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On tilted Giraud subcategories

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- Dedicated to Alberto Facchini on the occasion of his sixtieth birthday -

Abstract

Firstly we provide a technique to move torsion pairs in abelian categories via adjoint functors and in particular through Giraud subcategories. We apply this point in order to develop a correspondence between Giraud subcategories of an abelian category $\mathcal{C}$ and those of its tilt $\mathcal{H}(\mathcal{C})$ i.e., the heart of a t-structure on the derived category $D(\mathcal{C})$ induced by a torsion pair.

Keywords:
Abelian categories, Giraud subcategories, derived categories, t-structures, torsion pairs, tilting

Introduction

One of the most useful process in abelian category theory is the so-called localization of an abelian category $\mathcal{D}$ to a quotient category $\mathcal{D}/\mathcal{S}$ by means of a Serre class $\mathcal{S}$ in $\mathcal{D}$. When $\mathcal{S}$ is a localizing subcategory in the sense of [**?**], the canonical exact functor $\mathcal{D} \to \mathcal{D}/\mathcal{S}$ has a fully faithful right adjoint functor $S : \mathcal{D}/\mathcal{S} \to \mathcal{D}$ which allows to deal with $\mathcal{D}/\mathcal{S}$ as a full subcategory of $\mathcal{D}$, which is called a Giraud subcategory of $\mathcal{D}$. Dualizing the context, one get the notion of a co-Giraud subcategory. Giraud and co-Giraud subcategories very often appear in the literature in very different settings (see 1.3).

On the other side, in 1981 Beilinson, Bernstein and Deligne introduced the notion of t-structure on a triangulated category related to the study of the derived category of constructible sheaves on a stratified space. Actually the notion of t-structure is a generalization of the notion of torsion pair on an abelian category (see for example [**?**]). In their work [**?**] Happel, Reiten and Smalø related the study of torsion pairs to Tilting theory and t-structures. In particular given an abelian category $\mathcal{C}$ one can construct many non-trivial t-structures on its derived category $D^b(\mathcal{C})$ by the procedure of tilting at a torsion pair (see 4.5).

Inspired by the fundamental role of localizing subcategories in the study of problems of gluing abelian categories or even triangulated categories we propose in this work a bridge between the two previous abstract contexts. The main progress in the present paper is to show how the process of (co-) localizing moves from a basic abelian category to the level of its tilt, with respect to a torsion pair, and viceversa.

On the one side we deal with a (co-) Giraud subcategory $\mathcal{C}$ of $\mathcal{D}$, looking the way torsion pairs on $\mathcal{D}$ reflect on $\mathcal{C}$ and, conversely, torsion pairs on $\mathcal{C}$ extend to $\mathcal{D}$: in particular we find a one to one correspondence between arbitrary torsion pairs $(\mathcal{T}, \mathcal{F})$ on $\mathcal{C}$ and the torsion pairs $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{D}$ which are "compatible" with the (co-) localizing functor (Theorems 3.4 and 3.5).

On the other side, we compare this action of "moving" torsion pairs from $\mathcal{D}$ to $\mathcal{C}$ (and viceversa) with a "tilting context": more precisely, we look at the associated hearts $\mathcal{H}_\mathcal{D}$ and $\mathcal{H}_\mathcal{C}$ with respect to the torsion pairs $(\mathcal{T}, \mathcal{F})$ on $\mathcal{C}$ and $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{D}$, respectively, proving that $\mathcal{H}_\mathcal{C}$ is still a (co-) Giraud subcategory of $\mathcal{H}_\mathcal{D}$, and that the "tilted" torsion pairs in the two hearts are still related (Theorem 5.6). Here the ambient abelian category $\mathcal{D}$ is arbitrary, with the unique request that the inclusion functor of C into $\mathcal{D}$ admits a right derived functor as an absolute Kan extension (see Definition 5.2).

Finally given any abelian category $\mathcal{D}$ endowed with a torsion pair $(\mathcal{X}, \mathcal{Y})$, and considering any Giraud subcategory C' of the associated heart $\mathcal{H}_{\mathcal{D}}$ which is "compatible" with the "tilted" torsion pair on $\mathcal{H}_{\mathcal{D}}$, we prove in Theorem 5.9 how to recover a Giraud subcategory C of $\mathcal{D}$ such that C' is equivalent to the heart $\mathcal{H}_C$ (with respect to the induced torsion pair).

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1. Serre, Giraud and co-Giraud subcategories

We begin by fixing some notations on Serre, Giraud and co-Giraud subcategories. A complete account on quotient categories and Serre classes can be found in [**?** , Chapter 3] and [**?** , Section 1.11].

Definition 1.1. *Let $\mathcal{D}$ be an abelian category. A Serre class $\mathcal{S}$ in $\mathcal{D}$ is a full subcategory $\mathcal{S}$ of $\mathcal{D}$ such that for any short exact sequence $0\to X_1\to X_2\to X_3\to 0$ in $\mathcal{D}$ the middle term X_2 belongs to $\mathcal{S}$ if and only if X_1, X_3 belong to $\mathcal{S}$.*

The data of an abelian category $\mathcal{D}$ and a Serre class $\mathcal{S}$ of $\mathcal{D}$ allow to construct a new abelian category, denoted by $\mathcal{D}/\mathcal{S}$, called the *quotient category of $\mathcal{D}$ by $\mathcal{S}$* (see [**?**]). It turns out that $\mathcal{D}/\mathcal{S}$ is abelian and the canonical functor $T\colon \mathcal{D} \to \mathcal{D}/\mathcal{S}$ is exact. A Serre class $\mathcal{S}$ in $\mathcal{D}$ is called a *localizing subcategory* (resp. *co-localizing subcategory*) if the functor T admits a right adjoint (resp. left adjoint) *section functor* S. In this case, the left exact (resp. right exact) functor $S \circ T$ is called the *localization functor*. This localization functor is exact if and only if S is exact (see [**?** , Chapter 3]).

Definition 1.2. *An abelian category with a distinguished Giraud subcategory is the data $(\mathcal{D}, C, \ell, i)$ of two abelian categories $\mathcal{D}$ and C and two adjoint functors $C \underset{i}{\overset{\ell}{\leftrightarrows}} \mathcal{D}$ (with ℓ left adjoint of i) such that ℓ is exact and i fully faithful.*

Dually an abelian category with a distinguished co-Giraud subcategory is the data $(\mathcal{D}, C, j, r)$ of two abelian categories $\mathcal{D}$ and C and two adjoint functors $\mathcal{D} \underset{r}{\overset{j}{\leftrightarrows}} C$ (with j left adjoint of r) such that r is exact and j fully faithful.

Therefore a localizing Serre subcategory $\mathcal{S}$ of $\mathcal{D}$ defines a distinguished Giraud subcategory $(\mathcal{D}, \mathcal{D}/\mathcal{S}, T, S)$. Conversely, given a distinguished Giraud subcategory $(\mathcal{D}, C, \ell, i)$, the kernel of the functor ℓ, i.e., the full subcategory $\mathcal{S}$ of $\mathcal{D}$ whose objects S in $\mathcal{S}$ satisfy $\ell(S) \cong 0$, defines a localizing Serre subcategory $\mathcal{S} = \operatorname{Ker}(\ell)$ of $\mathcal{D}$ whose associated quotient category is (equivalent to) C.

Let us denote by $\eta\colon \mathrm{id}_D \to i \circ \ell$ (resp. $\varepsilon : \ell \circ i \to \mathrm{id}_C$) the unit (resp. the counit) of the adjunction (ℓ, i), and by $\mathcal{S}^\perp$ the full subcategory of $\mathcal{D}$ whose objects are defined by:

$$\mathcal{S}^\perp := \{D \in \mathcal{D} \,|\, \mathcal{D}(S, D) = 0, \forall S \in \mathcal{S}\}.$$

Let us notice that since i is fully faithful the counit of the adjunction ε is an isomorphism of functors. In particular for any $D \in \mathcal{D}$, we have that $\ell(\eta(D)) = \varepsilon^{-1}_{\ell(D)}$ is an isomorphism. It turns out that

$$\mathcal{S}^\perp = \{D \in \mathcal{D} \,|\, \eta_D : D \to i\ell(D) \text{ is a monomorphism}\}. \tag{1}$$

Indeed for any $D \in \mathcal{D}$ we have $\operatorname{Ker}(\eta_D) \in \mathcal{S}$ (since $\ell(\operatorname{Ker}(\eta_D)) = \operatorname{Ker}(\ell(\eta_D)) = 0$ because $\ell(\eta_D)$ is an isomorphism) and hence given $X \in \mathcal{S}^\perp$ the kernel map $\operatorname{Ker}(\eta_X) \hookrightarrow X$ is zero and so η_X is a monomorphism. On the other hand let us consider $D \in \mathcal{D}$ such that η_D is a monomorphism; then for any object S in $\mathcal{S}$ (i.e., $\ell(S) = 0$) we have $\mathcal{D}(S, D) \subseteq \mathcal{D}(S, i\ell(D)) \cong C(\ell(S), \ell(D)) = 0$ and so $D \in \mathcal{S}^\perp$.

Dually, starting from a distinguished co-Giraud subcategory $(\mathcal{D}, C, j, r)$, and denoting by $\varepsilon\colon j \circ r \to \mathrm{id}_D$ the counit of the adjunction (j, r), the kernel of the functor r defines a co-localizing subcategory $\mathcal{S} = \mathrm{Ker}(r)$ of D such that

$$\begin{aligned} {}^{\perp}\mathcal{S} :=& \ \{D \in \mathcal{D} \,|\, \mathcal{D}(D, S) = 0, \forall S \in \mathcal{S}\} \\ =& \ \{D \in \mathcal{D} \,|\, \varepsilon_D : jr(D) \to D \text{ is an epimorphism}\}. \end{aligned}$$

Moreover, since j is fully faithful, the unit of the adjunction $\eta : \mathrm{id}_C \to r \circ j$ is an isomorphism of functors.

