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Hamilton–Jacobi meet Möbius

Alon E. Faraggi[1] **and Marco Matone**[2]

[1] Department of Mathematical Sciences, University of Liverpool, Liverpool L69 7ZL, UK
[2] Dipartimento di Fisica, Universit'a di Padova, Via Marzolo 8, I–35131 Padova, Italy

E-mail: `alon.faraggi@liv.ac.uk; marco.matone@pd.infn.it`

Abstract.
Adaptation of the Hamilton–Jacobi formalism to quantum mechanics leads to a cocycle condition, which is invariant under D–dimensional Möbius transformations with Euclidean or Minkowski metrics. In this paper we aim to provide a pedagogical presentation of the proof of the Möbius symmetry underlying the cocycle condition. The Möbius symmetry implies energy quantization and undefinability of quantum trajectories, without assigning any prior interpretation to the wave function. As such, the Hamilton–Jacobi formalism, augmented with the global Möbius symmetry, provides an alternative starting point, to the axiomatic probability interpretation of the wave function, for the formulation of quantum mechanics and the quantum spacetime. The Möbius symmetry can only be implemented consistently if spatial space is compact, and correspondingly if there exist a finite ultraviolet length scale. Evidence for non–trivial space topology may exist in the cosmic microwave background radiation.

1. Introduction

The possibility that the Standard Model of particle physics provides a viable parameterisation of subatomic data up to the Planck scale received substantial support from the observation of a Higgs–like particle at the LHC. While the properties of this Higgs–like particle will be the subject of experimental scrutiny in the decades to come, it is clear that fundamental understanding of the Standard Model parameters can only be gained by incorporating gravity into the picture. Alas, the synthesis of gravity and quantum mechanics remains an enigma. The majority of contemporary efforts entail the quantisation of gravity, *i.e.* the quantisation of the metric in Einstein's theory of general relativity. The most developed approach in this endeavour is string theory [1]. While many alternative approaches exist, which in principle should be regarded on equal footing, the key advantage of string theory is that, while producing a consistent approach to quantum gravity, it gives rise to the matter and gauge structures that arise in the Standard Model. Several examples of quasi–realistic string models have been constructed over the years, and include the free fermionic standard–like models [2]. This class of models produce solely the states of the Minimal Supersymmetric Standard Model in the Standard Model charged sector, and produced a successful prediction of the top quark mass [3]. String theory provides an effective perturbative approach to quantum gravity. Important properties of string theory include the various perturbative [4] and non–perturbative dualities [5]. While string theory, due to its relation to observational data, can be regarded as an important step on the road to the synthesis of gravity and quantum mechanics, it does not provide a satisfactory final answer. What we would like to have is a formulation of quantum gravity, which follows from a basic physical hypothesis, akin to equivalence principle underlying general relativity.

An alternative approach to the quantisation of gravity may be pursued by seeking the geometrisation of quantum mechanics. This is the aim of the equivalence postulate (EP) approach to quantum mechanics [6, 7, 8, 9, 10, 11, 12, 13]. The EP formalism may be regarded as adaptation of the Hamilton–Jacobi (HJ) formalism to quantum mechanics. In the classical HJ theory the solution to the mechanical problem is obtained by performing canonical transformation to a system in which the phase space variables are constants of the motion. The Hamiltonian in the new system therefore vanishes identically and the Hamilton equations of motion are zero. The transformation from the old set of to the new set treat the phase space variables as independent variables. The solution to this problem is given by the Classical Hamilton–Jacobi Equation (CHJE). The functional relation between the phase space variables is then extracted from the solution of the CHJE $S(q)$. We may a pose a similar question, but rather than performing canonical transformations that treat the phase space variables as independent variables, we seek a trivialising coordinate transformation from the system q^a to the system q^b, and the induced transformation $p^a = \partial_a S^a(q^a) \to p^b = \partial_b S^b(q^b)$. Without loss of generality we further impose that $S^a(q^a) = S^b(q^b)$, *i.e.* that $S(q)$ transform as a scalar function under the transformation. By trivialising we mean that in the new coordinate system the kinetic and potential energy vanish. However, it is clear that this proposition would not make sense in classical mechanics. The reason is that in classical mechanics we have the solution $S(q) = \textit{constant}$ for the state with vanishing kinetic energy and vanishing potential. In this case we have $p \equiv 0$, and this will remain the case after the transformation. This does not make sense because if we insist that all the physical states should be connectable to the free state with vanishing energy and vanishing potential, consistency dictates that the inverse transformation should exist as well. This means that we have to remove the solution $S(q) = \textit{constant}$ from the space of allowed solution. We therefore seek a modification of the CHJE that will remove the state $S(q) = \textit{constant}$ from the space of allowed solutions. In the stationary case the CHJE is given by

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \quad + \quad V(q) \quad - \quad E \quad = \quad 0. \tag{1}$$

We consider the modification of the CHJE by adding the function $Q(q)$ to the CHJE, whose properties are yet to be determined. We further require that the all the physical systems labelled by $W(q) = V(q) - E$ can be connected by coordinate transformations to the trivial state $W(q) \equiv 0$. The condition that $S(q) \neq \textit{constant}$ and that $S(q) = \textit{constant}$ can be reached only in the classical limit, entails that the classical limit coincides with the limit $Q(q) \to 0$.

Imposing these conditions consistently leads to a cocyle condition. The modified Hamilton–Jacobi equation corresponds to the Quantum Hamilton–Jacobi Equation (QHJE), and is related to the Schrödinger equation. The cocycle condition is invariant under D–dimensional Möbius transformations. This invariance is the key property of quantum mechanics in the EP approach, and is equivalent to the condition that $S(q) \neq \textit{constant}$. The Möbius symmetry implies that spatial space is compact and the the decompactification limit coincides with the classical limit, *i.e.* the limit in which $Q(q) \to 0$. Furthermore, it implies the existence of a finite length scale in the ultraviolet and correspondingly a finite length scale in the infrared. Hence, the Möbius symmetry has far reaching implications on the structure of the geometry, and correspondingly on the structure of the quantum space time. In this paper our aim is to provide a pedagogical presentation of the proof of the Möbius invariance of the cocycle condition.

2. The Hamilton–Jacobi Theory

Physics is first and foremost an experimental science, and may be defined as the mathematical modelling of experimental observations. A successful mathematical model is the one that can account for a wider range of experimental data. One may contemplate the possibility that there exist a representation of the physical laws, which is devoid of any experimental input, and follows from rigorous mathematical reasoning. The relation between the circumference of a circle to its diameter may attest to this possibility. Given the finite period allocated to an experiment, its resolution is similarly limited. Infinite resolution requires infinite time and the proposition that physics may have a completely mathematically rigorous representation, without resource to experimental input, is theological.

The methodology of the mathematical modelling of experimental observations is constructed as follows. Start with some initial conditions of a set of physical observables. Build a mathematical model for how this set of variables may evolve in the course of the physical process. Measure the set of variables in some experimental apparatus. Confront the observations by using the experimental apparatus with the prediction of the mathematical model. The key to the experimental methodology is that, given a proper set of instructions for the preparation of the initial conditions and the experimental apparatus, the outcome of the experimental measurements will be identical.

