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The Möbius symmetry of quantum mechanics

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Abstract.
The equivalence postulate approach to quantum mechanics aims to formulate quantum mechanics from a fundamental geometrical principle. Underlying the formulation there exists a basic cocycle condition which is invariant under D–dimensional Möbius transformations with respect to the Euclidean or Minkowski metrics. The invariance under global Möbius transformations implies that spatial space is compact. Furthermore, it implies energy quantisation and undefinability of quantum trajectories without assuming any prior interpretation of the wave function. The approach may be viewed as conventional quantum mechanics with the caveat that spatial space is compact, as dictated by the Möbius symmetry, with the classical limit corresponding to the decompactification limit. Correspondingly, there exists a finite length scale in the formalism and consequently an intrinsic regularisation scheme. Evidence for the compactness of space may exist in the cosmic microwave background radiation.

1. Introduction

The mathematical modelling of observational data on the smallest and largest distance scales currently rests on two main theories, quantum mechanics and general relativity. On the smallest scales quantum mechanics, through its incarnation as the Standard Model of Particle Physics, accounts for all subatomic data with a high degree of precision. The vindication of the Standard Model received strong support with the observation of a scalar boson resonance at the Large Hadron Collider (LHC) with the expected Standard Model properties. Higgs studies will dominate the experimental particle physics program in the next few decades, *i.e.*: measuring its couplings to the Standard Model fermions and vector bosons; measuring its self–couplings; and constraining deviation from the Standard Model by constraining higher dimensional nonrenormalisable operators suppressed by a higher energy scale.

On the largest scales general relativity receives ample support from observations in celestial, galactical and cosmological data. In seeking extensions to the two theories, it is hard to compare the two sets of data as the particle physics experiments rely on billions of controlled events and a thorough understanding of the background, whereas the astrophysical and, in particular, the cosmological data, rely on a few, or a single, event, and poor understanding of the background. This distinction is particularly important when seeking to synthesise the two theories, and the weight placed on observations in the two domains. It is also vital in the consideration of future experimental facilities and the prospect that they will lead to improved understanding of fundamental physics.

While quantum mechanics and general relativity are successful in accounting for observational data in their respective domains, their synthesis is nothing but settled. Furthermore, the two formalisms follow conceptually distinct approaches. The principles of equivalence and covariance with respect to general diffeomorphism underly general relativity. That is, general relativity follows from a fundamental geometrical principle. On the other hand, quantum mechanics does not follow from a geometrical principle. The main tenant of the axiomatic formulation of quantum mechanics is the probability interpretation of the wave–function.

Thus, quantum mechanics and general relativity follow fundamentally distinct approaches. However, the two theories are also fundamentally incompatible. This incompatibility is most clearly seen in the treatment of the vacuum and the vacuum energy. In quantum mechanics we have to define a vacuum state on which the quantum operators operate and create the physical states of the Hilbert space. The vacuum is the state which is annihilated by all annihilation operators. The vacuum should be bounded from below. One issue arises in quantum field theories due to the existence of an infinite number of states and the normal ordering ambiguity[1]. The zero point energy of the vacuum state therefore leads to an infinite contribution to the vacuum energy. In particle physics experiment we only measure energy differences and absorb additional infinities in physical parameters that are measured experimentally. If the number of parameters needed to absorb the infinities is finite the theory is said to be renormalizable. Otherwise it is said to be nonrenomalisable. The triumph of the Standard Model is precisely because it contains a relatively small number of such parameters, and is able to account for a much larger number of experimental observations. This opens the possibility that the Standard Model is a valid description of the physical data not only at a scale which is within reach of contemporary experiments, but up to a much larger energy scale. It is clear then that the first priority of forthcoming experiments is to continue to test the validity of the Standard Model at increasing energy scales, by using effective field theory approach to parameterise possible deviations. Furthermore, the set of parameters associated with the Higgs sector are particularly ripe for experimental picking. Gravity on the other hand is known to be nonrenomalizable. Furthermore, gravitational measurements are sensitive to the absolute energy scale. Observations dictate that the vacuum energy is miniscule compared to what we would expect from particle physics. The dichotomy between the two theories motivates much of the contemporary research in theoretical particle physics. Different approaches are pursued to develop a consistent theory that synthesises quantum mechanics and gravity. These include: effective field theories [2]; Euclidean quantum gravity [3]; asymptotic safety [4]; causal dynamical triangulation [5]; twistor theory [6]; noncommutative geometry [7]; loop quantum gravity [8]; string theory [9]. What are we to learn from this very partial list? First, we note that all these approaches aim at quantising general relativity, *i.e.* quantising spacetime. Second, non of these theories produces an unequivocal signature that has been confirmed experimentally, and in that respect, all these attempts should be regarded on equal footing. Nevertheless, the most developed effort is undoubtedly that of string theory. The main successes of string theory is that: 1. it provides a viable perturbative approach to quantum gravity; 2. it produce the gauge and matter structures that underly the Standard Model. As such it provides a framework for the construction of phenomenologically realistic models, and is therefore relevant for experimental observations. In that respect, by unifying the gauge and matter sectors with gravity, string theory provides a framework to study how the parameters of the Standard Model may arise from a more fundamental theory, and goes beyond the field theoretic Grand Unified Theories. Moreover, string theory accommodates consistently one of the most intriguing properties of the observed particles, that of chirality and with it parity violation. It achieves that by its most important property, modular invariance, which underlies the anomaly cancellation. By achieving that string theory demonstrated its potential relevance for experimental data, though we may still be a long way off before we can describe in a detailed and rigorous way how this relevance

[1] *For lucid introduction to quantum field theories, see e.g.* [1].