Remark 1.3. Giraud and co-Giraud subcategories very often appear in the literature in very different settings. For example a well known result due to Popescu and Gabriel (see, for instance, [**?** , Chapter 10]) tells that any Grothendieck category is in a natural way a Giraud subcategory of the category R-Mod of all the left R-modules, for a suitable ring R. On the other hand, the Yoneda tensor-embedding $M \mapsto - \otimes_R M$ naturally makes R-Mod to be a co-Giraud subcategory of the Grothendieck category $(\mathrm{FP_R}, \mathrm{Ab})$ whose objects are the covariant functors from the finitely presented right R-modules to the abelian groups, and the morphisms are the natural transformations between them. This allows, for instance, to deal with the extensively studied notion of pure-injective module by means of injective objects in $(\mathrm{FP_R}, \mathrm{Ab})$, thanks to a result of Gruson and Jensen [**?**]. Dually, the Yoneda Hom-embedding $M \mapsto \mathrm{Hom}_R(-, M)$ naturally makes R-Mod to be a Giraud subcategory of the Grothendieck category of contravariant functors $({}_R\mathrm{FP}^{\mathrm{op}}, \mathrm{Ab})$. Auslander proposed to study the representation theory of R in terms of the ambient category $({}_R\mathrm{FP}^{\mathrm{op}}, \mathrm{Ab})$, and in [**?**] and [**?**] he and Reiten studied deeper the subcategory of the finitely presented objects of $({}_R\mathrm{FP}^{\mathrm{op}}, \mathrm{Ab})$, developing the powerful theory of almost split sequences for Artin algebras.

2. Torsion and Torsion-free Classes

Definition 2.1. *Given an abelian category C, a torsion class $\mathcal{T}$ in C is a full subcategory of C which is closed under taking (existing) coproducts, quotients and extensions. Dually a torsion free class $\mathcal{F}$ in C is a full subcategory of C which is closed under taking (existing) products, subobjects and extensions.*

A torsion pair $(\mathcal{T}, \mathcal{F})$ in C is the data of a torsion class $\mathcal{T}$ and a torsion free class $\mathcal{F}$ such that $C(\mathcal{T}, \mathcal{F}) = 0$ and any object $C \in C$ is the middle term of a short exact sequence $0 \to T \to C \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

A torsion class $\mathcal{T}$ *cogenerates* C when any object in C is a subobject of a suitable object in $\mathcal{T}$, and, dually, a torsion free class $\mathcal{F}$ *generates* C when any object in C is a factor of a suitable object in $\mathcal{F}$. Typically, cogenerating torsion classes arise from Tilting theory and generating torsion free classes arise from Cotilting theory (see, for instance, [**?** , Chapter I.3] and [**?** , Section 2]).

Remark 2.2. If C is a subcomplete abelian category in the sense of [**?**] (that is, C is an abelian category such that for any family $\{A_u \,|\, u \in U\}$ of subobjects of a fixed object A, the infinite sum $\sum_{u \in U} A_u$ and the infinite product $\prod_{u \in U}(A/A_u)$ exist in C), then any torsion class $\mathcal{T}$ (torsion-free class $\mathcal{F}$) on C induces a torsion pair $(\mathcal{T}, \mathcal{F})$ on C.

The reader is referred to [**?** , Chapter 1] for more details.

In what follows, our aim is to move torsion classes trough exact functors and subsequently trough a distinguished Giraud (resp. co-Giraud) subcategory C of $\mathcal{D}$. Since any torsion class (resp. torsion free class) is closed under coproducts and quotients (resp. products and subobjects), it seems to us natural to use the left (resp. right) adjoint functor ℓ (resp. i) in order to move torsion classes (resp. torsion free classes) from C to $\mathcal{D}$ (resp. from $\mathcal{D}$ to C).

Lemma 2.3. *(Dual to 2.4). Let C be an abelian category and $\mathcal{T}$ a torsion class on C. Let $\ell : \mathcal{D} \to C$ be a functor between abelian categories which respects arbitrary colimits. Then the class*

$$\ell^{\leftarrow}(\mathcal{T}) = \{D \in \mathcal{D} \mid \ell(D) \in \mathcal{T}\}$$

is a torsion class in $\mathcal{D}$.

Proof. Clearly, the class $\ell^{\leftarrow}(\mathcal{T})$ is closed under taking coproducts and quotients, because so is $\mathcal{T}$ and ℓ respects arbitrary colimits by assumption. Let us show that $\ell^{\leftarrow}(\mathcal{T})$ is closed under extensions. Consider a short exact sequence in $\mathcal{D}$

$$0 \longrightarrow X_1 \longrightarrow D \longrightarrow X_2 \longrightarrow 0$$

with $X_1, X_2 \in \ell^{\leftarrow}(\mathcal{T})$. By applying the functor ℓ (which is right exact) to this sequence we get an exact sequence in $\mathcal{C}$

$$\ell(X_1) \longrightarrow \ell(D) \longrightarrow \ell(X_2) \longrightarrow 0$$

with $\ell(X_1), \ell(X_2) \in \mathcal{T}$. Taking the kernel K of the morphism $\ell(D) \to \ell(X_2)$, we see that K is an epimorphic image of $\ell(X_1)$ and so $K \in \mathcal{T}$, therefore $\ell(D) \in \mathcal{T}$ as extension of objects in a torsion class. We conclude that $D \in \ell^{\leftarrow}(\mathcal{T})$. □

Lemma 2.4. *(Dual to 2.3). Let $\mathcal{C}$ be an abelian category and $\mathcal{F}$ a torsion-free class on $\mathcal{C}$. Let $r : \mathcal{D} \to \mathcal{C}$ be a functor between abelian categories which respects arbitrary limits. Then the class*

$$r^{\leftarrow}(\mathcal{F}) = \{D \in \mathcal{D} \mid r(D) \in \mathcal{F}\}$$

is a torsion-free class in $\mathcal{D}$.

3. Moving Torsion Pairs trough Giraud subcategories

Given an abelian category $\mathcal{D}$ with a distinguished Giraud subcategory $\mathcal{C}$, by Lemma 2.3, the class $\ell^{\leftarrow}(\mathcal{T}) := \{D \in \mathcal{D} \mid \ell(D) \in \mathcal{T}\}$ is a torsion class on $\mathcal{D}$.

Proposition 3.1. *Let $\mathcal{D}$ be an abelian category with a distinguished Giraud subcategory $\mathcal{C}$. Suppose that $\mathcal{C}$ is endowed with a torsion pair $(\mathcal{T}, \mathcal{F})$. Then the classes $(\hat{\mathcal{T}}, \hat{\mathcal{F}})$:*

$$\hat{\mathcal{T}} := \ell^{\leftarrow}(\mathcal{T}) = \{X \in \mathcal{D} \mid \ell(X) \in \mathcal{T}\}$$
$$\hat{\mathcal{F}} := \ell^{\leftarrow}(\mathcal{F}) \cap \mathcal{S}^{\perp} = \{Y \in \mathcal{D} \mid Y \in \mathcal{S}^{\perp} \text{ and } \ell(Y) \in \mathcal{F}\}$$

define a torsion pair on $\mathcal{D}$ such that $i(\mathcal{T}) \subseteq \hat{\mathcal{T}}$, $i(\mathcal{F}) \subseteq \hat{\mathcal{F}}$, $\ell(\hat{\mathcal{T}}) = \mathcal{T}$, $\ell(\hat{\mathcal{F}}) = \mathcal{F}$.

Proof. For any $T \in \mathcal{T}$ we have $\ell i(T) \cong T$, which proves that $i(\mathcal{T}) \subseteq \hat{\mathcal{T}}$. Moreover given $F \in \mathcal{F}$ it is clear that $i(F) \in \mathcal{S}^{\perp}$ and $\ell i(F) \cong F \in \mathcal{F}$, hence $i(\mathcal{F}) \subseteq \hat{\mathcal{F}}$. We deduce that $\mathcal{T} = \ell i(\mathcal{T}) \subseteq \ell(\hat{\mathcal{T}}) \subseteq \mathcal{T}$ and $\mathcal{F} = \ell i(\mathcal{F}) \subseteq \ell(\hat{\mathcal{F}}) \subseteq \mathcal{F}$, which prove that $\ell(\hat{\mathcal{T}}) = \mathcal{T}$ and $\ell(\hat{\mathcal{F}}) = \mathcal{F}$. Let us show that $(\hat{\mathcal{T}}, \hat{\mathcal{F}})$ is a torsion pair on $\mathcal{D}$.

Given $X \in \hat{\mathcal{T}}$ and $Y \in \hat{\mathcal{F}}$,

$$\mathcal{D}(X, Y) \hookrightarrow \mathcal{D}(X, i\ell(Y)) \cong \mathcal{C}(\ell(X), \ell(Y)) = 0$$

where the first inclusion holds since $Y \in \hat{\mathcal{F}} \subseteq \mathcal{S}^{\perp}$ and $\mathcal{S}^{\perp} = \{D \in \mathcal{D} \mid \eta_D : D \to i\ell(D) \text{ is a monomorphism}\}$ by (1). It remains to prove that for any D in $\mathcal{D}$ there exists a short exact sequence

$$0 \longrightarrow X \longrightarrow D \longrightarrow Y \longrightarrow 0$$

with $X \in \hat{\mathcal{T}}$ and $Y \in \hat{\mathcal{F}}$.

Given D in $\mathcal{D}$ there exist $T \in \mathcal{T}$ and $F \in \mathcal{F}$ such that the sequence

$$0 \longrightarrow T \longrightarrow \ell(D) \longrightarrow F \longrightarrow 0 \tag{2}$$

is exact in $\mathcal{C}$. Let define $X := i(T) \times_{i\ell(D)} D$; then we obtain the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i(T) & \longrightarrow & i\ell(D) & \longrightarrow & i(F) & & \\ & & \uparrow & & \uparrow{\scriptstyle \eta_D} & & \uparrow & & \\ 0 & \longrightarrow & X & \longrightarrow & D & \longrightarrow & D/X & \longrightarrow & 0 \end{array} \tag{3}$$

whose rows are exact (the first because the functor i is left exact since it is a right adjoint, while the second by definition) and the map $D/X \hookrightarrow i(F)$ is injective since the first square is cartesian.