The mathematical modelling therefore rests on identifying a set of variables that are to be measured in the experiment. In the Newtonian system these may be the position and velocities of some objects. Say, a ball is dropped at some height h, with zero initial velocity. Then measure its position and velocity as a function of time. In this type of experiment the position and velocity change with time, which makes the experiment difficult. What we would like to do is to measure quantities that do not change with time. For example, if we can measure the initial energy and we know that the energy does not change with time, then we can measure the energy at the end of the experiment and we should get the same value. We can model the physical process from the start of the experiment and how the total energy is distributed among the constituents of the mathematical model. Ultimately, whatever happens between the start and the end of the experiment the total energy should be the same. What we are after are the constants of the motion. The Hamilton–Jacobi (HJ) formulation of classical mechanics is completely equivalent to Newtonian mechanics, and provides a general method to extract the constants of the motion for any physical system.

To achieve this fiat in classical mechanics we have to make a change of variables from configuration space to phase space. In configuration space the set of variables are the positions and velocities. In phase space the set of variables are the coordinates and the momenta. What we are after is to get rid of the explicit time dependence of the variables, so that the constants of the motion can be more readily extracted.

The Hamiltonian in classical mechanics is a function of the phase space variables. In systems in which the Hamiltonian does not depend explicitly on time the Hamiltonian corresponds to the total energy of the physical system and is a constant of the motion. The Hamilton equations of motion are given by

$$\dot{q} \ = \ \frac{\partial H}{\partial p} \ \ , \ \ \dot{p} = -\frac{\partial H}{\partial q} \tag{2}$$

The Hamilton–Jacobi procedure then follows by transforming the Hamiltonian to a trivial Hamiltonian, with the result that the new phase space–variables are constants of the motion, *i.e.*

$$H(q,p) \ \ \longrightarrow \ \ K(Q,P) \equiv 0 \qquad \Longrightarrow \qquad \dot{Q} = \frac{\partial K}{\partial P} \equiv 0 \ , \ \dot{P} = -\frac{\partial K}{\partial Q} \equiv 0$$

The solution is the Classical Hamilton–Jacobi Equation

$$H(q,p) \ \longrightarrow \ K(Q,P) \ = \ H(q, p = \frac{\partial S}{\partial q}) \ + \ \frac{\partial S}{\partial t} \ = \ 0 \ \ \Rightarrow \ \ \text{CHJE},$$

which in the stationary case becomes eq. (1). The transformations from the old to the new phase space variables, $(q,p) \to (Q,P)$ are canonical, which treat the phase space variables as independent variables. Their functional relation is then extracted from the solution of the HJ equation as $p = \partial_q S_0(q)$. We may reverse this order. Namely, we seek a trivialising coordinate transformation, such that $V(q) = E = 0$, but keeping the relation $p = \partial_q S_0(q) =$ when performing the transformations. The transformation are reversible coordinate transformations. Generality, then demands that we should be able to perform the reverse of the transformation from the trivial state to the non–trivial state. Furthermore, from the trivial state, we should be able to transform to any non–trivial state. However, as we discussed above this is not consistent in classical mechanics. As seen from eq. (1), while the first term transforms as quadratic differential, the second term, in general, does not. Furthermore, the state $W(q) = V(q) - E = 0$, corresponding to the solution $S_0 = constant$ with $p = 0$, is a fixed point under the transformations. Therefore, consistent implementation of this trivialising procedure requires the modification of the CHJE. As we elaborate in the next section the modification leads to a cocycle condition which is invariant under D–dimensional Möbius transformations. Our aim is to provide a pedagogical presentation of this proof. The Möbius symmetry is the fundamental property underlying quantum mechanics in our approach. It provides an alternative to the axiomatic probability interpretation of the wave function, and by that it provides a framework for rigorous formulation of quantum mechanics. Furthermore, invariance under the Möbius transformations reveals the existence of an inherent fundamental length in the formalism [10]. Proper implementation of the classical limit then shows that this length scale may be identified with the Planck length [10, 11]. In turn, the existence of a minimal length scale and the Möbius symmetry implies the existence of a maximal length scale. Furthermore, the limit in which the maximal length scale goes to infinity, *i.e.* the decompactification limit, correspondingly corresponds to the classical limit. The Möbius symmetry underlying the Quantum Hamilton–Jacobi Theory (QHJT), therefore carries within it an intrinsic ultraviolet regularisation length and correspondingly an infrared finite scale. The QHJT, with its underlying Möbius symmetry, provides the arena for the proper understanding of the quantum spacetime.

3. The cocycle condition

We seek the v–transformations

$$(\, q \, , \, S_0(q) \, , \, p \, = \, \frac{\partial S_0}{\partial q}) \quad \longrightarrow \quad (\, q^v \, , \, {S^v}_0(q^v) \, , \, p^v \, = \, \frac{\partial {S^v}_0}{\partial q^v} \,)$$

Such that $W(q) \quad \longrightarrow \quad W^0(q^0) \, = \, 0$ exist for all $W(q)$. In the following we impose the conditions

$$S^v(q^v) = S(q), \tag{3}$$

and in particular

$$S^0(q^0) = S(q). \tag{4}$$

The CSHJE for a particle of mass m and energy E moving in N dimensions under the influence of a static velocity independent potential $V(q)$ is

$$\frac{1}{2m}\sum_{i=1}^{N}\left(\frac{\partial S}{\partial q_i}\right)^2 + W(q) = 0, \tag{5}$$

where $W \equiv V - E$, and where the q_i are Cartesian coordinates. Under a change of coordinates $q \to q^v$ we have by (3)

$$\frac{\partial S^v(q^v)}{\partial q_j^v} = \frac{\partial S(q)}{\partial q_j^v} = \sum_i \frac{\partial S(q)}{\partial q_i}\frac{\partial q_i}{\partial q_j^v}, \tag{6}$$

which we can write as

$$\boldsymbol{p}^v = \mathbf{J}^v \boldsymbol{p} \tag{7}$$

where

$$J_{ij}^v = \frac{\partial q_i}{\partial q_j^v}$$

is the Jacobian matrix connecting the coordinate systems q and q^v and $p_i = \frac{\partial S}{\partial q_i}$. Then

$$\begin{aligned}\sum_j \left(\frac{\partial S^v}{\partial q_j^v}\right)^2 &= |\boldsymbol{p}^v|^2 \\ &= \left(\frac{|\boldsymbol{p}^v|^2}{|\boldsymbol{p}|^2}\right)|\boldsymbol{p}|^2 \\ &= (p^v|p)|\boldsymbol{p}|^2, \qquad (8)\end{aligned}$$

where we defined

$$\begin{aligned}(p^v|p) &\equiv \frac{|\boldsymbol{p}^v|^2}{|\boldsymbol{p}|^2} \qquad (9)\\ &= \frac{\boldsymbol{p}^{v\mathsf{T}}\boldsymbol{p}^v}{\boldsymbol{p}^{\mathsf{T}}\boldsymbol{p}} \\ &= \frac{\boldsymbol{p}^{\mathsf{T}}\mathbf{J}^{v\mathsf{T}}\mathbf{J}^v\boldsymbol{p}}{\boldsymbol{p}^{\mathsf{T}}\boldsymbol{p}}. \qquad (10)\end{aligned}$$

Note also that by (7)

$$\begin{aligned}\boldsymbol{p} &= \mathbf{J}^{v-1}\boldsymbol{p}^v \qquad (11)\\ \Rightarrow \qquad p_i &= \sum_j (\mathbf{J}^{v-1})_{ij} p_j^v.\end{aligned}$$

But simply relabelling (6)

$$p_i = \frac{\partial S(q)}{\partial q_i} = \frac{\partial S^v(q^v)}{\partial q_i} = \sum_j \frac{\partial S^v(q^v)}{\partial q_j^v}\frac{\partial q_j^v}{\partial q_i} = \sum_j \frac{\partial q_j^v}{\partial q_i} p_j^v. \qquad (12)$$

