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REAL ANALYTIC DEPENDENCE OF SIMPLE AND DOUBLE LAYER POTENTIALS UPON PERTURBATION OF THE SUPPORT AND OF THE DENSITY

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ABSTRACT. We consider a hypersurface in Euclidean space $\mathbf{R}^n$ parametrized by a diffeomorphism of the unit sphere to $\mathbf{R}^n$, and a density function on the hypersurface, which we think as points in suitable Schauder spaces, and we consider the dependence of the corresponding simple and double layer potentials, which we also think as points in suitable Schauder spaces, upon variation of the diffeomorphism and of the density, and we show a real analyticity theorem for such dependence.

1. Introduction. In this paper, we plan to study the dependence of simple and of double layer potentials upon the hypersurface of integration, i.e., upon the support. We assume the hypersurface of integration to be of sphere-type. Namely, we denote by $\mathbf{B}_n \equiv \{x \in \mathbf{R}^n : |x| < 1\}$ the open unit ball in the Euclidean space $\mathbf{R}^n$, with $n \geq 2$, and we consider our hypersurface to be assigned by a diffeomorphism ϕ of $\partial\mathbf{B}_n$ onto $\phi(\partial\mathbf{B}_n) \subseteq \mathbf{R}^n$, such that $\phi(\partial\mathbf{B}_n)$ is an $(n-1)$-dimensional manifold imbedded in $\mathbf{R}^n$. Then we consider the set $\mathcal{A}_{\partial\mathbf{B}_n}$ of such admissible functions ϕ, see Lemma 2.5, and we consider the Schauder space $C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n)$. The set $C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}$ is open in $C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n)$, and we can think of ϕ as a point of such a set. If $f \in C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R})$, then the function $f \circ \phi^{(-1)}$ is defined on $\phi(\partial\mathbf{B}_n)$, and it makes sense to consider the simple and double layer potentials

$$v[\phi, f](\xi) \equiv \int_{\phi(\partial\mathbf{B}_n)} S_n(\xi-\eta)\, f\circ\phi^{(-1)}(\eta)\, d\sigma_\eta \qquad \forall\, \xi \in \phi(\partial\mathbf{B}_n),$$

$$w[\phi, f](\xi) \equiv \int_{\phi(\partial\mathbf{B}_n)} \frac{\partial}{\partial\nu_\phi(\eta)}\left[S_n(\xi-\eta)\right] f\circ\phi^{(-1)}(\eta)\, d\sigma_\eta \quad \forall\, \xi \in \phi(\partial\mathbf{B}_n),$$

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where $S_n(\cdot)$ denotes the fundamental solution of the Laplace operator in $\mathbf{R}^n$, and where ν_ϕ denotes the outward normal to the set $\mathcal{I}[\phi]$ bounded by $\phi\,(\partial\mathbf{B}_n)$, see Section 2. Then we can consider the functions

$$V[\phi,f](x) \equiv v[\phi,f]\circ\phi(x) \quad \forall\, x \in \partial\mathbf{B}_n, \tag{1.1}$$

$$W[\phi,f](x) \equiv w[\phi,f]\circ\phi(x) \quad \forall\, x \in \partial\mathbf{B}_n\,. \tag{1.2}$$

One can consider the nonlinear operator $V[\cdot,\cdot]$ of $(C^{m,\alpha}\,(\partial\mathbf{B}_n,\mathbf{R}^n)\cap\mathcal{A}_{\partial\mathbf{B}_n})\times C^{m-1,\alpha}\,(\partial\mathbf{B}_n,\mathbf{R})$ to $C^{m,\alpha}\,(\partial\mathbf{B}_n,\mathbf{R})$ which takes (ϕ,f) to the function $V[\phi,f]$ defined above, and the nonlinear operator $W[\cdot,\cdot]$ of $(C^{m,\alpha}\,(\partial\mathbf{B}_n,\mathbf{R}^n)\cap\mathcal{A}_{\partial\mathbf{B}_n})\times C^{m,\alpha}\,(\partial\mathbf{B}_n,\mathbf{R})$ to $C^{m,\alpha}\,(\partial\mathbf{B}_n,\mathbf{R})$ which takes (ϕ,f) to the function $W[\phi,f]$ defined above.

The purpose of this paper is to show that the operators $V[\cdot,\cdot]$, $W[\cdot,\cdot]$ defined above are real analytic, see Theorem 3.12. Then we also compute all order differentials of V, W, cf., Proposition 3.14. Problems such as those treated here are not new. A stability result for simple layer potentials has been proved by Keldish [**7**, Lemma VIII] in order to study stability properties of boundary value problems. Then we mention a continuity result of Verchota, see Meyer [**18**, Theorem 7] for simple layer potentials defined on Lipschitz hypersurfaces upon variation of the hypersurface. Also in connection with the work of this paper, we mention the study of the dependence of the Cauchy integral

$$C[\phi,f](\cdot) \equiv \frac{1}{2\pi i}\int_{\partial\mathbf{B}_2}\frac{f(t)\phi'(t)}{\phi(t)-\phi(\cdot)}\,dt$$

upon the pair (ϕ,f), and the contribution of Calderón, Coifman, Meyer, McIntosh, David, whose work implies the analyticity of operators related to C. For references to the various contributions of the above authors to this issue, we refer to Lanza and Preciso [**13**, Section 1]. In particular, at least in case $n=2$, a formula for the derivatives of W could also be deduced by known corresponding formulas for the derivatives of C. By Lanza and Preciso [**13**], the operator C is complex analytic from $(C^{m,\alpha}\,(\partial\mathbf{B}_n,\mathbf{C})\cap\mathcal{A}_{\partial\mathbf{B}_n})\times C^{m,\alpha}\,(\partial\mathbf{B}_n,\mathbf{C})$ to $C^{m,\alpha}\,(\partial\mathbf{B}_n,\mathbf{C})$. The analyticity of C finds application in various problems, see Lanza [**12**], Lanza and Preciso [**14, 15**] and Lanza and Rogosin [**16, 17**]. Operators such as V, W or C appear in the study of various problems, and one of the motivations to prove the real analyticity of V, W is to analyze regular and singular perturbation problems for integral and differential equations of the type of those treated in the above mentioned papers for C.

To prove our statement, we shall consider first the operator $V[\cdot,\cdot]$. Indeed, the corresponding result for $W[\cdot,\cdot]$ can be deduced by that for $V[\cdot,\cdot]$, and by well known formulas of classical potential theory.

We now briefly describe our strategy. Given a pair (ϕ_0, f_0) in the set $(C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}) \times C^{m-1,\alpha}(\partial\mathbf{B}_n, \mathbf{R})$, we show that there exists a $0 < \delta < 1$ and an open neighborhood $\mathcal{W}_0$ of ϕ_0 such that all functions ϕ in $\mathcal{W}_0$ can be extended to a diffeomorphism $\mathbf{E}_{\phi_0}[\phi]$ of the annulus

$$\mathbf{A}_\delta \equiv \{x \in \mathbf{R}^n : 1-\delta < |x| < 1+\delta\}$$

onto a neighborhood of $\phi(\partial\mathbf{B}_n)$. Next we set

$$\mathbf{A}_\delta^+ \equiv \{x \in \mathbf{R}^n : 1-\delta < |x| < 1\},$$
$$\mathbf{A}_\delta^- \equiv \{x \in \mathbf{R}^n : 1 < |x| < 1+\delta\},$$

and we show that $v[\phi, f]$ is uniquely determined by two harmonic functions v^+, v^- defined in $\mathbf{E}_{\phi_0}[\phi](\mathbf{A}_\delta^+)$, and in $\mathbf{E}_{\phi_0}[\phi](\mathbf{A}_\delta^-)$, respectively, and that the pair (v^+, v^-) is the unique solution of a coupled boundary value problem, cf. Theorem 3.2. Next we exploit the diffeomorphism $\mathbf{E}_{\phi_0}[\phi]$ in order to transform such a domain dependent problem for (v^+, v^-) into a coupled boundary value problem for a pair of functions (V^+, V^-) which determines uniquely $V[\phi, f]$ and such that V^+ and V^- are defined on $\mathbf{A}_\delta^+$ and on $\mathbf{A}_\delta^-$, respectively. Then we recast such a boundary value problem for (V^+, V^-) into an abstract functional equation in Banach space, which we analyze by means of the Implicit Function Theorem, see proof of Proposition 3.11, in order to deduce the real analyticity of (V^+, V^-), and thus of $V[\cdot,\cdot]$.

The paper is organized as follows. Section 2 is a section of preliminaries, at the end of which we construct the extension operator $\mathbf{E}_{\phi_0}[\phi]$. In Section 3, we first introduce some known properties of layer potentials, and then we introduce the boundary value problems for (v^+, v^-) and for (V^+, V^-), which we need to prove our main Theorem 3.12. Then in Proposition 3.14, we compute all order differentials of V, W.

2. Technical preliminaries and notation. We denote the norm on a (real) normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the product space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbf{R}^n$. For standard definitions of calculus in normed spaces, we refer to Prodi and Ambrosetti [**20**]. The symbol $\mathbf{N}$ denotes the set of natural numbers including 0. Throughout the paper, n is an element of $\mathbf{N} \setminus \{0, 1\}$. The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a complex-valued function g, or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. A dot '$\cdot$' denotes the inner product in $\mathbf{R}^n$, or the matrix product between matrices with real entries. Let A be a matrix. Then A^t denotes the transpose matrix of A, and $\operatorname{tr} A$ denotes the trace of A and A_{ij} denotes the (i, j) entry of A. If A is invertible, we set $A^{-t} \equiv \left(A^{-1}\right)^t$. The set of $r \times r$ matrices with real entries is denoted $M_r(\mathbf{R})$. Let $\mathbf{D} \subseteq \mathbf{R}^n$. Then $\operatorname{cl} \mathbf{D}$ denotes the closure of $\mathbf{D}$. For all $R > 0$, $x \in \mathbf{R}^n$, x_j denotes the jth coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbf{R}^n$, and $\mathbf{B}_n(x, R)$ denotes the ball $\{y \in \mathbf{R}^n : |x - y| < R\}$. For short, we set $\mathbf{B}_n \equiv \mathbf{B}_n(0, 1)$. Let Ω be an open subset of $\mathbf{R}^n$. The space of m times continuously differentiable real-valued functions on Ω is denoted by $C^m(\Omega, \mathbf{R})$, or more simply by $C^m(\Omega)$. $\mathcal{D}(\Omega)$ denotes the space of functions of $C^\infty(\Omega)$ with compact support. The dual $\mathcal{D}'(\Omega)$ denotes the space of distributions in Ω. Let $f \in (C^m(\Omega))^n$. The sth component of f is denoted f_s, and Df denotes the gradient matrix $(\partial f_s/\partial x_l)_{s,l=1,\ldots,n}$. Let $\eta \equiv (\eta_1, \ldots, \eta_n) \in \mathbf{N}^n$, $|\eta| \equiv \eta_1 + \cdots + \eta_n$. Then $D^\eta f$ denotes $\partial^{|\eta|} f/\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n}$. The subspace of $C^m(\Omega)$ of those functions f such that f and its derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\operatorname{cl} \Omega$ is denoted $C^m(\operatorname{cl} \Omega)$. The subspace of $C^m(\operatorname{cl} \Omega)$ whose functions have mth order derivatives that are Hölder continuous with exponent $\alpha \in \,]0, 1]$ is denoted $C^{m,\alpha}(\operatorname{cl} \Omega)$, cf., e.g., Gilbarg and Trudinger [**4**]. Let $\mathbf{D} \subseteq \mathbf{R}^n$. Then $C^{m,\alpha}(\operatorname{cl} \Omega, \mathbf{D})$ denotes $\{f \in (C^{m,\alpha}(\operatorname{cl} \Omega))^n : f(\operatorname{cl} \Omega) \subseteq \mathbf{D}\}$. $C^{m,\alpha}(\operatorname{cl} \Omega, M_r(\mathbf{R}))$ denotes the space of functions of $\operatorname{cl} \Omega$ to $M_r(\mathbf{R})$, whose components are of class $C^{m,\alpha}$. Now let Ω be a bounded open subset of $\mathbf{R}^n$. Then $C^m(\operatorname{cl} \Omega)$ endowed with the norm $\|f\|_m \equiv \sum_{|\eta| \leq m} \sup_{\operatorname{cl} \Omega} |D^\eta f|$ is a Banach space. If $f \in C^{0,\alpha}(\operatorname{cl} \Omega)$, then its Hölder quotient $|f : \Omega|_\alpha$ is defined as $\sup\{|f(x) - f(y)|/|x - y|^\alpha : x, y \in \operatorname{cl} \Omega, x \neq y\}$. The space $C^{m,\alpha}(\operatorname{cl} \Omega)$, endowed with its usual norm $\|f\|_{m,\alpha} = \|f\|_m + \sum_{|\eta|=m} |D^\eta f|_\alpha$, is well known to be a Banach space. We say that a bounded open subset of $\mathbf{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if it is a manifold with boundary imbedded in $\mathbf{R}^n$ of class C^m or $C^{m,\alpha}$, respectively, cf., e.g., Gilbarg and Trudinger [**4**, Section 6.2]. In order to compactify our notation, we find it convenient to set

$$C^{m,0} \equiv C^m. \tag{2.1}$$

Thus for example, by a manifold of class $C^{m,0}$, we just mean a manifold of class C^m.