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is realised in the material world. The state of the art in this endeavour is the derivation of the Minimal Supersymmetric Standard Model from heterotic–string theory [10]. These models demonstrate how the detailed structures of the Standard Model may arise from string theory. In particular, the models enabled the calculation of the Yukawa couplings of the top quark, bottom quark and tau lepton in terms of the gauge coupling at the unification scale. Assuming consistency with the low scale gauge data then enables extrapolation of the Yukawa coupling to the low scale and prediction of the top quark mass of order $O(175GeV)$ [11] in the vicinity of the observed value. These results unambiguously demonstrate the potential relevance of string theory for low scale experimental data. Furthermore, the string consistency conditions dictate that additional extra degrees of freedom beyond the Standard Model, are needed to obtain a consistent theory of perturbative quantum gravity, which is an unambiguous prediction of the theory. These may be interpreted as extra spacetime dimensions, and/or as additional gauge symmetries beyond those observed in the Standard Model. It is a secondary question whether we possess the technological tools to detect experimentally these additional degrees of freedom. The myriad of string solutions which may in principle be compatible with the low energy data places an additional hindrance on extracting unambiguous experimental signatures from string theory. Given the hierarchical separation between the strength of the gravitational and gauge couplings it may be a long while before such an unambiguous correlation may be extracted. In that respect progress in string theory is likely to be incremental and slow. Another limitation is that our current understanding of string theory is limited to asymptotic stable string solutions, and we lack a good understanding of it as a dynamical theory. This hinders the development of cosmological string scenarios and of a possible dynamical vacuum selection mechanism in string theory. One then has to resort to analysis in the effective low energy field theory limit of the string vacua, which is deficient because it misses the massive string spectrum and the possible roll that it may play in the string vacuum selection mechanism. However, as long as the low scale data does not indicate departure from perturbative Standard Model parameterisation of the experimental observations, string theory continues to provide the most detailed framework to calculate the Standard Model parameters from a more fundamental theory.

String theory therefore provides a viable perturbative framework to explore how the Standard Model parameters may arise from a fundamental theory and to develop a phenomenological approach to quantum gravity. However, string theory does not provide a fundamental physical principle, akin to the equivalence principle of general relativity, that we may use as the basic hypothesis, and formulate quantum gravity rigoursely staring from that hypothesis. For that we may need an entirely new approach. In that respect string theory may be viewed as an effective theory, perhaps in a similar spirit to the view of effective quantum field theories as effective theories. However, string theory may hint at a possible basic hypothesis from which string theory may be derived as an effective limit.

An important property of string theory is the relation that it exhibits between different vacua by perturbative and non–perturbative transformations, and the existence of self–dual states under the duality transformation. We may envision that the self–dual states, which represent enhanced symmetry points in the space of vacua, may play a role in the string vacuum selection mechanism. We may also imagine that the duality structures underlying string theory may provide a hint at the basic hypothesis from whence string theory may be derived. In that context, however, it is noted that the string dualities correspond to specific dualities between concrete vacua and detailed mathematical structures. In that respect they provide examples that possess too much structure and are not sufficiently rudimentary to provide a basic physical hypothesis. It therefore does not provide a sufficiently basic physical hypothesis. Rather, we may consider that the perturbative and non–perturbative dualities exhibited in string theory may all be regarded as dualities in an extended phase space.

Phase space represents the key transition from the classical to the modern description of physical experiments. In that respect we need to establish what is the usefulness of a physical theory. We should first prioritise physics as an experimental science. We may define physics as "mathematical modelling of experimental observations". The aim of a mathematical model is to predict the outcome of experiments. Starting with some initial conditions, which are fed into the mathematical model, the predicted outcomes are calculated, and are then confronted with experiments. Physics is a practical field and an accepted mathematical model is the one most successful in accounting for a wide range of experimental observations. From that perspective, as our technological tools develop, our capacity to make experimental observations advances with time. Consequently, our mathematical modelling evolves with time to accommodate the expanding body of experimental data. This process is traditionally termed as reductionism. Namely, as experimental tools evolve we can resolve more refined physical distances. From celestial in the Galilean era to the sub-nuclear in the LHC era. The process of adapting our mathematical models to the increased body of experimental data may be labelled as "unification", and is likewise a constant theme in the mathematical modelling of experimental data. Thus, for example, Newton unified terrestrial and celestial observations in Newtonian mechanics; Maxwell combined the electric and magnetic forces into electromagnetism; Einstein synthesised mechanics and electromagnetism in special relativity; Glashow, Salam and Weinberg meshed quantum electrodynamics with Marshak and Sudarshan's vector minus axial–vector theory of the weak interactions into the electroweak model. The Standard Model that describe all the subatomic interactions as gauge theories may also be regarded as unifying the subatomic interactions under a single physical principle.

The list above represent the steps in the mathematical modelling of experimental data in which diverse observations are described in a common framework. The next stops on this route are the ones for which we do not yet have experimental evidence. They may therefore not necessarily be realised in nature. These include Grand Unified Theories which unify the electroweak and strong interactions in a simple or semi–simple group. The main prediction of Grand Unified Theories is proton decay. Global supersymmetry which combines fermions and bosons in common multiplets. The generic prediction of the simplest realisations of supersymmetry is the existence of superpartners. However, non–linear realisations of supersymmetry may exist, in which this will not be the case. Local supersymmetry implies the existence of a spin $3/2$ particle, and consequently the existence of a spin 2 particle. Local supersymmetry therefore unifies gravity with the gauge interactions, albeit in a nonrenormalisable theory.

In Newtonian mechanics the mathematical modelling of the physical systems uses the position coordinates of a particle in space and their derivative with respect to time, *i.e.* the velocities. Given the position and velocity at some initial time $t = t_i$, and given some force field $\vec{F}(\vec{x})$, the evolution of the position and velocity coordinates is determined by Newton's equation of motion. A fully equivalent classical description is provided by exchanging the velocities of the particles with their momenta. The transformation is from the configuration space to the phase space. An equivalent representation of Newton's equations of motion in classical mechanics is given given in terms of the Euler–Lagrange equations of motion, which are derived by defining a Lagrangian function in configuration space. By transforming from the configuration space to the phase space, we transform the Lagrangian to the Hamiltonian function in phase space by a Legendre transformation. In the process we transform the second order Euler–Lagrange equations to the set of first order Hamilton equations of motion. The price that we have to pay is the doubling of the number of equations. Namely for every second equation we have two first order equations. The Hamilton equations of motion are invariant, up to a sign, under exchange of coordinates and momenta. Similar to the Newton equations of motion, the Hamilton equations of motion account for the time evolution of the phase space variables.

The relevance of a physical theory is established by confronting a mathematical model with experimental data. Typically the experimental setup and the variables of the mathematical model may evolve with time. Therefore, an important set of variables in the mathematical model are those that do not change as the physical system evolves. These are the constants of the motions and are related to the symmetries of the physical systems.