Thus $(\mathbf{J}^{v-1})_{ij} = \frac{\partial q_j^v}{\partial q_i}$ and we have

$$(\mathbf{J}^{v-1})_{ij} = (J_{ij}^v)^{-1}. \qquad (13)$$

Then using (12) in (9) we have

$$[(p^v|p)]^{-1} = (p|p^v) = \frac{\boldsymbol{p}^{v\mathsf{T}}\mathbf{J}^{v-1\mathsf{T}}\mathbf{J}^{v-1}\boldsymbol{p}^v}{\boldsymbol{p}^{v\mathsf{T}}\boldsymbol{p}^v}, \qquad (14)$$

where by (13)

$$\begin{aligned}[\mathbf{J}^{v-1\mathsf{T}}\mathbf{J}^{v-1}]_{jk} &= \sum_i \mathbf{J}^{v-1}_{ik}\mathbf{J}^{v-1}_{ij} \\ &= \sum_i \left[J_{ik}^v J_{ij}^v\right]^{-1}. \qquad (15)\end{aligned}$$

Having determined that the first term in (5) transforms as a quadratic differential under v-maps, we require that for consistency the second term transforms similarly That is:

$$W^v(q^v) = (p^v|p)W(q), \qquad (16)$$

and

$$W^0(q^0) = (p^0|p)W(q), \tag{17}$$

where $W^0 \equiv 0$ corresponds to the free particle at rest, with $V = 0$ and $E = 0$. Substituting $W^0 = 0$ in the left–hand side of eq. (17) yields

$$0 = (p^0|p)W(q). \tag{18}$$

Hence, W^0 is a fixed point under the v–maps and cannot be connected to other states. Therefore, classical mechanics in not compatible with the equivalence postulate.

As the CSHJE is not consistent with the equivalence postulate, we consider the modification

$$W \to W + Q,$$

The Modified Stationary Hamilton Jacobi Equation takes the form

$$\frac{1}{2m}\sum_i \left(\frac{\partial S}{\partial q_i}\right)^2 + W(q) + Q(q) = 0. \tag{19}$$

Covariance under v–map imposes the transformation property

$$W^v + Q^v = (p^v|p)\left[W + Q\right]. \tag{20}$$

In order that the v–map may connect W^0 to any other state, W must transform with an inhomogeneous term,

$$W^v = (p^v|p)W + (q; q^v), \tag{21}$$

and by eq, (20)

$$Q^v = (p^v|p)Q - (q; q^v). \tag{22}$$

We note that for an arbitrary state

$$\begin{aligned} W &= (p|p^0)W^0 + (q^0; q) \\ &= 0 + (q^0; q) \\ &= (q^0; q), \end{aligned} \tag{23}$$

so the inhomogeneous term for the trivialising map generates all other W states. Let us consider the properties of the inhomogeneous term. First, we note from (21) that in the case of the identity transformation $q \to q$

$$\begin{aligned} W &= (p|p)W + (q; q) \\ \Rightarrow (q; q) &= 0. \end{aligned} \tag{24}$$

Second, from eq. (21) we can write

$$W^b = (p^b|p^a)W^a + (q^a; q^b), \tag{25}$$

$$W^c = (p^c|p^a)W^a + (q^a; q^c), \tag{26}$$

and

$$W^c = (p^c|p^b)W^b + (q^b; q^c), \tag{27}$$

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for three states a, b, c. Substituting (25) into (27) and equating this with (26) we obtain

$$\begin{aligned} (p^c|p^a)W^a + (q^a; q^c) &= (p^c|p^b)\left[(p^b|p^a)W^a + (q^a; q^b)\right] + (q^b; q^c) \\ \Rightarrow (q^a; q^c) - (q^b; q^c) &= (p^c|p^b)(q^a; q^b), \end{aligned} \tag{28}$$

or

$$(q^a; q^b) = (p^b|p^c)\left[(q^a; q^c) - (q^b; q^c)\right]. \tag{29}$$

Eq. 29) is our celebrated *cocycle condition.* It expresses the essence of the quantum mechanics in the equivalence postulate approach. We further note that if we set $q^a = q^c$ in (29) we get

$$(q^a; q^b) = -(p^b|p^a)(q^b; q^a). \tag{30}$$

4. Higher dimensional Möbius group

The form of the cocycle condition has far–reaching implications. Here we reproduce the arguments introduced in [12] which reveal a symmetry of the inhomogeneous term $(q^a; q^b)$, under D–dimensional Möbius transformations. Our aim is to provide a more pedagogical presentation of the proof of these properties.

We denote by $q = (q_1, \cdots, q_D)$ an arbitrary point in R^D. A similarity is a D–dimensional transformation that includes translations, rotations and dilatations. Similarities are naturally extended to the compactified space $\hat{R}^D = R^D \cup \{\infty\}$. A similarity maps ∞ to itself. Setting

$$r^2 = q_1^2 + \cdots + q_D^2, \tag{31}$$

the last generator of the Möbius group is the *inversion* or reflection in the unit sphere S^{D-1}. If $q \neq 0$

$$q \longrightarrow q^* = \frac{q}{r^2}, \tag{32}$$

otherwise

$$0 \longrightarrow \infty, \qquad \infty \longrightarrow 0. \tag{33}$$

Summarising, an arbitrary Möbius transformation of an N-dimensional vector $q = (q_1, \ldots, q_i, \ldots, q_N)$ is made up of any combination of:

- a 'translation' $q \to q^B = q + B$ (B a vector in R^D),
- a 'dilatation' $q \to q^A = Aq$ (A a real number),
- a 'rotation' $q \to q^R = Rq$ (R an orthogonal $D \times D$matrix),
- an 'inversion' $q \to q^* = \frac{q}{r^2}$ ($r^2 \equiv \sum_i q_i^2$).

The Möbius transformations naturally extend to the compactified space $\hat{R}^D = R^D \cup \{\infty\}$. A similarity maps ∞ to itself. The Möbius group $M(\hat{R}^D)$ is defined as the set of transformations generated by all similarities together with the inversion. A general Möbius transformation is the combination of a number of reflections and inversions. A Möbius transformation is conformal with respect to the Euclidean metric. A theorem due to Liouville states that the conformal group. $M(\hat{R}^D)$ actually coincide[1] for $D > 2$.

[1] for review of the D–dimensional Möbius group, see *e.g.* [14].

4.1. Transformation Factors

In the subsections bellow we derive the transformations factors $(p^M|p)$ in the case of the Möbius transformations $q \to q^M$ discussed in section 4.

