Mediterr. j. math. 99 (9999), 1–15
1660-5446/99000-0, DOI 10.1007/s00009-003-0000

Mediterranean Journal of Mathematics

A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Neumann problem for the Laplace operator

Pier Domenico Lamberti and Massimo Lanza de Cristoforis

Abstract. Let Ω be an open connected subset of $\mathbb{R}^n$ for which the imbedding of the Sobolev space $W^{1,2}(\Omega)$ into the space $L^2(\Omega)$ is compact. We consider the Neumann eigenvalue problem for the Laplace operator in the open subset $\phi(\Omega)$ of $\mathbb{R}^n$, where ϕ is a Lipschitz continuous homeomorphism of Ω onto $\phi(\Omega)$. Then we prove a result of real analytic dependence for symmetric functions of the eigenvalues upon variation of ϕ.

Mathematics Subject Classification (2000). Primary 35P15; Secondary 47H30.

Keywords. Neumann eigenvalues and eigenvectors, Laplace operator, domain perturbation, special nonlinear operators.

1. Introduction.

This paper concerns the dependence of the Neumann eigenvalues for the Laplace operator upon domain perturbation, and one of our goals is to consider nonsmooth domains. Our main results, namely Theorems 2.2 and 2.5 find application in [8], [9].

To prove our results, we exploit an abstract Theorem in Hilbert space proved in [7] and concerning the dependence of the eigenvalues of selfadjoint operators upon perturbation both of the operator and of the scalar product in Hilbert space.

We fix a connected open subset Ω of $\mathbb{R}^n$ of finite measure. We consider the Sobolev space $W^{1,2}(\Omega)$ endowed with its usual norm (cf. (2.1)), and we assume that $W^{1,2}(\Omega)$ is compactly imbedded into $L^2(\Omega)$, an assumption which holds under weak regularity assumptions on Ω (cf. Burenkov [1]), and which implies the validity

This paper represents an extension of a part of the work performed by P.D. Lamberti in his Doctoral Dissertation at the University of Padova under the guidance of M. Lanza de Cristoforis.

of the Poincaré-Wirtinger inequality. We parametrize our perturbed domain by a homeomorphism ϕ of Ω onto $\phi(\Omega) \subseteq \mathbb{R}^n$. In Continuum Mechanics, ϕ plays the role of deformation of the body Ω. We shall assume that ϕ is Lipschitz continuous together with its inverse function $\phi^{(-1)}$. Then we consider the Neumann eigenvalue problem in $\phi(\Omega)$

$$\int_{\phi(\Omega)} DvDw\,dy = \lambda \int_{\phi(\Omega)} vw\,dy \qquad \forall w \in W^{1,2}(\phi(\Omega)) \tag{1.1}$$

in the unknowns $v \in W^{1,2}(\phi(\Omega))$ (the Neumann eigenfunctions), $\lambda \in \mathbb{R}$ (the Neumann eigenvalues.) If $\phi(\Omega)$ is of class C^1, such problem has the well known classical formulation

$$-\Delta v = \lambda v \qquad \text{in } \phi(\Omega), \qquad \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\phi(\Omega),$$

where $-\Delta$ is the Laplace operator, ν is the unit outer normal to $\partial\phi(\Omega)$. Problem (1.1) is well known to have an increasing sequence of eigenvalues

$$0 = \lambda_0[\phi] < \lambda_1[\phi] \leq \lambda_2[\phi] \leq \cdots, \tag{1.2}$$

where we write each eigenvalue as many times as its multiplicity. Here we are interested in the dependence of $\lambda_j[\phi]$ on ϕ for all $j \in \mathbb{N} \setminus \{0\}$. Now, by a standard procedure, problem (1.1) can be reduced to an eigenvalue problem for a compact selfadjoint operator in the space $W^{1,2}(\phi(\Omega))/\mathbb{R}$ endowed with the scalar product induced by the bilinear form

$$\int_{\phi(\Omega)} Dv_1 Dv_2^t\,dy \qquad \forall v_1,\, v_2 \in W^{1,2}(\phi(\Omega)). \tag{1.3}$$

Clearly, one of the main difficulties here is that the Sobolev space $W^{1,2}(\phi(\Omega))$ changes as ϕ is perturbed. In order to overcome this difficulty, as often done in domain perturbation problems (cf. *e.g.*, Pólya and Schiffer [13]), we plan to change variables in equation (1.1) and to 'transplant' our problem into the fixed domain Ω. Namely, we consider the transformation

$$v \mapsto u = v \circ \phi$$

which takes an element v of $W^{1,2}(\phi(\Omega))$ to an element u of $W^{1,2}(\Omega)$, and by means of such transformation we pull back our problem to the fixed domain Ω. In this way, problem (1.1) can be reduced to an eigenvalue problem for an operator $T_{\phi,N}$ in the space $W^{1,2}(\Omega)/\mathbb{R}$ (see Theorem 2.1.) In particular, all eigenvalues $\lambda_j[\phi]$, for $j \in \mathbb{N} \setminus \{0\}$, coincide with the reciprocals of the eigenvalues of such operator $T_{\phi,N}$. The operator $T_{\phi,N}$ turns out to be compact and selfadjoint in the space $W^{1,2}(\Omega)/\mathbb{R}$ with respect to the natural scalar product induced on $W^{1,2}(\Omega)/\mathbb{R}$ by the bilinear form

$$Q_\phi[u_1, u_2] \equiv \int_{\phi(\Omega)} D\left(u_1 \circ \phi^{(-1)}\right) D\left(u_2 \circ \phi^{(-1)}\right)^t dy \qquad \forall u_1, u_2 \in W^{1,2}(\Omega)\,, \tag{1.4}$$

which is obtained by pulling back (1.3) to Ω. Thus the domain of $T_{\phi,N}$ is fixed but the corresponding scalar product Q_ϕ depends on ϕ. As we shall see, both $T_{\phi,N}$ and Q_ϕ depend analytically on ϕ.

Then we can resort to the abstract scheme of [7] which is concerned with families of operators in Hilbert space with variable scalar product and we can prove our results. Namely, we fix a finite subset F of indices of $\mathbb{N}\setminus\{0\}$ of cardinality $|F|$, and we consider the set $\mathcal{A}_\Omega[F]$ of ϕ's such that the eigenvalues $\lambda_j[\phi]$ with $j\in F$ do not equal any of the eigenvalues $\lambda_l[\phi]$ with $l\notin F$. We show that $\mathcal{A}_\Omega[F]$ is open in the set of admissible ϕ's endowed with a Lipschitz norm (cf. Theorem 2.2.) Note that the eigenvalues of (1.1) indexed by $j\in F$ can well be multiple. Then we show that the elementary symmetric functions $\Lambda_{F,s}$ for $s=1,\dots,|F|$ of $\lambda_j[\phi]$ with $j\in F$ depend real analytically on ϕ, although each $\lambda_j[\phi]$ is well known not to be real analytic, or even differentiable on ϕ, unless the multiplicity is one (cf. Theorem 2.2.) Also, we note that real analyticity of $\Lambda_{F,s}[\phi]$ in ϕ is well known to be stronger than the real analyticity of $\Lambda_{F,s}[\phi_t]$ in t for all families $\{\phi_t\}_{t\in\mathbb{I}}$ in $\mathcal{A}_\Omega[F]$ depending real analytically on t in some open neighborhood $\mathbb{I}$ of 0 in $\mathbb{R}$. Then our result cannot be derived by the celebrated Theorem of Rellich and Nagy (cf. Rellich [14, Thm. 1, p. 33].)

At the end of the paper, we compute the first order derivatives of the functions $\Lambda_{F,s}$ at a point $\tilde{\phi}$ such that the eigenvalues $\lambda_j[\tilde{\phi}]$ coincide for all $j\in F$.

We point out that other authors have used different methods to 'transplant' equation (1.1) to the fixed domain Ω. Their methods lead to the rather simpler case of families of compact selfadjoint operators in a Hilbert space with a fixed scalar product (cf. *e.g.*, Micheletti [11].) However such methods require stronger regularity assumptions on the map ϕ, which we do not want to assume, as we are interested in nonsmooth domains.