2. The Hamilton–Jacobi Theory

A general method to solve a physical problem in classical mechanics is given by the Hamilton–Jacobi formalism. In the classical Hamilton–Jacobi formalism the solution of the physical problem is obtained by transforming the Hamiltonian from one set of phase–space variables to a new set of phase–space variables, such that Hamiltonian is mapped to a trivial Hamiltonian. We may refer to these transformations as trivialising transformations. Consequently, the new phase–space variables are constants of the motion, *i.e.*,

$$H(q,p) \quad \longrightarrow \quad K(Q,P)\equiv 0 \qquad \Longrightarrow \qquad \dot{Q}=\frac{\partial K}{\partial P}\equiv 0 ~,~ \dot{P}=-\frac{\partial K}{\partial Q}\equiv 0. \tag{1}$$

The solution to this problem is given by the Classical Hamilton–Jacobi Equation (CHJE), which in the stationary case is given by

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \quad + \quad V(q) \quad - \quad E \quad \equiv \quad \frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \quad + W(q) \quad = \quad 0. \tag{2}$$

The phase space variables are taken to be independent and their functional dependence is only extracted from the solution of the Hamilton–Jacobi equation via the relation

$$p=\frac{\partial S(q)}{\partial q}, \tag{3}$$

where the generating function $S(q)$ is Hamilton's principal function. The key property of quantum mechanics is that the phase space variables are not independent. Namely, represented as quantum operator they do no commute. We may therefore envision posing a similar problem to the Hamilton–Jacobi procedure, but one in which the phase–space variables are not treated as independent variables, but are rather related by (3).

3. The cocycle condition

We first consider the stationary problem in the one dimensional case [12, 13, 15, 16, 17]. This reveals the general properties of the formalism and paves the way for the general cases. We therefore assume that there always exist coordinate transformations such that $H\to K\equiv 0$, *i.e.* such that in the new system both the potential energy and the kinetic energy vanish. More generally, impose that there exist coordinate transformations such that in the transformed system $W(Q)=V(Q)-E\equiv 0$. We consider the transformations on

$$(~q~,~S_0(q)~,~p~=~\frac{\partial S_0}{\partial q}) \quad \longrightarrow \quad (~\tilde{q}~,~\tilde{S}_0(\tilde{q})~,~\tilde{p}~=~\frac{\partial \tilde{S}_0}{\partial \tilde{q}}~), \tag{4}$$

such that $W(q) \quad \longrightarrow \quad \tilde{W}(\tilde{q}) \; = \; 0$, exist for all $W(q)$. We refer to this proposition as the "equivalence postulate of quantum mechanics". It implies that all physical systems, labelled by a potential function $W(q)$, can be connected by coordinate transformations. More narrowly, we may regard it as adaptation of the classical Hamilton–Jacobi formalism to quantum mechanics. Irrespective of the concrete interpretation, it reveals the Möbius symmetry that underlies quantum mechanics. The equivalence postulate implies that the Hamilton–Jacobi (HJ) equation has to be covariant under coordinate transformations. However, from eq. (2) it is seen that the first term transforms as a quadratic differential, whereas the potential function $W(q)$, in general, does not. Furthermore, the state $W(q) \equiv 0$ is fixed under the transformations $q \to \tilde{q}(q)$. We therefore assume that the HJ equation is modified by adding a yet to be determined function $Q(q)$. The modified HJ equation then takes the form

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \quad + \quad W(q) \quad + \quad Q(q) \quad = \quad 0, \tag{5}$$

where under the transformations $q \to \tilde{q}(q)$ the two functions $W(q)$ and $Q(q)$ transform as

$$\begin{aligned}\tilde{W}(\tilde{q}) &= \left(\frac{\partial \tilde{q}}{\partial q}\right)^{-2} W(q) + (\tilde{q}; q),\\ \tilde{Q}(\tilde{q}) &= \left(\frac{\partial \tilde{q}}{\partial q}\right)^{-2} Q(q) - (\tilde{q}; q),\end{aligned}$$

with $\tilde{S}_0(\tilde{q}) \;=\; S_0(q)$. It is seen that each of the functions $W(q)$ and $Q(q)$ transforms as a quadratic differential up to an additive term and that the combination $W(q) + Q(q)$ transforms as a quadratic differential. Starting with the trivial state $W^0(q^0) \equiv 0$, all other physical states arise from the additive inhomogeneous term as $W(q) \equiv (q; q^0)$. Furthermore, considering the transformations $q^a \to q^b \to q^c$ and $q^a \to q^c$ and the induced transformations $W^a(q^a) \to W^b(q^b) \to W^c(q^c)$ and $W^a(q^a) \to W^c(q^c)$ gives rise to a cocycle condition on the inhomogeneous term given by

$$(q^a; q^c) \;=\; \left(\frac{\partial q^b}{\partial q^c}\right)^2 \Big[\, (q^a; q^b) \;-\; (q^c; q^b)\Big]. \tag{6}$$

It can then be shown that the cocycle condition is invariant under Möbius transformations $\gamma(q^a)$. In the one dimensional case the Möbius symmetry uniquely fixes the functional form of the inhomogeneous term to be given by the Schwarzian derivative, *i.e.* $(q^a; q^c) \;\sim\; \{q^a; q^c\}$,where the Schwarzian derivative is defined by

$$\{f(q), q\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{7}$$

4. The Quantum Hamilton–Jacobi Equation

Considering the Schwarzian identity

$$\left(\frac{\partial S_0}{\partial q}\right)^2 = \frac{\beta^2}{2}\left(\{\mathrm{e}^{\frac{i2S_0}{\beta}}; q\} - \{S_0; q\}\right), \tag{8}$$

we note that the quadratic differential on the left–hand side of the equation is written as the difference of two Schwarzian derivatives. As we will see, the cocycle condition eq. (6) and the Schwarzian identity eq. (8) lay down the key ingredients for the generalisations to higher dimensions. Equally fundamental is the invariance of the cocycle condition and of the Schwarzian derivative under Möbius transformations. It is proven rigoursely [18] that the corresponding cocycle condition in D–dimensions is invariant under D–dimensional Möbius transformations, with respect to the Euclidean or Minkowski metrics. Similarly, the generalisation of the