4.1.1. Translation The Jacobian for a translation is

$$J_{ij}^B = \frac{\partial q_i}{\partial q_j^B} = \frac{\partial q_i}{\partial (q_j + B_j)} = \frac{\partial q_i}{\partial q_j} \times \left(\frac{\partial (q_j + B_j)}{\partial q_j} \right)^{-1} = \delta_{ij}. \tag{34}$$

Hence, $\mathbf{J}^B = I_N$, where I_N is the $N \times N$ identity matrix. The transformation factor is therefore given by

$$(p^B|p) = \frac{\boldsymbol{p}^\mathsf{T} I_N^\mathsf{T} I_N \boldsymbol{p}}{\boldsymbol{p}^\mathsf{T} \boldsymbol{p}} = 1. \tag{35}$$

4.1.2. Dilatation The Jacobian for a dilatation is

$$J_{ij}^A = \frac{\partial q_i}{\partial q_j^A} = \frac{\partial q_i}{\partial (A q_j)} = A^{-1} \delta_{ij}. \tag{36}$$

Then $\mathbf{J}^A = A^{-1} I_N$ and

$$(p^A|p) = \frac{\boldsymbol{p}^\mathsf{T} A^{-1} I_N^\mathsf{T} A^{-1} I_N \boldsymbol{p}}{\boldsymbol{p}^\mathsf{T} \boldsymbol{p}} = A^{-2}, \tag{37}$$

or

$$(p|p^A) = A^2. \tag{38}$$

4.1.3. Rotation We have

$$\begin{array}{rrcl} & q^R & = & Rq \\ \Rightarrow & q & = & R^{-1} q^R \\ \text{and} & q_i & = & \sum_k \left(R^{-1} \right)_{ik} q_k^R, \\ \text{so that} & J_{ij}^R = \dfrac{\partial q_i}{\partial q_j^R} & = & \left(R^{-1} \right)_{ij}. \end{array}$$

Furthermore, since R is orthogonal

$$\begin{array}{rccc} & R^\mathsf{T} R = R R^\mathsf{T} & = & I_N \\ & \Rightarrow \quad R^\mathsf{T} & = & R^{-1} \\ \text{so that} & \mathbf{J}^R = R^\mathsf{T} & \text{and} & \mathbf{J}^{R^\mathsf{T}} = R. \end{array}$$

This leads to

$$(p^R|p) = \frac{\boldsymbol{p}^\mathsf{T} R R^\mathsf{T} \boldsymbol{p}}{\boldsymbol{p}^\mathsf{T} \boldsymbol{p}} = \frac{\boldsymbol{p}^\mathsf{T} I_N \boldsymbol{p}}{\boldsymbol{p}^\mathsf{T} \boldsymbol{p}} = 1. \tag{39}$$

4.1.4. Inversion This requires a modicum of care. First, observe that

$$\begin{aligned} \frac{\partial r^2}{\partial q_i} &= \frac{\partial}{\partial q_i}\sum_k q_k^2 \\ \Rightarrow \qquad 2r\frac{\partial r}{\partial q_i} &= 2q_i \\ \Rightarrow \qquad \frac{\partial r}{\partial q_i} &= \frac{q_i}{r}. \end{aligned} \tag{40}$$

For the Jacobian we get

$$\begin{aligned} J^*_{ij} = \frac{\partial q_i}{\partial q^*_j} &= \left(\frac{\partial q^*_j}{\partial q_i}\right)^{-1} \\ \Rightarrow \qquad \left(J^*_{ij}\right)^{-1} = \frac{\partial q^*_j}{\partial q_i} &= \frac{\partial}{\partial q_i}\left(\frac{q_j}{r^2}\right) \\ &= \frac{\delta_{ij}}{r^2} - \frac{2q_j}{r^3}\frac{\partial r}{\partial q_i} \\ &= \frac{\delta_{ij}}{r^2} - \frac{2}{r^4}q_i q_j. \end{aligned} \tag{41}$$

Now, using (14)

$$(p|p^*) = \frac{\boldsymbol{p}^{*\mathsf{T}}\mathbf{J}^{*-1^{\mathsf{T}}}\mathbf{J}^{*-1}\boldsymbol{p}^*}{\boldsymbol{p}^{*\mathsf{T}}\boldsymbol{p}^*}, \tag{42}$$

where

$$\begin{aligned} [\mathbf{J}^{*-1^{\mathsf{T}}}\mathbf{J}^{*-1}]_{kj} = \sum_i \left[J^*_{ik}J^*_{ij}\right]^{-1} &= \sum_i \left(\frac{\delta_{ik}}{r^2} - \frac{2}{r^4}q_i q_k\right)\left(\frac{\delta_{ij}}{r^2} - \frac{2}{r^4}q_i q_j\right) \\ &= \sum_i \frac{\delta_{ik}\delta_{ij}}{r^4} - \frac{2}{r^6}\sum_i q_i(q_j\delta_{ik} + q_k\delta_{ij}) + \frac{4}{r^8}\sum_i q_i^2 q_k q_j \\ &= \frac{\delta_{kj}}{r^4} - \frac{2}{r^6}q_j q_k - \frac{2}{r^6}q_k q_j + \frac{4r^2}{r^8}q_k q_j \\ &= \frac{\delta_{kj}}{r^4} \\ \Rightarrow \qquad \mathbf{J}^{*-1^{\mathsf{T}}}\mathbf{J}^{*-1} &= r^{-4}I_N. \end{aligned} \tag{43}$$

Substituting this into (42) gives us

$$(p|p^*) = \frac{\boldsymbol{p}^{*\mathsf{T}}r^{-4}I_N\boldsymbol{p}^*}{\boldsymbol{p}^{*\mathsf{T}}\boldsymbol{p}^*} = r^{-4} \qquad \Rightarrow \qquad (p^*|p) = r^4. \tag{44}$$

5. Inhomogeneous Term

In this section we prove the invariance of the inhomogeneous term, and hence of the cocycle condition under D–dimensional Möbius transformations.

5.1. Translation

The cocycle condition (29) can be written

$$(q^a;q^b) = (p^b|p^c)(q^a;q^c) + (q^c;q^b), \tag{45}$$

which tells us that

$$\begin{aligned}(q+B+C;q) &= (p|p^B)(q+B+C;q+B)+(q+B;q)\\ &= (q+B+C;q+B)+(q+B;q),\end{aligned}\tag{46}$$

or,

$$\begin{aligned}(q+B+C;q) &= (p|p^C)(q+B+C;q+C)+(q+C;q)\\ &= (q+C+B;q+C)+(q+C;q),\end{aligned}\tag{47}$$

where B,C are two arbitrary constant vectors. Now consider restricting B,C so that they each have only a single component along the j-axis:

$$B=(0,0,\ldots,0,B_j,0,\ldots,0) \qquad \text{and} \qquad C=(0,0,\ldots,0,C_j,0,\ldots,0)\ ; \qquad j\in\{1,\ldots,N\}.$$

We can then define a function

$$f(B,q)\equiv(q+B;q),\tag{48}$$

and, equating (46) with (47) we obtain

$$\begin{aligned}f(C,q+B)+f(B,q) &= f(B,q+C)+f(C,q)\\ \Rightarrow \qquad f(C,q+B) &= f(B,q+C)+f(C,q)-f(B,q).\end{aligned}\tag{49}$$

We can differentiate this with respect to B_j. Note first that

$$\frac{\partial(q_j+B_j)}{\partial q_j}=\frac{\partial(q_j+B_j)}{\partial B_j}=1,$$

so for any function P of the combination $q+B$ we have

$$\begin{aligned}\partial_{B_j}P(q+B) &= \frac{\partial P(q+B)}{\partial q_j}\times\left(\frac{\partial(q_j+B_j)}{\partial q_j}\right)^{-1}\times\frac{\partial(q_j+B_j)}{\partial B_j}\\ &= \partial_{q_j}P(q+B)\ ; \qquad \text{where} \qquad \partial_z\equiv\frac{\partial}{\partial z}.\end{aligned}\tag{50}$$