We summarize in the following statement some known properties of Schauder spaces, which we need in the sequel, cf., e.g., Gilbarg and Trudinger [**4**], Lanza [**10**, Section 2, Lemmas 3.1, 4.26, Theorem 4.28].

Lemma 2.1. *Let m, $r \in \mathbf{N}$, $r > 0$; α, $\beta \in]0,1]$. Let Ω, Ω_1 be bounded connected open subsets of $\mathbf{R}^n$ of class C^1. Then*

(i) *The pointwise product is continuous in $C^{m,\alpha}(\mathrm{cl}\,\Omega)$.*

(ii) *$C^{m+1}(\mathrm{cl}\,\Omega)$ is continuously imbedded in $C^{m,1}(\mathrm{cl}\,\Omega)$.*

(iii) *If $\alpha > \beta$, then $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ is compactly imbedded in $C^{m,\beta}(\mathrm{cl}\,\Omega)$.*

(iv) *If $m > 0$ and if $(\phi, \psi) \in C^{m,\alpha}(\mathrm{cl}\,\Omega_1) \times C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathrm{cl}\,\Omega_1)$, then $\phi \circ \psi \in C^{m,\alpha}(\mathrm{cl}\,\Omega)$.*

(v) *If $m > 0$, $\phi \in C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{R}^n)$, ϕ is injective and $\det D\phi \neq 0$ in $\mathrm{cl}\,\Omega$, then the inverse function $\phi^{(-1)} \in C^{m,\alpha}(\mathrm{cl}\,\phi(\Omega), \mathbf{R}^n)$.*

(vi) *The pointwise matrix product, which reduces to the pointwise product of functions when $r = 1$, is bilinear and continuous and henceforth real analytic from the space $C^{m,\alpha}(\mathrm{cl}\,\Omega, M_r(\mathbf{R})) \times C^{m,\alpha}(\mathrm{cl}\,\Omega, M_r(\mathbf{R}))$ to the space $C^{m,\alpha}(\mathrm{cl}\,\Omega, M_r(\mathbf{R}))$.*

(vii) *The map $F \mapsto F^{-1}$ is real analytic from the set $\{F \in C^{m,\alpha}(\mathrm{cl}\,\Omega, M_r(\mathbf{R})) : \det F \neq 0 \text{ on } \mathrm{cl}\,\Omega\}$ to itself, and its differential at the element F_0 is given by the map $M \mapsto -F_0^{-1} \cdot M \cdot F_0^{-1}$.*

We note that throughout the paper 'analytic' means 'real analytic.' For the definition and properties of analytic operators, we refer to Prodi and Ambrosetti [**20**, p. 89].

As was mentioned in the introduction, we shall deal with diffeomorphisms of open subsets of $\mathbf{R}^n$. Then we introduce the following lemma, see [**10**, Corollary 4.24, Proposition 4.29].

Lemma 2.2. *Let Ω be a bounded connected open subset of $\mathbf{R}^n$ of class C^1. Then the set*

$$\mathcal{A}_{\mathrm{cl}\,\Omega} \equiv \left\{\Phi \in C^1(\mathrm{cl}\,\Omega, \mathbf{R}^n) : \Phi \textit{ is injective}, \ \det D\Phi(x) \neq 0, \ \forall\, x \in \mathrm{cl}\,\Omega\right\}$$

is open in $C^1(\mathrm{cl}\,\Omega, \mathbf{R}^n)$.

Now let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in [0,1]$. As is well known, a subset M of $\mathbf{R}^n$ is a differential manifold of dimension s and of class $C^{m,\alpha}$ imbedded in $\mathbf{R}^n$, if, for every $P \in M$, there exist a neighborhood W of P in $\mathbf{R}^n$ and a parametrization $\psi \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_s, \mathbf{R}^n)$ such that ψ is a homeomorphism of $\mathbf{B}_s$ onto $W \cap M$, $\psi(0) = P$, and $D\psi$ has rank s at all points of cl $\mathbf{B}_s$. If we further assume that M is compact, then there exist $P_1, \dots, P_r \in M$, and parametrizations $\{\psi_i\}_{i=1,\dots,r}$, with $\psi_i \in C^{m,\alpha}(\mathrm{cl}\ \mathbf{B}_s, \mathbf{R}^n)$ such that $\cup_{i=1}^r \psi_i(\mathbf{B}_s) = M$. Then there exist $0 \leq \theta_i \in \mathcal{D}(\mathbf{R}^n)$ for $i = 1, \dots, r$ such that

$$\sum_{i=1}^r \theta_i(x) = 1 \ \forall\, x \in M, \ M \cap \mathrm{supp}\,\theta_i \subseteq \psi_i(\mathbf{B}_s), \quad i = 1, \dots, r. \tag{2.2}$$

We denote by $C^{m,\alpha}(M)$ the linear space of functions f of M to $\mathbf{R}$ such that $f \circ \psi_i \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_s)$ for all $i = 1, \dots, r$, and we set

$$\|f\|_{C^{m,\alpha}(M)} \equiv \sup_{i=1,\dots,r} \|f \circ \psi_i\|_{C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_s)} \quad \forall\, f \in C^{m,\alpha}(M).$$

It is well known that, by choosing a different finite family of parametrizations as $\{\psi_i\}_{i=1,\dots,r}$, we would obtain an equivalent norm. Also, the normed space $\left(C^{m,\alpha}(M), \|\cdot\|_{C^{m,\alpha}(M)}\right)$ is known to be complete. Then we have the following, cf., e.g., Troianiello [**23**, Theorem 1.3, Lemma 1.5].

Lemma 2.3. *Let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in [0,1]$, cf.* (2.1). *Let Ω be a bounded open connected subset of $\mathbf{R}^n$ of class $C^{m,\alpha}$. Then*

(i) *$\partial\Omega$ is a manifold of class $C^{m,\alpha}$ and codimension* 1, *and the restriction map is linear and continuous from $C^{k,\alpha}(\mathrm{cl}\ \Omega)$ to $C^{k,\alpha}(\partial\Omega)$ for all $k = 0, \dots, m$.*

(ii) *There exists a linear and continuous extension operator $\mathbf{F}$ of $C^{m,\alpha}(\partial\Omega)$ to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ such that $\mathbf{F}[f]_{|\partial\Omega} = f$ for all $f \in C^{m,\alpha}(\partial\Omega)$.*

(iii) *Let* $R > 0$ *be such that* $\mathrm{cl}\,\Omega \subseteq \mathbf{B}_n(0,R)$*. Then there exists a linear and continuous operator* $\mathbf{F}_R$ *of* $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_n(0,R))$ *such that* $\mathbf{F}_R[f]_{|\mathrm{cl}\,\Omega} = f$ *for all* $f \in C^{m,\alpha}(\mathrm{cl}\,\Omega)$*.*

We now introduce the following variant of [**10**, Proposition 4.29], which can be proved by a straightforward modification of the proof of Proposition 4.29 of [**10**].

Lemma 2.4. *Let* K *be a compact subset of* $\mathbf{R}^n$*. Let* $C^{0,1}(K,\mathbf{R}^n)$ *denote the space of Lipschitz continuous functions of* K *to* $\mathbf{R}^n$*. Let* $|f : K|_1$ *denote the Lipschitz constant of* $f \in C^{0,1}(K,\mathbf{R}^n)$*. Let*

$$l_K[f] \equiv \inf\left\{\frac{|f(x)-f(y)|}{|x-y|} : x,y \in K,\ x \neq y\right\} \quad \forall\, f \in C^{0,1}(K,\mathbf{R}^n).$$

Then $l[\cdot]$ *is continuous from* $C^{0,1}(K,\mathbf{R}^n)$ *endowed with the semi-norm* $|\cdot : K|_1$ *to* $[0,+\infty[$*.*

We now introduce the following variant of [**10**, Theorem 4.18, Proposition 4.29].

Lemma 2.5. *Let* $\phi \in C^1(\partial\mathbf{B}_n,\mathbf{R}^n)$*. Then* $l_{\partial\mathbf{B}_n}[\phi] > 0$ *holds if and only if* ϕ *is injective and the differential* $d\phi(p)$ *is injective for all* $p \in \partial\mathbf{B}_n$*. The map* $l_{\partial\mathbf{B}_n}[\cdot]$ *of* $C^1(\partial\mathbf{B}_n,\mathbf{R}^n)$ *to* $[0,+\infty[$ *is continuous and the set*

$$\mathcal{A}_{\partial\mathbf{B}_n} \equiv \left\{\phi \in C^1(\partial\mathbf{B}_n,\mathbf{R}^n) : l_{\partial\mathbf{B}_n}[\phi] > 0\right\}$$

is open in $C^1(\partial\mathbf{B}_n,\mathbf{R}^n)$*.*

Proof. As is well known, there exists $\Phi \in C^1(\mathbf{R}^n,\mathbf{R}^n)$ such that $\Phi_{|\partial\mathbf{B}_n} = \phi$, see, for example, the construction of the proof of Lemma 6.38 of Gilbarg and Trudinger [**4**]. The tangent space $T_p(\partial\mathbf{B}_n)$ to $\partial\mathbf{B}_n$ at the point p of $\partial\mathbf{B}_n$ is easily seen to coincide with the linear subspace of $\mathbf{R}^n$ generated by the elements $v \in \mathbf{R}^n$ such that $|v| = 1$ and such that there exist sequences $\{x_j\}_{j\in\mathbf{N}}$, $\{y_j\}_{j\in\mathbf{N}}$ in $\partial\mathbf{B}_n$ with $x_j \neq y_j$ for all j, $\lim_{j\to\infty} x_j = p = \lim_{j\to\infty} y_j$, $\lim_{j\to\infty}(x_j - y_j)/|x_j - y_j| = v$. Then condition $l_{\partial\mathbf{B}_n}[\phi] > 0$ can be seen to be equivalent to the injectivity of ϕ together with condition $D\Phi(p)\cdot v \neq 0$ for all $v \in T_p(\partial\mathbf{B}_n)$, see [**10**, Theorem 4.18]. Since $d\phi(p)[v] = D\Phi(p)\cdot v$ for all $v \in T_p(\partial\mathbf{B}_n)$, then the proof is complete. We now turn to the proof of the continuity of $l_{\partial\mathbf{B}_n}[\cdot]$. By Lemma 2.4 it suffices to note that $C^1(\partial\mathbf{B}_n, \mathbf{R}^n)$ is continuously imbedded in the space $C^{0,1}(\partial\mathbf{B}_n, \mathbf{R}^n)$ endowed with the semi-norm $|\cdot : \partial\mathbf{B}_n|_1$, a fact which follows by the well-known existence of a continuous extension operator of $C^1(\partial\mathbf{B}_n, \mathbf{R}^n)$ into $C^1(\mathrm{cl}\,\mathbf{B}_n, \mathbf{R}^n)$, and by Lemma 2.1 (ii). $\square$

We now note that if $\phi \in \mathcal{A}_{\partial\mathbf{B}_n}$, then by the Jordan-Leray separation theorem, cf., e.g., Deimling [**2**, Theorem 5.2], the set $\mathbf{R}^n \setminus \phi(\partial\mathbf{B}_n)$ has exactly two connected components. We denote by $\mathcal{I}[\phi]$ the bounded connected component, and by $\mathcal{E}[\phi]$ the unbounded connected component. Then we have the following.