Let us apply the functor ℓ to (3) remembering that ℓ is exact (so in particular it preserves pullbacks and exact sequences) and that $\ell \circ i \cong \mathrm{id}_C$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & \ell(D) & \longrightarrow & F & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle \mathrm{id}_{\ell(D)}} & & \uparrow{\scriptstyle\cong} & & \\ 0 & \longrightarrow & \ell(X) & \longrightarrow & \ell(D) & \longrightarrow & \ell(D/X) & \longrightarrow & 0. \end{array}$$

The first row coincides with (2) which is exact, $\ell(X) \cong T \times_{\ell(D)} \ell(D) \cong T \in \mathcal{T}$, which proves that $X \in \hat{\mathcal{T}}$ and so $\ell(D/X) \cong F \in \mathcal{F}$, and the third vertical arrow of (3) proves that $D/X \in \mathcal{S}^{\perp}$, thus $D/X \in \hat{\mathcal{F}}$. □

The following is a corollary of 2.4:

Corollary 3.2. *Let $\mathcal{D}$ be an abelian category with a distinguished Giraud subcategory C. Suppose that $\mathcal{D}$ is endowed with a torsion pair $(\mathcal{X}, \mathcal{Y})$. Then the class $i^{\leftarrow}(\mathcal{Y}) := \{C \in C \,|\, i(C) \in \mathcal{Y}\}$ is a torsion free class on C.*

Proposition 3.3. *Let $\mathcal{D}$ be an abelian category with a distinguished Giraud subcategory C. Suppose that $\mathcal{D}$ is endowed with a torsion pair $(\mathcal{X}, \mathcal{Y})$,and let*

$$\ell(\mathcal{X}) := \{T \in C \,|\, T \cong \ell(X), \exists\, X \in \mathcal{X}\}$$
$$\ell(\mathcal{Y}) := \{F \in C \,|\, F \cong \ell(Y), \exists\, Y \in \mathcal{Y}\}$$

Then $(\ell(\mathcal{X}), \ell(\mathcal{Y}))$ defines a torsion pair on C if and only if $i\ell(\mathcal{Y}) \subseteq \mathcal{Y}$. In this case, $i^{\leftarrow}(\mathcal{Y}) = \ell(\mathcal{Y})$.

Proof. First let us suppose that $i\ell(\mathcal{Y}) \subseteq \mathcal{Y}$. Then since $\ell \circ i \cong \mathrm{id}_C$ one has $i^{\leftarrow}(\mathcal{Y}) = \ell(\mathcal{Y})$ and by Corollary 2.4 this is a torsion free class on C. Given $T \in \ell(\mathcal{X})$ (i.e., $T \cong \ell(X)$, with $X \in \mathcal{X}$) and $F \in i^{\leftarrow}(\mathcal{Y})$, one has $C(T, F) = C(\ell(X), F) \cong \mathcal{D}(X, i(F)) = 0$, since $i(F) \in \mathcal{Y}$ by the definition of $i^{\leftarrow}(\mathcal{Y})$. Now let $C \in C$. There exist $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and a short exact sequence in $\mathcal{D}$

$$0 \longrightarrow X \longrightarrow i(C) \longrightarrow Y \longrightarrow 0.$$

Applying the functor ℓ to the previous sequence we get a short exact sequence in C

$$0 \longrightarrow \ell(X) \longrightarrow C \longrightarrow \ell(Y) \longrightarrow 0$$

where $\ell(X) \in \ell(\mathcal{X})$ and $\ell(Y) \in \ell(\mathcal{Y})$, which proves that $(\ell(\mathcal{X}), \ell(\mathcal{Y}))$ is a torsion pair on C.

Conversely, if $(\ell(\mathcal{X}), \ell(\mathcal{Y}))$ is a torsion pair on C then for every $X \in \mathcal{X}$ and every $Y \in \mathcal{Y}$ one has $0 = C(\ell(X), \ell(Y)) \cong \mathcal{D}(X, i\ell(Y))$, therefore $i\ell(Y) \in \mathcal{Y}$. □

From 3.1 and 3.3 we derive the following correspondence:

Theorem 3.4. *Let $\mathcal{D}$ be an abelian category with a distinguished Giraud subcategory C. There exists a one to one correspondence between torsion pairs $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{D}$ satisfying $i\ell(\mathcal{Y}) \subseteq \mathcal{Y} \subseteq \mathcal{S}^{\perp}$ and torsion pairs $(\mathcal{T}, \mathcal{F})$ on C.*

Proof. From one side, taking a torsion pair $(\mathcal{T}, \mathcal{F})$ in C, the torsion pair $(\hat{\mathcal{T}}, \hat{\mathcal{F}})$ (defined in Proposition 3.1) satisfies $i\ell(\hat{\mathcal{F}}) \subseteq \hat{\mathcal{F}}$ and one can easily verify that $(\ell(\hat{\mathcal{T}}), \ell(\hat{\mathcal{F}})) = (\mathcal{T}, \mathcal{F})$.

On the other side given $(\mathcal{X}, \mathcal{Y})$ a torsion pair on $\mathcal{D}$ satisfying $i\ell(\mathcal{Y}) \subseteq \mathcal{Y} \subseteq \mathcal{S}^{\perp}$, its corresponding torsion pair on C is $(\ell(\mathcal{X}), \ell(\mathcal{Y}))$ (by Proposition 3.3) for which it is clear that $\widehat{\ell(\mathcal{Y})} := \ell^{\leftarrow}(\ell(\mathcal{Y})) \cap \mathcal{S}^{\perp} = \mathcal{Y}$ (since $\mathcal{Y} \subseteq \mathcal{S}^{\perp}$) and so $(\mathcal{X}, \mathcal{Y}) = (\widehat{\ell(\mathcal{X})}, \widehat{\ell(\mathcal{Y})})$. □

Dually, one obtains:

Theorem 3.5. *Let $\mathcal{D}$ be an abelian category with a distinguished co-Giraud subcategory C. There exists a one to one correspondence between torsion pairs $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{D}$ satisfying $jr(\mathcal{X}) \subseteq \mathcal{X} \subseteq {}^{\perp}\mathcal{S}$ and torsion pairs $(\mathcal{T}, \mathcal{F})$ on C.*

4. t-structures induced by torsion pairs

Definition 4.1. *A t-structure on a triangulated category $\mathcal{D}$ is a pair $t = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of strictly full subcategories of $\mathcal{D}$ such that, setting $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$, one has*

(0) $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$ *and* $\mathcal{D}^{\geq 0} \supseteq \mathcal{D}^{\geq 1}$.

(i) $\mathcal{D}(X, Y) = 0$ *for every X in* $\mathcal{D}^{\leq 0}$ *and every Y in* $\mathcal{D}^{\geq 1}$.

(ii) *For any object $X \in \mathcal{D}$ there exists a distinguished triangle:*

$$A \to X \to B \to A[1]$$

in $\mathcal{D}$ such that $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

Proposition 4.2. [? , Proposition 1.3.3] *Let* $\mathrm{t} = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ *be a t-structure on a triangulated category $\mathcal{D}$.*

(i) *The inclusion of $\mathcal{D}^{\leq n}$ in $\mathcal{D}$ admits a right adjoint $\tau^{\leq n}$, and the inclusion of $\mathcal{D}^{\geq n}$ in $\mathcal{D}$ a left adjoint $\tau^{\geq n}$, called the truncation functors.*

(ii) *For every X in $\mathcal{D}$ there exists a unique morphism $d\colon \tau^{\geq 1}(X) \to \tau^{\leq 0}(X)[1]$ such that the triangle*

$$\tau^{\leq 0}(X) \to X \to \tau^{\geq 1}(X) \overset{d}{\to}$$

is distinguished. This triangle is (up to a unique isomorphism) the unique distinguished triangle (A, X, B) with A in $\mathcal{D}^{\leq 0}$ and B in $\mathcal{D}^{\geq 1}$.

(iii) *The category $\mathcal{H}_t := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is abelian, and the truncation functors induce a functor $\mathrm{H}_t\colon \mathcal{D} \to \mathcal{H}_t$, called the t-cohomological functor ($H^0_t(X) = \tau^{\geq 0}\tau^{\leq 0}(X) \cong \tau^{\leq 0}\tau^{\geq 0}(X)$ and for every $i \in \mathbb{Z}$, $\mathrm{H}^i_t(X) = \mathrm{H}^0_t(X[i])$), see* [? , Theorem 1.3.6]*).*

In particular, given an abelian category C its (unbounded) derived category $D(C)$ is a triangulated category which admits a canonical t-structure, called the *natural t-structure*, whose class $D(C)^{\leq 0}$ (resp. $D(C)^{\geq 0}$) is that of complexes without cohomology in positive (resp. negative) degrees. We will denote by $\mathrm{H}\colon D(C) \to C$ its cohomological functor and by $t^{\leq n}$ resp. $t^{\geq n}$ its truncation functors. As explained by A. Beligiannis and I. Reiten in their work [?], one can regard a t-structure on a triangulated category $\mathcal{D}$ as a generalization of a torsion pair, where the role of the torsion class is provided by $\mathcal{D}^{\leq 0}$, while that of the torsion free class is played by $\mathcal{D}^{\geq 1}$. Moreover, given a torsion pair on an abelian category C one can construct a t-structure on its derived category $D(C)$, as explained in [?] and [? , Proposition 1.8].

Proposition 4.3. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on an abelian category C. The classes*

$$\begin{aligned} t(\mathcal{T}) = \mathcal{D}^{\leq 0}_{\mathrm{t}} &= \{C^\bullet \in D(C) \mid H^0(C^\bullet) \in \mathcal{T},\ H^i(C^\bullet) = 0\ \forall i > 0\} \\ t(\mathcal{F}) = \mathcal{D}^{\geq 0}_{\mathrm{t}} &= \{C^\bullet \in D(C) \mid H^{-1}(C^\bullet) \in \mathcal{F},\ H^i(C^\bullet) = 0\ \forall i < -1\} \end{aligned}$$

define a t-structure on $D(C)$ which is called the t-structure induced by the torsion pair t.

Remark 4.4. Let C be an abelian category endowed with the trivial torsion pair $(C, 0)$. The t-structure associated to this trivial torsion pair is the natural t-structure on $D(C)$.