Differentiating (49) then:

$$\partial_{q_j}f(C,q+B)=\partial_{B_j}\left[f(B,q+C)-f(B,q)\right].\tag{51}$$

From (24)

$$f(0,q)\equiv(q;q)=0.\tag{52}$$

We can express f in a general series form which guarantees this. Writing

$$f(B,q)=\sum_{n=1}^{\infty}c_n(q)B_j^n.\tag{53}$$

and substituting (53) into (51) we have

$$\begin{aligned}\partial_{q_j}\sum_{n=1}^{\infty}c_n(q+B)C_j^n &= \partial_{B_j}\sum_{n=1}^{\infty}\left[c_n(q+C)-c_n(q)\right]B_j^n\\ &= \sum_{n=1}^{\infty}\left[c_n(q+C)-c_n(q)\right]nB_j^{n-1}.\end{aligned}\tag{54}$$

Using a Taylor expansion the term on the left hand side becomes

$$\begin{aligned}\partial_{q_j}\sum_{n=1}^{\infty}c_n(q+B)C_j^n &= \partial_{q_j}\sum_{n=1}^{\infty}\left[\sum_{m=0}^{\infty}\frac{1}{m!}\partial_{q_j}^m c_n(q)B_j^m\right]C_j^n \\ &= \sum_{n=1}^{\infty}\left[\sum_{m=0}^{\infty}\frac{1}{m!}\partial_{q_j}^{m+1}c_n(q)B_j^m\right]C_j^n \\ &= \sum_{n=1}^{\infty}\left[\sum_{l=1}^{\infty}\frac{1}{(l-1)!}\partial_{q_j}^l c_n(q)B_j^{l-1}\right]C_j^n\ , \qquad \text{where } \ l=m+1, \\ &= \sum_{l,n=1}^{\infty}\frac{1}{(n-1)!}\partial_{q_j}^n c_l(q)B_j^{n-1}C_j^l, \end{aligned} \tag{55}$$

where in the last line we have simply switched indices (n for l) under the double summation. The right hand side of (54) becomes:

$$\begin{aligned}\sum_{n=1}^{\infty}\left[c_n(q+C)-c_n(q)\right]nB_j^{n-1} &= \sum_{n=1}^{\infty}\left[\sum_{l=0}^{\infty}\frac{1}{l!}\partial_{q_j}^l c_n(q)C_j^l-\frac{1}{0!}\partial_{q_j}^0 c_n(q)C_j^0\right]nB_j^{n-1} \\ &= \sum_{l,n=1}^{\infty}\frac{1}{l!}\partial_{q_j}^l c_n(q)C_j^l nB_j^{n-1}. \end{aligned} \tag{56}$$

Equating (55) with (56) we find

$$\begin{aligned}\sum_{l,n=1}^{\infty}\frac{1}{(n-1)!}\partial_{q_j}^n c_l(q)B_j^{n-1}C_j^l &= \sum_{l,n=1}^{\infty}\frac{n}{l!}\partial_{q_j}^l c_n(q)B_j^{n-1}C_j^l \\ \Rightarrow \qquad \frac{1}{n!}\partial_{q_j}^n c_l(q) &= \frac{1}{l!}\partial_{q_j}^l c_n(q). \end{aligned} \tag{57}$$

Setting $l=1$ gives

$$\partial_{q_j}c_n(q)=\frac{1}{n!}\partial_{q_j}^n c_1(q). \tag{58}$$

If we differentiate (53) with respect to q_j and substitute in (58) we obtain

$$\begin{aligned}\partial_{q_j}f(B,q)=\sum_{n=1}^{\infty}\partial_{q_j}c_n(q)B_j^n &= \sum_{n=1}^{\infty}\frac{1}{n!}\partial_{q_j}^n c_1(q)B_j^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\partial_{q_j}^n c_1(q)B_j^n-c_1(q) \\ &= c_1(q+B)-c_1(q). \end{aligned} \tag{59}$$

Integrating this:

$$f(B,q)=c(q+B)-c(q)+g(B,\hat{q}), \tag{60}$$

where we define $c(q)$ such that $\partial_{q_j}c(q)=c_1(q)$ and where $g(B,\hat{q})$ is some function of anything *but* q_j (so $\hat{q}$ denotes all the q_i with $i\neq j$). Recall from (52) that $f(0,q)=0$. Thus:

$$f(0,q)=c(q)-c(q)+g(0,\hat{q})=g(0,\hat{q})=0. \tag{61}$$

Using the cocycle condition (46) once again:

$$\begin{aligned}(q+B+C;q) &= (q+B+C;q+B)+(q+B;q) \\ \Rightarrow \qquad f(B+C,q) &= f(C,q+B)+f(B,q), \end{aligned} \tag{62}$$

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that is

$$c(q+B+C)-c(q)+g(B+C,\hat{q}) = c(q+B+C)-c(q+B)+g(C,\hat{q}+\hat{B})+c(q+B)-c(q)+g(B,\hat{q}).$$

Also $\hat{B}=0$ since by construction $B_i=0$ for $i\neq j$. So we end up with

$$g(B+C,\hat{q}) = g(C,\hat{q})+g(B,\hat{q}). \tag{63}$$

We have shown that g is linear in its first argument, and that it vanishes when its first argument is set to zero. We can write g in a general form which guarantees these properties:

$$g(B,\hat{q}) = K(\hat{q})B_j. \tag{64}$$

Now differentiating $f(B,q)$, with respect to B_j this time, we find on the one hand

$$\begin{aligned} \partial_{B_j} f(B,q) &= \partial_{B_j}\left[c(q+B)-c(q)+g(B,\hat{q})\right] \\ &= \partial_{q_j} c(q+B)+\partial_{B_j} K(\hat{q})B_j \\ &= c_1(q+B)+K(\hat{q}), \end{aligned} \tag{65}$$

and on the other hand,

$$\partial_{B_j} f(B,q) = \sum_{n=1}^{\infty} n c_n(q) B_j^{n-1} = c_1(q)+2c_2(q)B_j+\ldots, \tag{66}$$

so setting $B=0$ and equating (65) with (66) we have

$$\begin{aligned} c_1(q)+K(\hat{q}) &= c_1(q) \\ \Rightarrow \qquad K(\hat{q}) &= 0. \end{aligned} \tag{67}$$

This means that

$$g(B,\hat{q}) = 0\times B_j = 0,$$

and so we arrive at the result

$$f(B,q) = c(q+B)-c(q). \tag{68}$$

Next, we consider a general constant vector D (which we need not restrict to having only a single component). We first define the function

$$G(D,q) \equiv (q+D;q). \tag{69}$$

Note that we must have

$$G(B,x) = f(B,x) \tag{70}$$

where as before B has only one component along the j-axis. Again we use the cocycle condition (46,47),

$$\begin{aligned} (q+B+D;q) &= (q+B+D;q+B)+(q+B;q) \\ \textit{and} \qquad (q+B+D;q) &= (q+D+B;q+D)+(q+D;q) \\ \Rightarrow (q+B+D;q+B)-(q+B+D;q+D) &= (q+D;q)-(q+B;q) \\ \text{so} \qquad G(D,q+B)-G(B,q+D) &= G(D,q)-G(B,q), \\ \text{that is} \qquad G(D,q+B)-f(B,q+D) &= G(D,q)-f(B,q) \\ \Rightarrow G(D,q+B)-c(q+D+B)+c(q+D) &= G(D,q)-c(q+B)+c(q). \end{aligned}$$

Differentiating this with respect to B_j we get

$$\partial_{q_j} G(D, q+B) - \partial_{q_j} c(q+D+B) = -\partial_{q_j} c(q+B), \tag{71}$$

which yields, for $B = 0$,

$$\partial_{q_j} G(D, q) = \partial_{q_j} \left[c(q+D) - c(q) \right]. \tag{72}$$