2. The real analyticity Theorem

We first introduce some technical preliminaries and notation. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be real Banach spaces. We denote by $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the Banach space of linear and continuous maps of $\mathcal{X}$ to $\mathcal{Y}$ endowed with its usual norm of the uniform convergence on the unit sphere of $\mathcal{X}$. We denote by $\mathcal{B}_s(\mathcal{X}\times\mathcal{Y},\mathcal{Z})$ the space of the bilinear symmetric and continuous maps of $\mathcal{X}\times\mathcal{Y}$ to $\mathcal{Z}$ endowed with the norm of the uniform convergence on the cross product of the unit sphere of $\mathcal{X}$ and of the unit sphere of $\mathcal{Y}$. We say that the space $\mathcal{X}$ is continuously imbedded in the space $\mathcal{Y}$ provided that $\mathcal{X}$ is a linear subspace of $\mathcal{Y}$, and that the inclusion map is continuous. We denote by $\mathbb{Z}$ the set of integer numbers, and by $\mathbb{N}$ the set of natural numbers including 0. The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function g, or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. We denote by $M_m(\mathbb{R})$ the set of $m\times m$ matrices with real entries, and by $S_m(\mathbb{R})$ the set of the symmetric elements of $M_m(\mathbb{R})$. If

$A \in M_m(\mathbb{R})$, we denote by A^t the transpose matrix of A. If A is invertible, we set $A^{-t} \equiv \left(A^{-1}\right)^t$. All elements of $\mathbb{R}^n$ are thought as row vectors.

Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\mathrm{cl}\Omega$ the closure of Ω and by $\partial\Omega$ the boundary of Ω. Throughout this paper, we shall consider only case $n \geq 2$. We denote by $L^2(\Omega)$ the space of square summable real valued measurable functions defined on Ω, and by $W^{1,2}(\Omega)$ the Sobolev space of distributions in Ω which have weak derivatives up to the first order in $L^2(\Omega)$, endowed with the norm defined by

$$\|u\|_{W^{1,2}(\Omega)} \equiv \left\{ \|u\|^2_{L^2(\Omega)} + \sum_{l=1}^{n} \|u_{x_l}\|^2_{L^2(\Omega)} \right\}^{1/2} \qquad \forall u \in W^{1,2}(\Omega)\,. \tag{2.1}$$

Now, we are interested in open connected subsets Ω of $\mathbb{R}^n$ of finite measure $|\Omega|$ such that

$$W^{1,2}(\Omega) \text{ is compactly imbedded in } L^2(\Omega)\,. \tag{2.2}$$

As is well known, if (2.2) holds, then the Poincaré-Wirtinger inequality holds in Ω (cf. *e.g.*, Evans [2, Proof of Thm. 1, p. 275].) Then we deform Ω by a Lipschitz continuous homeomorphism of the class $\mathcal{A}_\Omega$ which we now introduce. We denote by $\mathrm{Lip}(\Omega)$ the set of Lipschitz continuous functions of Ω to $\mathbb{R}$, and we set

$$\mathcal{A}_\Omega \equiv \Bigg\{ \phi \in (\mathrm{Lip}(\Omega))^n : \tag{2.3}$$
$$l_\Omega[\phi] \equiv \inf\left\{ \frac{|\phi(x)-\phi(y)|}{|x-y|} : x, y \in \Omega, x \neq y \right\} > 0 \Bigg\}.$$

We note that

$$l_\Omega[\phi] \leq |\det D\phi(x)|^{1/n}\ , \tag{2.4}$$

for almost all $x \in \Omega$ (cf. [10, Lem. 4.22].) Clearly, $\mathcal{A}_\Omega$ coincides with the set of injections ϕ of Ω into $\mathbb{R}^n$ such that both ϕ and $\phi^{(-1)}$ are Lipschitz continuous. Now it can be verified that if Ω satisfies (2.2) and if $\phi \in \mathcal{A}_\Omega$, then $\phi(\Omega)$ also satisfies (2.2) (cf. [5, §2].) Accordingly, the Neumann eigenvalue problem (1.1) has a sequence of eigenvalues as in (1.2). In this paper we are interested in the dependence of $\lambda_j[\phi]$ on ϕ for all $j \in \mathbb{N} \setminus \{0\}$. Thus we need to introduce a topology in $\mathcal{A}_\Omega$. As usual, we introduce the seminorm

$$|f|_1 \equiv \sup\left\{ \frac{|f(x)-f(y)|}{|x-y|} : x, y \in \Omega,\ x \neq y \right\} \qquad \forall f \in \mathrm{Lip}(\Omega)\,,$$

on $\mathrm{Lip}(\Omega)$. It is easily seen that $\mathcal{A}_\Omega$ is open in $(\mathrm{Lip}(\Omega))^n$ (cf. [10, Prop. 4.29], [7, Thm. 3.11].) As is well known, $(\mathrm{Lip}(\Omega), |\cdot|_1)$ is a complete seminormed space. However, we prefer to deal with a normed space, rather than with a seminormed space. Then we will state our results for an arbitrary Banach space $\mathcal{X}_\Omega$, continuously imbedded in $(\mathrm{Lip}(\mathbb{D}), |\cdot|_1)$. Alternatively, one could also endow $\mathrm{Lip}(\Omega)$ with a norm which renders $\mathrm{Lip}(\Omega)$ a Banach space continuously imbedded in $(\mathrm{Lip}(\Omega), |\cdot|_1)$, and take $\mathcal{X}_\Omega$ equal to such Banach space.

As we have said in the introduction, one of the main difficulties in analyzing problem (1.1) upon perturbation of ϕ is that the domain $\phi(\Omega)$, on which the

Sobolev space $W^{1,2}(\phi(\Omega))$ is defined, changes as ϕ is perturbed. In order to overcome this difficulty, we plan to change the variables in problem (1.1) by means of ϕ, and to obtain a problem in $W^{1,2}(\Omega)$. Actually, we will obtain a problem in $W^{1,2}(\Omega)/\mathbb{R}$, as we need to get rid of the constants which generate the eigenspace corresponding to the trivial eigenvalue $\lambda_0[\phi] = 0$. See Theorem 2.1.

To do so, we need to introduce some notation, under the assumption that Ω is a nonempty open connected subset of $\mathbb{R}^n$, and that $\phi \in \mathcal{A}_\Omega$. We denote by $\mathcal{I}$ the canonical imbedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$. We denote by $\mathcal{J}_\phi$ the operator of $L^2(\Omega)$ to the strong dual $\left(W^{1,2}(\Omega)\right)'$ which takes $u \in L^2(\Omega)$ to the the functional $\mathcal{J}_\phi[u] \in \left(W^{1,2}(\Omega)\right)'$ defined by

$$\mathcal{J}_\phi[u][w] \equiv \int_\Omega uw|\det D\phi|\, dx \qquad \forall w \in W^{1,2}(\Omega)\,. \tag{2.5}$$

We set

$$W^{1,2,0}_\phi(\Omega) \equiv \left\{ u \in W^{1,2}(\Omega) : \int_\Omega u|\det D\phi|\, dx = 0 \right\} .$$

Then we simply write $W^{1,2,0}(\Omega)$ in case ϕ is the identity. We also find convenient to denote by $w^{1,2,0}(\Omega)$ the space $W^{1,2,0}(\Omega)$ endowed with the energy scalar product

$$< u_1, u_2 >\equiv \int_\Omega Du_1 Du_2^t\, dx \qquad \forall u_1, u_2 \in W^{1,2,0}(\Omega)\,. \tag{2.6}$$

