SUM OF ONE PRIME AND TWO SQUARES OF PRIMES IN SHORT INTERVALS

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Abstract. Assuming the Riemann Hypothesis we prove that the interval $[N, N+H]$ contains an integer which is a sum of a prime and two squares of primes provided that $H \geq C(\log N)^4$, where $C > 0$ is an effective constant.

1. Introduction

The problem of representing an integer as a sum of a prime and of two prime squares is classical. Letting

$$\mathcal{A} = \{n \in \mathbb{N} : n \equiv 1 \bmod 2;\ n \not\equiv 2 \bmod 3\},$$

it is conjectured that every sufficiently large $n \in \mathcal{A}$ can be represented as $n = p_1 + p_2^2 + p_3^2$. Let now N be a large integer. Several results about the cardinality $E(N)$ of the set of integers $n \leq N$, $n \in \mathcal{A}$ which are not representable as a sum of a prime and two prime squares were proved during the last 75 years; we recall the papers of Hua [3], Schwarz [17], Leung-Liu [11], Wang [18], Wang-Meng [19], Li [12] and Harman-Kumchev [2]. Recently L. Zhao [20] proved that

$$E(N) \ll N^{1/3+\varepsilon}.$$

As a consequence we can say that every integer $n \in [1, N] \cap \mathcal{A}$, with at most $\mathcal{O}\bigl(N^{1/3+\varepsilon}\bigr)$ exceptions, is the sum of a prime and two prime squares. Letting

$$r(n) = \sum_{p_1+p_2^2+p_3^2=n} \log p_1 \log p_2 \log p_3, \tag{1}$$

in fact L. Zhao also proved that a suitable asymptotic formula for $r(n)$ holds for every $n \in [1, N] \cap \mathcal{A}$, with at most $\mathcal{O}\bigl(N^{1/3+\varepsilon}\bigr)$ exceptions.

In this paper we study the average behaviour of $r(n)$ over short intervals $[N, N+H]$, $H = o(N)$. Assuming that the Riemann Hypothesis (RH) holds, we prove that a suitable asymptotic formula for such an average of $r(n)$ holds in short intervals with no exceptions.

Theorem 1. *Assume the Riemann Hypothesis (RH). We have*

$$\sum_{n=N+1}^{N+H} r(n) = \frac{\pi}{4} HN + \mathcal{O}\Bigl(H^{1/2} N (\log N)^2 + H N^{3/4} (\log N)^3 + H^2 L^{3/2}\Bigr) \quad \text{as } N \to \infty,$$

uniformly for $\infty((\log N)^4) \leq H \leq o\bigl(NL^{-3/2}\bigr)$, *where* $f = \infty(g)$ *means* $g = o(f)$.

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Letting

$$r^*(n) = \sum_{p_1+p_2^2+p_3^2=n} 1,$$

a similar asymptotic formula holds for the average of $r^*(n)$ too.

In the unconditional case our proof yields a weaker result than Zhao's, namely, the asymptotic formula for the average of $r(n)$ holds just for $H \geq N^{7/12+\varepsilon}$; for this reason, here we are only concerned with the conditional one. It is worth remarking that, under the assumption of RH, the formula in Theorem 1 implies that every interval $[N, N+H]$ contains an integer which is a sum of a prime and two prime squares, where $CL^4 \leq H = o\big(NL^{-3/2}\big)$, $C > 0$ is a suitable large constant and $L = \log N$. We recall that the analogue results for the binary Goldbach problem are respectively $H \gg N^{c+\varepsilon}$ with $c = 21/800$, by Baker-Harman-Pintz and Jia, see [15], and $H \gg L^2$, under the assumption of RH; see, *e.g.*, [5]. Assuming RH, the expectation in Theorem 1 is the lower bound $H \gg L^2$ since the crucial error term should be $\ll H^{1/2} N \log N$; the loss of a factor L in such an error term is due to the lack of information about a truncated fourth-power average for $\widetilde{S}_2(\alpha)$: see Lemma 5 and (32) below.

The proof of Theorem 1 uses the original Hardy-Littlewood settings of the circle method, *i.e.*, with infinite series instead of finite sums over primes. This is due to the fact that for this problem both the direct and the finite sums approaches do not seem to be able to work in intervals shorter than $N^{1/2}$.

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2. Notation and Lemmas

Let $\ell \geq 1$ be an integer. The standard circle method approach requires to define

$$S_\ell(\alpha) = \sum_{1 \leq p^\ell \leq N} \log p \; e(p^\ell \alpha) \quad \text{and} \quad T_\ell(\alpha) = \sum_{1 \leq n^\ell \leq N} e(n^\ell \alpha),$$

where $e(x) = \exp(2\pi i x)$, and needs the following lemma which collects the results of Theorems 3.1-3.2 of [8].