Integrating *this* gives

$$G(D, q) = c(q+D) - c(q) + \hat{G}(D, \hat{q}), \tag{73}$$

where $\hat{G}(D, \hat{q})$ is some function of the q_i ($i \neq j$) which by (70) and (68) we know must vanish if D has only a single component along the j-axis. Putting this back into the cocycle condition again we get

$$\begin{aligned} G(D+B, q) &= G(B, q+D) + G(D, q) \qquad (74) \\ \Rightarrow c(q+D+B) - c(q) + \hat{G}(D+B, \hat{q}) &= c(q+D+B) - c(q+D) + \hat{G}(B, \hat{q}+\hat{D}) + \\ &\quad\; c(q+D) - c(q) + \hat{G}(D, \hat{q}) \\ \text{so} \qquad \hat{G}(D+B, \hat{q}) &= \hat{G}(B, \hat{q}+\hat{D}) + \hat{G}(D, \hat{q}), \end{aligned}$$

so we see that $\hat{G}(D, \hat{q})$ satisfies the same algebra as $G(D, q)$ (only with one fewer variable since $\hat{q}$ excludes q_j). We can therefore apply the same arguments to $\hat{G}(D, \hat{q})$ as we have used for $G(D, q)$: We'll end up with a relation like (73) for $\hat{G}(D, \hat{q})$, with a new function tacked onto the end which is now only a function of $N-2$ of the q_i. Then we put *this* back into the cocycle condition - and so on and so on...

Having worked our way recursively through all N components we obtain

$$G(D, q) \equiv (q+D; q) = F(q+D) - F(q) + H(D), \tag{75}$$

where $H(D)$ is some function which vanishes whenever D has only one component. Since $H(D)$ has no second argument, we find that upon substitution of (75) into (74) we are left with

$$H(D+C) = H(D) + H(C), \tag{76}$$

so H is linear. We can write H in a general form which guarantees this property:

$$H(D) = \sum_{i=1}^{N} a_i D_i \tag{77}$$

(just a linear combination of the components of D). However we require that $H = 0$ if $D_i = 0$ and $D_j \neq 0$ ($i \neq j$) for any j, so we must have that $a_j = 0$ for any j, which means that H is identically vanishing. We have determined the form for the inhomogeneous term under arbitrary translations to be

$$(q+D; q) = F(q+D) - F(q). \tag{78}$$

5.2. Dilatation

In a similar way to our previous treatment we define a function

$$h(A, q) \equiv (Aq; q), \tag{79}$$

and use the cocycle condition to examine its structure. From (45), and using (78), we have

$$\begin{aligned} (A[q+B]; q) &= (p|p^A)(A[q+B]; Aq) + (Aq; q) \\ &= A^2(Aq + AB; Aq) + (Aq; q) \\ &= A^2 \left[F(Aq+AB) - F(Aq) \right] + h(A, q), \end{aligned} \tag{80}$$

which may also be written as,

$$\begin{aligned}(A[q+B];q) &= (p|p^B)(A[q+B];q+B)+(q+B;q) \\ &= h(A,q+B)+F(q+B)-F(q). \end{aligned} \tag{81}$$

Equating Eqs. (80) and 81) we obtain,

$$h(A,q+B)-h(A,q)=A^2\left[F(Aq+AB)-F(Aq)\right]-F(q+B)+F(q), \tag{82}$$

and differentiating (82) with respect to B_j we find

$$\partial_{q_j}h(A,q+B)=A^2\partial_{q_j}F(Aq+AB)-\partial_{q_j}F(q+B). \tag{83}$$

Setting $B=0$ gives

$$\partial_{q_j}h(A,q)=\partial_{q_j}\left[A^2F(Aq)-F(q)\right], \tag{84}$$

which upon integration yields

$$h(A,q)=A^2F(Aq)-F(q)+g(A), \tag{85}$$

where $g(A)$ is some function which cannot depend on any of the q_i (as we could have chosen any j in (83)). Now, at the origin ($q=0$) a dilatation can have no effect, so $h(A,0)$ is independent of A and

$$\begin{aligned} h(A,0)=h(1,0)\equiv(q;q) &= 0 \\ \Rightarrow h(A,0)=A^2F(A\times 0)-F(0)+g(A) &= 0 \\ \Rightarrow \qquad g(A)=F(0)\left[1-A^2\right]. & \end{aligned}$$

Putting this back into (85) gives

$$h(A,q)=A^2\left[F(Aq)-F(0)\right]-\left[F(q)-F(0)\right], \tag{86}$$

or more concisely

$$h(A,q)\equiv(Aq;q)=A^2F(Aq)-F(q), \tag{87}$$

where we defined $F(0)=0$.

5.3. Rotation

Following similar steps, from the cocycle condition (80,81) we have

$$(R[q+B];q) = (p|p^R)(Rq+RB];Rq)+(Rq;q) \tag{88}$$

$$\text{and} \qquad (R[q+B];q) = (p|p^B)(R[q+B];q+B)+(q+B;q) \tag{89}$$

$$\Rightarrow (R[q+B];q+B)-(Rq;q) = (Rq+RB;Rq)-(q+B;q). \tag{90}$$

Using (78), $\qquad (R[q+B];q+B)-(Rq;q) = \{F(R[q+B])-F(q+B)\}-\{F(Rq)-F(q)\}.$

This is satisfied by

$$(Rq;q)=F(Rq)-F(q)+C\;; \qquad C \text{ a constant.} \tag{91}$$

A rotation can have no effect at the origin, so $(Rq;q)|_{q=0}\equiv(q;q)=0$ and

$$F(R\times 0)-F(0)+C=0 \qquad \Rightarrow \qquad C=0.$$

The form of the inhomogeneous term under rotations is therefore $(Rq;q)=F(Rq)-F(q)$.

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5.4. Inversion

To begin with we consider how lengths transform under the generators of the Möbius group.

$$r^{*2} = \sum_{i=1}^{N} (q^*)_i^2 = \sum_{i=1}^{N} \left(\frac{q_i}{r^2}\right)^2 = \frac{1}{r^4} r^2 = \frac{1}{r^2}, \tag{92}$$

$$r^{A2} = \sum_{i=1}^{N} (Aq)_i^2 = A^2 r^2, \tag{93}$$

and trivially, since lengths are preserved by rotations,

$$r^{R2} = r^2. \tag{94}$$

(92) fixes q as *involutive*:

$$(q^*)_i^* = \left(\frac{q}{r^2}\right)_i^* = \frac{q_i^*}{r^{*2}} = \frac{q_i}{r^2 \frac{1}{r^2}} = q_i. \tag{95}$$

From (94) we see that rotation commutes with inversion:

$$(Rq)_i^* = \frac{(Rq)_i}{r^{R2}} = \frac{\sum_k R_{ik} q_k}{r^2} = \sum_k R_{ik} q_k^* = (Rq^*)_i. \tag{96}$$

Finally, under dilatations we have, using (93),

$$(Aq)_i^* = \frac{(Aq)_i}{r^{A2}} = \frac{Aq_i}{A^2 r^2} = A^{-1} q_i^*. \tag{97}$$