We denote by π_ϕ the map of $W^{1,2}(\Omega)$ to $W^{1,2,0}_\phi(\Omega)$ defined by

$$\pi_\phi[u] = u - \frac{\int_\Omega u|\det D\phi|\, dx}{\int_\Omega |\det D\phi|\, dx} ,$$

for all $u \in W^{1,2}(\Omega)$. If ϕ is the identity, we write π instead of π_ϕ. We denote by $\pi^\sharp_\phi$ the map of $W^{1,2}(\Omega)/\mathbb{R}$ onto $W^{1,2,0}_\phi(\Omega)$ defined by equality $\pi_\phi = \pi^\sharp_\phi \circ p$, where p is the canonical projection of $W^{1,2}(\Omega)$ onto the quotient $W^{1,2}(\Omega)/\mathbb{R}$. We denote by $Q^\sharp_\phi$ the bilinear form

$$Q^\sharp_\phi[p[u_1], p[u_2]] \equiv Q_\phi[u_1, u_2] \qquad \forall u_1, u_2 \in W^{1,2}(\Omega),$$

which is clearly a scalar product on the quotient $W^{1,2}(\Omega)/\mathbb{R}$. We denote by $w^{1,2}_\phi(\Omega)/\mathbb{R}$ the quotient $W^{1,2}(\Omega)/\mathbb{R}$ endowed with $Q^\sharp_\phi$. Then we write simply $w^{1,2}(\Omega)/\mathbb{R}$ in case ϕ is the identity. Thus $w^{1,2}(\Omega)/\mathbb{R}$ is endowed with its usual energy scalar product. We denote by $\Delta_{\phi,N}$ the operator of $W^{1,2,0}_\phi(\Omega)$ to

$$\mathbf{F}(\Omega) \equiv \left\{ F \in \left(W^{1,2}(\Omega)\right)' : F[1] = 0 \right\} ,$$

which takes $u \in W^{1,2,0}_\phi(\Omega)$ to the element $\Delta_{\phi,N}[u]$ of $\mathbf{F}(\Omega)$ defined by equality

$$\Delta_{\phi,N}[u][w] = -Q_\phi[u, w] \qquad \forall w \in W^{1,2}(\Omega). \tag{2.7}$$

If ϕ is the identity, we write Δ_N instead of $\Delta_{\phi,N}$.

Then we have the following Theorem which states that the eigenvalues $\lambda_j[\phi]$ coincide with the reciprocals of the eigenvalues of a suitable compact selfadjoint operator $T_{\phi,N}$ in $w^{1,2}_{\phi}(\Omega)/\mathbb{R}$.

Theorem 2.1. *Let Ω be a nonempty open connected subset of $\mathbb{R}^n$ of finite measure. Let (2.2) hold. Let $\phi \in \mathcal{A}_{\Omega}$. Then the following statements hold.*

(i) *The operator $\Delta_{\phi,N}$ of $W^{1,2,0}_{\phi}(\Omega)$ onto $\mathbf{F}(\Omega)$ is a linear homeomorphism, and the operator $T_{\phi,N} \equiv -\left(\pi^{\sharp}_{\phi}\right)^{(-1)} \circ \Delta^{(-1)}_{\phi,N} \circ \mathcal{J}_{\phi} \circ \mathcal{I} \circ \pi^{\sharp}_{\phi}$ is compact and selfadjoint in $w^{1,2}_{\phi}(\Omega)/\mathbb{R}$.*

(ii) *The pair (λ, v) of the set $\mathbb{R} \times \left(w^{1,2,0}(\phi(\Omega)) \setminus \{0\}\right)$ satisfies equation (1.1) if and only if $\lambda > 0$ and the pair $\left(\mu \equiv \lambda^{-1}, \breve{u} \equiv p[v \circ \phi]\right)$ of the set $\mathbb{R} \times \left(\left(w^{1,2}_{\phi}(\Omega)/\mathbb{R}\right) \setminus \{0\}\right)$ satisfies equation*

$$\mu \breve{u} = T_{\phi,N} \breve{u} \,. \tag{2.8}$$

(iii) *Equation (2.8) has a decreasing sequence $\{\mu_j[\phi]\}_{j \in \mathbb{N} \setminus \{0\}}$ of eigenvalues in $]0, +\infty[$, and $\mu_j[\phi] = \lambda^{-1}_j[\phi]$ for all $j \in \mathbb{N} \setminus \{0\}$, where $\{\lambda_j[\phi]\}_{j \in \mathbb{N} \setminus \{0\}}$ is the sequence of all the nonzero eigenvalues of (1.1) counted with their multiplicity. Each eigenvalue of (1.1), or of (2.8) has finite multiplicity.*

A proof of Theorem 2.1 can be obtained by a standard argument (see [5, Thm. 2.8, Prop. 2.10].) Finally, we shall also need the scalar product $\hat{Q}_{\phi}$ on $W^{1,2}(\Omega)$ defined by

$$\hat{Q}_{\phi}[u_1, u_2] \equiv \int_{\Omega} u_1 u_2 |\det D\phi| \, dx + Q_{\phi}[u_1, u_2] \qquad \forall u_1, u_2 \in W^{1,2}(\Omega) \,.$$

We are now ready to prove the following.

Theorem 2.2. *Let Ω be a nonempty open connected subset of $\mathbb{R}^n$ of finite measure satisfying (2.2). Let $\mathcal{X}_{\Omega}$ be a normed space continuously imbedded in* $\mathrm{Lip}(\Omega)$. *Let F be a finite nonempty subset of $\mathbb{N} \setminus \{0\}$. Let*

$$\mathcal{A}_{\Omega}[F] \equiv \{\phi \in \mathcal{A}_{\Omega} \cap \mathcal{X}^n_{\Omega} : \lambda_l[\phi] \notin \{\lambda_j[\phi] : j \in F\} \, \forall l \in \mathbb{N} \setminus (F \cup \{0\})\} \,.$$

Then the following statements hold.

(i) *The set $\mathcal{A}_{\Omega}[F]$ is open in $\mathcal{X}^n_{\Omega}$. The map $\hat{P}_F$ of the set $\mathcal{A}_{\Omega}[F]$ to the space $\mathcal{L}\left(W^{1,2}(\Omega), W^{1,2}(\Omega)\right)$ which takes $\phi \in \mathcal{A}_{\Omega}[F]$ to the orthogonal projection $\hat{P}_F[\phi]$ of $\left(W^{1,2}(\Omega), \hat{Q}_{\phi}\right)$ onto the (finite dimensional) subspace $\hat{E}[\phi, F]$ generated by the set*

$$\Big\{ u \in W^{1,2,0}_{\phi}(\Omega) : \, -\Delta_N \left[u \circ \phi^{(-1)}\right] = \lambda_j[\phi] \mathcal{J} \circ \mathcal{I} \left[u \circ \phi^{(-1)}\right] \text{ for some } j \in F \Big\}$$

is real analytic.

(ii) *Let* $s \in \{1, \ldots, |F|\}$*. The function* $\Lambda_{F,s}$ *of* $\mathcal{A}_\Omega[F]$ *to* $\mathbb{R}$ *defined by*

$$\Lambda_{F,s}[\phi] \equiv \sum_{j_1,\ldots,j_s \in F \; j_1<\cdots<j_s} \lambda_{j_1}[\phi] \cdots \lambda_{j_s}[\phi] \qquad \forall \phi \in \mathcal{A}_\Omega[F]$$

is real analytic.

Proof. We shall prove the statement by exploiting an abstract result of [7, §2]. To do so, we need to introduce some notation. We denote by $\mathcal{B}_s\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2, \mathbb{R}\right)$ the normed space of symmetric bilinear and continuous functions of $\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2$ to $\mathbb{R}$. Then we set

$$\mathcal{Q}\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2, \mathbb{R}\right) \equiv \left\{B \in \mathcal{B}_s\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2, \mathbb{R}\right) : \right.$$
$$\left. \eta\,[B] \equiv \inf\left\{\frac{B[\breve{u},\breve{u}]}{\|\breve{u}\|^2_{w^{1,2}(\Omega)/\mathbb{R}}} : \breve{u} \in \left(w^{1,2}(\Omega)/\mathbb{R}\right) \setminus \{0\}\right\} > 0\right\}.$$