Lemma 1. *Let N be a large integer, $\ell > 0$ be a real number and ε be an arbitrarily small positive constant. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on ℓ, such that*

$$\int_{-1/H}^{1/H} |S_\ell(\alpha) - T_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell N^{2/\ell - 1}\Big(\exp\Big(-c_1\Big(\frac{L}{\log L}\Big)^{1/3}\Big) + \frac{HL^2}{N}\Big),$$

uniformly for $N^{1-5/(6\ell)+\varepsilon} \leq H \leq N$. Assuming further RH we get

$$\int_{-1/H}^{1/H} |S_\ell(\alpha) - T_\ell(\alpha)|^2 \,\mathrm{d}\alpha \ll_\ell \frac{N^{1/\ell} L^2}{H} + HN^{2/\ell - 2} L^2,$$

uniformly for $N^{1-1/\ell} \leq H \leq N$.

So it is clear that this approach works only when the lower bound $H \geq N^{1-1/\ell}$ holds. Such a limitation comes from the fact that Gallagher's lemma translates the mean-square average of an exponential sum in a short interval problem. When ℓ-powers are involved, this leads to $p^\ell \in [N, N+H]$ which is a non-trivial condition only when $H \geq N^{1-1/\ell}$.

So, when $\ell = 2$, the standard circle method approach works only if $H \geq N^{1/2}$; on the other hand we can easily show that the direct attack works, under RH, only for $H = \infty(N^{1/2}L^2)$. Therefore, to have the chance to reach smaller H-values, we will use the original Hardy and Littlewood [1] circle method setting, *i.e.*, the weighted exponential sum

$$\widetilde{S}_\ell(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n^\ell/N} e(n^\ell \alpha),$$

since it lets us avoid the use of Gallagher's lemma, see Lemmas 2-3 below.

The first ingredient we need is the following explicit formula which generalizes and slightly sharpens what Linnik [13] proved: see also eq. (4.1) of [14]. Let

$$z = 1/N - 2\pi i \alpha. \tag{2}$$

We remark that

$$|z|^{-1} \ll \min\bigl(N, |\alpha|^{-1}\bigr). \tag{3}$$

Lemma 2. *Let $\ell \geq 1$ be an integer, $N \geq 2$ and $\alpha \in [-1/2, 1/2]$. Then*

$$\widetilde{S}_\ell(\alpha) = \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma\Bigl(\frac{\rho}{\ell}\Bigr) + \mathcal{O}_\ell(1), \tag{4}$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$.

Proof. We recall that Linnik proved this formula in the case $\ell = 1$, with an error term $\ll 1 + \log^3(N|\alpha|)$.

Following the line of Lemma 4 in Hardy and Littlewood [1] and of §4 in Linnik [13], we have that

$$\widetilde{S}_\ell(\alpha) = \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma\Bigl(\frac{\rho}{\ell}\Bigr) - \frac{\zeta'}{\zeta}(0) - \frac{1}{2\pi i} \int_{(-\sqrt{3}/2)} \frac{\zeta'}{\zeta}(\ell w) \Gamma(w) z^{-w} \, \mathrm{d}w. \tag{5}$$

Now we estimate the integral in (5). Writing $w = -\sqrt{3}/2 + it$, we have $|(\zeta'/\zeta)(\ell w)| \ll_\ell \log(|t|+2)$, $z^{-w} = |z|^{\sqrt{3}/2} \exp(t \arg(z))$, where $|\arg(z)| \leq \pi/2$. Furthermore the Stirling formula implies that $\Gamma(w) \ll |t|^{-(\sqrt{3}+1)/2} \exp(-\pi|t|/2)$. Hence

$$\begin{aligned}\int_{(-\sqrt{3}/2)} \frac{\zeta'}{\zeta}(\ell w) \Gamma(w) z^{-w} \, \mathrm{d}w &\ll_\ell |z|^{\sqrt{3}/2} \int_0^1 \log(t+2) \, \mathrm{d}t \\ &\quad + |z|^{\sqrt{3}/2} \int_1^{\infty} \log(t+2) t^{-(\sqrt{3}+1)/2} \exp\Bigl(\bigl(\arg(z) - \frac{\pi}{2}\bigr)t\Bigr) \, \mathrm{d}t \\ &\ll_\ell |z|^{\sqrt{3}/2} + |z|^{\sqrt{3}/2} \int_1^{\infty} \log(t+2) t^{-(\sqrt{3}+1)/2} \, \mathrm{d}t \ll_\ell |z|^{\sqrt{3}/2}.\end{aligned}$$

This is $\ll_\ell 1$ as stated since $z \ll 1$ by (2). Hence the lemma is proved. □

We explicitly remark that Lemma 2 is stronger than the corresponding Lemma 1 of [9] (or Lemma 1 of [7]) because in this case α is bounded.