Schwarzian identity eq. (8) entails writing the quadratic differential on the left–hand side as a generalised identity on the right–hand side. Making the identifications

$$\begin{aligned} W(q) &= -\frac{\beta^2}{4m}\{\mathrm{e}^{\frac{i2S_0}{\beta}};q\} = V(q)-E, \\ Q(q) &= \frac{\beta^2}{4m}\{S_0;q\}, \end{aligned}$$

the modified Hamilton–Jacobi equation becomes,

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0;q\} = 0. \tag{9}$$

The key property of quantum mechanics is gleaned from eq (9). In the CSHJE admits the solution $S_0 = constant$ for the state $W^0(q^0) \equiv 0$. The Quantum Stationary Hamilton–Jacobi Equation (QSHJE), eq. (9), admits the solutions,

$$S_0 = \pm\frac{\beta}{2}\log q \neq Aq + B,$$

where A and B are constants. Thus, in quantum mechanics S_0 is never a constant, and more generally is never a linear function of q. This is a key property of quantum mechanics in this formalism, and is intimately related to consistency of phase–space duality. Furthermore, from the properties of the Schwarzian derivative we know that the function $W(q) = V(q) - E$ is a potential of a second order differential equation given by

$$\left(-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) - E\right)\Psi(q) = 0, \tag{10}$$

which we identify as the Schrödinger equation, and $\beta = \hbar$ as the covariantising parameter of the Hamilton–Jacobi equation. The general solutions of (10) and (9) are given by

$$\Psi(q) = \psi_1 + \psi_2 = \frac{1}{\sqrt{S_0'}}\left(A\mathrm{e}^{+\frac{i}{\hbar}S_0} + B\mathrm{e}^{-\frac{i}{\hbar}S_0}\right) \tag{11}$$

and

$$\mathrm{e}^{+\frac{i2S_0}{\hbar}} = \mathrm{e}^{i\alpha}\,\frac{w+i\bar{\ell}}{w-i\ell} \tag{12}$$

where $w = \psi_1/\psi_2$, $\ell = \ell_1 + i\,\ell_2$, and $\ell_1 \neq 0$, $\alpha \in R$. The condition $\ell_1 \neq 0$ is synonymous to the condition that $S_0 \neq constant$.

It is noted that the Schrödinger equation serves as a tool to solve the Quantum Hamilton–Jacobi Equation (QHJE). Hence, the more fundamental equation is the QHJE, and captures the symmetry properties that underly quantum mechanics. In that respect the solutions of the Schrödinger equation, eq. (11), facilitate the solution of the QHJE via eq. (12).

It is further seen that consistency of the formalism necessitates that both solutions of the Schrödinger equation are used. This is evident in eq. (12) from which we see that the ratio of the two solutions of the Schrödinger equation is used to extract the solution for the the QHJE. It is a reflection of the Möbius symmetry that underlies the formalism, and has implication on the global geometry that underlies quantum mechanics and quantum gravity. The basic point is that the Möbius symmetry includes a symmetry under inversions. In the case of D–dimensional spacetime this can be implemented via inversions with respect to the unit sphere.

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More generally, the global geometry, whatever it may be, has to be invariant under the global Möbius transformations. In the case of spatial space, *i.e.* with Euclidean signature of the metric, the implication is that spatial space must be compact. Otherwise the invariance under the Möbius symmetry cannot be applied consistently. Therefore, at the basic level what we face in comparison to conventional quantum mechanics, is a question of boundary conditions. Namely, in the case of bound physical systems in conventional quantum mechanics the solutions of the Schrödinger equation include a square integrable solution and a solution that diverges at infinity. Therefore, the physical solution is retained, whereas the solution that diverges at infinity is non–physical and hence discarded. In essence, however, it is a question of boundary conditions. Namely, if space is infinite then the diverging solution may be discarded. However, if space is compact, as implied by the Möbius symmetry, then the diverging solution cannot be discarded and must be retained. The wave function is necessarily a combination of the two solutions, albeit the coefficient of the diverging solution may be unobservationally small. In a sense therefore the equivalence postulate approach may be regarded as conventional quantum mechanics plus the condition that spatial space is compact, as dictated by the Möbius symmetry that underlies quantum mechanics. Consistency of the formalism therefore entails that the transformation $W(q) = V(q) - E \longrightarrow \tilde{W}(\tilde{q}) = 0$ always exists, and is given by $q \to \tilde{q} = \psi_1/\psi_2$. Applying the transformation then entails that

$$\left(-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) - E\right)\psi(q) = 0 \quad \to \quad \frac{\partial^2}{\partial \tilde{q}^2}\tilde{\psi}(\tilde{q}) = 0 \tag{13}$$

where

$$\tilde{\psi}(\tilde{q}) = \left(\frac{dq}{d\tilde{q}}\right)^{-\frac{1}{2}}\psi(q).$$

5. Energy quantisation

In conventional quantum mechanics the probability interpretation implies that the wave function and its derivative are continuous and is square integrable. Consequently, for bound states the the energy level are quantised. The question is therefore, how is it replicated in the equivalence postulate approach? As the trivialising transformation is given by

$$q^0 = w = \frac{\psi_1}{\psi_2} = \frac{\psi^D}{\psi} \tag{14}$$

and is a solution of

$$\{w, q\} = -\frac{4m}{\hbar^2}(V(q) - E) \tag{15}$$

we have that $w \neq const$; $w \in C^2(R)$ *and* w'' differentiable on R. From the properties of the Schwarzian derivative under inversions it follows that $\{w, q^{-1}\} = q^4\{w, q\}$. Hence, the consistency conditions on the trivialising transformation must be imposed not merely on the real line, but on the extended real line, *i.e.* on the real line plus the point at infinity. That is,

$$w \neq const \;\; ; \;\; w \in C^2(\hat{R}) \;\; and \;\; w'' \text{ differentiable on } \hat{R}, \quad \text{where} \quad \hat{R} = R \cup \{\infty\}.$$

Consequently, the equivalence postulate implies the continuity of the two solutions of the Schrödinger equation and their derivatives. Furthermore, a general theorem that states that if the potential function $W(q)$ is bounding in some interval, then the ratio $w = \psi_1/\psi_2$ is continuous on the extended real line if and only if the Schrödinger equation admits a square integrable solution, is proven rigoursely in [15, 17, 19]. Rather than go through the theorem it is instructive to consider the simple problem of a particle in a potential well with

$$V(q) = \begin{cases} 0, & |q| \leq L, \\ V_0, & |q| > L. \end{cases} \tag{16}$$