Clearly, $\mathcal{Q}\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2, \mathbb{R}\right)$ is the set of scalar products on $w^{1,2}(\Omega)/\mathbb{R}$ which induce on $w^{1,2}(\Omega)/\mathbb{R}$ the topology of $w^{1,2}(\Omega)/\mathbb{R}$. Also, it is can be readily checked that $\mathcal{Q}\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2, \mathbb{R}\right)$ is an open subset of $\mathcal{B}_s\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2, \mathbb{R}\right)$ (cf. [7, §2].) Then we set

$$\mathcal{O}_\Omega \equiv \left\{(Q,T) \in \mathcal{Q}\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2, \mathbb{R}\right) \times \mathcal{L}\left(w^{1,2}(\Omega)/\mathbb{R}, w^{1,2}(\Omega)/\mathbb{R}\right) : \right. \tag{2.9}$$
$$\left. T \text{ is compact and selfadjoint in } \left(w^{1,2}(\Omega)/\mathbb{R}, Q\right)\right\}.$$

If T is compact and selfadjoint in $\left(w^{1,2}(\Omega)/\mathbb{R}, Q\right)$ for a $Q \in \mathcal{Q}\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2, \mathbb{R}\right)$, then the spectrum $\sigma[T]$ of T is finite or countable and each element of $\sigma[T]$ different form zero is an eigenvalue of T of finite multiplicity. We denote by $j^+[T]$ the (possibly infinite) number of elements of $\sigma[T] \cap]0, +\infty[$, each counted with its multiplicity, and we denote by $j^-[T]$ the (possibly infinite) number of elements of $\sigma[T] \cap]-\infty, 0[$, each counted with its multiplicity. We need to enumerate the eigenvalues of T in a suitable way. To do so, we also set $J^+[T] \equiv \{j \in \mathbb{Z} : 1 \le j \le j^+[T]\}$, $J^-[T] \equiv \{j \in \mathbb{Z} : -j^-[T] \le j \le -1\}$. Then there exists a uniquely determined function $j \mapsto \mu_j[T]$ of $J[T] \equiv J^-[T] \cup J^+[T]$ to $\mathbb{R} \setminus \{0\}$ such that $j \mapsto \mu_j[T]$ is decreasing on $J^-[T]$ and on $J^+[T]$, and such that

$$\sigma[T] \setminus \{0\} = \{\mu_j[T] : j \in J[T]\},$$

and such that each eigenvalue $\mu_j[T]$ is repeated as many times as its multiplicity. Then we consider the set

$$\mathcal{A}[F] \equiv \{(Q,T) \in \mathcal{O}_\Omega : F \subseteq J[T], \tag{2.10}$$
$$\mu_l[T] \notin \{\mu_j[T] : j \in F\} \;\; \forall l \in J[T] \setminus F\}.$$

By [7, §2], the set $\mathcal{A}[F]$ is open in $\mathcal{O}_\Omega$. The gradient operator D is obviously linear and continuous from $(\mathrm{Lip}(\Omega))^n$ to $(L^\infty(\Omega))^{n^2}$. Then we note that the operator $\Delta_{\phi,N}[\cdot]$ can be defined on all of $W^{1,2}(\Omega)$ by use of equality (2.7). Since linear and bilinear continuous operators are real analytic, and $\mathcal{X}^n_\Omega$ is continuously imbedded in $(\mathrm{Lip}(\Omega))^n$, we conclude that the maps which take $\phi \in \mathcal{A}_\Omega \cap \mathcal{X}^n_\Omega$ to $\Delta_{\phi,N}$, and to $\mathcal{J}_\phi$, and to $Q^\sharp_\phi$, and to $\pi^\sharp_\phi$ are real analytic from $\mathcal{A}_\Omega \cap \mathcal{X}^n_\Omega$ to the space $\mathcal{L}\left(W^{1,2}(\Omega), \mathbf{F}(\Omega)\right)$, and to the space $\mathcal{L}\left(L^2(\Omega), \left(W^{1,2}(\Omega)\right)'\right)$, and to $\mathcal{Q}\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2, \mathbb{R}\right)$, and to the space $\mathcal{L}\left(w^{1,2}(\Omega)/\mathbb{R}, W^{1,2}(\Omega)\right)$, respectively. Then $\Delta_{\phi,N} \circ \pi^\sharp_\phi$ is real analytic in ϕ from $\mathcal{A}_\Omega \cap \mathcal{X}^n_\Omega$ to $\mathcal{L}\left(w^{1,2}(\Omega)/\mathbb{R}, \mathbf{F}(\Omega)\right)$, and $\mathcal{J}_\phi \circ \mathcal{I} \circ \pi^\sharp_\phi$ is real analytic in ϕ from $\mathcal{A}_\Omega \cap \mathcal{X}^n_\Omega$ to $\mathcal{L}\left(w^{1,2}(\Omega)/\mathbb{R}, \left(W^{1,2}(\Omega)\right)'\right)$, and thus to $\mathcal{L}\left(w^{1,2}(\Omega)/\mathbb{R}, \mathbf{F}(\Omega)\right)$ (cf. (2.5).) Since the map which takes an operator into its inverse is real analytic on the set of invertible operators in $\mathcal{L}\left(w^{1,2}(\Omega)/\mathbb{R}, \mathbf{F}(\Omega)\right)$ (cf. *e.g.*, Hille and Phillips [3, Thms. 4.3.2, 4.3.4]), it follows that the map $\phi \mapsto \left(Q^\sharp_\phi, T_{\phi,N}\right)$ is real analytic from $\mathcal{A}_\Omega \cap \mathcal{X}^n_\Omega$ to $\mathcal{O}_\Omega$. By Theorem 2.1 (iii), the set $\mathcal{A}_\Omega[F]$ coincides with the set

$$\left\{\phi \in \mathcal{A}_\Omega \cap \mathcal{X}^n_\Omega : \left(Q^\sharp_\phi, T_{\phi,N}\right) \in \mathcal{A}[F]\right\} .$$

Since $\mathcal{A}[F]$ is open in $\mathcal{O}_\Omega$, and $\left(Q^\sharp_\phi, T_{\phi,N}\right)$ is continuous in $\phi \in \mathcal{A}_\Omega \cap \mathcal{X}^n_\Omega$, we conclude that $\mathcal{A}_\Omega[F]$ is open in $\mathcal{X}^n_\Omega$. Now let $\tilde{\phi} \in \mathcal{A}_\Omega[F]$ be fixed. By [7, Thm. 2.18], there exists an open neighborhood $\tilde{\mathcal{W}}$ of $\left(Q^\sharp_{\tilde{\phi}}, T_{\tilde{\phi},N}\right)$ in $\mathcal{Q}\left(\left(w^{1,2}(\Omega)/\mathbb{R}\right)^2, \mathbb{R}\right) \times \mathcal{L}\left(w^{1,2}(\Omega)/\mathbb{R}, w^{1,2}(\Omega)/\mathbb{R}\right)$, and a real analytic operator $P^\sharp_F$ of $\tilde{\mathcal{W}}$ to the space $\mathcal{L}\left(w^{1,2}(\Omega)/\mathbb{R}, w^{1,2}(\Omega)/\mathbb{R}\right)$ such that $P^\sharp_F[Q,T]$ equals the orthogonal projection $P_F[Q,T]$ of $\left(w^{1,2}(\Omega)/\mathbb{R}, Q\right)$ onto the subspace $E[T,F]$ generated by the set

$$\left\{\breve{u} \in w^{1,2}(\Omega)/\mathbb{R} : T\breve{u} = \mu\breve{u}, \text{ for some } \mu \in \{\mu_j[T] : j \in F\}\right\} ,$$

for all $(Q,T) \in \tilde{\mathcal{W}} \cap \mathcal{A}[F]$. Now let $\mathcal{W}_1$ be an open neighborhood of $\tilde{\phi}$ in $\mathcal{A}_\Omega[F]$ such that $(Q^\sharp_\phi, T_{\phi,N}) \in \tilde{\mathcal{W}}$ for all $\phi \in \mathcal{W}_1$. Such $\mathcal{W}_1$ exists by continuity of $(Q^\sharp_\phi, T_{\phi,N})$ in the variable ϕ. Then we have $P_F\left[Q^\sharp_\phi, T_{\phi,N}\right] = P^\sharp_F\left[Q^\sharp_\phi, T_{\phi,N}\right]$ for all $\phi \in \mathcal{W}_1$, and thus the map $\phi \mapsto P_F\left[Q^\sharp_\phi, T_{\phi,N}\right]$ is real analytic from $\mathcal{W}_1$ to $\mathcal{L}\left(w^{1,2}(\Omega)/\mathbb{R}, w^{1,2}(\Omega)/\mathbb{R}\right)$. Now we note that