The second lemma is an L^2-estimate of the remainder term in (4) which generalizes a result of Languasco and Perelli [5]; we will follow their proof inserting many details since the presence of ℓ changes the shape of the involved estimates at several places. In fact we will use Lemma 3 just for $\ell = 1, 2$ but we take this occasion to describe the more general case since it may be useful for future works.

Lemma 3. *Assume RH. Let $\ell \ge 1$ be an integer and N be a sufficiently large integer. For $0 \le \xi \le 1/2$, we have*

$$\int_{-\xi}^{\xi} \Big| \widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} \Big|^2 \mathrm{d}\alpha \ll_\ell N^{1/\ell} \xi L^2.$$

Proof. Since $z^{-\rho/\ell} = |z|^{-\rho/\ell} \exp\big(-i(\rho/\ell) \arctan 2\pi N\alpha\big)$, by RH and Stirling's formula we have that

$$\frac{1}{\ell} \sum_\rho z^{-\rho/\ell} \Gamma\Big(\frac{\rho}{\ell}\Big) \ll_\ell \sum_\rho |z|^{-1/(2\ell)} |\gamma|^{(1-\ell)/(2\ell)} \exp\Big(\frac{\gamma}{\ell} \arctan 2\pi N\alpha - \frac{\pi}{2\ell}|\gamma|\Big).$$

If $\gamma\alpha \le 0$ or $|\alpha| \le 1/N$ we get $\sum_\rho z^{-\rho/\ell}\Gamma(\rho/\ell) \ll_\ell N^{1/(2\ell)}$, where, in the first case, ρ runs over the zeros with $\gamma\alpha \le 0$. Hence

$$I(N, \xi, \ell) := \int_{-\xi}^{\xi} \Big| \widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} \Big|^2 \mathrm{d}\alpha \ll_\ell N^{1/\ell} \xi \tag{6}$$

if $0 \le \xi \le 1/N$, and

$$I(N, \xi, \ell) \ll_\ell \int_{1/N}^{\xi} \Big| \sum_{\gamma > 0} z^{-\rho/\ell} \Gamma\Big(\frac{\rho}{\ell}\Big) \Big|^2 \mathrm{d}\alpha + \int_{-\xi}^{-1/N} \Big| \sum_{\gamma < 0} z^{-\rho/\ell} \Gamma\Big(\frac{\rho}{\ell}\Big) \Big|^2 \mathrm{d}\alpha + N^{1/\ell}\xi \tag{7}$$

if $\xi > 1/N$. We will treat only the first integral on the right hand side of (7), the second being completely similar. Clearly

$$\int_{1/N}^{\xi} \Big| \sum_{\gamma > 0} z^{-\rho/\ell} \Gamma\Big(\frac{\rho}{\ell}\Big) \Big|^2 \mathrm{d}\alpha = \sum_{k=1}^{K} \int_{\eta}^{2\eta} \Big| \sum_{\gamma > 0} z^{-\rho/\ell} \Gamma\Big(\frac{\rho}{\ell}\Big) \Big|^2 \mathrm{d}\alpha + \mathcal{O}(1) \tag{8}$$

where $\eta = \eta_k = \xi/2^k$, $1/N \le \eta \le \xi/2$ and K is a suitable integer satisfying $K = \mathcal{O}(L)$. Writing $\arctan 2\pi N\alpha = \pi/2 - \arctan(1/2\pi N\alpha)$ and using the Saffari-Vaughan technique we have

$$\begin{aligned}\int_{\eta}^{2\eta} \Big| \sum_{\gamma > 0} z^{-\rho/\ell} \Gamma\Big(\frac{\rho}{\ell}\Big) \Big|^2 \mathrm{d}\alpha &\le \int_1^2 \Big(\int_{\delta\eta/2}^{2\delta\eta} \Big| \sum_{\gamma > 0} z^{-\rho/\ell} \Gamma\Big(\frac{\rho}{\ell}\Big) \Big|^2 \mathrm{d}\alpha \Big) \mathrm{d}\delta \\ &= \sum_{\gamma_1 > 0} \sum_{\gamma_2 > 0} \Gamma\Big(\frac{\rho_1}{\ell}\Big) \overline{\Gamma\Big(\frac{\rho_2}{\ell}\Big)}\, e^{\frac{\pi}{2\ell}(\gamma_1 + \gamma_2)} \cdot J, \end{aligned} \tag{9}$$

say, where

$$J = J(N, \eta, \gamma_1, \gamma_2) = \int_1^2 \Big(\int_{\delta\eta/2}^{2\delta\eta} f_1(\alpha) f_2(\alpha)\ \mathrm{d}\alpha \Big) \mathrm{d}\delta, \quad w = \frac{1}{\ell} + \frac{i}{\ell}(\gamma_1 - \gamma_2),$$

$$f_1(\alpha) = |z|^{-w} \quad \text{and} \quad f_2(\alpha) = \exp\Big(-\frac{\gamma_1 + \gamma_2}{\ell} \arctan \frac{1}{2\pi N\alpha}\Big).$$