Lemma 2.6. *Let $m \in \mathbf{N}\setminus\{0\}$, $\alpha \in [0,1]$, cf. (2.1). If $\phi \in C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}$, then $\mathcal{I}[\phi]$ is a bounded open connected set of class $C^{m,\alpha}$, and $\partial\mathcal{I}[\phi] = \phi(\partial\mathbf{B}_n) = \partial\mathcal{E}[\phi]$.*

Proof. We first show that $\phi(\partial\mathbf{B}_n)$ is a manifold of class $C^{m,\alpha}$ imbedded in $\mathbf{R}^n$ of codimension 1. Let $p \in \phi(\partial\mathbf{B}_n)$, $q \in \partial\mathbf{B}_n$, $p = \phi(q)$. Let $\psi \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_{n-1}, \mathbf{R}^n)$ be a local parametrization for $\partial\mathbf{B}_n$ around q. By assumption, $\phi\circ\psi \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_{n-1}, \mathbf{R}^n)$, $\phi\circ\psi(0) = p$, and $D(\phi\circ\psi)$ has rank $n-1$ at all points of $\mathrm{cl}\,\mathbf{B}_{n-1}$. Since $\partial\mathbf{B}_n$ is compact and ϕ is a continuous bijection of $\partial\mathbf{B}_n$ onto $\phi(\partial\mathbf{B}_n)$, then ϕ is a homeomorphism of $\partial\mathbf{B}_n$ onto $\phi(\partial\mathbf{B}_n)$. Accordingly, $\phi\circ\psi$ is a homeomorphism of $\mathbf{B}_{n-1}$ onto a neighborhood of p in $\phi(\partial\mathbf{B}_n)$. Since $\phi(\partial\mathbf{B}_n)$ is homeomorphic to $\partial\mathbf{B}_n$, then we have $\partial\mathcal{I}[\phi] = \phi(\partial\mathbf{B}_n) = \partial\mathcal{E}[\phi]$, cf., e.g., Dugundji [**3**, Theorem 2.4, Chapter XVII]. We now show that if $p \in \phi(\partial\mathbf{B}_n)$, then $\mathcal{I}[\phi]$ is locally around p an open set of class $C^{m,\alpha}$. We can write $p \equiv (p', p_n)$ with $p' \in \mathbf{R}^{n-1}$, $p \in \mathbf{R}$. Since $\partial\mathcal{I}[\phi] = \phi(\partial\mathbf{B}_n)$ is a manifold embedded in $\mathbf{R}^n$, $\partial\mathcal{I}[\phi]$ is locally around p a graph. We can assume that there exist $r > 0$, $\delta > 0$, $g \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_{n-1}(p', r))$ with nonvanishing gradient such that

$$
\begin{aligned}
&\{(x', x_n) \in \mathbf{B}_{n-1}(p', r)\times]-\delta + p_n, \delta + p_n[: g(x') = x_n\} \\
&\qquad\qquad = (\mathbf{B}_{n-1}(p', r)\times]-\delta + p_n, \delta + p_n[) \cap \partial\mathcal{I}[\phi].
\end{aligned}
$$

Clearly, the sets

$$A_1 \equiv \{(x', x_n) \in \mathbf{B}_{n-1}(p', r) \times] - \delta + p_n, \delta + p_n[: g(x') < x_n\} ,$$
$$A_2 \equiv \{(x', x_n) \in \mathbf{B}_{n-1}(p', r) \times] - \delta + p_n, \delta + p_n[: g(x') > x_n\} ,$$

are both connected and must be contained either in the connected component $\mathcal{I}[\phi]$ or in the connected component $\mathcal{E}[\phi]$. However, they cannot be both contained in the same component, otherwise equality $\partial\mathcal{I}[\phi] = \phi(\partial\mathbf{B}_n) = \partial\mathcal{E}[\phi]$ would be violated. Then $\mathcal{I}[\phi]$ is locally around p an open set of class $C^{m,\alpha}$. □

Next we show that each 'admissible' ϕ can be extended to a diffeomorphism of an annular neighborhood of $\partial\mathbf{B}_n$.

Proposition 2.7. *Let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in [0, 1]$, cf. (2.1). If ϕ belongs to $C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}$, then there exist $\delta \in]0, 1[$ and $\Phi \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}_{\mathrm{cl}\,\mathbf{A}_\delta}$ such that*

(i) *$\Phi(x) = \phi(x)$ for all $x \in \partial\mathbf{B}_n$.*

(ii) *$\Phi\left(\mathbf{A}_\delta^+\right)$ is a bounded open connected subset of $\mathbf{R}^n$ of class $C^{m,\alpha}$ contained in $\mathcal{I}[\phi]$, and $\partial\Phi\left(\mathbf{A}_\delta^+\right) = \Phi\left((1-\delta)\partial\mathbf{B}_n\right) \cup \Phi\left(\partial\mathbf{B}_n\right)$.*

(iii) *$\Phi\left(\mathbf{A}_\delta^-\right)$ is a bounded open connected subset of $\mathbf{R}^n$ of class $C^{m,\alpha}$ contained in $\mathcal{E}[\phi]$, and $\partial\Phi\left(\mathbf{A}_\delta^-\right) = \Phi\left(\partial\mathbf{B}_n\right) \cup \Phi\left((1+\delta)\partial\mathbf{B}_n\right)$.*

Proof. We denote by $\nu_\phi(P)$ the exterior normal to $\mathcal{I}[\phi]$ at the point P of $\phi(\partial\mathbf{B}_n)$. By classical differential topology, one may think of defining $\Phi(x)$ as $\phi(x/|x|) + (|x| - 1)\nu_\phi(\phi(x/|x|))$ for a sufficiently small δ. For such a definition, $\Phi(\mathbf{A}_\delta)$ would be a so-called tubular neighborhood of $\phi(\partial\mathbf{B}_n)$. However, the problem with such a construction is that the normal field ν_ϕ is only of class $C^{m-1,\alpha}$ when ϕ is of class $C^{m,\alpha}$, while we need Φ of class $C^{m,\alpha}$. Then we proceed as follows. We note that $\nu_\phi \circ \phi \in C^0(\partial\mathbf{B}_n, \mathbf{R}^n)$, and that accordingly, by exploiting the local parametrizations of $\partial\mathbf{B}_n$, a partition of unity as in (2.2) and Weierstrass Approximation Theorem, $\nu_\phi \circ \phi$ can be approximated uniformly by a vector field a of class $C^\infty(\partial\mathbf{B}_n, \mathbf{R}^n)$. We take $a \in C^\infty(\partial\mathbf{B}_n, \mathbf{R}^n)$ such that $|a(x)| = 1$ for all $x \in \partial\mathbf{B}_n$ and such that $a(x)\cdot\nu_\phi(\phi(x)) > 1/2$ for all $x \in \partial\mathbf{B}_n$. Obviously, the function Φ defined by $\Phi(x) \equiv \phi(x/|x|) + (|x| - 1)a(x/|x|)$ for all $x \in \mathrm{cl}\,\mathbf{A}_\delta$, is of class

$C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n)$ for all $0<\delta<1$. We now show that the Jacobian of Φ is nonzero at all points q of $\partial\mathbf{B}_n$. It suffices to show that if $v_1, \dots, v_{n-1}$ is a basis for the tangent space $T_q(\partial\mathbf{B}_n)$, and if $v_n \equiv q/|q|$ is the outward normal to $\partial\mathbf{B}_n$ at q, then the vectors $\{d\Phi(q)[v_i]\}_{i=1,\dots,n}$ are linearly independent. By simple computations, we have $d\Phi(q)[v_i] = d\phi(q)[v_i]$ for $1 \le i \le n-1$, and $d\Phi(q)[v_n] = a(q)$. Then we conclude that the vectors $\{d\Phi(q)[v_i]\}_{i=1,\dots,n}$ are linearly independent, otherwise $a(q)$ belongs to the space generated by the vectors $\{d\Phi(q)[v_i]\}_{i=1,\dots,n-1}$, i.e., $a(q)$ belongs to the tangent space to $\phi(\partial\mathbf{B}_n)$ at $p=\phi(q)$, in contradiction to assumption $a(q)\cdot\nu_\phi(\phi(q)) > 1/2$. By taking $0<\delta<1$ sufficiently small, we can assume that $\det(D\Phi(x)) \neq 0$ for all $x \in \mathrm{cl}\,\mathbf{A}_\delta$. We now show that by possibly shrinking δ, we can assume that Φ is injective. Assume by contradiction that for all $j \in \mathbf{N}$ there exist x_j, $y_j \in \mathrm{cl}\,\mathbf{A}_{2^{-j}}$ with $x_j \neq y_j$, $\Phi(x_j) = \Phi(y_j)$. By possibly extracting subsequences, we can assume that the sequences $\{x_j\}_{j\in\mathbf{N}}$, $\{y_j\}_{j\in\mathbf{N}}$ have limits x, y in $\mathbf{R}^n$, respectively. Obviously, $x, y \in \partial\mathbf{B}_n$ and $\Phi(x)=\Phi(y)$. Since $\Phi_{|\partial\mathbf{B}_n} = \phi$, we must have $x=y$. By taking j sufficiently large, we can assume that $x_j, y_j \in \mathbf{B}_n(x,\delta)$. Then for such js we have

$$0 = \frac{\Phi(x_j)-\Phi(y_j)}{|x_j-y_j|} = \int_0^1 D\Phi(x_j+(y_j-x_j)t)\cdot\frac{x_j-y_j}{|x_j-y_j|}\,dt. \tag{2.3}$$

By further selecting subsequences, we can assume that $\lim_{j\to\infty}(x_j-y_j)/|x_j-y_j|$ exists equal to some $v \in \partial\mathbf{B}_n$. Then by taking the limit as $j\to\infty$ in (2.3), we deduce that $D\Phi(x)\cdot v = 0$, in contradiction with assumption $\det(D\Phi(x)) \neq 0$. Thus we have proved that for $0<\delta<1$ sufficiently small, $\Phi \in \mathcal{A}_{\mathrm{cl}\,\mathbf{A}_\delta}$. Then, by a standard argument, we have that $\Phi\left(\mathbf{A}_\delta^\pm\right)$ are bounded open connected subsets of $\mathbf{R}^n$ of class $C^{m,\alpha}$ and that $\partial\Phi\left(\mathbf{A}_\delta^+\right) = \Phi((1-\delta)\partial\mathbf{B}_n) \cup \Phi(\partial\mathbf{B}_n)$, $\partial\Phi\left(\mathbf{A}_\delta^-\right) = \Phi(\partial\mathbf{B}_n)\cup\Phi((1+\delta)\partial\mathbf{B}_n)$, see Lamberti and Lanza [**9**, Lemmas 2.2, 2.4]. Moreover, $\Phi\left(\mathbf{A}_\delta^\pm\right)$ are connected and contained in $\mathbf{R}^n \setminus \phi(\partial\mathbf{B}_n)$. Then each of such two sets must be contained either in $\mathcal{I}[\phi]$ or in $\mathcal{E}[\phi]$. If both $\Phi\left(\mathbf{A}_\delta^+\right)$ and $\Phi\left(\mathbf{A}_\delta^-\right)$ are contained in the same connected component of $\mathbf{R}^n\setminus\phi(\partial\mathbf{B}_n)$, say in $\mathcal{I}[\phi]$, then the points of $\phi(\partial\mathbf{B}_n)$, which are interior to $\Phi(\mathbf{A}_\delta) = \Phi(\mathbf{A}_\delta^+)\cup\phi(\partial\mathbf{B}_n)\cup\Phi(\mathbf{A}_\delta^-)$, would not be boundary points of $\mathcal{E}[\phi]$, contrary to Lemma 2.6. Then $\Phi(\mathbf{A}_\delta^+)$ and $\Phi(\mathbf{A}_\delta^-)$ cannot be contained in the same component of $\mathbf{R}^n\setminus\phi(\partial\mathbf{B}_n)$. We now show that with our choice of $a(\cdot)$, one must necessarily have $\Phi(\mathbf{A}_\delta^-) \subseteq \mathcal{E}[\phi]$. Assume by contradiction that $\Phi(\mathbf{A}_\delta^-) \subseteq \mathcal{I}[\phi]$. Then we take $q \in \partial \mathbf{B}_n$. Since $\nu_\phi(\phi(q))$ is the exterior normal to $\mathcal{I}[\phi]$ at $\phi(q)$, then there exists $0 < \eta_0 < \delta$ such that $\phi(q) + \eta\nu_\phi(\phi(q)) \in \mathcal{E}[\phi]$ for $0 < \eta < \eta_0$. Moreover, we must have

$$\phi\left(\frac{q+\eta q}{|q+\eta q|}\right) + (|q+\eta q| - 1)a\left(\frac{q+\eta q}{|q+\eta q|}\right) = \phi(q) + \eta a(q) \in \mathcal{I}[\phi] \quad \forall\, \eta \in]0, \delta_0[\,.$$