Remark 4.5. Let C be an abelian category with a torsion pair $(\mathcal{T}, \mathcal{F})$. The heart associated to the t-structure $(\mathcal{D}^{\leq 0}_{\mathrm{t}}, \mathcal{D}^{\geq 0}_{\mathrm{t}})$ on $D(C)$ is the full subcategory $\mathcal{H}_C := t(\mathcal{T}) \cap t(\mathcal{F})$ of $D(C)$ called the *tilt* of C by the torsion pair $(\mathcal{T}, \mathcal{F})$. It is shown in [?] that $\mathcal{H}_C$ is an abelian category where short exact sequences are induced by distinguished triangles in $D(C)$. The objects of $\mathcal{H}_C$ are represented, up to isomorphism, by complexes of the form

$$X:\ X^{-1} \overset{x}{\longrightarrow} X^0, \quad \text{with } \mathrm{Ker}(x) \in \mathcal{F} \text{ and } \mathrm{Coker}(x) \in \mathcal{T},$$

while a morphism $\phi : X \to Y$ in $\mathcal{H}_C$ is a formal fraction $\phi = (s)^{-1} \circ f$, where:

1. $X \xrightarrow{f} Z$ is a representative of a homotopy class of maps of complexes

$$\begin{array}{ccc} X^{-1} & \xrightarrow{x} & X^0 \\ {\scriptstyle f^{-1}}\downarrow & & \downarrow{\scriptstyle f^0} \\ Z^{-1} & \xrightarrow{z} & Z^0 \end{array}$$

where we recall that $X \xrightarrow{f} Z$ is null-homotopic if there is a map $r^0 : X^0 \to Z^{-1}$ such that

$$f^0 = zr^0 \text{ and } f^{-1} = r^0 x$$

2. $Y \xrightarrow{s} Z$ is a representative of a homotopy class of maps of complexes and it is a quasi-isomorphism, i.e., it is a map of complexes which induces isomorphism in cohomology:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Ker}(y) & \longrightarrow & Y^{-1} & \xrightarrow{y} & Y^0 & \longrightarrow & \mathrm{Coker}(y) & \longrightarrow & 0 \\ & & \cong\downarrow & & {\scriptstyle s^{-1}}\downarrow & & \downarrow{\scriptstyle s^0} & & \downarrow\cong & & \\ 0 & \longrightarrow & \mathrm{Ker}(z) & \longrightarrow & Z^{-1} & \xrightarrow{z} & Z^0 & \longrightarrow & \mathrm{Coker}(z) & \longrightarrow & 0 \end{array}$$

Every distinguished triangle $X_1^\bullet \to X_2^\bullet \to X_3^\bullet \overset{[+1]}{\to} X_1^\bullet[1]$ in $D(C)$ provides a long exact sequence of t-cohomology in the heart $\mathcal{H}_C$:

$$\cdots H_t^{-1}(X_3) \to H_t^0(X_1) \to H_t^0(X_2) \to H_t^0(X_3) \to H_t^1(X_1) \cdots$$

Moreover given an object C in C, its t-cohomology objects in $\mathcal{H}_C$ are $H_t^i(C) = 0$ for any $t < 0, t > 1$; $H_t^0(C) = t(C)[0]$ is the torsion part of C (with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$) placed in degree zero, while $H_t^1(C) = \frac{C}{t(C)}[1]$. The tilted pair $(\mathcal{F}[1], \mathcal{T}[0])$ is a torsion pair in $\mathcal{H}_C$ with category equivalences $\mathcal{F}[1] \cong \mathcal{F}$ and $\mathcal{T}[0] \cong \mathcal{T}$ (see [? , Corollary 2.2]).

Remark 4.6. In [?] the authors introduced the notion of a tilting object for an arbitrary abelian category, proving that for any ring R and for any faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ in R-Mod the heart $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ of the t-structure in $D(R)$ associated to $(\mathcal{X}, \mathcal{Y})$ is (an abelian category) with a tilting object $T = R[1]$. Then, again the first author with Gregorio and Mantese in [?] showed that the heart is a prototype for these categories, in the sense that an abelian category $\mathcal{D}$ admits a tilting object T if and only if $\mathcal{D}$ is equivalent to the category $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ for a suitable torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-End(T) which is “tilted” by T, and with Gregorio in [?] proved that $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ is a Grothendieck category if and only if the torsion pair is cogenerated by a cotilting module in the sense of [?]. This allows us to deal with a more general notion of a“tilting context”: given an abelian category $\mathcal{D}$ endowed with a faithful torsion pair $(\mathcal{X}, \mathcal{Y})$ (i.e., such that $\mathcal{Y}$ generates $\mathcal{D}$), we get a new abelian category $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ endowed with a torsion pair $(\mathcal{Y}[1], \mathcal{X}[0])$ which is “tilting”, in the sense that the torsion class $\mathcal{Y}[1]$ cogenerates the category $\mathcal{H}(\mathcal{X}, \mathcal{Y})$ and there are category equivalences $\mathcal{Y}[1] \cong \mathcal{Y}$ and $\mathcal{X}[0] \cong \mathcal{X}$ induced by exact functors.

Let C be an abelian category endowed with a torsion pair $(\mathcal{T}, \mathcal{F})$. Since we need to use different torsion pairs we will use the notation $(t(\mathcal{T}), t(\mathcal{F}))$ instead of $(\mathcal{D}_t^{\leq 0}, \mathcal{D}_t^{\geq 0})$ to denote the t-structure associated to the torsion pair $(\mathcal{T}, \mathcal{F})$. For the same reason when we need to clarify the torsion pair we will denote by $\tau_{t(\mathcal{T})}, \tau_{t(\mathcal{F})}$ the truncation functors instead of $\tau_t^{\leq 0}, \tau_t^{\geq 1}$.

As showed in [?], there exists an injective function between the poset of torsion pairs in C and that of t-structures in $D(C)$. Moreover one can recover those t-structures on $D(C)$ which are induced by torsion pairs by means of the following fact proved by Polishchuk in [? , Lemma 1.2.2].

Theorem 4.7. *Given C an abelian category. There exists a bijection between*

1. *torsion pairs on C*
2. *t-structures $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ on $D(C)$ such that $D(C)^{\leq -1} \subset \mathcal{T}^{\leq 0} \subset D(C)^{\leq 0}$.*

5. Tilted Giraud subcategories

For sake of simplicity, in this section we deal with the case of Giraud subcategories, although the case of co-Giraud subcategories can be proved by a dual argument.

In the sequel we will use the notion of derived functor in the stronger sense due to Maltsiniotis [**?**] (see also [**?**]) via the notion of *absolute Kan extension*.

Definition 5.1. [**?** , 2.1 and 2.5] *Let $P : C \to C'$ be a functor. A* left Kan extension *of a functor $F : C \to \mathcal{D}$ along P is a pair (F', α) where $F' : C' \to \mathcal{D}$ is a functor and $\alpha : F \to F' \circ P$ a natural transformation satisfying the following universal property: for any other pair (G, β) (with $G : C' \to \mathcal{D}$ a functor and $\beta : F \to G \circ P$ a natural transformation) there exists a unique natural transformation $\gamma : F' \to G$ such that $\beta = (\gamma \star P)\,\alpha$. In particular, when a left Kan extension exists, it is unique up to a unique isomorphism.*
An absolute left Kan extension *of F along P is a pair (F', α) where $F' : C' \to \mathcal{D}$ is a functor and $\alpha : F \to F' \circ P$ a natural transformation such that for any functor $H : \mathcal{D} \to \mathcal{E}$ the pair $(H \circ F', H \star \alpha)$ is a left Kan extension of $H \circ F$ along P. Any absolute left Kan extension is a left Kan extension.*

The previous definitions applied to the derived categories provide the following:

Definition 5.2. [**?** , 3.2] *Let $F : K(C) \to K(\mathcal{D})$ be a triangulated functor between the homotopy categories of two abelian categories $C, \mathcal{D}$. Let us denote by p_C (resp. $p_{\mathcal{D}}$) the canonical localization functor from the homotopy category $K(C)$ (resp. $K(\mathcal{D})$) to its derived category $D(C)$ (resp. $D(\mathcal{D})$).*

- *The* total right derived functor *of F is a left Kan extension $(\mathbf{R}F, \alpha)$ of $p_{\mathcal{D}} \circ F$ along p_C.*
- *The* absolute total right derived functor *of F is an absolute left Kan extension $(\mathbf{R}F, \alpha)$ of $p_{\mathcal{D}} \circ F$ along p_C.*

In the present paper, we will deal with absolute total right derived functors referring to them simply as *ab-tot right derived functors*.

Remark 5.3. Whenever $F : C \to \mathcal{D}$ is an exact functor, by abuse of notation we may denote by $F : D(C) \to D(\mathcal{D})$ the induced functor between the derived categories, which coincides with its ab-tot right derived functor. Moreover, by [**?**], when the category C has an injective model structure, then any left exact functor $C \to \mathcal{D}$ admits an ab-tot right derived functor, which can be computed in the usual way via injective replacements.

Lemma 5.4. *Let $\mathcal{D}$ and C be abelian categories and $\ell\colon \mathcal{D} \to C$ be an exact functor. Suppose that $\mathcal{D}$ is endowed with a torsion pair $(\mathcal{X}, \mathcal{Y})$ and that $(\mathcal{T}, \mathcal{F}) = (\ell(\mathcal{X}), \ell(\mathcal{Y}))$ defines a torsion pair on C. Then $\ell \circ \tau_{t(\mathcal{Y})} = \tau_{t(\mathcal{F})} \circ \ell$ and $\ell \circ \tau_{t(\mathcal{X})} = \tau_{t(\mathcal{T})} \circ \ell$.*

Proof. Since ℓ is exact it admits an ab-tot derived functor $\ell\colon D(\mathcal{D}) \to D(C)$ which is computed simply applying the functor ℓ termwise. Moreover, from $\ell(\mathcal{X}) = \mathcal{T}$ and $\ell(\mathcal{Y}) = \mathcal{F}$ we derive that $\ell(t(\mathcal{X})) \subseteq t(\mathcal{T})$ and $\ell(t(\mathcal{Y})) \subseteq t(\mathcal{F})$, i.e., ℓ is an exact functor for the t-structures $(t(\mathcal{X}), t(\mathcal{Y}))$ on $D(\mathcal{D})$ and $(t(\mathcal{T}), t(\mathcal{F}))$ on $D(C)$ (see [**?** , 1.3.16]). Let $D^\bullet \in D(C)$ and

$$\tau_{t(\mathcal{X})}(D^\bullet) \longrightarrow D^\bullet \longrightarrow \tau_{t(\mathcal{Y})}(D^\bullet) \xrightarrow{+1} \tag{4}$$

its distinguished triangle, with $\tau_{t(\mathcal{X})}(D^\bullet) \in t(\mathcal{X})$ and $\tau_{t(\mathcal{Y})}(D^\bullet) \in t(\mathcal{Y})$. By applying the functor ℓ to (4) we get the distinguished triangle in $D(C)$

$$\ell(\tau_{t(\mathcal{X})}(D^\bullet)) \longrightarrow \ell(D^\bullet) \longrightarrow \ell(\tau_{t(\mathcal{Y})}(D^\bullet)) \xrightarrow{+1} \tag{5}$$

with $\ell(\tau_{t(\mathcal{X})}(D^\bullet)) \in t(\mathcal{T})$ and $\ell(\tau_{t(\mathcal{Y})}(D^\bullet)) \in t(\mathcal{F})$, so (5) is the distinguished triangle associated to $\ell(D^\bullet)$, which proves that $\ell \circ \tau_{t(\mathcal{Y})} = \tau_{t(\mathcal{F})} \circ \ell$ and $\ell \circ \tau_{t(\mathcal{X})} = \tau_{t(\mathcal{T})} \circ \ell$. □

Remark 5.5. Under the assumptions of Lemma 5.4 and following the notations of Remark 4.5 we have $l(\mathcal{H}_{\mathcal{D}}) \subseteq \mathcal{H}_C$ and so l induces an functor $\ell_{\mathcal{H}}\colon \mathcal{H}_{\mathcal{D}} \to \mathcal{H}_C$ which is exact since ℓ sends distinguished triangles in $D(\mathcal{D})$ into distinguished triangles in $D(C)$. Moreover, ℓ commutes with the t-cohomological functors H_t^i.