Setting $k = \sqrt{2mE}/\hbar$ and $K = \sqrt{2m(V_0 - E)}/\hbar$, the general solutions in and outside the potential well are given by

$$\begin{array}{lll} |q| \leq L & \Psi_1^1 = \cos kq & \Psi_2^1 = \sin kq \\ q > L & \Psi_1^2 = \mathrm{e}^{-Kq} & \Psi_2^2 = \mathrm{e}^{Kq} \end{array}$$

The solutions at $q < -L$ are fixed by parity. For $q > L$ we can choose the solution to be Ψ_1^2 or Ψ_2^2 or a linear combination. Continuity across the boundary at $q = L$ implies that $\Psi_i^1(L) = \Psi_j^2(L)$ and $\partial_q \Psi_i^1(L) = \partial_q \Psi_j^2(L)$. Denoting such solutions as (i, j), in the $(1, 1)$ case imposing continuity on the solution and its derivative yields the quantisation condition $k\ \tan kL = K$. In this case

$$\psi = \begin{cases} \cos(kL)\exp[K(q+L)], & q < -L, \\ \cos(kq), & |q| \leq L, \\ \cos(kL)\exp[-K(q-L)], & q > L, \end{cases} \tag{17}$$

and a linearly independent solution is given by

$$\psi^D = [2k\sin(kL)]^{-1} \cdot \begin{cases} \cos(2kL)\exp[K(q+L)] - \exp[-K(q+L)], & q < -L, \\ 2\sin(kL)\sin(kq), & |q| \leq L, \\ \exp[K(q-L)] - \cos(2kL)\exp[-K(q-L)], & q > L. \end{cases} \tag{18}$$

The trivialising map $w = \psi^D/\psi$ associated with the $(1, 1)$ solution is therefore given by

$$\frac{\psi^D}{\psi} = [k\sin(2kL)]^{-1} \cdot \begin{cases} \cos(2kL) - \exp[-2K(q+L)], & q < -L, \\ \sin(2kL)\tan(kq), & |q| \leq L, \\ \exp[2K(q-L)] - \cos(2kL), & q > L. \end{cases} \tag{19}$$

In this case $\lim_{q\longrightarrow\pm\infty} \psi^D/\psi = \pm\infty$. Hence, in the $(1, 1)$ case the trivialising map is continuous on $\hat{R}$ as required by consistency of the equivalence postulate. The solutions imposed by the continuity conditions $k\ \tan kL = K$ are therefore physical energy levels. Next, considering the case $(1, 2)$ the constraint that $\Psi,\ \Psi'$ are continuous across at $q = L$ imposes the condition $k\ \tan(kL) = -K$. Applying similar analysis to the $(1, 1)$ case, the trivialising transformation w is give by

$$\frac{\psi^D}{\psi} = [k\sin(2kL)]^{-1} \cdot \begin{cases} \cos(2kL) - \exp[2K(q+L)], & q < -L, \\ \sin(2kL)\tan(kq), & |q| \leq L, \\ \exp[-2K(q-L)] - \cos(2kL), & q > L, \end{cases} \tag{20}$$

whose asymptotic behaviour is

$$\lim_{q\longrightarrow\pm\infty} \frac{\psi^D}{\psi} = \mp k^{-1}\cot(2kL). \tag{21}$$

The only possibility of having $w(-\infty) = w(+\infty)$ is that $k^{-1}\cot(2kL) = 0$. However, this is not compatible with the condition $k\ \tan(kL) = -K$. Hence, we have $w(-\infty) \neq w(+\infty)$. It follows that the energy eigenvalues associated with this solution are not consistent with the equivalence postulate and are therefore not physical. Therefore, the same physical eigenstates that are selected in convential quantum mechanics by the probability interpretation of the wave function, are selected in the equivalence postulate approach by mathematical consistency. Essentially respecting the Möbius symmetry that underlies quantum mechanics. We further note that the requirement that the trivialising transformation is continuous on the extended real line, amounts to the requirement that the real line is compact. Therefore, energy quantisation and square integrability arises from the consistency of the equivalence postulate and the compactness of space, which is mandated by the Möbius symmetry underlying quantum mechanics. However, the probabilistic nature of quantum mechanics, rather than a deterministic parameterisation of a particle propagation is indicated by many experiments. A viable question is whether such parameterisation is consistent with the quantum Hamilton–Jacobi formalism and the Möbius symmetry that underlies it.

6. Time parameterisation

There are two primary approaches to define parameterisation of trajectories in quantum mechanics. The first is Bohmian mechanics [20, 21] in which time parameterisation is defined by identifying the conjugate momentum with the mechanical momentum, *i.e.* $p = \partial_q S = m\dot{q}$, where S is the solution of the QHJE. The second is Floyd's definition of time parameterisation by using Jacobi's theorem, $t = \partial_E S_0^{QM}$, where S_0^{QM} is the solution of the QSHJE. In classical mechanics the two definitions are compatible. Namely, setting $p = \partial_q \mathcal{S}_0^{\rm cl} = m\dot{q}$ gives

$$t - t_0 = m \int_{q_0}^{q} \frac{dx}{\partial_x \mathcal{S}_0^{\rm cl}} = \int_{q_0}^{q} dx \frac{\partial}{\partial E} \partial_x \mathcal{S}_0^{\rm cl} = \frac{\partial \mathcal{S}_0^{\rm cl}}{\partial E}, \tag{22}$$

and by inverting we can solve the equation of motion for $q = q(t)$. However, in quantum mechanics the two definitions are not compatible as

$$t - t_0 = \frac{\partial S_0^{\rm qm}}{\partial E} = \frac{\partial}{\partial E} \int_{q_0}^{q} dx \partial_x S_0^{\rm qm} = \left(\frac{m}{2}\right) \int_{q_0}^{q} dx \frac{1 - \partial_E Q}{(E - V - Q)^{1/2}}. \tag{23}$$