Now we proceed to the estimation of J. Integrating twice by parts and denoting by F_1 a primitive of f_1 and by G_1 a primitive of F_1, we get

$$J = \frac{1}{2\eta}\Big(G_1(4\eta) f_2(4\eta) - G_1(2\eta) f_2(2\eta)\Big) - \frac{2}{\eta}\Big(G_1(\eta) f_2(\eta) - G_1\Big(\frac{\eta}{2}\Big) f_2\Big(\frac{\eta}{2}\Big)\Big)$$

$$-2\int_1^2 G_1(2\delta\eta)f_2'(2\delta\eta)\mathrm{d}\delta+2\int_1^2 G_1\Big(\frac{\delta\eta}{2}\Big)f_2'\Big(\frac{\delta\eta}{2}\Big)\mathrm{d}\delta+\int_1^2\Big(\int_{\delta\eta/2}^{2\delta\eta}G_1(\alpha)f_2''(\alpha)\,\mathrm{d}\alpha\Big)\mathrm{d}\delta. \quad (10)$$

If $\alpha > 1/N$ we have

$$f_2'(\alpha)\ll_\ell \frac{1}{\alpha}\Big(\frac{\gamma_1+\gamma_2}{N\alpha}\Big)f_2(\alpha)$$
$$f_2''(\alpha)\ll_\ell \frac{1}{\alpha^2}\Big\{\Big(\frac{\gamma_1+\gamma_2}{N\alpha}\Big)+\Big(\frac{\gamma_1+\gamma_2}{N\alpha}\Big)^2\Big\}f_2(\alpha),$$

hence from (10) we get

$$J\ll_\ell \frac{1}{\eta}\max_{\alpha\in[\eta/2,4\eta]}|G_1(\alpha)|\Big\{1+\Big(\frac{\gamma_1+\gamma_2}{N\eta}\Big)^2\Big\}\exp\Big(-c\Big(\frac{\gamma_1+\gamma_2}{N\eta}\Big)\Big), \quad (11)$$

where $c=c(\ell)>0$ is a suitable constant.

In order to estimate $G_1(\alpha)$ we use the substitution

$$u=u(\alpha)=\Big(\frac{1}{N^2}+4\pi^2\alpha^2\Big)^{1/2}, \quad (12)$$

thus getting

$$F_1(\alpha)=\frac{1}{2\pi}\int\frac{u^{1-w}}{(u^2-N^{-2})^{1/2}}\,\mathrm{d}u.$$

By partial integration we have

$$F_1(\alpha)=\frac{1}{2\pi(2-w)}\Big\{\frac{u^{2-w}}{(u^2-N^{-2})^{1/2}}+\int\frac{u^{3-w}}{(u^2-N^{-2})^{3/2}}\,\mathrm{d}u\Big\}. \quad (13)$$

From (12) and (13) we get

$$G_1(\alpha)=\frac{1}{2\pi(2-w)}\Big\{A(\alpha)+\int B(\alpha)\,\mathrm{d}\alpha\Big\}, \quad (14)$$

where

$$A(\alpha)=\frac{1}{2\pi}\int\frac{u^{3-w}}{u^2-N^{-2}}\,\mathrm{d}u \quad\text{and}\quad B(\alpha)=\int\frac{u^{3-w}}{(u^2-N^{-2})^{3/2}}\,\mathrm{d}u.$$

Again by partial integration we obtain

$$A(\alpha)=\frac{1}{2\pi(4-w)}\Big\{\frac{u^{4-w}}{u^2-N^{-2}}+2\int\frac{u^{5-w}}{(u^2-N^{-2})^2}\mathrm{d}u\Big\}$$

and

$$B(\alpha)=\frac{1}{4-w}\Big\{\frac{u^{4-w}}{(u^2-N^{-2})^{3/2}}+3\int\frac{u^{5-w}}{(u^2-N^{-2})^{5/2}}\,\mathrm{d}u\Big\}.$$

Hence by (12) we have for $\alpha\in[\eta/2,4\eta]$ that

$$A(\alpha)\ll_\ell\frac{u^{2-1/\ell}}{1+|\gamma_1-\gamma_2|}\ll\frac{\alpha^{2-1/\ell}}{1+|\gamma_1-\gamma_2|}\quad\text{and}\quad B(\alpha)\ll_\ell\frac{\alpha^{1-1/\ell}}{1+|\gamma_1-\gamma_2|}, \quad (15)$$

where $A(\alpha)$ and $B(\alpha)$ satisfy $A(\eta/4)=B(\eta/4)=0$, and from (14)-(15) we obtain

$$G_1(\alpha)\ll_\ell\frac{\alpha^{2-1/\ell}}{1+|\gamma_1-\gamma_2|^2} \quad (16)$$