Theorem 5.6. *Let $\mathcal{D}$ be an abelian category endowed with a torsion pair $(\mathcal{X}, \mathcal{Y})$ and let $\mathcal{H}_{\mathcal{D}}$ be the corresponding heart with respect to the t-structure on $D(\mathcal{D})$ induced by $(\mathcal{X}, \mathcal{Y})$. Let $\mathcal{S}$ be a Serre subcategory of $\mathcal{D}$ and $\ell : \mathcal{D} \to C := \mathcal{D}/\mathcal{S}$ be its corresponding quotient functor, and let us suppose that $(\ell(\mathcal{X}), \ell(\mathcal{Y}))$ is a torsion pair on C. Then:*

1. *The class $\mathcal{S}_{\mathcal{H}} = \{D^\bullet \in \mathcal{H}_{\mathcal{D}} \mid \ell_{\mathcal{H}}(D^\bullet) = 0\}$ (for the definition of $\ell_{\mathcal{H}}$ see Remark 5.5) is a Serre subcategory of $\mathcal{H}_{\mathcal{D}}$.*
2. *Denote by $C' = \mathcal{H}_{\mathcal{D}}/\mathcal{S}_{\mathcal{H}}$ the quotient category and by $\ell' : \mathcal{H}_{\mathcal{D}} \to C'$ the quotient functor. Then ℓ' is exact and the classes $(\ell'(\mathcal{Y}[1]), \ell'(\mathcal{X}[0]))$ define a torsion pair in C'.*
3. *There is a canonical functor $\varphi : C' \to \mathcal{H}_C$ such that $\ell_{\mathcal{H}} = \varphi \circ \ell'$. The functor φ is an equivalence of categories.*
4. *Moreover in the case in which $(\mathcal{D}, C, \ell, i)$ is a distinguished Giraud subcategory such that i admits an ab-tot right derived functor $\mathbf{R}i$ there exists a distinguished Giraud subcategory $(\mathcal{H}_{\mathcal{D}}, \mathcal{H}_C, \ell_{\mathcal{H}}, i_{\mathcal{H}})$ such that $i_{\mathcal{H}}(\ell_{\mathcal{H}}(\mathcal{X}[0])) \subseteq \mathcal{X}[0]$.*

Proof. 1. The first statement follows from the exactness of the functor $\ell_{\mathcal{H}}$ (see Remark 5.5). Moreover we observe that $\mathcal{S}_{\mathcal{H}} \cap \mathcal{X}[0] = (\mathcal{S} \cap \mathcal{X})[0]$ and $\mathcal{S}_{\mathcal{H}} \cap \mathcal{Y}[1] = (\mathcal{S} \cap \mathcal{Y})[1]$.

2. Let us show that the classes $(\ell'(\mathcal{Y}[1]), \ell'(\mathcal{X}[0]))$ define a torsion pair on C'. First of all, since any object of C' may be regarded as an object of $\mathcal{H}_{\mathcal{D}}$ and the functor ℓ' is exact, it is clear that any object $C^\bullet \in C'$ is the middle term of a short exact sequence $0 \to T^\bullet \to C^\bullet \to F^\bullet \to 0$ with $T^\bullet \in \ell'(\mathcal{Y}[1])$ and $F^\bullet \in \ell'(\mathcal{X}[0])$. It remains to show that $C'(T^\bullet, F^\bullet) = 0$ for every $T^\bullet \in \ell'(\mathcal{Y}[1])$ and every $F^\bullet \in \ell'(\mathcal{X}[0])$. So let $T^\bullet \in \ell'(\mathcal{Y}[1])$ and $F^\bullet \in \ell'(\mathcal{X}[0])$. A morphism $f : T^\bullet \to F^\bullet$ in C' may be viewed as the class of a morphism $U^\bullet \to F^\bullet/V^\bullet$ in $\mathcal{H}_{\mathcal{D}}$, where $T^\bullet/U^\bullet$ and $V^\bullet$ are in $\mathcal{S}_{\mathcal{H}}$. Let $t(U^\bullet)$ be the torsion part of $U^\bullet$ (viewed as an object of $\mathcal{H}_{\mathcal{D}}$) with respect to the torsion pair $(\mathcal{Y}[1], \mathcal{X}[0])$ in $\mathcal{H}_{\mathcal{D}}$ and $F^\bullet/W^\bullet$ be the torsion-free quotient of $F^\bullet/V^\bullet$. We show that the composite morphism $t(U^\bullet) \to U^\bullet \to F^\bullet/V^\bullet \to F^\bullet/W^\bullet$ also represents the morphism f in C', i.e., $T^\bullet/t(U^\bullet) \in \mathcal{S}_{\mathcal{H}}$ and $W^\bullet \in \mathcal{S}_{\mathcal{H}}$ and hence $f = 0$, since it is a morphism from a torsion to a torsion-free object. The short exact sequence in $\mathcal{H}_{\mathcal{D}}$

$$0 \to \frac{U^\bullet}{t(U^\bullet)} \to \frac{T^\bullet}{t(U^\bullet)} \to \frac{T^\bullet}{U^\bullet} \to 0$$

defines a distinguished triangle in $D(\mathcal{D})$. By applying the cohomological functor for the natural t-structure on $D(\mathcal{D})$ to this distinguished triangle, we obtain the following long exact sequence in $\mathcal{D}$:

$$\cdots \to 0 \to H^{-1}\left(\tfrac{U^\bullet}{t(U^\bullet)}\right) \to H^{-1}\left(\tfrac{T^\bullet}{t(U^\bullet)}\right) \to H^{-1}\left(\tfrac{T^\bullet}{U^\bullet}\right) \to H^{0}\left(\tfrac{U^\bullet}{t(U^\bullet)}\right) \to H^{0}\left(\tfrac{T^\bullet}{t(U^\bullet)}\right) \to H^{0}\left(\tfrac{T^\bullet}{U^\bullet}\right) \to 0 \to \cdots$$

Now, $U^\bullet/t(U^\bullet) \in \mathcal{X}[0]$ while $T^\bullet/t(U^\bullet), T^\bullet/U^\bullet \in \mathcal{Y}[1]$ (as quotients in $\mathcal{H}_{\mathcal{D}}$ of a torsion object), so the previous sequence reduces to the short exact sequence in $\mathcal{D}$:

$$0 \to \tfrac{T^\bullet}{t(U^\bullet)}[-1] \to \tfrac{T^\bullet}{U^\bullet}[-1] \to \tfrac{U^\bullet}{t(U^\bullet)}[0] \to 0$$

and since the middle term $\frac{T^\bullet}{U^\bullet}[-1] \in (\mathcal{S}_{\mathcal{H}} \cap \mathcal{Y}[1])[-1] = \mathcal{S} \cap \mathcal{Y}$ we deduce that $\frac{T^\bullet}{t(U^\bullet)}[-1] \in \mathcal{S} \cap \mathcal{Y} = (\mathcal{S}_{\mathcal{H}} \cap \mathcal{Y}[1])[-1]$ and so $\frac{T^\bullet}{t(U^\bullet)} \in \mathcal{S}_{\mathcal{H}}$. A dual argument shows that $W^\bullet \in \mathcal{S}_{\mathcal{H}}$.
For later purposes, we remark here that $\mathrm{Ker}(l') = \mathcal{S}_{\mathcal{H}}$.

3. First of all, the exactness of the functor $\ell_{\mathcal{H}}$ permits to apply [**?** , Corollary 2, page 368] which claims that there exists a unique functor $\varphi : C' \to \mathcal{H}_C$ such that $\ell_{\mathcal{H}} = \varphi \circ \ell'$ and moreover by [**?** , Corollary 3, page 369] the functor φ is exact. To prove that φ is an equivalence, we will show that it is faithful, essentially surjective and full.

Given a morphism $C'_1 \overset{\alpha}{\to} C'_2$ in C' we can suppose that $\alpha = \ell'(\beta)$ for a suitable morphism $D^\bullet_1 \overset{\beta}{\to} D^\bullet_2$ in $\mathcal{H}_{\mathcal{D}}$; hence $\varphi(\alpha) = \varphi(\ell'(\beta)) = \ell_{\mathcal{H}}(\beta) = 0$ if and only if $\mathrm{Im}(\varphi(\alpha)) = \ell_{\mathcal{H}}(\mathrm{Im}(\beta)) = 0$ which is equivalent to ask $\mathrm{Im}(\beta) \in \mathcal{S}_{\mathcal{H}}$ and so $\mathrm{Im}(\alpha) = \mathrm{Im}(\ell'(\beta)) = \ell'(\mathrm{Im}(\beta)) = 0$ which proves that $\alpha = 0$. This shows that φ is faithful.

Let us prove that $\ell_{\mathcal{H}}$ is essentially surjective from which we deduce that φ is essentially surjective too. Let us consider $C^\bullet = [C^{-1} \overset{y}{\longrightarrow} C^0]$ an object in $\mathcal{H}_C$ (and hence $\mathrm{Ker}(y) \in \mathcal{F} = \ell(\mathcal{Y})$ and $\mathrm{Coker}(y) \in \mathcal{T} = \ell(\mathcal{X})$). We can suppose that $y = \ell(x)$ where $x : D^{-1} \to D^0$ (and so $C^{-1} = \ell(D^{-1})$ and $C^0 = \ell(D^0)$). Hence $\ell(\mathrm{Ker}(x)) = \mathrm{Ker}(y) \in \ell(\mathcal{Y})$ which means that there exists an epimorphism (since $\mathcal{Y}$ is closed by subobjects) $\phi : \mathrm{Ker}(x) \twoheadrightarrow Y$ such that $\mathrm{Ker}(\phi) \in \mathcal{S}$ and $Y \in \mathcal{Y}$.