$$\pi^\sharp_\phi[E[T_{\phi,N}, F]] = \hat{E}[\phi, F], \qquad \hat{P}_F[\phi] = \pi^\sharp_\phi \circ P_F[Q^\sharp_\phi, T_{\phi,N}] \circ p\,, \tag{2.11}$$

for all $\phi \in \mathcal{W}_1$ (see also [5, (2.37) p. 131]). Since $\pi^\sharp_\phi$ depends real analytically on ϕ, we conclude that statement (i) follows. We now prove statement (ii). Thus we consider the functions $M_{F,s}[\cdot]$ of $\mathcal{A}[F]$ to $\mathbb{R}$ defined by

$$M_{F,s}[T] \equiv \sum_{j_1,\ldots,j_s \in F,\ j_1<\cdots<j_s} \mu_{j_1}[T]\cdots\mu_{j_s}[T] \qquad \forall s \in \{1,\ldots,|F|\}\,, \tag{2.12}$$

for all $(Q,T) \in \mathcal{A}[F]$. Possibly shrinking $\tilde{\mathcal{W}}$, [7, Thm. 2.30] ensures that there exist real analytic functions $M^{\sharp}_{F,s}[\cdot,\cdot]$ for $s = 1,\dots,|F|$ of $\tilde{\mathcal{W}}$ to $\mathbb{R}$ such that

$$M^{\sharp}_{F,s}[Q,T] = M_{F,s}[T] \tag{2.13}$$

for all $(Q,T) \in \tilde{\mathcal{W}} \cap \mathcal{A}[F]$, and for all $s = 1,\dots,|F|$. Since $\mu_j[T_{\phi,N}] = \mu_j[\phi] = \lambda^{-1}[\phi]$, we have

$$\Lambda_{F,s}[\phi] = \frac{M_{F,|F|-s}[T_{\phi,N}]}{M_{F,|F|}[T_{\phi,N}]} \qquad s = 1,\dots,|F|\,, \tag{2.14}$$

where $M_{F,0}[T_{\phi,N}] \equiv 1$, and statement (ii) follows. □

Then we have the following immediate Corollary.

Corollary 2.3. *Let Ω be a nonempty open connected subset of $\mathbb{R}^n$ of finite measure satisfying (2.2). Let F be a finite nonempty subset of $\mathbb{N}\setminus\{0\}$. Let*

$$\Theta_\Omega[F] \equiv \{\phi \in \mathcal{A}_\Omega[F] : \lambda_j[\phi] \text{ have a common value } \lambda_F[\phi]\ \forall j \in F\}\ . \tag{2.15}$$

Then the real analytic functions

$$\left(\binom{|F|}{1}^{-1}\Lambda_{F,1}[\cdot]\right)^{\frac{1}{1}},\dots,\left(\binom{|F|}{|F|}^{-1}\Lambda_{F,|F|}[\cdot]\right)^{\frac{1}{|F|}},$$

of $\mathcal{A}_\Omega[F]$ to $\mathbb{R}$ coincide on $\Theta_\Omega[F]$ with the function which takes ϕ to $\lambda_F[\phi]$.

Roughly speaking, the previous Corollary says that multiple eigenvalues depend analytically on the domain, as long as their multiplicity do not change, which is a known fact for simple eigenvalues (in case of smooth domains and Dirichlet boundary conditions, we refer to [6] for a similar result.)

We conclude this section by computing the first order derivatives of the functions $\Lambda_{F,s}[\cdot]$ at a point $\tilde{\phi} \in \Theta_\Omega[F]$.

Let Ω be an open subset of $\mathbb{R}^n$. As customary, we denote by $W^{2,2}(\Omega)$ the Sobolev space of distributions in Ω with derivatives of order less or equal to 2 in $L^2(\Omega)$, and by $W^{1,\infty}(\Omega)$ the space of distributions in Ω with derivatives of order less or equal to 1 in $L^\infty(\Omega)$. Then we have the following technical Lemma.

Lemma 2.4. *Let Ω be a nonempty open connected subset of $\mathbb{R}^n$ of finite measure satisfying (2.2). Let $\mathcal{X}_\Omega$ be a normed space continuously imbedded in $\mathrm{Lip}(\Omega)$. Let F be a finite nonempty subset of $\mathbb{N}\setminus\{0\}$. Let $\tilde{\phi} \in \Theta_\Omega[F]$. Let $\tilde{u}_1$, $\tilde{u}_2 \in W^{1,2}(\Omega)$ be such that $p[\tilde{u}_1]$, $p[\tilde{u}_2]$ be two eigenvectors corresponding to the eigenvalue $\lambda_F^{-1}[\tilde{\phi}]$*

of the operator $T_{\tilde{\phi},N}$. *Then we have that*

$$Q^\sharp_{\tilde{\phi}}\left[\left\{d_{|\phi=\tilde{\phi}}\left[T_{\phi,N}\right][\psi]\right\}[p[\tilde{u}_1]]\,,p[\tilde{u}_2]\right]= \tag{2.16}$$
$$\lambda_F^{-1}[\tilde{\phi}]\left\{\int_{\tilde{\phi}(\Omega)}D\tilde{v}_1\left[D\left(\psi\circ\tilde{\phi}^{(-1)}\right)+D\left(\psi\circ\tilde{\phi}^{(-1)}\right)^t\right]D\tilde{v}_2^t\,dy\right.$$
$$\left.-\int_{\tilde{\phi}(\Omega)}D\tilde{v}_1D\tilde{v}_2^t\mathrm{div}\left(\psi\circ\tilde{\phi}^{(-1)}\right)\,dy\right\}+\int_{\tilde{\phi}(\Omega)}\pi[\tilde{v}_1]\pi[\tilde{v}_2]\mathrm{div}\left(\psi\circ\tilde{\phi}^{(-1)}\right)\,dy\,,$$

for all $\psi\in\mathcal{X}^n_\Omega$, *where* $\tilde{v}_1\equiv\tilde{u}_1\circ\tilde{\phi}^{(-1)}$, $\tilde{v}_2\equiv\tilde{u}_2\circ\tilde{\phi}^{(-1)}$. *If we further assume that* $\tilde{v}_1$, $\tilde{v}_2\in W^{2,2}(\tilde{\phi}(\Omega))$, *then the right hand side of (*2.16*) equals*

$$-\lambda_F^{-1}[\tilde{\phi}]\int_{\tilde{\phi}(\Omega)}\mathrm{div}\left[\left(D\tilde{v}_1D\tilde{v}_2^t-\lambda_F[\tilde{\phi}]\pi[\tilde{v}_1]\pi[\tilde{v}_2]\right)\left(\psi\circ\tilde{\phi}^{(-1)}\right)\right]\,dy\,, \tag{2.17}$$

for all $\psi\in(\mathrm{Lip}(\Omega)\cap L^\infty(\Omega))^n$.