for $\alpha\in[\eta/2,4\eta]$. From (11) and (16) we get

$$J\ll_\ell \eta^{1-1/\ell}\frac{1+\ (\frac{\gamma_1+\gamma_2}{N\eta})^2}{1+|\gamma_1-\gamma_2|^2}\exp\Big(-c\Big(\frac{\gamma_1+\gamma_2}{N\eta}\Big)\Big),$$

hence from (9) and Stirling's formula we have

$$\int_{\eta}^{2\eta}\Big|\sum_{\gamma>0} z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big)\Big|^2 \mathrm{d}\alpha$$

$$\ll_\ell \eta^{1-1/\ell}\sum_{\gamma_1>0}\sum_{\gamma_2>0}|\gamma_1|^{(1-\ell)/(2\ell)}|\gamma_2|^{(1-\ell)/(2\ell)}\frac{1+(\frac{\gamma_1+\gamma_2}{N\eta})^2}{1+|\gamma_1-\gamma_2|^2}\exp\Big(-c\Big(\frac{\gamma_1+\gamma_2}{N\eta}\Big)\Big). \quad (17)$$

But sorting imaginary parts it is clear that

$$|\gamma_1|^{(1-\ell)/(2\ell)}|\gamma_2|^{(1-\ell)/(2\ell)}\Big\{1+\Big(\frac{\gamma_1+\gamma_2}{N\eta}\Big)^2\Big\}\exp\Big(-c\Big(\frac{\gamma_1+\gamma_2}{N\eta}\Big)\Big) \ll_\ell |\gamma_1|^{(1-\ell)/\ell}\exp\Big(-\frac{c}{2}\frac{\gamma_1}{N\eta}\Big),$$

hence (17) becomes

$$\ll_\ell \eta^{1-1/\ell}\sum_{\gamma_1>0}|\gamma_1|^{(1-\ell)/\ell}\exp\Big(-\frac{c}{2}\frac{\gamma_1}{N\eta}\Big)\sum_{\gamma_2>0}\frac{1}{1+|\gamma_1-\gamma_2|^2} \ll_\ell N^{1/\ell}\eta L^2, \quad (18)$$

since the number of zeros $\rho_2 = 1/2 + i\gamma_2$ with $n \le |\gamma_1-\gamma_2| \le n+1$ is $\mathcal{O}(\log(n+|\gamma_1|))$.

From (6)-(8) and (18) we get

$$\int_{-\xi}^{\xi}\Big|\sum_{\gamma>0} z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big)\Big|^2 \mathrm{d}\alpha \ll_\ell N^{1/\ell}\xi L^2, \quad (19)$$

and Lemma 3 follows from (19). □

We will also need the following result based on the Laplace formula for the Gamma function, see [10]. In fact we will need it just for $\mu = 2$ but, as before, we write the more general case.

Lemma 4. *Let N be a positive integer, $z = 1/N - 2\pi i\alpha$, and $\mu > 0$. Then*

$$\int_{-1/2}^{1/2} z^{-\mu} e(-n\alpha)\,\mathrm{d}\alpha = e^{-n/N}\frac{n^{\mu-1}}{\Gamma(\mu)} + \mathcal{O}_\mu\Big(\frac{1}{n}\Big),$$

uniformly for $n \ge 1$.

Proof. We start with the identity

$$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{iDu}}{(a+iu)^s}\,\mathrm{d}u = \frac{D^{s-1}e^{-aD}}{\Gamma(s)},$$

which is valid for $\sigma = \Re(s) > 0$ and $a \in \mathbb{C}$ with $\Re(a) > 0$ and $D > 0$. Letting $u = -2\pi\alpha$ and taking $s = \mu$, $D = n$ and $a = N^{-1}$ we find

$$\int_{\mathbb{R}}\frac{e(-n\alpha)}{(N^{-1}-2\pi i\alpha)^\mu}\,\mathrm{d}\alpha = \int_{\mathbb{R}} z^{-\mu}e(-n\alpha)\,\mathrm{d}\alpha = \frac{n^{\mu-1}e^{-n/N}}{\Gamma(\mu)}.$$

For $0 < X < Y$ let

$$I(X,Y) = \int_X^Y \frac{e^{iDu}}{(a+iu)^\mu}\,\mathrm{d}u.$$

An integration by parts yields

$$I(X,Y) = \Big[\frac{1}{iD}\frac{e^{iDu}}{(a+iu)^\mu}\Big]_X^Y + \frac{\mu}{D}\int_X^Y\frac{e^{iDu}}{(a+iu)^{\mu+1}}\,\mathrm{d}u.$$

Since $a > 0$, the first summand is $\ll_\mu D^{-1}X^{-\mu}$, uniformly. The second summand is

$$\ll \frac{\mu}{D}\int_X^Y \frac{\mathrm{d}u}{u^{\mu+1}} \ll_\mu D^{-1}X^{-\mu}.$$

The result follows. □

We remark that if $\mu \in \mathbb{N}$, $\mu \geq 2$, Lemma 4 can be proved in an easier way using the Residue Theorem (see, *e.g.*, Languasco [4] or Languasco and Zaccagnini [6]).