Let define $D'^{-1} := Y \oplus_{\mathrm{Ker}(x)} D^{-1}$; then by the following commutative diagram

$$\begin{array}{ccccccc} & & \mathrm{Ker}(\phi) & \xrightarrow{id} & \mathrm{Ker}(\phi) & & \\ & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \ker(x) & \hookrightarrow & D^{-1} & \xrightarrow{x} & D^0 \\ & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow & & \big\downarrow{\scriptstyle id} \\ 0 & \longrightarrow & Y & \longrightarrow & D'^{-1} & \xrightarrow{x'} & D^0 \end{array} \tag{6}$$

we deduce that $\ell(x') = y$ but now $\mathrm{Ker}(x') \in \mathcal{Y}$. Dually $\ell(\mathrm{Coker}(x')) = \mathrm{Coker}(y) \in \ell(\mathcal{X})$ which means that there exists a monomorphism $\psi : X \hookrightarrow \mathrm{Coker}(x')$ such that $\mathrm{Coker}(\psi) \in \mathcal{S}$ and $X \in \mathcal{X}$. Dually to (6) let define $D'^0 := X \times_{\mathrm{Coker}(x)} D^0$; then by the following commutative diagram

$$\begin{array}{ccccc} D'^{-1} & \xrightarrow{\bar{x}} & D'^0 & \twoheadrightarrow & X \\ \big\downarrow{\scriptstyle id} & & \big\downarrow & & \big\downarrow{\scriptstyle \psi} \\ D'^{-1} & \xrightarrow{x'} & D^0 & \twoheadrightarrow & \mathrm{Coker}(x') \\ & & \big\downarrow & & \big\downarrow \\ & & \mathrm{Coker}(\psi) & \xrightarrow{id} & \mathrm{Coker}(\psi) \end{array} \tag{7}$$

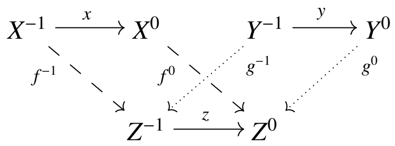
we deduce that $\ell(\bar{x}) = y$ but now $\mathrm{Ker}(\bar{x}) \in \mathcal{Y}$ and $\mathrm{Coker}(\bar{x}) \in \mathcal{X}$ which proves that $D^\bullet := [D'^{-1} \xrightarrow{\bar{x}} D'^0]$ is an object in $\mathcal{H}_\mathcal{D}$ such that $\ell_\mathcal{H}(D^\bullet) = C^\bullet$. This proves that $\ell_\mathcal{H}$ is essentially surjective and so is φ.
It remains to prove that $\varphi : \mathcal{C}' \to \mathcal{H}_\mathcal{C}$ is a full functor. The proof of this statement is based on a lifting argument inspired by Gabriel [? , Corollary 1, page 368].
First of all let us note that the pull-back push-out arguments of diagrams (6) and (7) prove that:

1. given $A \xrightarrow{\alpha} B$ a morphism such that: $\mathrm{Coker}(\alpha) \in \mathcal{X}$ and $\ell(A) \xrightarrow{\ell(\alpha)} \ell(B)$ in $\mathcal{H}_\mathcal{C}$; then there exist an object $A' \xrightarrow{\alpha'} B$ in $\mathcal{H}_\mathcal{D}$ and a morphism in $K(\mathcal{D})$
$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \big\downarrow & & \big\| \\ A' & \xrightarrow{\alpha'} & B \end{array} \tag{8}$$
which induces an isomorphism $\mathrm{Coker}(\alpha) \cong \mathrm{Coker}(\alpha')$ and it gives rise to a quasi-isomorphism in $K(\mathcal{C})$;
2. dually given $C \xrightarrow{\beta} D$ a morphism such that: $\mathrm{Ker}(\beta) \in \mathcal{Y}$ and $\ell(C) \xrightarrow{\ell(\beta)} \ell(D)$ in $\mathcal{H}_\mathcal{C}$; then there exist an object $C \xrightarrow{\beta'} D'$ in $\mathcal{H}_\mathcal{D}$ and a morphism in $K(\mathcal{D})$
$$\begin{array}{ccc} C & \xrightarrow{\beta'} & D' \\ \big\| & & \big\downarrow \\ C & \xrightarrow{\beta} & D \end{array} \tag{9}$$
which induces an isomorphism $\mathrm{Ker}(\beta') \cong \mathrm{Ker}(\beta)$ and it gives rise to a quasi-isomorphism in $K(\mathcal{C})$.

Moreover we note that the previous quasi-isomorphisms in $K(\mathcal{C})$ induce isomorphisms in $\mathcal{H}_\mathcal{C}$ which proves that their kernels and cokernels in $\mathcal{H}_\mathcal{D}$ lie in $\mathcal{S}_\mathcal{H}$ and so also $\ell'(\alpha) \cong \ell'(\alpha')$ and $\ell'(\beta') \cong \ell'(\beta)$ in $\mathcal{C}'$.

Since $\mathcal{C}' = \mathcal{H}_\mathcal{D}/\mathcal{S}_\mathcal{H}$ we have that the objects of $\mathcal{C}'$ are those of $\mathcal{H}_\mathcal{D}$ (via the functor ℓ'). So we have to prove that given a morphism $\gamma : \varphi(\ell'(X^\bullet)) \to \varphi(\ell'(Y^\bullet))$ in $\mathcal{H}_\mathcal{C}$ there exists a morphism $\delta : \ell'(X^\bullet) \to \ell'(Y^\bullet)$ in $\mathcal{C}'$ such that

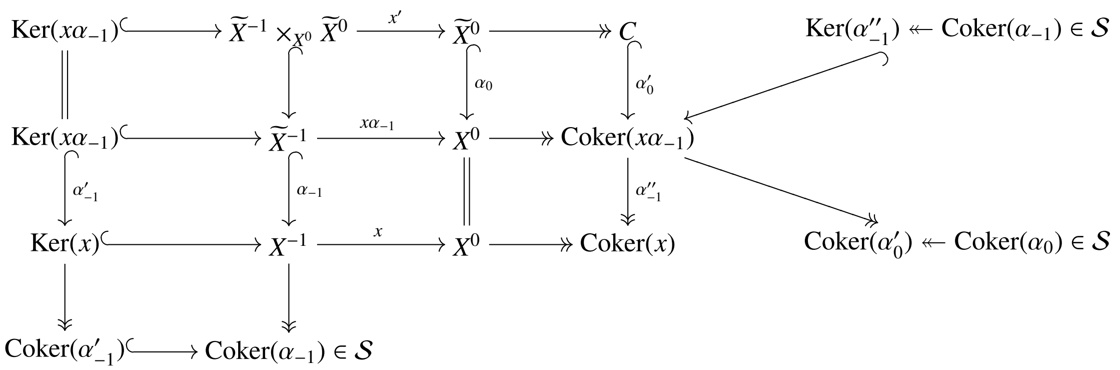
$\varphi(\delta) = \gamma$ where $X^\bullet, Y^\bullet$ are objects in $\mathcal{H}_\mathcal{D}$. The heart $\mathcal{H}_\mathcal{C}$ is a full subcategory of the derived category $D(\mathcal{C})$ hence, as stated in Remark 4.5 the morphism γ can be represented by the following diagram



(10)

where the dashed arrows f^{-1} and f^0 are morphisms in the quotient category $\mathcal{C} := \mathcal{D}/\mathcal{S}$ such that the left parallelogram is commutative, while the dotted arrows g^{-1} and g^0 are morphisms in the quotient category $\mathcal{C}$ such that the right parallelogram is commutative and they form a quasi-isomorphism in $D(\mathcal{C})$ between $Y^\bullet$ and $Z^\bullet$. First of all we will lift (10) in a commutative diagram in $\mathcal{D}$ and next we will adjust the rows following the arguments of (8) and (9) providing a lift in $\mathcal{C}'$.

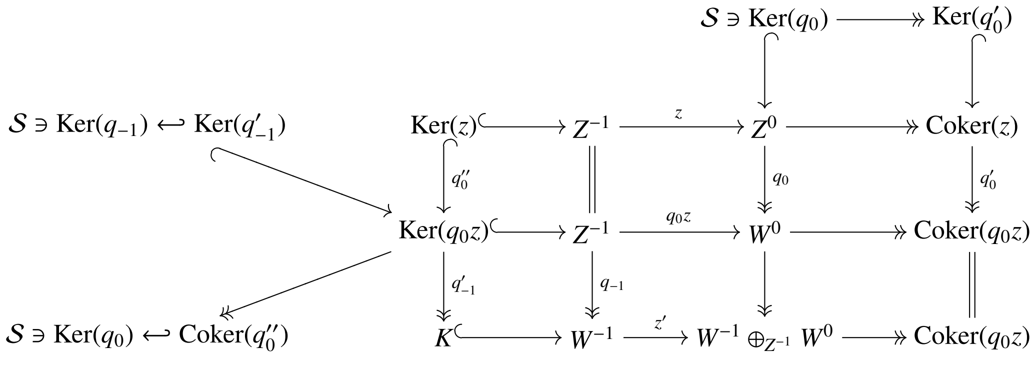
By definition of a morphism in the quotient category $\mathcal{C}$ the morphisms f^i are defined by $f^i : \widetilde{X}^i \to \widetilde{Z}^i$ with $\alpha_i : \widetilde{X}^i \hookrightarrow X^i$ and $\beta_i : Z^i \twoheadrightarrow \widetilde{Z}^i$ such that $\mathrm{Coker}(\alpha_i), \mathrm{Ker}(\beta_i) \in \mathcal{S}$ for $i \in \{-1, 0\}$. Analogously the morphisms g^i are defined by $g^i : \bar{Y}^i \to \bar{Z}^i$ with $j_i : \bar{Y}^i \hookrightarrow Y^i$ and $p_i : Z^i \twoheadrightarrow \bar{Z}^i$ such that $\mathrm{Coker}(j_i), \mathrm{Ker}(p_i) \in \mathcal{S}$ and $i \in \{-1, 0\}$. Let denote by $W^i = \widetilde{Z}^i \oplus_{Z^i} \bar{Z}^i$ and note that $\mathrm{Ker}(\beta_i) \oplus \mathrm{Ker}(p_i) \twoheadrightarrow \mathrm{Ker}(Z^i \twoheadrightarrow W^i) \in \mathcal{S}$. The following commutative diagram with exact rows



(11)

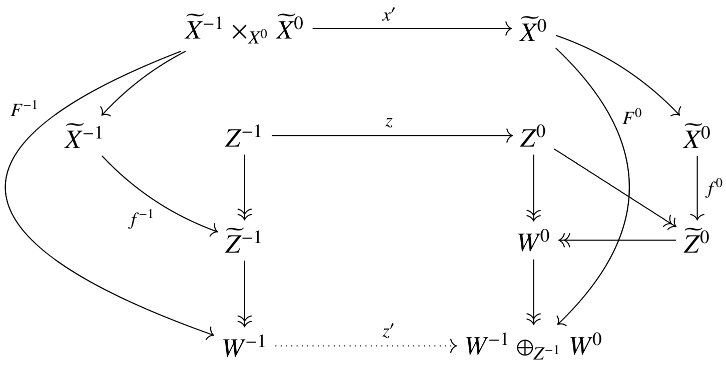
provides a morphism in $K(\mathcal{D})$ between $\widetilde{X}^{-1} \times_{X^0} \widetilde{X}^0 \xrightarrow{x'} \widetilde{X}^0$ and $X^{-1} \xrightarrow{x} X^0$ which induces a quasi-isomorphism in $K(\mathcal{C})$ (since it is a composition of two quasi-isomorphisms in $K(\mathcal{C})$).