Proof. To shorten our notation, we set $\tilde{\lambda}\equiv\lambda_F[\tilde{\phi}]$. By standard Calculus in Banach space, and by the obvious equality $\Delta_{\tilde{\phi},N}\pi^\sharp_{\tilde{\phi}}[p[\tilde{u}_i]]=-\tilde{\lambda}\mathcal{J}_{\tilde{\phi}}\circ\mathcal{I}\circ\pi^\sharp_{\tilde{\phi}}[p[\tilde{u}_i]]$, for $i=1,2$, and by the symmetry of $Q^\sharp_{\tilde{\phi}}$, and by the definition of $\Delta_{\tilde{\phi},N}$, the left hand side of (2.16) equals

$$Q^\sharp_{\tilde{\phi}}\left[\left(\pi^\sharp_{\tilde{\phi}}\right)^{(-1)}\circ\left(\Delta_{\tilde{\phi},N}\right)^{(-1)}\circ\left(d_{|\phi=\tilde{\phi}}\left(\Delta_{\phi,N}\circ\pi^\sharp_{\phi}\right)[\psi]\right)\circ\right. \tag{2.18}$$
$$\left.\circ\left(\pi^\sharp_{\tilde{\phi}}\right)^{(-1)}\circ\left(\Delta_{\tilde{\phi},N}\right)^{(-1)}\circ\mathcal{J}_{\tilde{\phi}}\circ\mathcal{I}\circ\pi^\sharp_{\tilde{\phi}}[p[\tilde{u}_1]],p[\tilde{u}_2]\right]$$
$$-Q^\sharp_{\tilde{\phi}}\left[\left(\pi^\sharp_{\tilde{\phi}}\right)^{(-1)}\circ\left(\Delta_{\tilde{\phi},N}\right)^{(-1)}\circ\left(d_{|\phi=\tilde{\phi}}\left(\mathcal{J}_\phi\circ\mathcal{I}\circ\pi^\sharp_\phi\right)[\psi]\right)[p[\tilde{u}_1]],p[\tilde{u}_2]\right]$$
$$=-\left\{\left(d_{|\phi=\tilde{\phi}}\left(\Delta_{\phi,N}\circ\pi^\sharp_\phi\right)[\psi]\right)\circ\left(\pi^\sharp_{\tilde{\phi}}\right)^{(-1)}\circ\right.$$
$$\left.\circ\left(\Delta_{\tilde{\phi},N}\right)^{(-1)}\circ\mathcal{J}_{\tilde{\phi}}\circ\mathcal{I}\circ\pi^\sharp_{\tilde{\phi}}[p[\tilde{u}_1]]\right\}\left[\pi^\sharp_{\tilde{\phi}}[p[\tilde{u}_2]]\right]$$
$$+\left\{\left(d_{|\phi=\tilde{\phi}}\left(\mathcal{J}_\phi\circ\mathcal{I}\circ\pi^\sharp_\phi\right)[\psi]\right)[p[\tilde{u}_1]]\right\}\left[\pi^\sharp_{\tilde{\phi}}[p[\tilde{u}_2]]\right]$$
$$=\tilde{\lambda}^{-1}\left\{d_{|\phi=\tilde{\phi}}\left(\Delta_{\phi,N}\circ\pi_\phi[\tilde{u}_1]\right)[\psi]\right\}\left[\pi_{\tilde{\phi}}[\tilde{u}_2]\right]$$
$$+\left\{d_{|\phi=\tilde{\phi}}\left(\mathcal{J}_\phi\circ\mathcal{I}\circ\pi_\phi[\tilde{u}_1]\right)[\psi]\right\}\left[\pi_{\tilde{\phi}}[\tilde{u}_2]\right]\,,$$

for all $\psi\in\mathcal{X}^n_\Omega$. We now compute $d_{|\phi=\tilde{\phi}}\left\{\left(\mathcal{J}_\phi\circ\mathcal{I}\circ\pi_\phi[\tilde{u}_1]\right)[\psi]\right\}\left[\pi_{\tilde{\phi}}[\tilde{u}_2]\right]$. By standard calculus, it is easy to see that

$$\left[\left(d_{|\phi=\tilde{\phi}}\left(\det D\phi\right)[\psi]\right)\circ\tilde{\phi}^{(-1)}\right]\det D\tilde{\phi}^{(-1)}=\mathrm{div}\left(\psi\circ\tilde{\phi}^{(-1)}\right)\,, \tag{2.19}$$

and that the map of $A \equiv \{f \in L^\infty(\Omega) : \operatorname{ess\,inf}_\Omega |f| > 0\}$ to $L^\infty(\Omega)$ which takes f to $|f|$ is differentiable, and that for all $f \in A$, its differential at f is the map of $L^\infty(\Omega)$ to itself which maps h to $\operatorname{sgn}(f)h$. Then by (2.19), and by changing the variables with the map $\tilde{\phi}$ (cf. Reshetnyak [15, Thm. 2.2, p. 99]), and by equality $\int_\Omega \pi_{\tilde{\phi}}[\tilde{u}_2] |\mathrm{det} D\tilde{\phi}|\, dx = 0$, we obtain

$$\begin{aligned} & d_{|\phi=\tilde{\phi}} \left\{ (\mathcal{J}_\phi \circ \mathcal{I} \circ \pi_\phi[\tilde{u}_1]) [\psi] \right\} \left[\pi_{\tilde{\phi}}[\tilde{u}_2] \right] \qquad (2.20) \\ & \quad = \int_\Omega \pi_{\tilde{\phi}}[\tilde{u}_1] \pi_{\tilde{\phi}}[\tilde{u}_2] d_{|\phi=\tilde{\phi}} \left(|\mathrm{det} D\phi| \right) [\psi]\, dx \\ & \quad = \int_{\tilde{\phi}(\Omega)} \left(\pi_{\tilde{\phi}}[\tilde{u}_1] \circ \tilde{\phi}^{(-1)} \right) \left(\pi_{\tilde{\phi}}[\tilde{u}_2] \circ \tilde{\phi}^{(-1)} \right) \mathrm{div} \left(\psi \circ \tilde{\phi}^{(-1)} \right)\, dy \,. \end{aligned}$$

We now compute $\left\{ d_{|\phi=\tilde{\phi}} \left(\Delta_{\phi,N} \circ \pi_\phi[\tilde{u}_1] \right) [\psi] \right\} \left[\pi_{\tilde{\phi}}[\tilde{u}_2] \right]$. To shorten our notation, we find convenient to set $G_\phi \equiv (D\phi)^{-1} (D\phi)^{-t}$. Then by definition of $\Delta_{\phi,N}$, we obtain

$$\begin{aligned} & \left\{ d_{|\phi=\tilde{\phi}} \left(\Delta_{\phi,N} \circ \pi_\phi[\tilde{u}_1] \right) [\psi] \right\} \left[\pi_{\tilde{\phi}}[\tilde{u}_2] \right] = \qquad (2.21) \\ & \qquad - \int_\Omega D\tilde{u}_1 \left(d_{|\phi=\tilde{\phi}} G_\phi[\psi] \right) D\tilde{u}_2^t |\mathrm{det} D\tilde{\phi}|\, dx \\ & \qquad - \int_\Omega D\tilde{u}_1 G_{\tilde{\phi}} D\tilde{u}_2^t d_{|\phi=\tilde{\phi}} \left(|\mathrm{det} D\phi| \right) [\psi]\, dx \,. \end{aligned}$$

By equality (2.19), we have

$$\int_\Omega D\tilde{u}_1 G_{\tilde{\phi}} D\tilde{u}_2^t d_{|\phi=\tilde{\phi}} \left(|\mathrm{det} D\phi| \right) [\psi]\, dx = \int_{\tilde{\phi}(\Omega)} D\tilde{v}_1 D\tilde{v}_2^t \mathrm{div} \left(\psi \circ \tilde{\phi}^{(-1)} \right)\, dy \,. \qquad (2.22)$$

We now observe that

$$\begin{aligned} & \left[d_{|\phi=\tilde{\phi}} G_\phi[\psi] \right] \circ \tilde{\phi}^{(-1)} = \qquad (2.23) \\ & \quad -D \left(\tilde{\phi}^{(-1)} \right) \left[D \left(\psi \circ \tilde{\phi}^{(-1)} \right) + D \left(\psi \circ \tilde{\phi}^{(-1)} \right)^t \right] D \left(\tilde{\phi}^{(-1)} \right)^t \,. \end{aligned}$$

Then, by another change of variables, we obtain

$$\begin{aligned} & \int_\Omega D\tilde{u}_1 \left(d_{|\phi=\tilde{\phi}} G_\phi[\psi] \right) D\tilde{u}_2^t |\mathrm{det} D\tilde{\phi}|\, dx \qquad (2.24) \\ & \quad = - \int_{\tilde{\phi}(\Omega)} D\tilde{v}_1 \left[D \left(\psi \circ \tilde{\phi}^{(-1)} \right) + D \left(\psi \circ \tilde{\phi}^{(-1)} \right)^t \right] D\tilde{v}_2^t\, dy \,. \end{aligned}$$