In the following we will also need a fourth-power average of $\widetilde{S}_2(\alpha)$.

Lemma 5. *We have*

$$\int_{-1/2}^{1/2} |\widetilde{S}_2(\alpha)|^4 \,\mathrm{d}\alpha \ll NL^2.$$

Proof. Let $\mathcal{P}^2 = \{p^j : j \geq 2, p \text{ prime}\}$ and $r_0(m)$ be the number of representations of m as a sum of two squares. We have

$$\begin{aligned}
&\int_{-1/2}^{1/2} |\widetilde{S}_2(\alpha)|^4 \,\mathrm{d}\alpha \\
&\quad= \sum_{n_1,n_2,n_3,n_4\geq 2} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\Lambda(n_4)\, e^{-(n_1^2+n_2^2+n_3^2+n_4^2)/N} \int_{-1/2}^{1/2} e((n_1^2+n_2^2-n_3^2-n_4^2)\alpha)\,\mathrm{d}\alpha \\
&\quad\ll \sum_{p_1,p_2\geq 2} \log p_1 \log p_2\, e^{-2(p_1^2+p_2^2)/N} \sum_{\substack{p_3,p_4\geq 2\\ p_1^2+p_2^2=p_3^2+p_4^2}} \log p_3 \log p_4 \\
&\qquad\quad + \sum_{\substack{n_1,n_2\geq 2\\ n_1\in\mathcal{P}^2}} \Lambda(n_1)\Lambda(n_2)\, e^{-2(n_1^2+n_2^2)/N} \sum_{\substack{n_3,n_4\geq 2\\ n_1^2+n_2^2=n_3^2+n_4^2}} \Lambda(n_3)\Lambda(n_4) \\
&\quad= \Sigma_1 + \Sigma_2,
\end{aligned} \tag{20}$$

say. In Σ_1 we separately consider the contribution of the cases $p_1p_2 = p_3p_4$ and $p_1p_2 \neq p_3p_4$; hence $\Sigma_1 \ll S_1 + S_2$ where, by partial summation and the Prime Number Theorem, we have

$$S_1 = 2\Big(\sum_{p\geq 2}(\log p)^2 e^{-2p^2/N}\Big)^2 \ll \Big(1 + \int_2^{+\infty} \frac{u^2}{N}(\log u)\, e^{-2u^2/N}\,\mathrm{d}u\Big)^2 \ll NL^2,$$

and, by a dissection argument and Satz 3 on page 94 of Rieger [16], we also obtain

$$\begin{aligned}
S_2 &\ll \sum_{y\geq 1}\sum_{1\leq x\leq y} y^2x^2 e^{-2^{2y+1}/N} e^{-2^{2x+1}/N}\Big(\sum_{2^y\leq p_1<2^{y+1}}\ \sum_{2^x\leq p_2<2^{x+1}}\ \sum_{\substack{p_3,p_4\geq 2\\ p_1^2+p_2^2=p_3^2+p_4^2\\ p_1p_2\neq p_3p_4}} 1\Big) \\
&\ll \sum_{y\geq 1} y^4 e^{-2^{2y+1}/N}\Big(\sum_{p_1,p_2<2^{y+1}}\ \sum_{\substack{p_3,p_4\geq 2\\ p_1^2+p_2^2=p_3^2+p_4^2\\ p_1p_2\neq p_3p_4}} 1\Big)\Big(\sum_{1\leq x\leq y} e^{-2^{2x+1}/N}\Big) \\
&\ll \sum_{y\geq 1} y2^y e^{-2^{2y+1}/N}\Big(\int_1^y e^{-2^t/N}\mathrm{d}t\Big) \ll \sum_{y\geq 1} y^2 2^y e^{-2^{2y+1}/N} \\
&\ll \int_2^{+\infty} (\log u)^2 e^{-u/N}\,\mathrm{d}u \ll NL^2.
\end{aligned}$$