Now, applying (30), we see from (95) that

$$(q^*; q) = -(p|p^*)(q; q^*) = -\frac{1}{r^4}([q^*]^*; q^*), \tag{98}$$

which vanishes when evaluated at any q such that $q = q^*$:

$$(q^*; q)|_{q=q_0} = 0 \; ; \qquad q_0 = q_0^*. \tag{99}$$

Next, bearing in mind (87), we revisit the cocycle condition (45):

$$\begin{aligned} ([Aq]^*; q) &= (p|p^A)([Aq]^*; Aq) + (Aq; q) \\ &= A^2([Aq]^*; Aq) + A^2 F(Aq) - F(q), \end{aligned} \tag{100}$$

but using (97), we can equally write

$$\begin{aligned} ([Aq]^*; q) &= (p|p^*)([Aq]^*; q^*) + (q^*; q) &\quad (101) \\ &= \frac{1}{r^4}(A^{-1}q^*; q^*) + (q^*; q) \\ &= \frac{1}{r^4}\left[(A^{-1})^2 F(A^{-1}q^*) - F(q^*)\right] + (q^*; q). &\quad (102) \end{aligned}$$

So, equating (100) with (102) gives:

$$A^2([Aq]^*; Aq) + A^2 F(Aq) - F(q) = \frac{1}{r^4}\left[A^{-2} F(A^{-1}q^*) - F(q^*)\right] + (q^*; q). \tag{103}$$

If we choose a point q_0 on the surface of the unit sphere, so that $r_0^2 = \sum_i (q_{0i})^2 = 1$ and $q_0 = q_0^*$, then (by (99)) (103) reduces to

$$\begin{aligned} A^2([Aq_0]^*; Aq_0) &= A^{-2}F(A^{-1}q_0^*) - F(q_0^*) - A^2F(Aq_0) + F(q^0) \\ \Rightarrow \qquad ([Aq_0]^*; Aq_0) &= \frac{1}{A^4}F(A^{-1}q_0^*) - F(Aq_0). \end{aligned} \tag{104}$$

Lastly, we note that any vector q can be expressed in the polar form $r\hat{q}$ where r is the length of q and where $\hat{q}$ is a vector of unit length parallel to q. Clearly $\hat{q}$ automatically has the property required of q_0. The mapping $\hat{q} \to r\hat{q}$ is simply a dilatation of $\hat{q}$ with $A = r$, so we can put $Aq_0 = q$ (and from (97) $A^{-1}q_0^* = [Aq_0]^* = q^*$) in (104) to obtain the final result,

$$(q^*; q) = \frac{1}{r^4}F(q^*) - F(q). \tag{105}$$

6. Summary

To summarise the quantum Hamilton–Jacobi, eq. (19), and the cocycle condition, eq. (29), imply that $(q^a; q^b)$ vanishes if q^a and q^b are related by a Möbius transformation, that is

$$(q + B; q) = 0, \tag{106}$$
$$(Aq; q) = 0, \tag{107}$$
$$(\Lambda q; q) = 0, \tag{108}$$
$$(q^*; q) = 0. \tag{109}$$

These equations are equivalent to $(\gamma(q); q) = 0$, where $\gamma(q)$ is a general Möbius transformation. Moreover, from eq. (29) we have

$$(\gamma(q^a); q^b) = (q^a; q^b), \qquad (q^a; \gamma(q^b)) = (p^{\gamma(b)}|p^b)(q^a; q^b). \tag{110}$$

Considering the Jacobian factor $(p^{\gamma(b)}|p^b)$ we0 observe that the Möbius transformation is conformal with respect to the Euclidean metric, *i.e.*00

$$ds^2 = \sum_{j=1}^{D} d\gamma(q)_j d\gamma(q)_j = \sum_{j,k,l=1}^{D} \frac{\partial\gamma(q)_j}{\partial q_k}\frac{\partial\gamma(q)_j}{\partial q_l} dq_k dq_l = \sum_{j=1}^{D} e^{\phi_\gamma(q)} dq_j dq_j. \tag{111}$$

Hence,

$$(p^{\gamma(b)}|p^b) = e^{-\phi_\gamma(q^b)}. \tag{112}$$

We note that in the case of rotations and translations the conformal re–scaling is the identity. For dilatations $\exp\phi^A = A^2$, whereas for the inversion $\exp\phi^* = r^{-4}$.

We remark that this conformal structure is obtained by setting $S_0^v(q^v) = S_0(q)$. We would like to emphasise that this is not a restriction on the formalism, but merely a convenient choice. Any transformation that we may impose, other than $S_0^v(q^v) = S_0(q)$, would yield the same results. The freedom in setting $S_0^v(q^v) = S_0(q)$ results from the fact that q and q^v represent the spatial coordinates in their own systems. The condition $S_0^v(q^v) = S_0(q)$ can be seen just as the simplest way to set the coordinate transformations from the system with reduced action S_0^v to the one with reduced action S_0. Since the physics is determined by the functional structure of S_0^v, we can denote the coordinate as we like. However, this is not the case in classical mechanics, as for a free particle with vanishing energy we have $S_0(q) = cnst$, and imposing $S_0^v(q^v) = S_0(q)$ does not make sense. Requiring that this is well–defined for any system is synonymous to imposing the equivalence postulate, and the definability of phase space duality for all physical states. The existence of the conformal structure, manifested by the invariance of the inhomogeneous term under Möbius transformations, which in $D \geq 3$ coincides with the conformal group, is at the core of quantum mechanics.

7. Quadratic identities

In the previous sections we discussed the cocycle condition, which is obtained by requiring the QHJE equation retain its form under coordinate transformations. The transformation properties of the kinetic term fix those of the classical and added potential to transform as quadratic differentials. Furthermore, in the one dimensional case the Möbius symmetry uniquely fixes the functional form of the inhomogeneous term to that of the Schwarzian derivative. An identity of Schwarzian derivatives follows from these transformation properties and is given by

$$\left(\frac{\partial S_0}{\partial q}\right)^2 = \frac{\beta^2}{2}\left(\{e^{\frac{i2S_0}{\beta}}; q\} - \{S_0; q\}\right)$$

With the identifications

$$W(q) = -\frac{\beta^2}{4m}\{e^{\frac{i2S_0}{\beta}}; q\} = V(q) - E \tag{113}$$

$$Q(q) = \frac{\beta^2}{4m}\{S_0; q\}, \tag{114}$$

the modified Hamilton–Jacobi equations becomes

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0; q\} = 0.$$

We therefore note that for the state $W(q) \equiv 0$ the Modified HJ equation admits the solutions

$$S_0 = \pm\frac{\beta}{2}\ln q^0 \neq Aq^0 + B.$$

Hence, quantum mechanics enables consistency of the equivalence postulate by removing the linear solutions from the space of solutions. As we discussed before, since $Q(q)$ is never vanishing quantum mechanics carries within it its own regularisation scheme. Furthermore, from eq. (113) and the properties of the Schwarzian derivative, the function $W(q) = V(q) - E$ is a potential of a second order differential equation given by,

$$\left(-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) - E\right)\Psi(q) = 0,$$

which is the Schrödinger equation and we may identify the covariantising parameter β with the Planck constant $\hbar$. The general solutions is given by

$$\Psi(q) = (A\psi_1 + B\psi_2) = \frac{1}{\sqrt{S_0'}}\left(Ae^{+\frac{i}{\hbar}S_0} + Be^{-\frac{i}{\hbar}S_0}\right).$$

The solution of the Schwarzian equation eq. (113) is then given in terms of ratio of the solutions of the Schrödinger equation, *i.e.*

$$e^{+\frac{i2S_0}{\hbar}} = e^{i\alpha}\frac{w + i\bar{\ell}}{w - i\ell} \qquad \text{where} \qquad w = \frac{\psi_1}{\psi_2},$$

where, due to the symmetries of the Schwarzian derivative, the solution is given up to a Möbius transformation. Furthermore, from the condition that $S_0(q) \neq \textit{constant}$ we have that the constants ℓ and α satisfy $\ell = \ell_1 + i\ell_2$, $\ell_1 \neq 0$ and $\alpha \in R$.