and the mechanical momentum is given by

$$m\frac{dq}{dt} = m \left(\frac{dt}{dq}\right)^{-1} = \frac{\partial_q S_0^{\rm qm}}{(1 - \partial_E \mathcal{V})} \neq \partial_q S_0^{\rm qm}, \tag{24}$$

where $\mathcal{V}$ denotes the combined potential $\mathcal{V} = V(q) + Q(q)$. Thus in quantum mechanics the definition of time by using Jacobi's theorem does not coincide with the Bohmian definition of time by identifying the conjugate momentum with the mechanical momentum. Furthermore, the Bohmian time definition is not compatible with the Möbius symmetry that underlies quantum mechanics, and the compactness of space. In Bohmian mechanics the wave function is set as

$$\psi(\vec{q}, t) = R(q) {\rm e}^{iS/\hbar}, \tag{25}$$

where $R(q)$ and $S(q)$ are the two real functions of the QHJE, and $\psi(q)$ is a solution of the Schrödinger equation. The conjugate momentum is given by

$$p = \hbar {\rm Im} \frac{\nabla \psi}{\psi},$$

which is used to define trajectories by setting $p = m\dot{q}$. However, the Möbius and compactness of space dictate that the wave function cannot be identified by (25) but must be a linear combination of the two solutions of the Schrödinger equation,

$$\psi = R(q) \left(A{\rm e}^{\frac{i}{\hbar}S} + B{\rm e}^{-\frac{i}{\hbar}S}\right). \tag{26}$$

Consequently, in this case

$$\nabla S \neq \hbar {\rm Im} \frac{\nabla \psi}{\psi}$$

and the Bohmian definition of trajectories is invalid [23].

Floyd proposed to define time by using Jacobi's theorem [22], *i.e.*

$$t - t_0 = \frac{\partial S_0^{\rm qm}}{\partial E}, \tag{27}$$

which provides a time parameterisation of the trajectories by inverting $t(q) \to q(t)$. However, if space is compact, as dictated by the Möbius symmetry that underlies the QHJE, then the energy levels are always quantised. Therefore, differentiation with respect to energy is ill defined, and the definition of time parameterisation of trajectories by using Jacobi's theorem is inconsistent in the Quantum Hamilton–Jacobi formalism. In quantum mechanics time can only be used as a classical background parameter. The trajectory representation can only be used in a semi–classical approximation and in that context provides a useful tool to study different physical systems [21].

7. Generalisations

The discussion so far focussed on the one dimensional stationary case. This led to the cocycle condition, eq. (6), and the Schwarzian identity, eq. (8), and their invariance under M¨obius transformations. These are the key ingredients that pave the way for the generalisations to higher dimensions in Euclidean or Minkowski space, *i.e.* for the non-relativistic Schrödinger equation or the relativistic Klein–Gordon equation [18]. Furthermore, these generalisations are invariant under Möbius transformations in those spaces [18]. Thus, the Möbius symmetry captures the global symmetry structure that underlies quantum mechanics and dictates that spatial space is compact. Considering the transformations between two D–dimensional coordinate systems given by

$$q \to q^v = v(q) \tag{28}$$

and the induced transformations on the conjugate momenta

$$p_k = \frac{\partial S_0}{\partial q_k}, \tag{29}$$

with $S_0^v(q^v) = S_0(q)$. Consequently,

$$p_k \to p_k^v = \sum_{i=1}^{D} J_{ki} p_i \tag{30}$$

and the Jacobian matrix J is given by

$$J_{ki} = \frac{\partial q_i}{\partial q_j^v}. \tag{31}$$

Adopting the notation

$$(p^v|p) = \frac{\sum_k (p_k^v)^2}{\sum_k p_k^2} = \frac{p^t J^t J p}{p^t p}. \tag{32}$$

the cocycle condition takes the form

$$(q^a; q^c) = (p^c|p^b)\left[(q^a; q^b) - (q^c; q^b)\right]. \tag{33}$$

The cocycle condition, eq. (33) is invariant under D–dimensional Möbius transformations, which include translations, rotations, dilatations, and reflections with respect to the unit sphere [18].

In the case of Minkowski spacetime the invariance is with respect to the $D+1$–dimensional conformal group, where $q \equiv (ct, q_1, \cdots, q_D)$. The second key ingredient of the one dimensional formalism is the Schwarzian identity, equation (8). The generalisation of this identity provides the key for the extension of the formalism to higher dimensions in Euclidean and Minkowski spaces. In the non–relativistic case the identity is given by

$$\alpha^2(\nabla S_0)^2 = \frac{\Delta(R\mathrm{e}^{\alpha S_0})}{R\mathrm{e}^{\alpha S_0}} - \frac{\Delta R}{R} - \frac{\alpha}{R^2}\nabla\cdot(R^2\nabla S_0), \tag{34}$$

which holds for any two real functions R and S_0 and any constant α. Similarly, in the relativistic case the generalisation is given by [18]

$$\alpha^2(\partial\mathcal{S})^2 = \frac{\Box(Re^{\alpha\mathcal{S}})}{Re^{\alpha\mathcal{S}}} - \frac{\Box R}{R} - \frac{\alpha}{R^2}\partial\cdot(R^2\partial\mathcal{S}), \tag{35}$$

whereas if we include minimal coupling in the form of a vector potential the identity takes the form

$$\alpha^2(\partial S - eA)\cdot(\partial S - eA) = \frac{D^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \frac{\alpha}{R^2}\partial\cdot\Big(R^2(\partial S - eA)\Big), \tag{36}$$

where $D^\mu = \partial^\mu - \alpha e A^\mu$. Similarly to the one dimensional case, the higher dimensional cases are related to the conventional quantum mechanical equations. For example, in the case of eq. (35), setting $\alpha = i/\hbar$, we note that

$$\partial(R^2\cdot\partial\mathcal{S}) = 0, \tag{37}$$

and

$$\frac{1}{2m}(\partial\mathcal{S})^2 = -\frac{\hbar^2}{2m}\frac{\Box(Re^{\frac{i}{\hbar}\mathcal{S}})}{Re^{\frac{i}{\hbar}\mathcal{S}}} + \frac{\hbar^2}{2m}\frac{\Box R}{R}. \tag{38}$$

In analogy to the one dimensional identity we set

$$\mathcal{W}_{rel} = \frac{\hbar^2}{2m}\frac{\Box(Re^{\frac{i}{\hbar}\mathcal{S}})}{Re^{\frac{i}{\hbar}\mathcal{S}}}, \tag{39}$$

that by (38) implies

$$Q_{rel} = -\frac{\hbar^2}{2m}\frac{\Box R}{R}. \tag{40}$$

Setting $\mathcal{W}_{rel} = 1/2mc^2$ reproduces the Klein–Gordon equation and Eq. (38) corresponds to the Relativistic Quantum Hamilton–Jacobi Equation (RQHJE) [18]. We further remark that the two particle case was considered in [24].