Summing up

$$\Sigma_1 \ll NL^2. \tag{21}$$

Recalling that $r_0(m) \ll m^{\varepsilon}$, it is also easy to see that

$$\begin{aligned}
\Sigma_2 &\ll \sum_{\substack{n_1,n_2\geq 2\\ n_1\in\mathcal{P}^2}} \Lambda(n_1)\Lambda(n_2)(\log(n_1^2+n_2^2))^2\, r_0(n_1^2+n_2^2)\, e^{-2(n_1^2+n_2^2)/N}\\
&\ll \sum_{\substack{n_1,n_2\geq 2\\ n_1\in\mathcal{P}^2}} n_1^{\varepsilon} n_2^{\varepsilon}\, e^{-2(n_1^2+n_2^2)/N} \ll \Big(\sum_{j\geq 2}\sum_{p\geq 2} p^{j\varepsilon} e^{-2p^{2j}/N}\Big)\Big(\sum_{n\geq 2} n^{\varepsilon} e^{-2n^2/N}\Big)\\
&\ll \Big(\sum_{j\geq 2} e^{-2^{2j}/N}\int_2^{+\infty} t^{j\varepsilon} e^{-t^{2j}/N}\mathrm{d}t\Big)\Big(N^{1/2+\varepsilon}\int_0^{+\infty} u^{\varepsilon-1/2}e^{-u}\,\mathrm{d}u\Big)\\
&\ll N^{1/2+2\varepsilon}\sum_{j\geq 2} N^{1/(2j)} e^{-2^{2j}/N}\\
&\ll N^{1/2+2\varepsilon}\Big(N^{1/4}\log N + \sum_{j>(1/2)\log N} e^{-2^{2j}/N}\Big) \ll N^{3/4+3\varepsilon}.
\end{aligned} \tag{22}$$

Combining (20)-(22), Lemma 5 follows.

□

3. Proof of Theorem 1

Let $H \geq 2$, $H = o(N)$ be an integer. We recall that we set $L = \log N$ for brevity. Recalling (1) and letting

$$R(n) = \sum_{a+b^2+c^2=n} \Lambda(a)\Lambda(b)\Lambda(c),$$

we have (see, *e.g.*, page 14 of [20]) that

$$r(n) = R(n) + \mathcal{O}\big(n^{3/4}(\log n)^3\big). \tag{23}$$

Then, for every $n \leq 2N$, we can write

$$r(n) = R(n) + \mathcal{O}\big(n^{3/4}(\log n)^3\big) = e^{n/N}\int_{-1/2}^{1/2} \widetilde{S}_1(\alpha)\widetilde{S}_2(\alpha)^2 e(-n\alpha)\,\mathrm{d}\alpha + \mathcal{O}\big(n^{3/4}(\log n)^3\big).$$

From this equation, the Cauchy-Schwarz inequality, Lemma 5 and the Prime Number Theorem, for every $n \leq 2N$ we also have

$$\begin{aligned}
r(n) &\ll \int_{-1/2}^{1/2} |\widetilde{S}_1(\alpha)||\widetilde{S}_2(\alpha)|^2\mathrm{d}\alpha + N^{3/4}L^3\\
&\ll \Big(\int_{-1/2}^{1/2}|\widetilde{S}_1(\alpha)|^2\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-1/2}^{1/2}|\widetilde{S}_2(\alpha)|^4\mathrm{d}\alpha\Big)^{1/2} + N^{3/4}L^3 \ll NL^{3/2}.
\end{aligned} \tag{24}$$

We need now to choose a suitable weighted average of $r(n)$. We further set

$$U(\alpha, H) = \sum_{m=1}^{H} e(m\alpha)$$

and, moreover, we also have the usual numerically explicit inequality

$$|U(\alpha,H)| \le \min\Big(H;\frac{1}{|\alpha|}\Big). \tag{25}$$

With these definitions and (23), we may write

$$\widetilde{S}(N,H) := \sum_{n=N+1}^{N+H} e^{-n/N} r(n) = \int_{-1/2}^{1/2} \widetilde{S}_1(\alpha)\widetilde{S}_2(\alpha)^2 U(-\alpha,H) e(-N\alpha)\,\mathrm{d}\alpha + \mathcal{O}\big(HN^{3/4}L^3\big).$$

Using Lemma 2 with $\ell = 1,2$ and recalling that $\Gamma(1)=1$, $\Gamma(1/2)=\pi^{1/2}$, we can write

$$\begin{aligned}\widetilde{S}(N,H) &= \int_{-1/2}^{1/2} \frac{\pi}{4z^2} U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \int_{-1/2}^{1/2} \frac{1}{z}\Big(\widetilde{S}_2(\alpha)^2 - \frac{\pi}{4z}\Big) U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{-1/2}^{1/2} \Big(\widetilde{S}_1(\alpha) - \frac{1}{z}\Big)\widetilde{S}_2(\alpha)^2 U(-\alpha,H)e(-N\alpha)\,\mathrm{d}\alpha + \mathcal{O}\big(HN^{3/4}L^3\big) \\ &= I_1 + I_2 + I_3 + \mathcal{O}\big(HN^{3/4}L^3\big),\end{aligned} \tag{26}$$

say. From now on, we denote

$$\widetilde{E}_\ell(\alpha) := \widetilde{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}}.$$