The following diagram, dual to (11), proves that

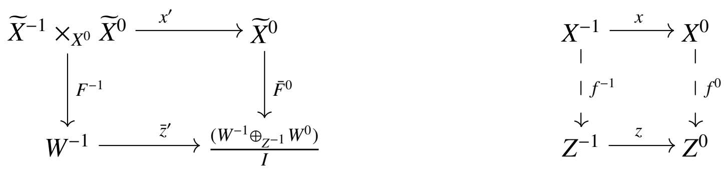


(12)

the morphism of complexes in $K(\mathcal{D})$ between $Z^{-1} \xrightarrow{z} Z^0$ and $W^{-1} \to W^{-1} \oplus_{Z^{-1}} W^0$ is a quasi-isomorphism in $K(\mathcal{C})$. Let consider the morphisms F^{-1} (which represents f^{-1}) and F^0 (which represents f^0) defined by the following diagram

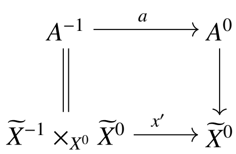


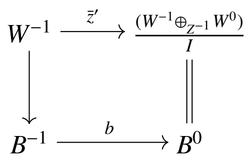
which is commutative in $\mathcal{D}$ by definition except for the dotted arrow. Nevertheless $F^0 \circ x' = z' \circ F^{-1}$ in the quotient $\mathcal{C} := \mathcal{D}/\mathcal{S}$ hence, working in the category $\mathcal{D}$, the image $I := \mathrm{Im}(F^0 \circ x' - z' \circ F^{-1}) \in \mathcal{S}$ so by taking the quotient by I in the target $W^{-1} \oplus_{Z^{-1}} W^0$ we obtain the following commutative diagram (in $\mathcal{D}$) on the left which lifts the commutative one (in $\mathcal{C}$) on the right

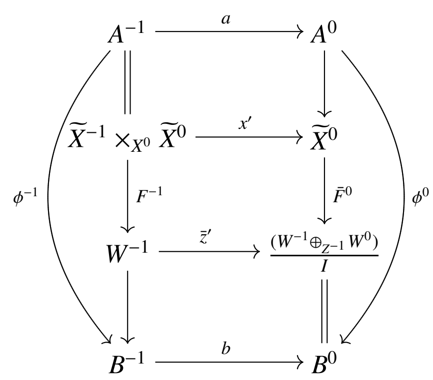


(13)

By diagram (11) we have $\mathrm{Ker}(x') \cong \mathrm{Ker}(x\alpha_{-1}) \subseteq \mathrm{Ker}(x) \in \mathcal{Y}$ and so $\mathrm{Ker}(x') \in \mathcal{Y}$ and $\ell(x') \cong \ell(x) \in \mathcal{H}_{\mathcal{C}}$ hence the morphism x' satisfies the hypotheses of (9). By diagram (12) we have: $\mathrm{Coker}(\bar{z}') \twoheadleftarrow \mathrm{Coker}(z') \cong \mathrm{Coker}(q_0 z) \twoheadleftarrow \mathrm{Coker}(z) \in \mathcal{X}$ and so $\mathrm{Coker}(\bar{z}') \in \mathcal{X}$ and $\ell(\bar{z}') \cong \ell(z') \cong \ell(z) \in \mathcal{H}_{\mathcal{C}}$ hence $\bar{z}'$ satisfies the hypotheses of (8). These remarks prove that there exist commutative squares (in $\mathcal{D}$)



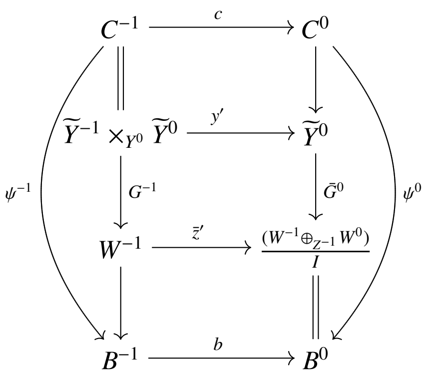


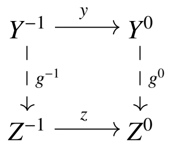


where $A^{-1} := \widetilde{X}^{-1} \times_{X^0} \widetilde{X}^0$ and $B^0 := \frac{(W^{-1} \oplus_{Z^{-1}} W^0)}{I}$ and the right diagram provides a lifting of (f^{-1}, f^0) in $\mathcal{D}$ with a, b objects in $\mathcal{H}_{\mathcal{D}}$.

Applying the same argument to the right parallelogram of (10) we obtain a lift in $\mathcal{D}$ (on the left) of the diagram on the right (which is commutative in $\mathcal{H}_{\mathcal{C}}$ and we can suppose to have the same I otherwise if we have an image $J \neq I$

we will pass to the quotient by $I + J$):





This concludes the proof since we have built a morphism

$$\delta : \ell'(X^\bullet) \cong \ell'(A^\bullet) \to \ell'(B^\bullet) \cong \ell'(C^\bullet) \cong \ell'(Y^\bullet)$$

such that $\varphi(\delta) = \gamma$.

4. First we remark that since ℓ and i are additive, they extend to an adjunction $K(\mathcal{C}) \underset{i}{\overset{\ell}{\leftrightarrows}} K(\mathcal{D})$ between the homotopy categories. By [**?**], the ab-tot derived functors ℓ and $\mathbf{R}i$ form an adjoint pair $D(\mathcal{C}) \underset{Ri}{\overset{\ell}{\leftrightarrows}} D(\mathcal{D})$ (with ℓ left adjoint of $\mathbf{R}i$), and $\ell \circ \mathbf{R}i \cong id_{D(\mathcal{C})}$ by the very definition of absolute total derived functor (taking $H = \ell$).

Next, the fact that $(\ell, \mathbf{R}i)$ is an adjoint pair of functors assures that $\mathbf{R}i$ is left t-exact both for the natural t-structure on $D(\mathcal{C})$ and the natural t-structure on $D(\mathcal{D})$, and for the t-structures $(t(\mathcal{T}), t(\mathcal{F}))$ in $D(\mathcal{C})$ and $(t(\mathcal{X}), t(\mathcal{Y}))$ in $D(\mathcal{D})$ (see [**?** , 1.3.16]). To prove the latter statement it is enough to show that fixed $F^\bullet \in t(\mathcal{F})$ for any $X^\bullet \in t(\mathcal{X})[1]$ we have $D(\mathcal{D})(X^\bullet, \mathbf{R}i(F^\bullet)) = 0$. This follows from the isomorphism:

$$D(\mathcal{D})(X^\bullet, \mathbf{R}i(F^\bullet)) \cong D(\mathcal{C})(\ell(X^\bullet), F^\bullet) = 0$$

since $\ell(X^\bullet) \in \ell(t(\mathcal{X}))[1] \subseteq t(\mathcal{T})[1]$ and $F^\bullet \in t(\mathcal{F})$. The first statement is proved similarly. Now let $\tau_{t(\mathcal{X})} \colon D(\mathcal{D}) \to t(\mathcal{X})$ be the right adjoint of the inclusion $t(\mathcal{X}) \to D(\mathcal{D})$ (see [**?** , Proposition 1.3.3.(i)]). Then the restriction of the composition $\tau_{t(\mathcal{X})} \circ \mathbf{R}i$ to $\mathcal{H}_\mathcal{C}$ gives a functor $i_\mathcal{H} \colon \mathcal{H}_\mathcal{C} \to \mathcal{H}_\mathcal{D}$ and it is easy to see that $\ell_\mathcal{H}$ is left adjoint of $i_\mathcal{H}$ by composing the previous adjunctions.

Next, using Lemma 5.4 we have that

$$\begin{aligned} \ell_\mathcal{H} \circ i_\mathcal{H} = {} & \ell \circ \tau_{t(\mathcal{X})} \circ \mathbf{R}i_{|\mathcal{H}_\mathcal{C}} \cong \tau_{t(\mathcal{T})} \circ \ell \circ \mathbf{R}i_{|\mathcal{H}_\mathcal{C}} \\ \cong {} & \tau_{t(\mathcal{T})} \circ id_{\mathcal{H}_\mathcal{C}} \cong id_{\mathcal{H}_\mathcal{C}} \end{aligned}$$

and from this we conclude that $i_\mathcal{H}$ is fully faithful.

Finally,

$$i_\mathcal{H} \circ \ell_\mathcal{H}(\mathcal{X}[0]) \subseteq \tau_{t(\mathcal{X})} \circ (\mathbf{R}i \circ \ell)(D^{\geq 0}(\mathcal{D})) \subseteq \tau_{t(\mathcal{X})}(D^{\geq 0}(\mathcal{D})) \subseteq \mathcal{X}[0].$$

□

Remark 5.7. Following the previous notations, point 3 of Theorem 5.6 yields the following formula:

$$\mathcal{H}_\mathcal{D} / \mathcal{S}_\mathcal{H} \simeq \mathcal{H}_{\mathcal{D}/\mathcal{S}}.$$

Remark 5.8. Let us explain two examples in which one can apply Theorem 5.6. As a first example let $\mathcal{C}$ be an abelian category satisfying $AB4^*$ (that is, small products exist in $\mathcal{C}$ and such products are exact in $\mathcal{C}$) and with enough injectives, and $i \colon \mathcal{C} \to \mathcal{D}$ is an additive functor. Then, by Remark 5.3, i admits an ab-tot right derived functor. Another interesting case is the one in which the category $\mathcal{C}$ admits enough i-acyclic objects. In this case one can use the same argument as in Theorem 5.6 restricted to the bounded below derived categories in order to obtain an analogous result.

The next result shows that the context described in Theorem 5.6 is as general as possible.

