of bivariate polynomial interpolation at the Morrow-Patterson points has the following upper bound*

$$\Lambda_n^{MP} \leq \frac{1}{6\sqrt{10}} \sqrt{(n+1)(n+2)(n+3)(n+4)(2n^2+10n+15)} = \mathcal{O}(n^3) \,. \tag{39}$$

Proof. As observed above, the hyperinterpolation operator at the Morrow-Patterson points is just the interpolation operator. Then we obtain (39), which is valid for *any* hyperinterpolation operator with respect to the product Chebyshev measure of the second kind, in view of Corollary 1 and (33), recalling that $\mu([-1,1]^2) = \pi^2/4$. □

On the other hand, we have numerical evidence that (39) is an overestimate of the order of growth, see Figure 1. Indeed, the numerical results in [6] show that the values of Λ_n^{MP} in the range $n = 2, 4, 6, \ldots, 60$ are well-fitted by the quadratic polynomial $(0.7n+1)^2$, so it can be conjectured that the actual order of growth is $\Lambda_n^{MP} = \mathcal{O}(n^2)$.

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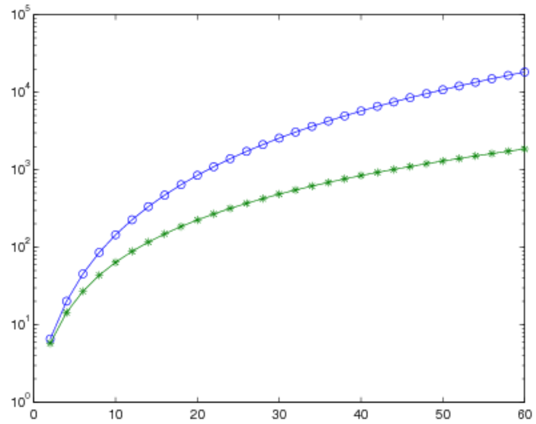

Figure 1: The upper bound (39) (∘) and the numerically evaluated Lebesgue constant (∗) of interpolation at the Morrow-Patterson points (log scale).

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