Then the segment $]\Phi(q) + \eta\nu_\phi(\phi(q)), \Phi(q) + \eta a(q)[$ must contain a point $\phi(q_\eta)$ with $q_\eta \in \partial \mathbf{B}_n$ for all $0 < \eta < \eta_0$. If equality $q_\eta = q$ holds for at least one η, then 0 belongs to the segment $]\eta\nu_\phi(\phi(q)), \eta a(q)[$, in contradiction with the positivity of $a(q) \cdot \nu_\phi(\phi(q))$. Then we have $q \neq q_\eta$ for all $0 < \eta < \eta_0$. Possibly choosing a smaller η_0, we have

$$\frac{\phi(q_\eta) - \phi(q)}{|q_\eta - q|} = \int_0^1 D\Phi(q + t(q_\eta - q)) \cdot \frac{q_\eta - q}{|q_\eta - q|}\, dt,$$

for all $0 < \eta < \eta_0$. Clearly, there exists a decreasing sequence $\{\eta_k\}_{k\in\mathbf{N}}$ in $]0, \eta_0[$ with 0 limiting value such that $\tilde{q} \equiv \lim_{k\to\infty}(q_{\eta_k} - q)/|q_{\eta_k} - q|$ exists in $\partial \mathbf{B}_n$. Clearly $\lim_{k\to\infty} \phi(q_{\eta_k}) = \phi(q)$, and $\lim_{k\to\infty} q_{\eta_k} = q$. Then $\lim_{k\to\infty}(\phi(q_{\eta_k}) - \phi(q))/|q_{\eta_k} - q| = D\Phi(q) \cdot \tilde{q}$, and thus we have

$$\lim_{k\to\infty} \frac{\phi(q_{\eta_k}) - \phi(q)}{|\phi(q_{\eta_k}) - \phi(q)|} = \frac{D\Phi(q) \cdot \tilde{q}}{|D\Phi(q) \cdot \tilde{q}|}.$$

Since $D\Phi(q) \cdot \tilde{q}/|D\Phi(q) \cdot \tilde{q}|$ belongs to the tangent space to $\phi(\partial \mathbf{B}_n)$ at $\phi(q)$, we must have

$$\lim_{k\to\infty} \frac{\phi(q_{\eta_k}) - \phi(q)}{|\phi(q_{\eta_k}) - \phi(q)|} \cdot \nu_\phi(\phi(q)) = 0. \tag{2.4}$$

Now by definition of q_η, there exists $\tau_\eta \in]0, 1[$ such that

$$\phi(q_\eta) = \phi(q) + \eta a(q) + \tau_\eta \left(\eta\nu_\phi(\phi(q)) - \eta a(q)\right) \tag{2.5}$$

However, by (2.5), we have

$$\begin{aligned}
&\lim_{k\to\infty} \frac{\phi(q_{\eta_k}) - \phi(q)}{|\phi(q_{\eta_k}) - \phi(q)|} \cdot \nu_\phi(\phi(q)) \\
&\qquad = \lim_{k\to\infty} \frac{a(q) + \tau_{\eta_k}(\nu_\phi(\phi(q)) - a(q))}{|a(q) + \tau_{\eta_k}(\nu_\phi(\phi(q)) - a(q))|} \cdot \nu_\phi(\phi(q)).
\end{aligned}$$

We also have $[a(q)+\tau_{\eta_k}(\nu_\phi(\phi(q))-a(q))]\cdot\nu_\phi(\phi(q)) \geq 1/2$. Furthermore, $|a(q)+\tau_{\eta_k}(\nu_\phi(\phi(q))-a(q))| \leq 1$, and accordingly

$$\lim_{k\to\infty} \frac{a(q)+\tau_{\eta_k}(\nu_\phi(\phi(q))-a(q))}{|a(q)+\tau_{\eta_k}(\nu_\phi(\phi(q))-a(q))|} \cdot \nu_\phi(\phi(q)) \geq \frac{1}{2},$$

contrary to (2.4). Thus the proof is complete. □

Now we have the following, which shows the existence of a local extension operator for diffeomorphisms.

Proposition 2.8. *Let* $m \in \mathbf{N}\setminus\{0\}$, $\alpha \in [0,1]$, *cf.* (2.1). *If* ϕ_0 *belongs to* $C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}$, *then there exist* $\delta \in]0,1[$, *and* $\Phi_0 \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}_{\mathrm{cl}\,\mathbf{A}_\delta}$ *satisfying* (i), (ii), (iii) *of Proposition* 2.7, *and an open neighborhood* $\mathcal{W}_0$ *of* ϕ_0 *contained in* $C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}$, *and a linear and continuous extension operator* $\mathbf{F}_0$ *of* $C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n)$ *such that the map* $\mathbf{E}_0$ *of* $C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n)$ *defined by*

$$\mathbf{E}_0[\phi] \equiv \Phi_0 + \mathbf{F}_0[\phi - \phi_0] \quad \forall\, \phi \in C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \tag{2.6}$$

maps $\mathcal{W}_0$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}_{\mathrm{cl}\,\mathbf{A}_\delta}$, *and satisfies the following conditions.*

(i) $\mathbf{E}_0[\phi]_{|\partial\mathbf{B}_n} = \phi$, *for all* $\phi \in \mathcal{W}_0$.

(ii) $\mathbf{E}_0[\phi](x) = \Phi_0(x)$, *for all* $x \in (1-\delta)\partial\mathbf{B}_n \cup (1+\delta)\partial\mathbf{B}_n$ *and for all* $\phi \in \mathcal{W}_0$.

(iii) $\mathbf{E}_0[\phi]\left(\mathbf{A}_\delta^+\right)$ *is a bounded open connected subset of* $\mathbf{R}^n$ *of class* $C^{m,\alpha}$ *contained in* $\mathcal{I}[\phi]$, *and* $\partial\left(\mathbf{E}_0[\phi]\left(\mathbf{A}_\delta^+\right)\right) = \Phi_0\left((1-\delta)\partial\mathbf{B}_n\right) \cup \phi\left(\partial\mathbf{B}_n\right)$, *for all* $\phi \in \mathcal{W}_0$.

(iv) $\mathbf{E}_0[\phi]\left(\mathbf{A}_\delta^-\right)$ *is a bounded open connected subset of* $\mathbf{R}^n$ *of class* $C^{m,\alpha}$ *contained in* $\mathcal{E}[\phi]$, *and* $\partial\left(\mathbf{E}_0[\phi]\left(\mathbf{A}_\delta^-\right)\right) = \phi\left(\partial\mathbf{B}_n\right) \cup \Phi_0\left((1+\delta)\partial\mathbf{B}_n\right)$, *for all* $\phi \in \mathcal{W}_0$.

Proof. Let δ and Φ_0 be as in Proposition 2.7 for the map ϕ_0. By Lemma 2.3, there exists a linear and continuous operator $\mathbf{F}$ of $C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n)$ to $C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_n(0,1+\delta), \mathbf{R}^n)$ such that $\mathbf{F}[\phi]_{|\partial\mathbf{B}_n} = \phi$ for all $\phi \in C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n)$. Possibly multiplying $\mathbf{F}$ by a function of

$C^\infty(\mathbf{R}^n)$ with compact support contained in $\mathbf{A}_\delta$ and equal to 1 on $\partial\mathbf{B}_n$, we can clearly assume that $\mathbf{F}[\phi](x) = 0$ for all $x \in \partial\mathbf{A}_\delta$ and $\phi \in C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n)$. Now we take $\mathbf{F}_0[\phi] \equiv \mathbf{F}[\phi]_{|\mathrm{cl}\,\mathbf{A}_\delta}$ for all $\phi \in C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n)$. Since $\mathbf{E}_0$ is continuous and $C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}_{\mathrm{cl}\,\mathbf{A}_\delta}$ is open in $C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n)$, we conclude that there exists an open neighborhood $\mathcal{W}_0$ of ϕ_0 such that $\mathbf{E}_0[\mathcal{W}_0] \subseteq C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}_{\mathrm{cl}\,\mathbf{A}_\delta}$. Properties (i) and (ii) hold by our choice of $\mathbf{F}_0$ and by (2.6). By a simple topological argument, cf., e.g., Lamberti and Lanza [**9**, Lemmas 2.2, 2.4], to prove statement (iii), it suffices to show that $\mathbf{E}_0[\phi]\left(\mathbf{A}_\delta^+\right)$ is included in $\mathcal{I}[\phi]$. By Lemma 2.6 and by Proposition 2.7 (ii), we have $\Phi_0((1-\delta)\partial\mathbf{B}_n) \subseteq \mathcal{I}[\phi_0]$. Since the norm in $C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n)$ is stronger than that of the uniform convergence, possibly shrinking $\mathcal{W}_0$, we can assume that $\Phi_0((1-\delta)\partial\mathbf{B}_n) \subseteq \mathcal{I}[\phi]$ for all $\phi \in \mathcal{W}_0$. Since $\mathbf{E}_0[\phi]\left(\mathbf{A}_\delta^+\right)$ is an open connected subset of $\mathbf{R}^n \setminus \{\phi(\partial\mathbf{B}_n)\}$, we either have $\mathbf{E}_0[\phi]\left(\mathbf{A}_\delta^+\right) \subseteq \mathcal{I}[\phi]$, or $\mathbf{E}_0[\phi]\left(\mathbf{A}_\delta^+\right) \subseteq \mathcal{E}[\phi]$. Since $\partial\left(\mathbf{E}_0[\phi]\left(\mathbf{A}_\delta^+\right)\right) = \Phi_0((1-\delta)\partial\mathbf{B}_n) \cup \phi(\partial\mathbf{B}_n) \subseteq \mathrm{cl}\,\mathcal{I}[\phi]$, we must have $\mathbf{E}_0[\phi]\left(\mathbf{A}_\delta^+\right) \subseteq \mathcal{I}[\phi]$, and the proof of statement (iii) is complete. The proof of statement (iv) is similar. $\square$

In the sequel, we shall also need the following, which can be proved by a straightforward argument based on the connectivity of $\mathbf{A}_\delta^\pm$ and on Lemma 2.6, see also Lamberti and Lanza [**9**, Lemmas 2.2, 2.4].