By the above equalities, it follows that (2.16) holds. We now consider the case in which we further assume that $\tilde{v}_1, \tilde{v}_2 \in W^{2,2}(\tilde{\phi}(\Omega))$. To shorten our notation, we set $\omega \equiv (\omega_s)_{s=1,\dots,n}$ where $\omega_s = \psi_s \circ \tilde{\phi}^{(-1)}$ and $\psi = (\psi_s)_{s=1,\dots,n}$. Since $l_\Omega[\tilde{\phi}] > 0$, then $\tilde{\phi}^{(-1)}$ is Lipschitz continuous on $\tilde{\phi}(\Omega)$, and thus ω is also Lipschitz continuous on $\tilde{\phi}(\Omega)$, and the functions ω_s have essentially bounded first order distributional derivatives. Since $\psi \in (L^\infty(\Omega))^n$, the Lipschitz continuity of $\tilde{\phi}^{(-1)}$ in

$\tilde{\phi}(\Omega)$ ensures that $\omega \in \left(L^\infty(\tilde{\phi}(\Omega))\right)^n$. Then we have $\omega \in \left(W^{1,\infty}(\tilde{\phi}(\Omega))\right)^n$, and $\omega D\tilde{v}_r^t \in W^{1,2}(\tilde{\phi}(\Omega))$ for $r = 1, 2$. Now we note that

$$\begin{aligned} & D\tilde{v}_1 \left(D\omega + D\omega^t\right) D\tilde{v}_2^t \\ & \qquad = \mathrm{div}\left((\omega D\tilde{v}_1^t)D\tilde{v}_2 + (\omega D\tilde{v}_2^t)D\tilde{v}_1 - (D\tilde{v}_1 D\tilde{v}_2^t)\omega\right) \\ & \qquad\quad - \left[(\omega D\tilde{v}_1^t)\Delta\tilde{v}_2 + (\omega D\tilde{v}_2^t)\Delta\tilde{v}_1\right] + (D\tilde{v}_1 D\tilde{v}_2^t)\mathrm{div}\omega\,, \end{aligned} \tag{2.25}$$

where Δ denotes the Laplacian (in the sense of distributions) applied to a function of $W^{2,2}(\tilde{\phi}(\Omega))$. Since $\omega D\tilde{v}_r^t \in W^{1,2}(\tilde{\phi}(\Omega))$, and $\pi[\tilde{v}_r]$ is an eigenvector corresponding to the eigenvalue $\tilde{\lambda}$ of problem (1.1) for $\phi = \tilde{\phi}$ and for $r = 1, 2$, we have

$$\int_{\tilde{\phi}(\Omega)} D\left(\omega D\tilde{v}_r^t\right) D\tilde{v}_s^t\, dy = \tilde{\lambda} \int_{\tilde{\phi}(\Omega)} \left(\omega D\tilde{v}_r^t\right) \pi[\tilde{v}_s]\, dy\,, \tag{2.26}$$

for $r, s = 1, 2$. Since $\tilde{v}_s \in W^{2,2}(\tilde{\phi}(\Omega))$ and $\Delta\tilde{v}_s = -\tilde{\lambda}\pi[\tilde{v}_s]$ in $\tilde{\phi}(\Omega)$ for $s = 1, 2$ equality (2.26) and Leibnitz rule for derivation implies that

$$\int_{\tilde{\phi}(\Omega)} \mathrm{div}\left[\left(\omega D\tilde{v}_r^t\right) D\tilde{v}_s\right]\, dy = 0 \qquad\qquad r, s = 1, 2\,. \tag{2.27}$$

Then by equalities (2.25) and (2.27), we have

$$\begin{aligned} & \int_{\tilde{\phi}(\Omega)} D\tilde{v}_1 \left(D\omega + D\omega^t\right) D\tilde{v}_2^t \\ & \qquad = -\int_{\tilde{\phi}(\Omega)} \mathrm{div}\left[(D\tilde{v}_1 D\tilde{v}_2^t)\omega\right]\, dy + \tilde{\lambda}\int_{\tilde{\phi}(\Omega)} \left[(\omega D\tilde{v}_1^t)\pi[\tilde{v}_2] + (\omega D\tilde{v}_2^t)\pi[\tilde{v}_1]\right]\, dy \\ & \qquad\quad + \int_{\tilde{\phi}(\Omega)} D\tilde{v}_1 D\tilde{v}_2^t \mathrm{div}\omega\, dy\,. \end{aligned} \tag{2.28}$$

Then by equality (2.28), the right hand side of equality (2.16) equals

$$\begin{aligned} & -\tilde{\lambda}^{-1}\int_{\tilde{\phi}(\Omega)} \mathrm{div}\left[(D\tilde{v}_1 D\tilde{v}_2^t)\omega\right]\, dy \\ & \qquad + \int_{\tilde{\phi}(\Omega)} (\omega D\tilde{v}_1^t)\pi[\tilde{v}_2] + (\omega D\tilde{v}_2^t)\pi[\tilde{v}_1] + \pi[\tilde{v}_1]\pi[\tilde{v}_2]\mathrm{div}\omega\, dy\,. \end{aligned} \tag{2.29}$$

Now we note that

$$\begin{aligned} & \int_{\tilde{\phi}(\Omega)} (\omega D\tilde{v}_1^t)\pi[\tilde{v}_2]\, dy \\ & \quad = \int_{\tilde{\phi}(\Omega)} \mathrm{div}\left[\pi[\tilde{v}_1]\pi[\tilde{v}_2]\omega\right]\, dy - \int_{\tilde{\phi}(\Omega)} \pi[\tilde{v}_1]\pi[\tilde{v}_2]\mathrm{div}\omega\, dy - \int_{\tilde{\phi}(\Omega)} (\omega D\tilde{v}_2^t)\pi[\tilde{v}_1] dy\,. \end{aligned} \tag{2.30}$$

Then equality (2.17) follows by (2.29).

□

Then we have the following.

Theorem 2.5. *Let Ω be a nonempty open connected subset of $\mathbb{R}^n$ of finite measure satisfying (2.2). Let $\mathcal{X}_\Omega$ be a normed space continuously imbedded in* $\mathrm{Lip}(\Omega)$*. Let F be a finite nonempty subset of $\mathbb{N}\setminus\{0\}$. Let $\Theta_\Omega[F]$ be as in (2.15). Let $\tilde\phi \in \Theta_\Omega[F]$. Let $\tilde v_1,\dots,\tilde v_{|F|}$ be an orthonormal basis of the eigenspace associated to the eigenvalue $\lambda_F[\tilde\phi]$ of $-\Delta_N$ in $w^{1,2,0}(\tilde\phi(\Omega))$, where the orthonormality is taken with respect to the scalar product of (1.3). Let $s \in \{1,\dots,|F|\}$. Then we have*

$$
\begin{aligned}
& d_{|\phi=\tilde\phi}\left(\Lambda_{F,s}\right)[\psi] \qquad (2.31)\\
&= -\lambda_F^s[\tilde\phi] \begin{pmatrix} |F|-1 \\ s-1 \end{pmatrix} \sum_{l=1}^{|F|} \Bigg\{ \int_{\tilde\phi(\Omega)} \left[\lambda_F[\tilde\phi]\tilde v_l^2 - |D\tilde v_l|^2\right] \mathrm{div}\left(\psi\circ\tilde\phi^{(-1)}\right) dy \\
&\quad + \int_{\tilde\phi(\Omega)} D\tilde v_l \left[D\left(\psi\circ\tilde\phi^{(-1)}\right) + D\left(\psi\circ\tilde\phi^{(-1)}\right)^t \right] D\tilde v_l^t \, dy \Bigg\},
\end{aligned}
$$

for all $\psi \in \mathcal{X}_\Omega^n$. If we further assume that $\tilde v_l \in W^{2,2}(\tilde\phi(\Omega))$ for $l = 1,\dots,|F|$, then the right hand side of (2.31) equals

$$
\lambda_F^s[\tilde\phi] \begin{pmatrix} |F|-1 \\ s-1 \end{pmatrix} \sum_{l=1}^{|F|} \int_{\tilde\phi(\Omega)} \mathrm{div}\left[\left(|D\tilde v_l|^2 - \lambda_F[\tilde\phi]\tilde v_l^2\right)\left(\psi\circ\tilde\phi^{(-1)}\right)\right] dy\,, \qquad (2.32)
$$

for all $\psi \in (\mathrm{Lip}(\Omega)\cap L^\infty(\Omega))^n$.