8. Phase space duality and Legendre transformations

We noted from eq. (5) that the modified Hamilton–Jacobi equation allows for non–trivial solutions for the physical system with $W(q) \equiv 0$. The QHJE therefore enables all physical states labelled by the potential function $W(q)$ to be connected to the trivial state via coordinate transformations, and facilitates the covariance of the QHJE. These properties are intimately related to phase space duality [12], which is implemented by the involutive nature of Legendre transformations. Manifest phase–space duality may therefore provide the fundamental physical principle that is sought as the axiomatic principle for formulating quantum gravity. In this respect we note that perturbative and nonperturbative dualities play an important role in attempts to develop a fundamental understanding of string theory, with T–duality being an important perturbative property of string theory [25]. T–duality in toroidal spaces exchanges momentum modes with winding modes. We may therefore view T–duality as phase–space duality in compact space. Furthermore, we may question whether T–duality reflects a property of string theory which is also valid in non–toroidal spaces. An additional important property of T duality in string theory is the existence of self–dual states under T–duality.

Manifest phase–space duality is implemented by the involutive nature of Legendre transformations. Recalling the relation between the momenta and coordinates via a generating function $p = \partial_q S$, we define a dual relation via a new generating function $T(p)$ as $p = \partial T$, where the generating functions are related by Legendre transformations as

$$S \; = \; p\frac{\partial T}{\partial p} \; - \; T \qquad , \qquad T \; = \; q\frac{\partial S}{\partial q} \; - \; S, \tag{41}$$

which in the stationary case reduces to

$$S_0 = p\frac{\partial T_0}{\partial p} - T_0 \qquad , \qquad T_0 = q\frac{\partial S_0}{\partial q} - S_0. \tag{42}$$

Remarkably, the left–hand side of eq. (42) is invariant under Möbius transformations

$$q \;\longrightarrow\; q^v \;=\; \frac{Aq+B}{Cq+D},$$

with the induced transformations on p and T_0

$$\begin{array}{ccccc} p & \longrightarrow & p^v & = & \rho^{-1}(Cq+D)^2 p \;\;,\quad \rho \;= AD - BC \\ T_0 & \longrightarrow & T_0^v(p^v) & = & T_0(p) \;+\; \rho^{-1}(ACq^2 + 2BCq + BD)p. \end{array}$$

We define a general coordinate transformation $q \to q^v = v(q)$ by the property that S_0 is a scalar function under v, *i.e.* $S_0^v(q^v) = S_0(q)$. With each Legendre transformation we associate a second order differential equation [12, 17], which in the stationary case is given by

$$\left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p}\end{pmatrix} \;= 0 \tag{43}$$

where $U(S_0)$ is given by the Schwarzian derivative of q with respect to S_0,

$$U(S_0) = \frac{1}{2}\{q, S_0\} = \frac{1}{2}\left(\frac{q'''}{q'} - \frac{3}{2}\left(\frac{q''}{q'}\right)^2\right).$$

We may associate a second order differential equation with the involutive Legendre transformation. We therefore obtain manifest $p \leftrightarrow q - S_0 \leftrightarrow T_0$ duality with

$$\begin{array}{cccccc} p & = & \dfrac{\partial S_0}{\partial q} & \qquad & q \;=\; \dfrac{\partial T_0}{\partial p} \\ S_0 & = & p\dfrac{\partial T_0}{\partial p} - T_0 & \qquad & T_0 \;=\; q\dfrac{\partial S_0}{\partial q} - S_0 \end{array}$$

$$\left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p}\end{pmatrix} = 0 \qquad \left(\frac{\partial^2}{\partial T_0^2} + \mathcal{V}(T_0)\right)\begin{pmatrix} p\sqrt{q} \\ \sqrt{q}\end{pmatrix} = 0$$

However, the crucial point is the existence of self–dual states, with the property that $pq = \gamma = constant$, which are simultaneous solutions of the two pictures. In these cases $S_0 = -T_0 + constant$, and

$$S_0(q) = \gamma \ln \gamma_q q \qquad\qquad T_0(p) = \gamma \ln \gamma_p . p$$

Hence, $S_0 + T_0 = pq = \gamma$, where $\gamma_q\gamma_p\gamma = \mathrm{e}$ and γ_q, γ_p are constants. The remarkable point is that the self–dual states coincide with the $W^0(q^0) \equiv 0$ states of the Quantum Hamilton–Jacobi Equation (QHJE), which render the consistency of the equivalence postulate for all physical states, and the compatibility of quantum mechanics with the underlying Möbius symmetry of the QHJE. In the case of the Classical Hamilton–Jacobi Equation (CHJE), it was noted that the solution in the case with $W^0(q^0) \equiv 0$ correspond to $S_0 = constant$, or more generally $S_0 = Aq^0 + B$, with constants A and B. The Legendre transformation is not defined for linear functions and therefore classically in these case the phase–space duality would not be well defined. The existence of the self–dual quantum mechanical solutions, *i.e.* in the case with $W^{sd} = W^0 = 0$ and with $\gamma^{sd} = \pm\hbar/(2i)$, facilitate the consistency of phase–space duality for all physical states, as well as the consistency of the equivalence postulate, and compatibility with the underlying Möbius symmetry. It is crucial to appreciate that these properties merely accommodate the basic quantum mechanical properties, and in that sense they are not esoteric at all. Namely, in this approach the emergence of $\hbar$ as the basic quantum mechanical parameter, arises as the covariantising parameter in the QHJE, and enables the consistency of the formalism. Furthermore, it is noted that distinction between the classical and quantum mechanical cases in this approach is primarily reflected in the distinction of the $W^0 \equiv 0$ state. In this respect it will not be surprising if the formalism offers an intrinsic regularisation scheme. Furthermore, this intrinsic regularisation scheme is rooted in the global Möbius symmetry that underlies quantum mechanics. This is again not a surprise because the Möbius symmetry implies that spatial space is compact. In turn this implies the existence of a finite length scale in the formalism and therefore an intrinsic regularisation scale. Therefore, a deeper understanding of the implications of the Möbius symmetry that underlies quantum mechanics, brings forth the fertile soil on which the seeds of quantum gravity may grow.