Lemma 2.9. *Let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in [0,1]$, cf. (2.1), $\delta \in]0,1[$. Then the following statements hold.*

(i) *If $\Phi \in \mathcal{A}_{\mathrm{cl}\,\mathbf{A}_\delta}$, then $\phi \equiv \Phi_{|\partial\mathbf{B}_n} \in \mathcal{A}_{|\partial\mathbf{B}_n}$.*

(ii) *The set $\mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta} \equiv \left\{\Phi \in \mathcal{A}_{\mathrm{cl}\,\mathbf{A}_\delta} : \Phi(\mathbf{A}_\delta^+) \subseteq \mathcal{I}[\Phi_{|\partial\mathbf{B}_n}]\right\}$ is open in $\mathcal{A}_{\mathrm{cl}\,\mathbf{A}_\delta}$ and $\Phi(\mathbf{A}_\delta^-) \subseteq \mathcal{E}[\Phi_{|\partial\mathbf{B}_n}]$ for all $\Phi \in \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}$.*

(iii) *If $\Phi \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}$, then both $\Phi(\mathbf{A}_\delta^+)$ and $\Phi(\mathbf{A}_\delta^-)$ are open sets of class $C^{m,\alpha}$, and $\partial\Phi\left(\mathbf{A}_\delta^+\right) = \Phi((1-\delta)\partial\mathbf{B}_n) \cup \Phi(\partial\mathbf{B}_n)$, $\partial\Phi\left(\mathbf{A}_\delta^-\right) = \Phi((1+\delta)\partial\mathbf{B}_n) \cup \Phi(\partial\mathbf{B}_n)$.*

We note that, by definition, the operator $\mathbf{E}_0[\cdot]$ of Proposition 2.8 has values in $C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}$.

3. Introduction of a modified problem and real analyticity of the layer potentials. Let S_n be the function of $\mathbf{R}^n \setminus \{0\}$ to $\mathbf{R}$ defined by

$$S_n(\xi) \equiv \begin{cases} s_n^{-1} \log |\xi| & \forall\, \xi \in \mathbf{R}^n \setminus \{0\}, \text{ if } n = 2, \\ (2-n)^{-1} s_n^{-1} |\xi|^{2-n} & \forall\, \xi \in \mathbf{R}^n \setminus \{0\}, \text{ if } n > 2, \end{cases}$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial \mathbf{B}_n$. S_n is well known to be the fundamental solution of the Laplace operator. Clearly, we have $DS_n(\xi) = s_n^{-1}\xi/|\xi|^n$ for $n \geq 2$. We collect in the following statement some known facts in classical potential theory.

Theorem 3.1. *Let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbf{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.*

(i) *If $\mu \in C^{m-1,\alpha}(\partial\Omega)$, then the function v of $\mathbf{R}^n$ to $\mathbf{R}$ defined by*

$$v(\xi) \equiv \int_{\partial\Omega} S_n(\xi - \eta)\mu(\eta)\, d\sigma_\eta \quad \forall\, \xi \in \mathbf{R}^n$$

is continuous in $\mathbf{R}^n$ and harmonic in $\mathbf{R}^n \setminus \partial\Omega$. The function $v^+ \equiv v_{|\mathrm{cl}\,\Omega}$ belongs to $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, and the function $v^- \equiv v_{|\mathbf{R}^n \setminus \Omega}$ belongs to $C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_n(0,R) \setminus \Omega)$ for all $R > 0$ such that $\mathrm{cl}\,\Omega \subseteq \mathbf{B}_n(0,R)$. Moreover

$$Dv^+(\xi) \cdot \nu_\Omega(\xi) - Dv^-(\xi) \cdot \nu_\Omega(\xi) = -\mu(\xi), \quad \forall\, \xi \in \partial\Omega, \tag{3.1}$$

where $\nu_\Omega(x)$ denotes the exterior normal to $\partial\Omega$ at x.

(ii) *If $\mu \in C^{m,\alpha}(\partial\Omega)$, then the function w of $\mathbf{R}^n \setminus \partial\Omega$ to $\mathbf{R}$ defined by*

$$w(\xi) \equiv \int_{\partial\Omega} \frac{\partial}{\partial \nu_\Omega(\eta)} \left[S_n(\xi - \eta)\right] \mu(\eta)\, d\sigma_\eta \quad \forall\, \xi \in \mathbf{R}^n \setminus \partial\Omega \tag{3.2}$$

is harmonic. The restriction $w_{|\Omega}$ can be extended uniquely to an element w^+ of $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, and the restriction $w_{|\mathbf{R}^n \setminus \mathrm{cl}\,\Omega}$ can be extended uniquely to an element w^- of $C^{m,\alpha}(\mathbf{R}^n \setminus \Omega)$, and we have

$$w^+ - w^- = \mu(\xi), \qquad Dw^+ \cdot \nu_\Omega(\xi) - Dw^- \cdot \nu_\Omega(\xi) = 0 \quad \forall\, \xi \in \partial\Omega. \tag{3.3}$$

If $\xi \in \partial\Omega$, then

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(\eta)}\left[S_n(\xi-\eta)\right]\mu(\eta)\,d\sigma_\eta &= w^+(\xi) - \frac{1}{2}\mu(\xi),\\ \int_{\partial\Omega} \frac{\partial}{\partial\nu_\Omega(\eta)}\left[S_n(\xi-\eta)\right]\,d\sigma_\eta &= \frac{1}{2}.\end{aligned} \tag{3.4}$$

(iii) *If $\mu \in C^{0,\alpha}(\partial\Omega)$, and w is as in (3.2), then we have*

$$w(\xi) = -\sum_{j=1}^{n} \frac{\partial}{\partial\xi_j}\left\{\int_{\partial\Omega} \mu(\eta)(\nu_\Omega)_j(\eta)S_n(\xi-\eta)\,d\sigma_\eta\right\}, \tag{3.5}$$

for all $\xi \equiv (\xi_1,\dots,\xi_n) \in \mathbf{R}^n \setminus \partial\Omega$.

(iv) *If $\mu \in C^{m,\alpha}(\partial\Omega)$, U is an open neighborhood of $\partial\Omega$ in $\mathbf{R}^n$, $\tilde{\mu} \in C^m(U)$, $\tilde{\mu}_{|\partial\Omega} = \mu$, and w is as in (3.2), then the following holds*

(3.6)

$$\frac{\partial w}{\partial\xi_i} = \sum_{j=1}^{n} \frac{\partial}{\partial\xi_j}\left\{\int_{\partial\Omega}\left[(\nu_\Omega)_i(\eta)\frac{\partial\tilde{\mu}}{\partial\eta_j}(\eta) - (\nu_\Omega)_j(\eta)\frac{\partial\tilde{\mu}}{\partial\eta_i}(\eta)\right]S_n(\xi-\eta)d\sigma_\eta\right\},$$

for all $\xi \equiv (\xi_1,\dots,\xi_n) \in \mathbf{R}^n \setminus \partial\Omega$.

Proof. Statement (iii) follows by differentiation under the integral sign. We prove statement (iv) by the following argument of Kupradze et al. [**8**, p. 315] for $n = 3$. Clearly,

$$\begin{aligned}&\frac{\partial}{\partial\xi_i}\left[\sum_{j=1}^{n}(\nu_\Omega)_j(\eta)\frac{\partial}{\partial\eta_j}\Big(S_n(\xi-\eta)\Big)\right]\\ &\qquad = \sum_{j=1}^{n}\left[(\nu_\Omega)_j(\eta)\frac{\partial}{\partial\eta_i} - (\nu_\Omega)_i(\eta)\frac{\partial}{\partial\eta_j}\right]\left(\frac{\partial}{\partial\xi_j}\left(S_n(\xi-\eta)\right)\right),\end{aligned} \tag{3.7}$$

for all $\xi \equiv (\xi_1,\dots,\xi_n) \in \mathbf{R}^n \setminus \partial\Omega$ and $\eta \equiv (\eta_1,\dots,\eta_n) \in \partial\Omega$. Now we fix a $\xi \in \mathbf{R}^n \setminus \partial\Omega$, and we take $\varphi \in \mathcal{D}(\mathbf{R}^n)$, $\operatorname{supp}\varphi \subseteq U$, $\varphi = 1$ in a neighborhood of $\partial\Omega$, $\varphi = 0$ in a neighborhood of ξ, and we set $\psi(\eta) \equiv \varphi(\eta)\tilde{\mu}(\eta)(\partial/\partial\xi_j)\left(S_n(\xi-\eta)\right)$. Clearly $\psi \in C^1(\mathbf{R}^n)$, and thus by approximating ψ on $\mathrm{cl}\,\Omega$ with smooth functions, and by applying the Divergence Theorem, we can conclude that

$$\int_{\partial\Omega}\left[(\nu_\Omega)_j(\eta)\frac{\partial\psi(\eta)}{\partial\eta_i}-(\nu_\Omega)_i(\eta)\frac{\partial\psi(\eta)}{\partial\eta_j}\right]d\sigma_\eta=0. \tag{3.8}$$

Then statement (iv) follows by (3.7), by (3.8), and by differentiation under the integral sign. We now turn to the first two statements. The continuity of v in $\mathbf{R}^n$, and the validity of (3.1), (3.3) and (3.4) are well known in classical Potential Theory, cf., e.g., Hackbusch [**5**, Theorems 8.1.9, 8.2.8, 8.2.13, 8.2.15]. Now let $\mu\in C^{m-1,\alpha}(\partial\Omega)$. Clearly,

$$\frac{\partial v}{\partial\xi_i}=\int_{\partial\Omega}\frac{\partial}{\partial\xi_i}\left(S_n(\xi-\eta)\right)\mu(\eta)\,d\sigma_\eta,\quad\forall\,\xi\equiv(\xi_1,\ldots,\xi_n)\in\mathbf{R}^n\setminus\partial\Omega.$$

Then by Miranda [**19**, p. 307], we can assert that $\partial v^+/\partial\xi_i\in C^{m-1,\alpha}(\mathrm{cl}\,\Omega)$, $\partial v^-/\partial\xi_i\in C^{m-1,\alpha}(\mathbf{R}^n\setminus\Omega)$, for all $i=1,\ldots,n$, and the proof of (i) is complete. The membership of w^+ in $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ and of w^- in $C^{m,\alpha}(\mathbf{R}^n\setminus\Omega)$ for $\mu\in C^{m,\alpha}(\partial\Omega)$ is an immediate consequence of statements (i), (iii) and (iv). $\square$

We now show that the pair (v^+,v^-) is the unique solution of a coupled boundary value problem by means of the following.

Theorem 3.2. *Let $m\in\mathbf{N}\setminus\{0\}$, $\alpha\in\,]0,1[$, $\delta\in\,]0,1[$. Let $\Phi\in C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta,\mathbf{R}^n)\cap\mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}$, $\phi\equiv\Phi_{|\partial\mathbf{B}_n}$. Let ν_ϕ denote the exterior normal to $\phi(\partial\mathbf{B}_n)$. If $f\in C^{m-1,\alpha}(\partial\mathbf{B}_n)$, then the boundary value problem*

(3.9)

$$\begin{cases}\Delta v^+=0 & \text{in } \Phi\left(\mathbf{A}_\delta^+\right),\\ \Delta v^-=0 & \text{in } \Phi\left(\mathbf{A}_\delta^-\right),\\ v^+-v^-=0 & \text{on } \phi(\partial\mathbf{B}_n),\\ Dv^+\cdot\nu_\phi-Dv^-\cdot\nu_\phi=-f\circ\phi^{(-1)} & \text{on } \phi(\partial\mathbf{B}_n),\\ v^+(\xi)=\int_{\phi(\partial\mathbf{B}_n)}S_n(\xi-\eta)f\circ\phi^{(-1)}(\eta)\,d\sigma_\eta & \forall\,\xi\in\Phi((1-\delta)\partial\mathbf{B}_n),\\ v^-(\xi)=\int_{\phi(\partial\mathbf{B}_n)}S_n(\xi-\eta)f\circ\phi^{(-1)}(\eta)\,d\sigma_\eta & \forall\,\xi\in\Phi((1+\delta)\partial\mathbf{B}_n),\end{cases}$$

where the Laplacian is understood in the sense of distributions, has one and only one solution $(v^+,v^-)\in C^{m,\alpha}\left(\mathrm{cl}\,\Phi\left(\mathbf{A}_\delta^+\right)\right)\times C^{m,\alpha}\left(\mathrm{cl}\,\Phi\left(\mathbf{A}_\delta^-\right)\right)$.