Proof. We set $\tilde u_l = \tilde v_l \circ \tilde\phi$, for all $l = 1,\dots,|F|$. Clearly $p[\tilde u_l]$, $l = 1,\dots,|F|$ is an orthonormal basis of the eigenspace associated to the eigenvalue $\lambda_F^{-1}[\tilde\phi]$ of $T_{\tilde\phi,N}$ in $w^{1,2}_{\tilde\phi}(\Omega)/\mathbb{R}$. We first consider case $|F| > 1$. Let $M_{F,s}$, $M^\sharp_{F,s}$ be as in the proof of Theorem 2.2 (see (2.13).) Then by [7, Thm. 2.30], we have

$$
d_T M^\sharp_{F,s}[Q^\sharp_{\tilde\phi}, T_{\tilde\phi,N}](\dot T) = \begin{pmatrix} |F|-1 \\ s-1 \end{pmatrix} \lambda_F^{1-s}[\tilde\phi] \sum_{l=1}^{|F|} Q^\sharp_{\tilde\phi}\left[\dot T[p[\tilde u_l]], p[\tilde u_l]\right], \qquad (2.33)
$$

for all compact and selfadjoint operators $\dot T$ in $\left(w^{1,2}(\Omega)/\mathbb{R}, Q^\sharp_{\tilde\phi}\right)$, and for all $s = 1,\dots,|F|$. Then by (2.14), and by (2.33), we have

$$
\begin{aligned}
d_{|\phi=\tilde\phi}\left(\Lambda_{F,s}\right)[\psi] &= \Big\{ d_{|\phi=\tilde\phi} M_{F,|F|-s}\left[T_{\phi,N}\right][\psi] M_{F,|F|}\left[T_{\tilde\phi,N}\right] \qquad (2.34)\\
&\qquad - M_{F,|F|-s}\left[T_{\tilde\phi,N}\right] d_{|\phi=\tilde\phi} M_{F,|F|}\left[T_{\phi,N}\right][\psi] \Big\} \lambda_F^{2|F|}[\tilde\phi] \\
&= \left[\begin{pmatrix} |F|-1 \\ |F|-s-1 \end{pmatrix} \lambda_F^{s+1-2|F|}[\tilde\phi] - \begin{pmatrix} |F| \\ s \end{pmatrix} \begin{pmatrix} |F|-1 \\ |F|-1 \end{pmatrix} \lambda_F^{s+1-2|F|}[\tilde\phi] \right] \\
&\qquad \cdot \lambda_F^{2|F|}[\tilde\phi] \sum_{l=1}^{|F|} Q^\sharp_{\tilde\phi}\left[\left\{ d_{|\phi=\tilde\phi}\left[T_{\phi,N}\right][\psi] \right\}[p[\tilde u_l]], p[\tilde u_l] \right] \\
&= -\lambda_F^{s+1}[\tilde\phi] \begin{pmatrix} |F|-1 \\ s-1 \end{pmatrix} \sum_{l=1}^{|F|} Q^\sharp_{\tilde\phi}\left[\left\{ d_{|\phi=\tilde\phi}\left[T_{\phi,N}\right][\psi] \right\}[p[\tilde u_l]], p[\tilde u_l] \right].
\end{aligned}
$$

Then we can conclude by Lemma 2.4, and by the obvious equalities $\pi[\tilde{v}_l] = \tilde{v}_l$ for $l = 1, \dots, |F|$. Case $|F| = 1$ can be treated similarly. □

Concerning the statement of Lemma 2.4, we note that if $\tilde{\phi}(\Omega)$ is of class $C^{1,1}$, then by standard elliptic regularity theory, we have $\tilde{v}_r \in W^{2,2}(\tilde{\phi}(\Omega))$ for $r = 1, 2$ (cf. *e.g.*, Troianiello [16, Thm. 3.29, p. 195].) Moreover, by the Divergence Theorem, the integral in (2.17) would equal

$$\int_{\partial\tilde{\phi}(\Omega)} \left(D\tilde{v}_1 D\tilde{v}_2^t - \lambda_F[\tilde{\phi}]\pi[\tilde{v}_1]\pi[\tilde{v}_2] \right) \left(\psi \circ \tilde{\phi}^{(-1)} \right) \cdot \nu^t \, d\sigma \,,$$

where ν denotes the exterior normal to $\partial \left(\tilde{\phi}(\Omega) \right)$, and $d\sigma$ denotes the $(n-1)$-dimensional area element of $\partial \left(\tilde{\phi}(\Omega) \right)$. A corresponding remark holds of course for Theorem 2.5 and formula (2.32).

Furthermore, we note that if we assume that Ω is of class $C^{1,1}$, and that $\tilde{\phi} \in \mathcal{A}_\Omega$ has continuous partial derivatives in Ω satisfying a Lipschitz condition in Ω, then $\tilde{\phi}(\Omega)$ is of class $C^{1,1}$ (cf. *e.g.*, [6, Lem. 2.4].)

References

[1] V.I. Burenkov, *Sobolev spaces on domains*, B.G. Teubner, Stuttgart, 1998.

[2] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, RI, 1998.

[3] E. Hille and R.S. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Publ., **31**, 1957.

[4] P.D. Lamberti, *A few spectral perturbation problems*, Doctoral Dissertation, University of Padova, Italy, 2002.

[5] P.D. Lamberti and M. Lanza de Cristoforis, *A global Lipschitz continuity result for a domain dependent Neumann eigenvalue problem for the Laplace operator*, J. Differential Equations, **216**, (2005), pp. 109–133.

[6] P.D. Lamberti and M. Lanza de Cristoforis, *An analyticity result for the dependence of multiple eigenvalues and eigenspaces of the Laplace operator upon perturbation of the domain*, Glasgow Math. J., **44**, (2002), pp. 29–43.

[7] P.D. Lamberti and M. Lanza de Cristoforis, *A real analyticity result for symmetric functions of the eigenvalues of a domain dependent Dirichlet problem for the Laplace operator*, Journal of Nonlinear and Convex Analysis, **5**, (2004), pp. 19–42.

[8] P.D. Lamberti and M. Lanza de Cristoforis, *Critical points of the symmetric functions of the eigenvalues of the Laplace operator and overdetermined problems*, J. Math. Soc. Japan, **58** (2006), pp. 231–245.

[9] P.D. Lamberti and M. Lanza de Cristoforis, *Persistence of eigenvalues and multiplicity in the Neumann problem for the Laplace operator on nonsmooth domains*, Rend. Circ. Mat. Palermo (2) Suppl. **76**, (2005), pp. 413–427.

[10] M. Lanza de Cristoforis, *Properties and pathologies of the composition and inversion operator in Schauder spaces*, Rend. Accad. Naz. delle Scienze detta dei XL, Memorie di Matematica, **15**, (1991), pp. 93–109.

[11] A.M. Micheletti, *Perturbazione dello spettro dell'operatore di Laplace, in relazione ad una variazione del campo*, Ann. Scuola Norm. Sup. Pisa (3), **26**, (1972), pp. 151–169.

[12] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et C., Paris, 1967.

[13] G. Pólya and M. Schiffer, *Convexity of functionals by transplantation*, J. Analyse Math., **3**, (1954), pp. 245–346.

[14] F. Rellich, *Perturbation theory of eigenvalue problems*, Gordon and Breach Science Publ., New York, 1969.

[15] Yu.G. Reshetnyak, *Space mappings with bounded distortion*, Translations of Mathematical Monographs, **73**, American Mathematical Society, Providence, RI, 1989.

[16] G.M. Troianiello, *Elliptic differential equations and obstacle problems*, Plenum Press, New York and London, 1987.

Pier Domenico Lamberti
Dipartimento di Matematica Pura ed Applicata
Università di Padova
Via Trieste 63
35121 Padova
Italy
e-mail: `lamberti@math.unipd.it`

Massimo Lanza de Cristoforis
Dipartimento di Matematica Pura ed Applicata
Università di Padova
Via Trieste 63
35121 Padova
Italy
e-mail: `mldc@math.unipd.it`