Using Lemma 4 we immediately get

$$I_1 = \frac{\pi}{4}\sum_{n=N+1}^{N+H} n e^{-n/N} + \mathcal{O}\Big(\frac{H}{N}\Big) = \frac{\pi HN}{4e} + \mathcal{O}\big(H^2\big). \tag{27}$$

Now we estimate I_2. Using the identity $f^2 - g^2 = 2f(f-g) - (f-g)^2$ we obtain

$$I_2 \ll \int_{-1/2}^{1/2} |\widetilde{E}_2(\alpha)| \frac{|U(\alpha,H)|}{|z|^{3/2}}\,\mathrm{d}\alpha + \int_{-1/2}^{1/2} |\widetilde{E}_2(\alpha)|^2 \frac{|U(\alpha,H)|}{|z|}\,\mathrm{d}\alpha = J_1 + J_2, \tag{28}$$

say. Using (3), (25), Lemma 3 and a partial integration argument we obtain

$$\begin{aligned} J_2 &\ll HN\int_{-1/N}^{1/N} |\widetilde{E}_2(\alpha)|^2\,\mathrm{d}\alpha + H\int_{1/N}^{1/H} |\widetilde{E}_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha} + \int_{1/H}^{1/2} |\widetilde{E}_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha^2} \\ &\ll HN^{1/2}L^2 + HN^{1/2}L^2\Big(1 + \int_{1/N}^{1/H} \frac{\mathrm{d}\xi}{\xi}\Big) + N^{1/2}L^2\Big(H + \int_{1/H}^{1/2} \frac{\mathrm{d}\xi}{\xi^2}\Big) \\ &\ll HN^{1/2}L^3. \end{aligned} \tag{29}$$

Using the Cauchy-Schwarz inequality and arguing as for J_2 we get

$$\begin{aligned} J_1 &\ll HN^{3/2}\Big(\int_{-1/N}^{1/N} \mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-1/N}^{1/N} |\widetilde{E}_2(\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} + H\Big(\int_{1/N}^{1/H} \frac{\mathrm{d}\alpha}{\alpha^2}\Big)^{1/2}\Big(\int_{1/N}^{1/H} |\widetilde{E}_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2} \\ &\qquad + \Big(\int_{1/H}^{1/2} \frac{\mathrm{d}\alpha}{\alpha^4}\Big)^{1/2}\Big(\int_{1/H}^{1/2} |\widetilde{E}_2(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha}\Big)^{1/2} \\ &\ll HN^{3/4}L + HN^{3/4}L\Big(1 + \int_{1/N}^{1/H} \frac{\mathrm{d}\xi}{\xi}\Big)^{1/2} + H^{3/2}N^{1/4}L\Big(1 + \int_{1/H}^{1/2} \frac{\mathrm{d}\xi}{\xi}\Big)^{1/2} \\ &\ll HN^{3/4}L^{3/2} + H^{3/2}N^{1/4}L^{3/2} \ll HN^{3/4}L^{3/2}. \end{aligned} \tag{30}$$

Combining (28)-(30) we finally obtain

$$I_2 \ll HN^{3/4}L^{3/2}. \tag{31}$$

Now we estimate I_3. By the Cauchy-Schwarz inequality, (25) and Lemma 5 we obtain

$$\begin{aligned} I_3 &\ll \Big(\int_{-1/2}^{1/2} |\widetilde{S}_2(\alpha)|^4 \,\mathrm{d}\alpha\Big)^{1/2} \Big(\int_{-1/2}^{1/2} |\widetilde{E}_1(\alpha)|^2 |U(\alpha,H)|^2 \,\mathrm{d}\alpha\Big)^{1/2} \\ &\ll N^{1/2}L\Big(H^2\int_{-1/H}^{1/H} |\widetilde{E}_1(\alpha)|^2 \,\mathrm{d}\alpha + \int_{1/H}^{1/2} |\widetilde{E}_1(\alpha)|^2 \frac{\mathrm{d}\alpha}{\alpha^2}\Big)^{1/2} \\ &\ll H^{1/2}NL^2, \end{aligned} \tag{32}$$

where in the last step we used Lemma 3 and a partial integration argument.

By (26)-(27), (31) and (32), we can finally write

$$\widetilde{S}(N,H) = \frac{\pi}{4e}HN + \mathcal{O}\big(H^{1/2}NL^2 + HN^{3/4}L^3 + H^2\big).$$

Theorem 1 follows since the exponential weight $e^{-n/N} = e^{-1} + \mathcal{O}(H/N)$ for $n \in [N+1, N+H]$ and hence by (24) it can be removed at the cost of inserting an extra factor $\mathcal{O}\big(H^2L^{3/2}\big)$ in the error term. The corollary about the existence in short intervals follows by remarking that $\widetilde{S}(N,H) > 0$ if $L^4 \ll H = o\big(NL^{-3/2}\big)$. □

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