Theorem 5.9. *Let $\mathcal{D}$ be an abelian category endowed with a torsion pair $(\mathcal{X}, \mathcal{Y})$ and let $\mathcal{H}_\mathcal{D}$ be the corresponding heart with respect to the t-structure on $D(\mathcal{D})$ induced by $(\mathcal{X}, \mathcal{Y})$. Let $\mathcal{S}'$ be a Serre subcategory of $\mathcal{H}_\mathcal{D}$ and $\ell' : \mathcal{H}_\mathcal{D} \to \mathcal{C}' := \mathcal{H}_\mathcal{D}/\mathcal{S}'$ be its corresponding quotient functor, and let us suppose that $(\ell'(\mathcal{Y}[1]), \ell'(\mathcal{X}[0]))$ is a torsion pair on $\mathcal{C}'$. Then:*

1. *The class $\mathcal{S} = \{D \in \mathcal{D} \mid \ell'(H^i_t(D)) = 0\ \forall i \in \mathbb{Z}\}$ is a Serre subcategory of $\mathcal{D}$.*
2. *Denoted by $\mathcal{C} := \mathcal{D}/\mathcal{S}$ the quotient category and by $\ell : \mathcal{D} \to \mathcal{C}$ the quotient functor. Then ℓ is exact and the classes $(\ell(\mathcal{X}), \ell(\mathcal{Y}))$ define a torsion pair on $\mathcal{C}$.*
3. *There is a canonical functor $\varphi : \mathcal{C}' \to \mathcal{H}_\mathcal{C}$ such that $\ell_\mathcal{H} = \varphi \circ \ell'$ (for the definition of $\ell_\mathcal{H}$ see Remark 5.5). The functor φ is an equivalence of categories.*
4. *Moreover in the case in which the torsion-free class $\mathcal{Y}$ generates $\mathcal{D}$ (or dually if the torsion class $\mathcal{X}$ cogenerates $\mathcal{D}$) and if $(\mathcal{H}_\mathcal{D}, \mathcal{C}', \ell', i')$ is a distinguished Giraud subcategory such that i' admits an ab-tot right derived functor, then the functor ℓ admits a right adjoint i such that the $(\mathcal{D}, \mathcal{C}, \ell, i)$ is a distinguished Giraud subcategory of $\mathcal{D}$ which induces the distinguished Giraud subcategory $\mathcal{C}'$ of $\mathcal{H}_\mathcal{D}$.*

Proof. 1. We have to prove that given a short exact sequence $0 \to S_1 \to S \to S_2 \to 0$ in $\mathcal{D}$ the middle term S belongs to $\mathcal{S}$ if and only if $S_1, S_2 \in \mathcal{S}$ where $\mathcal{S}$ is defined as $\mathcal{S} = \{D \in \mathcal{D} \mid \ell'(H^i_t(D)) = 0\ \forall i \in \mathbb{Z}\}$. Now, any short exact sequence on $\mathcal{D}$ defines a distinguished triangle in $D(\mathcal{D})$ and so one obtain the long exact sequence in $\mathcal{H}_\mathcal{D}$

$$\cdots H^{-1}_t(S_2) \to H^0_t(S_1) \to H^0_t(S) \to H^0_t(S_2) \to H^1_t(S_1) \to H^1_t(S) \to H^1_t(S_2) \to H^2_t(S_1) \cdots \tag{14}$$

By 4.5, $H^{-1}_t(S_2) = 0 = H^2_t(S_1)$ and for any $D \in \mathcal{D}$ one has $H^0(D) = t(D)[0]$ as a complex concentrated in degree 0 while $H^1(D) = \frac{D}{t(D)}[1]$. So the sequence (14) reduces to the sequence in $\mathcal{H}_\mathcal{D}$

$$0 \to t(S_1)[0] \to t(S)[0] \to t(S_2)[0] \to \frac{S_1}{t(S_1)}[1] \to \frac{S}{t(S)}[1] \to \frac{S_2}{t(S_2)}[1] \to 0. \tag{15}$$

Let us recall that the class

$$\mathcal{S}' = \{E \in \mathcal{H}_\mathcal{D} \mid \ell'(E) = 0\} \tag{16}$$

is a Serre subcategory of $\mathcal{H}_\mathcal{D}$. So from one side it is clear that if $S_1, S_2 \in \mathcal{S}$ then $t(S_i)[0]$, $\frac{S_i}{t(S_i)}[1] \in \mathcal{S}'$ for any $i \in \{1, 2\}$, which implies that $t(S)[0]$ and $\frac{S}{t(S)}[1]$ belong to $\mathcal{S}'$, and so $S \in \mathcal{S}$.

On the other side if $S \in \mathcal{S}$ then $t(S)[0]$, $\frac{S}{t(S)}[1] \in \mathcal{S}'$, and by applying the functor ℓ' (which is exact by hypothesis) to (14) we obtain the exact sequence in $\mathcal{C}'$

$$0 \to \ell'(t(S_1)[0]) \to 0 \to \ell'(t(S_2)[0]) \to \ell'\left(\frac{S_1}{t(S_1)}[1]\right) \to 0 \to \ell'\left(\frac{S_2}{t(S_2)}[1]\right) \to 0.$$

This proves that $t(S_1)[0]$, $\frac{S_2}{t(S_2)}[1] \in \mathcal{S}'$ and $\ell'(t(S_2)[0]) \cong \ell'\left(\frac{S_1}{t(S_1)}[1]\right) \in \ell'(\mathcal{X}[0]) \cap \ell'(\mathcal{Y}[1]) = 0$ which proves that $t(S_2)[0]$, $\frac{S_1}{t(S_1)}[1] \in \mathcal{S}'$ and so $S_2 \in \mathcal{S}$ and $S_1 \in \mathcal{S}$.

2. The proof relies on the same argument we have used in the proof of point 2 of Theorem 5.6.

3. First applying Remark 5.5 we see that the functor ℓ previously defined induces an exact functor $\ell_\mathcal{H} : \mathcal{H}_\mathcal{D} \to \mathcal{H}_\mathcal{C}$. Let us observe that given x a morphism in $\mathcal{D}$ and considering $X^\bullet := [X^{-1} \xrightarrow{x} X^0]$ as a complex with X^0 placed in degree 0 we have: $\mathrm{Ker}(x) \in \mathcal{S} \cap \mathcal{Y}$ if and only if $\mathrm{Ker}(x)[1] = H^1_t(X^\bullet) \in \mathcal{S}'$ and $\mathrm{Coker}(x) \in \mathcal{S} \cap \mathcal{X}$ if and only if $\mathrm{Coker}(x)[0] = H^0_t(X^\bullet) \in \mathcal{S}'$. We will use this argument in order to prove that $\mathcal{S}' = \mathcal{S}_\mathcal{H}$.

So given $X^\bullet := [X^{-1} \xrightarrow{x} X^0] \in \mathcal{S}_\mathcal{H}$ we have $\ell(X^{-1}) \xrightarrow{\ell(x)} \ell(X^0)$ is zero in $\mathcal{H}_\mathcal{C}$, that is: $\mathrm{Ker}(\ell(x)) = \ell(\mathrm{Ker}(x)) = 0$ and $\mathrm{Coker}(\ell(x)) = \ell(\mathrm{Coker}(x)) = 0$. This proves that $\mathrm{Ker}(x) \in \mathcal{S} \cap \mathcal{Y}$ and so $\mathrm{Ker}(x)[1] \in \mathcal{S}'$ and also $\mathrm{Coker}(x) \in \mathcal{S} \cap \mathcal{X}$ and so $\mathrm{Coker}(x)[0] \in \mathcal{S}'$. The short exact sequence $0 \to \mathrm{Ker}(x)[1] \to X^\bullet \to \mathrm{Coker}(x)[0] \to 0$ in $\mathcal{H}_\mathcal{D}$ proves that $X^\bullet \in \mathcal{S}'$ and hence $\mathcal{S}_\mathcal{H} \subseteq \mathcal{S}'$. On the other side if $X^\bullet := [X^{-1} \xrightarrow{x} X^0] \in \mathcal{S}'$ the short exact sequence $0 \to \mathrm{Ker}(x)[1] \to X^\bullet \to \mathrm{Coker}(x)[0] \to 0$ proves that both $\mathrm{Ker}(x)[1]$ and $\mathrm{Coker}(x)[0]$ belong to $\mathcal{S}'$ (since $\mathcal{S}'$ is a Serre class in $\mathcal{H}_\mathcal{D}$) and hence

$\mathrm{Ker}(x) \in \mathcal{S} \cap \mathcal{Y}; \mathrm{Coker}(x) \in \mathcal{S} \cap \mathcal{X}$ which proves that $\ell(x)$ is an isomorphism and so $\mathcal{S}' \subseteq \mathcal{S}_{\mathcal{H}}$.
Now by point 3 of Theorem 5.6 we have $C' := \mathcal{H}_{\mathcal{D}}/\mathcal{S}' = \mathcal{H}_{\mathcal{D}}/\mathcal{S}_{\mathcal{H}} \simeq \mathcal{H}_{\mathcal{D}/\mathcal{S}} = \mathcal{H}_{C}$.

4. Let us suppose that the torsion-free class $\mathcal{Y}$ generates $\mathcal{D}$. Then it is clear that $\ell(\mathcal{Y})$ generates the quotient category $\mathcal{D}/\mathcal{S}$ and so by [**?** , Theorem 8.2] the double heart $\mathcal{H}_{\mathcal{H}_{\mathcal{D}}}$ is equivalent to $\mathcal{D}$ and $\mathcal{H}_{\mathcal{H}_{C}} \cong C$.
If, moreover, $(\mathcal{H}_{\mathcal{D}}, C', \ell', i')$ is a distinguished Giraud subcategory such that i' admits an ab-tot derived functor, then we can apply Theorem 5.6 in order to obtain a distinguished Giraud subcategory on the associated hearts. This proves that the functor $\ell \cong \ell_{\ell_{\mathcal{H}}}$ admits a right adjoint i such that $(\mathcal{D}, C, \ell, i)$ is a distinguished Giraud subcategory of $\mathcal{D}$ which induces the distinguished Giraud subcategory C' of $\mathcal{H}_{\mathcal{D}}$. □

Remark 5.10. Following the previous notations, point 3 of Theorem 5.9 states that any quotient category C' of the heart $\mathcal{H}_{\mathcal{D}}$ (satisfying the condition that the quotient functor $\ell' : \mathcal{H}_{\mathcal{D}} \to C'$ moves the torsion pair) is equivalent to the heart of a quotient category of C. Moreover we remark that if we take $(C', \ell') = (\mathcal{H}_{C}, \ell_{\mathcal{H}})$ and we apply Theorem 5.9 we reconstruct exactly the category C.
Hence there is a one to one correspondence between quotient categories of $\mathcal{D}$ which moves the torsion pair $(\mathcal{X}, \mathcal{Y})$ and quotient categories of $\mathcal{H}_{\mathcal{D}}$ which moves the torsion pair $(\mathcal{Y}[1], \mathcal{X}[0])$.