Furthermore, we have

$$
\begin{aligned}
v^{+}(\xi) &= \int_{\phi(\partial \mathbf{B}_n)} S_n(\xi-\eta) f\circ\phi^{(-1)}(\eta)\, d\sigma_\eta \quad \forall\, \xi \in \Phi\left(\mathbf{A}_\delta^{+}\right), \\
v^{-}(\xi) &= \int_{\phi(\partial \mathbf{B}_n)} S_n(\xi-\eta) f\circ\phi^{(-1)}(\eta)\, d\sigma_\eta \quad \forall\, \xi \in \Phi\left(\mathbf{A}_\delta^{-}\right).
\end{aligned}
\tag{3.10}
$$

Proof. By known properties of layer potentials, see Theorem 3.1, the boundary value problem (3.9) admits the solution defined by the righthand side of (3.10). We now turn to prove the uniqueness. Assume that (v_1^+, v_1^-) and (v_2^+, v_2^-) belong to $C^{m,\alpha}\left(\mathrm{cl}\,\Phi\left(\mathbf{A}_\delta^+\right)\right) \times C^{m,\alpha}\left(\mathrm{cl}\,\Phi\left(\mathbf{A}_\delta^-\right)\right)$ and solve (3.9). Then we consider the function u of $\mathrm{cl}\,\Phi\left(\mathbf{A}_\delta\right)$ to $\mathbf{R}$ defined by setting $u \equiv v_1^+ - v_2^+$ on $\mathrm{cl}\,\Phi\left(\mathbf{A}_\delta^+\right)$, $u \equiv v_1^- - v_2^-$ on $\mathrm{cl}\,\Phi\left(\mathbf{A}_\delta^-\right)$ (note that $v_1^+ - v_2^+ = v_1^- - v_2^-$ on $\phi\left(\partial \mathbf{B}_n\right)$). By (3.9), u and $Du \cdot \nu_\phi$ have zero jump across $\phi\left(\partial \mathbf{B}_n\right)$. Then by a standard argument based on the Divergence Theorem, one can easily show that the continuous function u solves the Laplace equation in the sense of distributions in $\Phi\left(\mathbf{A}_\delta\right)$. Since u vanishes on $\partial \Phi\left(\mathbf{A}_\delta\right)$, we conclude that $u = 0$. □

Now the problem with (3.9) is that it is defined on a domain which depends on Φ. To transform such a problem into a problem defined on the fixed domain $\mathbf{A}_\delta$, we need to change the variable in (3.9) by means of the function Φ. To do so, we first of all need to know how the normals and the hypersurface area elements change, and we see it in the following Lemma 3.3. However, we will have to face another problem. Namely, if $m = 1$, the map Φ is only one time continuously differentiable, and accordingly, we have to explain how we plan to change the variables in the Laplace operators which appear in (3.9). We do so by means of Lemma 3.4 below. Both lemmas are of immediate verification.

Lemma 3.3. *Let Ω be a bounded open subset of $\mathbf{R}^n$ of class C^1. Let ν_Ω be the exterior normal to $\partial\Omega$. Let $\Phi \in C^1\left(\mathrm{cl}\,\Omega, \mathbf{R}^n\right) \cap \mathcal{A}_{\mathrm{cl}\,\Omega}$. Let ν_Φ denote the exterior normal to $\partial\Phi(\Omega)$. Then we have the following.*

(i) $\nu_\Phi(\Phi(x)) = (D\Phi(x))^{-t} \cdot \nu_\Omega(x) / |(D\Phi(x))^{-t} \cdot \nu_\Omega(x)|$ *for all* $x \in \partial\Omega$.

(ii) *If* $\omega \in L^1(\Phi(\partial\Omega))$, *then*

$$\int_{\Phi(\partial\Omega)} \omega(\eta)\, d\sigma_\eta = \int_{\partial\Omega} \omega \circ \Phi(y) \sigma_n[\Phi](y)\, d\sigma_y,$$

where $\sigma_n[\Phi] = |\det D\Phi| |(D\Phi)^{-t} \cdot \nu_\Omega|$.

(iii) *If* $u \in C^1(\mathrm{cl}\,\Phi(\Omega))$, $x \in \partial\Omega$, *then we have*

$$\frac{\partial u}{\partial \nu_\Phi}(\Phi(x)) = D(u \circ \Phi)(x) \cdot (D\Phi(x))^{-1} \cdot \frac{(D\Phi(x))^{-t} \cdot \nu_\Omega}{\left|(D\Phi(x))^{-t} \cdot \nu_\Omega\right|},$$

for all $x \in \partial\Omega$.

Lemma 3.4. *Let* $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in]0,1[$. *Let* Ω *be a bounded open connected subset of* $\mathbf{R}^n$. *The set*

$$\mathcal{Y}^{m-1,\alpha}(\Omega) \equiv \Bigg\{ (w_1, \dots, w_n) \in C^{m-1,\alpha}(\mathrm{cl}\,\Omega, \mathbf{R}^n) : \\ \sum_{j=1}^n \int_\Omega w_j \frac{\partial \psi}{\partial x_j}\, dx = 0 \ \forall\, \psi \in \mathcal{D}(\Omega) \Bigg\}$$

is a closed linear subspace of $C^{m-1,\alpha}(\mathrm{cl}\,\Omega, \mathbf{R}^n)$. *Let* Π_Ω *denote the canonical projection of* $C^{m-1,\alpha}(\mathrm{cl}\,\Omega, \mathbf{R}^n)$ *onto the quotient (Banach) space*

$$\mathcal{Z}^{m-1,\alpha}(\Omega) \equiv C^{m-1,\alpha}(\mathrm{cl}\,\Omega, \mathbf{R}^n) / \mathcal{Y}^{m-1,\alpha}(\Omega).$$

Let $A[\cdot,\cdot]$ *be the map of the set* $(C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{R}^n) \cap \mathcal{A}_{\mathrm{cl}\,\Omega}) \times C^{m,\alpha}(\mathrm{cl}\,\Omega)$ *to the space* $C^{m-1,\alpha}(\mathrm{cl}\,\Omega, \mathbf{R}^n)$ *defined by*

$$A[\Phi, v] \equiv Dv\, (D\Phi)^{-1} (D\Phi)^{-t} |\det D\Phi|$$

for all $(\Phi, v) \in (C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{R}^n) \cap \mathcal{A}_{\mathrm{cl}\,\Omega}) \times C^{m,\alpha}(\mathrm{cl}\,\Omega)$. *Let* (Φ, v) *belong to* $(C^{m,\alpha}(\mathrm{cl}\,\Omega, \mathbf{R}^n) \cap \mathcal{A}_{\mathrm{cl}\,\Omega}) \times C^{m,\alpha}(\mathrm{cl}\,\Omega)$, $f \equiv (f_1, \dots, f_n) \in C^{m-1,\alpha}(\mathrm{cl}\,\Omega, \mathbf{R}^n)$. *Then we have*

$$\Pi_\Omega A[\Phi, v] = \Pi_\Omega[f] \tag{3.11}$$

if and only if

$$\Delta\big(v\circ\Phi^{(-1)}\big) = \operatorname{div}\Big\{\big(f\circ\Phi^{(-1)}\big)\,\big(D\Phi\big(\Phi^{(-1)}\big)\big)^{t}|\det D\big(\Phi^{(-1)}\big)|\Big\}, \tag{3.12}$$

in the sense of distributions in $\Phi(\Omega)$*, i.e., in* $\mathcal{D}'(\Phi(\Omega))$*.*

Proof. By exploiting a simple argument based on the convolution with a family of mollifiers, one can easily see that equation (3.11) is equivalent to equation

$$\int_\Omega A[\Phi, v]\cdot\Big(D\big(\varphi\circ\Phi\big)\Big)^t\,dx = \int_\Omega f\cdot\Big(D\big(\varphi\circ\Phi\big)\Big)^t\,dx \quad \forall\,\varphi\in\mathcal{D}(\Phi(\Omega)).$$

Then one can easily conclude the equivalence of such an equation to equation (3.12) by the rule of change of variables in integrals with the function Φ. □

By Theorem 3.2, and by Lemmas 3.3 and 3.4, we immediately deduce the validity of the following.

Theorem 3.5. *Let* $m\in\mathbf{N}\setminus\{0\}$*,* $\alpha\in\,]0,1[$*,* $\delta\in\,]0,1[$*. Let* $\Phi\in C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta,\mathbf{R}^n)\cap\mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}$*. Let* $f\in C^{m-1,\alpha}(\partial\mathbf{B}_n)$*. Then the pair* (v^+,v^-) *of* $C^{m,\alpha}\big(\mathrm{cl}\,\Phi\big(\mathbf{A}_\delta^+\big)\big)\times C^{m,\alpha}\big(\mathrm{cl}\,\Phi\big(\mathbf{A}_\delta^-\big)\big)$ *satisfies* (3.9) *if and only if* $(V^+,V^-)\equiv(v^+\circ\Phi, v^-\circ\Phi)$ *of* $C^{m,\alpha}\big(\mathrm{cl}\,\mathbf{A}_\delta^+\big)\times C^{m,\alpha}\big(\mathrm{cl}\,\mathbf{A}_\delta^-\big)$ *satisfies the boundary value problem*

(3.13)

$$\begin{cases}
\Pi_{\mathbf{A}_\delta^+}\left[A\left[\Phi,V^+\right]\right]=0 & \text{in } \mathcal{Z}^{m-1,\alpha}\big(\mathbf{A}_\delta^+\big),\\
\Pi_{\mathbf{A}_\delta^-}\left[A\left[\Phi,V^-\right]\right]=0 & \text{in } \mathcal{Z}^{m-1,\alpha}\big(\mathbf{A}_\delta^-\big),\\
V^+-V^-=0 & \text{on } \partial\mathbf{B}_n,\\
B_1\left[V^+,V^-,\Phi\right]\equiv DV^+\cdot(D\Phi)^{-1} & \\
\cdot[(D\Phi)^{-t}\cdot\nu_{\mathbf{B}_n}/|(D\Phi)^{-t}\cdot\nu_{\mathbf{B}_n}|]-DV^-\cdot(D\Phi)^{-1} & \\
\cdot[(D\Phi)^{-t}\cdot\nu_{\mathbf{B}_n}/|(D\Phi)^{-t}\cdot\nu_{\mathbf{B}_n}|]=-f & \text{on } \partial\mathbf{B}_n,\\
V^+(x)=\int_{\partial\mathbf{B}_n}S_n\left(\Phi(x)-\Phi(y)\right)f(y)\sigma_n\left[\Phi\right](y)\,d\sigma_y & \forall\,x\in(1-\delta)\partial\mathbf{B}_n,\\
V^-(x)=\int_{\partial\mathbf{B}_n}S_n\left(\Phi(x)-\Phi(y)\right)f(y)\sigma_n\left[\Phi\right](y)d\sigma_y & \forall\,x\in(1+\delta)\partial\mathbf{B}_n,
\end{cases}$$

where $\sigma_n[\cdot]$*,* $A[\cdot,\cdot]$*,* $\Pi_{\mathbf{A}_\delta^+}$*,* $\Pi_{\mathbf{A}_\delta^-}$ *have been introduced in Lemmas* 3.3 *and* 3.4*.*

Now the boundary value problem (3.13) is actually defined on a fixed domain. As we have announced in the introduction, our next step will now be to recast (3.13) into an abstract equation in Banach space and to analyze it by means of the Implicit Function Theorem. To do so, however, we need to prove that the assumptions of the Implicit Function Theorem are satisfied. In particular, we need to show that the righthand sides of the last two equations in (3.13) depend analytically upon Φ and f. To do so, we introduce the following three technical lemmas, which are variants of known results. The first lemma has the purpose of reducing the problem on the manifold $\partial\mathbf{B}_n$ to a problem on $\mathbf{B}_{n-1}$ by means of the manifold parametrizations, and it can be proved by exploiting the definition of norm in the space $C^{m,\alpha}$.