9. Intrinsic length scale

The Schrödinger equation in the physical state with $W^0(q^0) = 0$ is given by

$$\frac{\partial^2\Psi}{\partial q^2} \; = \; 0 \; ,$$

and has two solutions $\psi_1 = q^0$ and $\psi_2 = constant$, which by the Möbius symmetry must both be included in the formalism. The duality, manifested by the invariance under the Möbius transformations, therefore imply the existence of a length scale in the formalism. The corresponding solution of the QHJE is given by [12, 17, 16]

$$\mathrm{e}^{\frac{2i}{\hbar}S_0^0} = \mathrm{e}^{i\alpha}\frac{q^0 + i\bar{\ell}_0}{q^0 - i\ell_0}, \tag{44}$$

where the constant ℓ_0 has the dimension of length [17, 16], and the conjugate momentum takes the form

$$p_0 = \partial_{q^0} S_0^0 = \pm\frac{\hbar(\ell_0 + \bar{\ell}_0)}{2|q^0 - i\ell_0|^2}. \tag{45}$$

It is seen that p_0 is null only when $q^0 \to \pm\infty$. As we noted in section 4 the condition that $\mathrm{Re}\ell_0 \neq 0$ is synonymous to the condition that $S_0 \neq constant$, which is the basic quantum mechanics property in the equivalence postulate formalism. We can represent this nonvanishing length parameter as a combination of some fundamental constants in nature. *e.g.* $\hbar$, c and G. The requirement that $\lim_{\hbar\to 0} p_0 = 0$ in the classical limit implies that we can identify $\mathrm{Re}\,\ell_0$ with the Planck length, [16, 17]

$$\mathrm{Re}\,\ell_0 = \lambda_p = \sqrt{\frac{\hbar G}{c^3}}, \tag{46}$$

The reason being that this identification has the correct scaling properties to reproduce the correct classical limit. Furthermore, we note from eq. (45) that the condition $\mathrm{Re}\ell_0 \neq 0$ serves as an ultraviolet cutoff, *i.e.* p_0 is maximal for $q^0 = -\mathrm{Im}\ell_0$, with

$$\mathrm{Max}|p_0| \;=\; \frac{\hbar}{\mathrm{Re}\ell_0}.$$

Therefore, the consistency of the equivalence postulate formalism with the underlying Möbius symmetry, implies the existence of an intrinsic regularisation scale in quantum mechanics. Additionally, we may identify the quantum potential as an intrinsic curvature term of elementary particles. This provides further evidence that in this approach quantum mechanics regularises itself and a possible connection with theories of extended objects. This is a mere reflection of the Möbius symmetry and the compactness of spatial space.

10. Quantum potential as a curvature term

Using the property of the Schwarzian derivative

$$\{S_0; q\} = -\left(\frac{\partial S_0}{\partial q}\right)^2 \{q; S_0\},$$

we can rewrite the Quantum Stationary Hamilton Jacobi Equation as,

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial \hat{q}}\right)^2 + V(\hat{q}) - E = 0,$$

where

$$\hat{q} = \int^q \frac{dx}{\sqrt{1 - \frac{\hbar^2}{2}\{q; S_0\}}}.$$

Flanders [26] have shown that the Schwarzian derivative can be interpreted as a curvature term of an equivalence problem for curves in P^1. Thus, the quantum potential, which is never vanishing, can be regarded as an intrinsic curvature term of elementary particle, and a deformation of the space geometry. Furthermore, we note from eq. (36) that the quantum potential as a universal character, which distinguishes it from the gauge interactions that are dependent on the gauge charges. In higher dimensions the quantum potential corresponds to the curvature of the function $R(q)$ as $Q(q) \sim \Delta R(q)/R$. We can estimate the scale of the quantum potential [27]. In the case $W^0(q^0) \equiv 0$ and using eq. (44) we obtain

$$Q^0 \;=\; \frac{\hbar^2}{4m}\{S_0^0, q^0\} \;=\; -\frac{\hbar^2(\mathrm{Re}\,\ell_0)^2}{2m}\frac{1}{|q^0 - i\ell_0|^4}. \tag{47}$$

Taking $m \sim 100GeV$; $\mathrm{Re}\ell_0 = \lambda_p \approx 10^{-35}m$ and q^0 as the size of the observable universe $q^0 \sim 93\mathrm{Ly}$, gives $|Q| \sim 10^{-202}eV$. The expected contribution of the quantum potential to the vacuum energy is very small. Nevertheless, we see from eq. (47) that the classical limit $Q(q) \to 0$ in fact correspond to the decompactification limit $q^0 \to \infty$.

11. Conclusions

The observation of a scalar resonance at the the LHC reinforces the picture that the Standard Model provides a viable effective description of all subatomic data up to the Planck scale. The synthesis of gravity with quantum mechanics remains an open problem. An alternative approach to quantising general relativity is to formulate a geometric approach to quantum mechanics. This is precisely the aim in the equivalence postulate approach to quantum mechanics. What is revealed is the key role of the Möbius symmetry that underlies quantum mechanics in this formalism. In turn the Möbius symmetry provides the key to a proper understanding of the geometry of the quantum spacetime. In this respect the formalism is intimately related to phase space duality manifested by the involutive nature of Legendre transformations. Recalling their role in thermodynamics we may envision that the Legendre transformations merely transfer from one set of variables to another, and neither set should be thought of as more fundamental. In this context the ubiquity of the variable themselves allows us to consider transformations between the space coordinates and the wave function itself [28], without considering one as being primary and the other secondary. Their only merit is their usefulness for a particular physical measurement. In this respect possible observational signatures of the Möbius symmetry underlying quantum mechanics may exist in the microwave background radiation [29]. Additionally, the quantum potential may lead to modified dispersion relations [30] with possible observational consequences [31].

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