While appearance of the Schwarzian derivative in the one dimensional case may seem a bit esoteric, the multi–dimensional case reveals more clearly the simplicity of the formalism. Consider applying the Laplacian to the function

$$\psi = R(q)\mathrm{e}^{\alpha S_0(q)}, \qquad i.e. \quad \Delta\left(R(q)\mathrm{e}^{\alpha S_0(q)}\right) .$$

Proper application of the chain rule then leads to a quadratic identity given by

$$\alpha^2(\nabla S_0)\cdot(\nabla S_0) = \frac{\Delta(Re^{\alpha S_0})}{Re^{\alpha S_0}} - \frac{\Delta R}{R} - \frac{\alpha}{R^2}\nabla\cdot\left(R^2\nabla S_0\right). \tag{115}$$

Setting $\alpha = i/\hbar$ the imaginary part of eq. (115) gives a continuity equation. The first term on the right–hand side of eq. (115) is identified with the classical potential, yielding the D–dimensional nonrelativistic Schrödinger equation, with the general solution given by

$$\Psi(q) = (A\psi_1 + B\psi_2) = R(q)\left(A\mathrm{e}^{+\frac{i}{\hbar}S_0} + B\mathrm{e}^{-\frac{i}{\hbar}S_0}\right), \tag{116}$$

where q_i are now the D–dimensional coordinates. These identifications produce the D–dimensional stationary nonrelativistic Quantum Hamilton–Jacobi Equation given by

$$\frac{1}{2m}(\nabla S_0)\cdot(\nabla S_0) + V(q) - E - \frac{\hbar^2}{2m}\frac{\Delta R}{R} = 0.$$

8. Time parameterisation

We note that the QHJE is reminiscent of Bohm's approach to quantum mechanics [15]. However, there is a crucial difference, which is precisely related to the Möbius symmetry underlying quantum mechanics in the equivalence postulate approach. As is well known Bohm's approach argues for the existence of a trajectory representation of quantum mechanics, in which time parameterisation of trajectories is obtained. In Bohm's approach the wave function is identified with $A = 0$ and $B \neq 0$ in eq. (116). In that case one identifies the conjugate momentum as

$$p = \hbar\mathrm{Im}\frac{\nabla\psi}{\psi}, \tag{117}$$

which can be used to define a trajectory representation by identifying the conjugate momentum the mechanical momentum, *i.e.* by setting $p = m\dot{q}$. However, the choice $A = 0$ and $B \neq 0$ is not consistent with the Möbius symmetry underlying quantum mechanics, which necessitates that $A \neq$ **and** $B \neq 0$. Alternatively, the Möbius symmetry implies that space is compact, in which case the boundary conditions are not compatible with the choice $A = 0$ and $B \neq 0$ but impose that both must be included in the solution. Therefore, the Möbius symmetry of the QHJE implies that

$$\nabla S \neq \hbar\mathrm{Im}\frac{\nabla\psi}{\psi}$$

and Bohm's definition of trajectories is not valid [16]. An alternative proposal [17] to define trajectories in quantum mechanics suggests to use Jacobi's theorem that identifies time as the derivative of the S_0 with respect to E, and by replacing the solution of the CHJE, with the solution of the QHJE, *i.e.*

$$t - t_0 = \frac{\partial S_0^{\mathrm{qm}}}{\partial E}. \tag{118}$$

Time parameterisation of the trajectory can then be obtained by inverting $t(q) \to q(t)$. However, the Möbius symmetry that underlies the QHJE dictates that the energy levels are always quantised [16]. Hence, differentiation with respect to the energy is not well defined. Time in quantum mechanics can be thought of as a classical background parameter, but not as a fundamental quantum variable. At the quantum level trajectories may only have a probabilistic interpretation rather than a deterministic representation. It should be stressed, however, that while a deterministic time parameterisation is not consistent with the Möbius symmetry that underlies the QHJE, time parameterisation à la Bohm [15] or via the bi–polar representation [18, 19] provides a useful semi–classical approximation.

The compactness of space, imposed by the Möbius symmetry underlying the QHJE, implies that the energy levels are always quantised. However, in quantum mechanics this does not suffice. The probability interpretation of the wave function implies that the wave function for bound states is square integrable. In general, the differential equations associated with the quantum mechanical problems for bound states admit solutions that are not square integrable. In the one dimensional case it was proven rigoursly that trivialising transformations $q \to q^0 = \psi_1/psi_2$, where ψ_1 and ψ_2 are the two solutions of the Schrödinger equation, has to be continuous on the extended real line, *i.e.* the real line plus the point at infinity. This requirement is synonymous to the requirement that the Möbius symmetry is preserved. It is then shown [9, 11] that this condition is satisfied iff the corresponding Schrödinger equation admits a square integrable solution. Thus, the same physical states that are selected in conventional quantum mechanics by the axiomatic probability interpretation of the wave function, are selected by consistency in the EP approach. The EP approach may therefore be regarded as reproducing the basic properties of conventional quantum mechanics, with the caveat that spatial space is compact.

9. Conclusions

The requirement that the HJ equation retain its form under coordinate transformation led to the cocycle condition and the QHJE. In turn this led to the removal of the classical solution $S_0(q) = const$ from the space of admissible solutions and consequently the requirement that $p \neq 0$. These properties are intimately related to a duality in phase space that is defined in terms of the involutive nature of the Legendre transformations [6, 11]. Now, the Legendre transformations are not defined for linear functions. The solutions admitted by the QHJE in the case of the trivial classical potential correspond to the self–dual states under the phase space duality. Thus, the QHJE enables the consistency of the phase space duality for the entire space of solutions.

The Möbius symmetry underlying the QHJE is the fundamental property of quantum mechanics in the equivalence postulate approach. It provides an alternative to the axiomatic formulation of quantum mechanics. It necessitates the existence of the quantum potential, which is never vanishing. The quantum potential may be interpreted as an internal curvature term of elementary particles [8, 20], which can therefore be seen to be a direct consequence of the Möbius symmetry underlying the QHJE. In turn, the Möbius symmetry, and the duality structure that it enforces, implies the existence of a fundamental length scale in the formalism [10, 11]. compatibility with the classical limit then implies that the fundamental length scale may be identified with the Planck length. In turn, the existence of a fundamental length scale implies the admission of an ultraviolet cutoff. Similarly, the Möbius symmetry dictates that spatial space is compact. Thus, the Möbius symmetry provides an intrinsic quantum mechanical regularisation scheme in the ultraviolet, as well as the infrared. Furthermore, it leads to the phenomenological characteristics of quantum mechanics [9, 11] without assuming any prior interpretation of the wave function. The HJ formalism, augmented with the Möbius symmetry, therefore provides an alternative starting point to the axiomatic formulation of of quantum mechanics, based on the probability interpretation of the wave function. In that respect, while a fundamental appreciation of the geometrical role of the wave function is yet to be developed, a key guide may lie in duality relations between the wave function and the space coordinates [21]. We further note that the universality of the quantum potential implies that it corresponds to a universal force acting on elementary particles [22]. The Möbius symmetry underlying quantum mechanics implies that spatial space is compact, which may have left a remnant in the Cosmic Microwave Background Radiation [23]. Additionally, it leads to modified energy dispersion relations [24], which may affect the propagation of gamma rays over astrophysical distances [25].

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