Lemma 3.6. *Let $\mathcal{X}$ be a Banach space, $\mathcal{O}$ an open subset of $\mathcal{X}$. Let $k, m \in \mathbf{N}$, $m \geq 1$, $0 \leq k \leq m$, $\alpha \in]0,1]$. Let M be a compact manifold of dimension $1 \leq s \leq n$ and class $C^{m,\alpha}$ imbedded in $\mathbf{R}^n$. Let N be a map of $\mathcal{O}$ to $C^{k,\alpha}(M)$. Let $\{\psi_i\}_{i=1,\dots,r}$ be parametrizations of M with $\psi_i \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_s, \mathbf{R}^n)$ and $\cup_{i=1}^r \psi_i(\mathbf{B}_s) = M$. Let C_{ψ_i} be the composition operator of $C^{k,\alpha}(M)$ to $C^{k,\alpha}(\mathrm{cl}\,\mathbf{B}_s)$ defined by*

$$C_{\psi_i}[w] \equiv w \circ \psi_i \quad \forall\, w \in C^{k,\alpha}(M),$$

for all $i = 1, \dots, r$. Let $h \in \mathbf{N} \cup \{\infty\}$. Then N is of class C^h or real analytic if and only if the operators $C_{\psi_i} \circ N$ for $i = 1, \dots, r$ are of class C^h or real analytic, respectively.

We now have the following lemma on the analyticity of superposition operators, which is just a variant of Böhme and Tomi [**1**, p. 10], Henry [**6**, p. 29], Valent [**24**, Theorem 5.2, p. 44].

Lemma 3.7. *Let $k, m, n, n_1, s \in \mathbf{N}$, $m \geq 1$, $0 \leq k \leq m$, $n_1 \geq 1$, $1 \leq s \leq n$, $\alpha \in]0,1]$. Let M be a compact manifold imbedded in $\mathbf{R}^n$ of dimension s and of class $C^{m,\alpha}$. Let Ω be an open subset of $\mathbf{R}^{n_1}$ and F a real analytic map of Ω to $\mathbf{R}$. Then the set $\mathcal{O} \equiv \left\{\Phi \in C^{k,\alpha}(M, \mathbf{R}^{n_1}) : \Phi(M) \subseteq \Omega\right\}$ is open in $C^{k,\alpha}(M, \mathbf{R}^{n_1})$ and* *the superposition operator T_F of $\mathcal{O}$ to $C^{k,\alpha}(M)$ defined by*

$$T_F[\Phi] \equiv F \circ \Phi \quad \forall\, \Phi \in \mathcal{O},$$

is real analytic.

Proof. The set $\mathcal{O}$ is open because the norm of $C^{k,\alpha}(M, \mathbf{R}^{n_1})$ is stronger than that of the uniform convergence. Let $\{\psi_j\}_{j=1,\dots,r}$ be local parametrizations of M as in the statement of Lemma 3.6. By Lemma 3.6 it suffices to show that the operators $C_{\psi_j} \circ T_F$ defined by

$$C_{\psi_j} \circ T_F[\Phi] \equiv F \circ \Phi \circ \psi_j \quad \forall\, \Phi \in \mathcal{O},$$

are real analytic. Now the map of $C^{k,\alpha}(M, \mathbf{R}^{n_1})$ to $C^{k,\alpha}(\mathrm{cl}\,\mathbf{B}_s, \mathbf{R}^{n_1})$ which takes Φ to $\Phi \circ \psi_j$ is obviously linear and continuous. Then it suffices to show that the composition operator of $\big\{\Psi \in C^{k,\alpha}(\mathrm{cl}\,\mathbf{B}_s, \mathbf{R}^{n_1}) : \Psi(\mathrm{cl}\,\mathbf{B}_s) \subseteq \Omega\big\}$ to $C^{k,\alpha}(\mathrm{cl}\,\mathbf{B}_s)$ which takes Ψ to $F \circ \Psi$ is real analytic, a known fact, see Böhme and Tomi [**1**, p. 10], Henry [**6**, p. 29] and Valent [**24**, Theorem 5.2, p. 44]. □

Then we have the following lemma on integral operators.

Lemma 3.8. *Let k, m, n_1, n_2, s_1, $s_2 \in \mathbf{N}$, $m \geq 1$, $0 \leq k \leq m$, $1 \leq s_1 \leq n_1$, $1 \leq s_2 \leq n_2$, $\alpha \in]0,1]$. Let M_1 and M_2 be two compact manifolds of class $C^{m,\alpha}$ imbedded in $\mathbf{R}^{n_1}$ and in $\mathbf{R}^{n_2}$ and of dimension s_1 and s_2, respectively. Then the bilinear map K of $C^{k,\alpha}(M_1 \times M_2) \times L^1(M_2)$ to $C^{k,\alpha}(M_1)$ defined by*

$$K[G,f](x) \equiv \int_{M_2} G(x,y) f(y)\, d\sigma_y \quad \forall\, x \in M_1,$$

for all $(G,f) \in C^{k,\alpha}(M_1 \times M_2) \times L^1(M_2)$, is continuous.

Proof. Let $\{\varphi_i\}_{i=1,\dots,r_1}$ be parametrizations of M_1 with $\varphi_i \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_{s_1}, \mathbf{R}^{n_1})$ and $\cup_{i=1}^{r_1} \varphi_i(\mathbf{B}_{s_1}/2) = M_1$. Let $\{\psi_j\}_{j=1,\dots,r_2}$ be parametrizations of M_2 with $\psi_j \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_{s_2}, \mathbf{R}^{n_2})$ and $M_2 = \cup_{j=1}^{r_2} \psi_j(\mathbf{B}_{s_2}/2)$. Let $\{\theta_j\}_{j=1}^{r_2}$ be a partition of unity as in (2.2) for the atlas $\{\psi_j\}_{j=1,\dots,r_2}$. Let π_1 and π_2 be the canonical projections of $\mathbf{R}^{s_1} \times \mathbf{R}^{s_2}$ onto $\mathbf{R}^{s_1}$ and onto $\mathbf{R}^{s_2}$, respectively. Clearly,

$(\varphi_i \circ \pi_1, \psi_j \circ \pi_2) \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{B}_{s_1+s_2}, \mathbf{R}^{n_1+n_2})$. As is well known, and of immediate verification, the collection of maps $\{(\varphi_i \circ \pi_1, \psi_j \circ \pi_2) : i = 1, \dots, r_1,\ j = 1, \dots, r_2\}$ is an atlas for $M_1 \times M_2$. Thus it suffices to show that there exists a constant $c > 0$ such that

$$\begin{aligned}\sup_{i=1,\dots,r_1} &\|K[G,f] \circ \varphi_i\|_{C^{k,\alpha}(\mathrm{cl}\,\mathbf{B}_{s_1})} \\ &\leq c \sup_{\substack{i=1,\dots,r_1 \\ j=1,\dots,r_2}} \|G(\varphi_i \circ \pi_1, \psi_j \circ \pi_2)\|_{C^{k,\alpha}(\mathrm{cl}\,\mathbf{B}_{s_1+s_2})} \\ &\quad \cdot \left\{ \sum_{j=1}^{r_2} \int_{\mathbf{B}_{s_2}} |f(\psi_j(\omega))\theta_j(\psi_j(\omega))| \left| \left(D\psi_j^t \cdot D\psi_j\right)(\omega)\right|^{1/2} d\omega \right\},\end{aligned}$$

for all $(G, f) \in C^{k,\alpha}(M_1 \times M_2) \times L^1(M_2)$. Now

$$\begin{aligned}K[G,f] \circ \varphi_i(\xi) = \sum_{j=1}^{r_2} \int_{\mathbf{B}_{s_2}} &G(\varphi_i(\xi), \psi_j(\omega)) f(\psi_j(\omega))\theta_j(\psi_j(\omega)) \\ &\cdot \left| \left(D\psi_j^t \cdot D\psi_j\right)(\omega)\right|^{1/2} d\omega.\end{aligned} \tag{3.14}$$

By a standard differentiation theorem for an integral depending on a parameter, and by the definition of norm in $C^{k,\alpha}(M_1 \times M_2)$, one can easily deduce the existence of $c > 0$ as above. □

We are now ready to show that the lefthand sides of the last two equations in (3.13) depend real analytically upon Φ and f.

Lemma 3.9. *Let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in]0,1[$, $\delta \in]0,1[$, $r \in [1-\delta, 1+\delta] \setminus \{1\}$. Let $\Phi \in C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}$. Then the map V_r of $\left(C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}\right) \times C^{m-1,\alpha}(\partial \mathbf{B}_n)$ to $C^{m,\alpha}(r\partial\mathbf{B}_n)$ defined by*

$$V_r[\Phi, f](x) \equiv \int_{\partial\mathbf{B}_n} S_n(\Phi(x) - \Phi(y)) f(y) \sigma_n[\Phi](y)\, d\sigma_y \quad \forall\, x \in r\partial\mathbf{B}_n,$$

for all $(\Phi, f) \in \left(C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}\right) \times C^{m-1,\alpha}(\partial\mathbf{B}_n)$ is real analytic.

Proof. The map of $C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}$ to $C^{m,\alpha}((r\partial\mathbf{B}_n) \times \partial\mathbf{B}_n, \mathbf{R}^n \setminus \{0\})$ which takes Φ to the function Ψ of $(r\partial\mathbf{B}_n) \times \partial\mathbf{B}_n$ to $\mathbf{R}^n \setminus \{0\}$

defined by $\Psi(x,y) \equiv \Phi(x) - \Phi(y)$ for all $(x,y) \in (r\partial\mathbf{B}_n) \times \partial\mathbf{B}_n$ is real analytic. By continuity of the pointwise product from $\left(C^{m-1,\alpha}(\partial\mathbf{B}_n)\right)^2$ to $L^1(\partial\mathbf{B}_n)$, the map of $\left(C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}\right) \times C^{m-1,\alpha}(\partial\mathbf{B}_n)$ to $L^1(\partial\mathbf{B}_n)$ which takes (Φ, f) to $f\sigma_n[\Phi]_{|\partial\mathbf{B}_n}$ is real analytic. Then the conclusion follows by Lemma 3.7 and by Lemma 3.8 with $M_1 = r\partial\mathbf{B}_n$, $M_2 = \partial\mathbf{B}_n$, $M = M_1 \times M_2$, $F = S_n$, $\Omega = \mathbf{R}^n \setminus \{0\}$. □

Finally, we have the following known technical lemma, which we will exploit in the proof of our main statement, in order to show solvability of the linearized problem of (3.13).

Lemma 3.10. *Let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let Ω be a bounded open connected subset of $\mathbf{R}^n$ of class $C^{m,\alpha}$. Let $b_1, \ldots, b_n \in C^{m-1,\alpha}(\mathrm{cl}\,\Omega)$, $g \in C^{m,\alpha}(\partial\Omega)$. Then there exists one and only one $u \in C^{m,\alpha}(\mathrm{cl}\,\Omega)$ such that*

$$\Delta u = \sum_{j=1}^{n} \frac{\partial b_j}{\partial x_j} \quad \text{in} \quad \mathcal{D}'(\Omega), \qquad u = g \quad \text{on} \quad \partial\Omega.$$

Proof. By Lemma 2.3, there exists $\tilde{g} \in C^{m,\alpha}(\mathrm{cl}\,\Omega)$ such that $\tilde{g}_{|\partial\Omega} = g$. Then case $m \geq 2$ follows by Gilbarg and Trudinger [**4**, Theorems 6.14, 6.19], while case $m = 1$ follows by Gilbarg and Trudinger [**4**, Theorem 8.34]. □

We now introduce some notation. We set

$$\begin{aligned} v^+[\phi,f](\xi) &\equiv \int_{\phi(\partial\mathbf{B}_n)} S_n(\xi-\eta) f\circ\phi^{(-1)}(\eta)\,d\sigma_\eta \quad \forall\,\xi \in \mathcal{I}[\phi], \\ v^-[\phi,f](\xi) &\equiv \int_{\phi(\partial\mathbf{B}_n)} S_n(\xi-\eta) f\circ\phi^{(-1)}(\eta)\,d\sigma_\eta \quad \forall\,\xi \in \mathcal{E}[\phi], \end{aligned} \tag{3.15}$$

for each $\phi \in C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}$, and $f \in C^{m-1,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n)$,

(3.16)

$$w^+[\phi,f](\xi) \equiv \int_{\phi(\partial\mathbf{B}_n)} \frac{\partial}{\partial\nu_\phi(\eta)}\left[S_n(\xi-\eta)\right] f\circ\phi^{(-1)}(\eta)\,d\sigma_\eta \quad \forall\,\xi \in \mathcal{I}[\phi],$$

$$w^-[\phi,f](\xi) \equiv \int_{\phi(\partial\mathbf{B}_n)} \frac{\partial}{\partial\nu_\phi(\eta)}\left[S_n(\xi-\eta)\right] f\circ\phi^{(-1)}(\eta)\,d\sigma_\eta \quad \forall\,\xi \in \mathcal{E}[\phi],$$

for each $\phi \in C^{m,\alpha}(\partial\mathbf{B}_n,\mathbf{R}^n)\cap\mathcal{A}_{\partial\mathbf{B}_n}$, and $f\in C^{m,\alpha}(\partial\mathbf{B}_n,\mathbf{R}^n)$. Then by Theorem 3.1, we know that $v^+[\phi,f]$, $w^+[\phi,f]$, and $v^-[\phi,f]$, $w^-[\phi,f]$ can be extended with continuity to $\mathrm{cl}\,\mathcal{I}[\phi]$ and to $\mathrm{cl}\,\mathcal{E}[\phi]$, respectively. We denote the corresponding extensions by the same symbol. Then we have the following proposition, which is preliminary to our main result, cf. Theorem 3.12.

Proposition 3.11. *Let $m\in\mathbf{N}\setminus\{0\}$, $\alpha\in\,]0,1[$, $\delta\in\,]0,1[$. Then the following statements hold.*

(i) *Let $V^+[\Phi,f]$ and $V^-[\Phi,f]$ denote the continuous extensions to $\mathrm{cl}\,\mathbf{A}_\delta^+$ and to $\mathrm{cl}\,\mathbf{A}_\delta^-$ of $v^+[\Phi_{|\partial\mathbf{B}_n},f]\circ\Phi_{|\mathbf{A}_\delta^+}$ and of $v^-[\Phi_{|\partial\mathbf{B}_n},f]\circ\Phi_{|\mathbf{A}_\delta^-}$, respectively, for all $(\Phi,f)\in\left(C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta,\mathbf{R}^n)\cap\mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}\right)\times C^{m-1,\alpha}(\partial\mathbf{B}_n)$. Then the maps of $\left(C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta,\mathbf{R}^n)\cap\mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}\right)\times C^{m-1,\alpha}(\partial\mathbf{B}_n)$ to $C^{m,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+\right)$ and to $C^{m,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^-\right)$ which take (Φ,f) to $V^+[\Phi,f]$ and to $V^-[\Phi,f]$ are real analytic, respectively.*

(ii) *Let $W^+[\Phi,f]$ and $W^-[\Phi,f]$ denote the continuous extensions to $\mathrm{cl}\,\mathbf{A}_\delta^+$ and to $\mathrm{cl}\,\mathbf{A}_\delta^-$ of $w^+[\Phi_{|\partial\mathbf{B}_n},f]\circ\Phi_{|\mathbf{A}_\delta^+}$ and of $w^-[\Phi_{|\partial\mathbf{B}_n},f]\circ\Phi_{|\mathbf{A}_\delta^-}$, respectively, for all $(\Phi,f)\in\left(C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta,\mathbf{R}^n)\cap\mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}\right)\times C^{m,\alpha}(\partial\mathbf{B}_n)$. Then the maps of $\left(C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta,\mathbf{R}^n)\cap\mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}\right)\times C^{m,\alpha}(\partial\mathbf{B}_n)$ to $C^{m,\alpha}\left(\mathrm{cl}\mathbf{A}_\delta^+\right)$ and to $C^{m,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^-\right)$ which take (Φ,f) to $W^+[\Phi,f]$ and to $W^-[\Phi,f]$ are real analytic, respectively.*

Proof. We first prove statement (i). Let $\mathcal{X}\equiv C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta,\mathbf{R}^n)\times C^{m-1,\alpha}(\partial\mathbf{B}_n)$, $\mathcal{Y}\equiv C^{m,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+\right)\times C^{m,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^-\right)$. Let Λ be the non-linear operator of $\mathcal{U}\equiv\left(C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta,\mathbf{R}^n)\cap\mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}\right)\times C^{m-1,\alpha}(\partial\mathbf{B}_n)\times\mathcal{Y}$ to the Banach space

$$\begin{aligned}\mathcal{Z}\equiv\mathcal{Z}^{m-1,\alpha}\left(\mathbf{A}_\delta^+\right)\times\mathcal{Z}^{m-1,\alpha}\left(\mathbf{A}_\delta^-\right)\times C^{m,\alpha}(\partial\mathbf{B}_n)\times C^{m-1,\alpha}(\partial\mathbf{B}_n)\\ \times C^{m,\alpha}((1-\delta)\partial\mathbf{B}_n)\times C^{m,\alpha}((1+\delta)\partial\mathbf{B}_n),\end{aligned}$$

defined by

$$\begin{aligned}\Lambda\left[\Phi,f,V^+,V^-\right]\equiv\Big(&\Pi_{\mathbf{A}_\delta^+}\left[A\left[\Phi,V^+\right]\right],\Pi_{\mathbf{A}_\delta^-}\left[A\left[\Phi,V^-\right]\right],\\ &V^+-V^-,B_1\left[V^+,V^-,\Phi\right]+f,\\ &V^+_{|(1-\delta)\partial\mathbf{B}_n}-V_{1-\delta}[\Phi,f],V^-_{|(1+\delta)\partial\mathbf{B}_n}-V_{1+\delta}[\Phi,f]\Big),\end{aligned}$$

for all $(\Phi, f, V^+, V^-) \in \mathcal{U}$, where $B_1[V^+, V^-, \Phi]$ and $V_{1\pm\delta}[\Phi, f]$ have been introduced in (3.13) and in Lemma 3.9, respectively. By Theorem 3.5, the graph of the operator $(V^+[\cdot,\cdot], V^-[\cdot,\cdot])$ of $(C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}) \times C^{m-1,\alpha}(\partial\mathbf{B}_n)$ to $\mathcal{Y}$ coincides with the set of zeros of Λ. Thus we can deduce the real analyticity of the operator $(V^+[\cdot,\cdot], V^-[\cdot,\cdot])$ by showing that we can apply the Implicit Function Theorem for real analytic operators, cf., e.g., Prodi and Ambrosetti [**20**, Theorem 11.6] to equation $\Lambda[\Phi, f, V^+, V^-] = 0$ around $(\Phi_1, f_1, V^+[\Phi_1, f_1], V^-[\Phi_1, f_1])$, for all $(\Phi_1, f_1) \in (C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}) \times C^{m-1,\alpha}(\partial\mathbf{B}_n)$. The domain $\mathcal{U}$ of Λ is clearly open in $\mathcal{X} \times \mathcal{Y}$. By definition, $\Pi_{\mathbf{A}_\delta^+}$, $\Pi_{\mathbf{A}_\delta^-}$ are linear and continuous. Then by Lemma 2.1 $A[\Phi, V^\pm]$ and $B_1[V^+, V^-, \Phi] + f$ depend real analytically on (Φ, f, V^+, V^-). Then by Lemma 3.9, and by the linearity and continuity of the trace on the boundary, see Lemma 2.3, we deduce that Λ is real analytic. Thus it suffices to show that the differential $d_{(V^+,V^-)}\Lambda[\Phi_1, f_1, V^+[\Phi_1, f_1], V^-[\Phi_1, f_1]]$ is a linear homeomorphism. Since $d_{(V^+,V^-)}\Lambda[\Phi_1, f_1, V^+[\Phi_1, f_1], V^-[\Phi_1, f_1]]$ is continuous, then by the Open Mapping Theorem it suffices to show that it is a bijection. Thus we now turn to show that for all $(F^+, F^-, g, g_1, h^+, h^-) \in \mathcal{Z}$ there exists one and only one $(X^+, X^-) \in \mathcal{Y}$ such that the following system holds

$$
\begin{cases}
\Pi_{\mathbf{A}_\delta^+}\left[A\left[\Phi_1, X^+\right]\right] = F^+ & \text{in } \mathcal{Z}^{m-1,\alpha}\left(\mathbf{A}_\delta^+\right), \\
\Pi_{\mathbf{A}_\delta^-}\left[A\left[\Phi_1, X^-\right]\right] = F^- & \text{in } \mathcal{Z}^{m-1,\alpha}\left(\mathbf{A}_\delta^-\right), \\
X^+ - X^- = g & \text{on } \partial\mathbf{B}_n, \\
B_1\left[X^+, X^-, \Phi_1\right] = g_1 & \text{on } \partial\mathbf{B}_n, \\
X^+ = h^+ & \text{on } (1-\delta)\partial\mathbf{B}_n, \\
X^- = h^- & \text{on } (1+\delta)\partial\mathbf{B}_n.
\end{cases}
\tag{3.17}
$$

By the surjectivity of $\Pi_{\mathbf{A}_\delta^+}$ and of $\Pi_{\mathbf{A}_\delta^-}$, there exist an element f^+ in $C^{m-1,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+, \mathbf{R}^n\right)$, and an element f^- in $C^{m-1,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^-, \mathbf{R}^n\right)$ such that $\Pi_{\mathbf{A}_\delta^+}[f^+] = F^+$, $\Pi_{\mathbf{A}_\delta^-}[f^-] = F^-$. Now we set $\phi_1 \equiv \Phi_{1|\partial\mathbf{B}_n}$. By changing variables with the map Φ_1, see Lemmas 3.3 and 3.4, we can

easily see that (3.17) is equivalent to the following system
(3.18)

$$\begin{cases} \Delta\left(X^+\circ\Phi_1^{(-1)}\right)=\sum_{j=1}^n \partial b_j^+/\partial\xi_j & \text{in } \mathcal{D}'\left(\Phi_1\left(\mathbf{A}_\delta^+\right)\right), \\ \Delta\left(X^-\circ\Phi_1^{(-1)}\right)=\sum_{j=1}^n \partial b_j^-/\partial\xi_j & \text{in } \mathcal{D}'\left(\Phi_1\left(\mathbf{A}_\delta^-\right)\right), \\ X^+\circ\Phi_1^{(-1)}-X^-\circ\Phi_1^{(-1)}=g\circ\Phi_1^{(-1)} & \text{on } \phi_1\left(\partial\mathbf{B}_n\right), \\ D\left(X^+\circ\Phi_1^{(-1)}\right)\cdot\nu_{\phi_1} & \\ -D\left(X^-\circ\Phi_1^{(-1)}\right)\cdot\nu_{\phi_1}=g_1\circ\Phi_1^{(-1)} & \text{on } \phi_1\left(\partial\mathbf{B}_n\right), \\ X^+\circ\Phi_1^{(-1)}=h^+\circ\Phi_1^{(-1)} & \text{on } \Phi_1\left((1-\delta)\partial\mathbf{B}_n\right), \\ X^-\circ\Phi_1^{(-1)}=h^-\circ\Phi_1^{(-1)} & \text{on } \Phi_1\left((1+\delta)\partial\mathbf{B}_n\right). \end{cases}$$

where

$$b^+\equiv(b_1^+,\dots,b_n^+)\equiv\left\{\left(f^+\circ\Phi_1^{(-1)}\right)\left(D\Phi_1(\Phi_1^{(-1)})\right)^t\left|\det\left(D(\Phi_1^{(-1)})\right)\right|\right\}$$

and

$$b^-\equiv(b_1^-,\dots,b_n^-)\equiv\left\{\left(f^-\circ\Phi_1^{(-1)}\right)\left(D\Phi_1(\Phi_1^{(-1)})\right)^t\left|\det\left(D(\Phi_1^{(-1)})\right)\right|\right\}.$$

We first show existence for system (3.18). Let $(F^+,F^-,g,g_1,h^+,h^-)\in\mathcal{Z}$. By Lemma 2.1, we can easily see that $b^+\in C^{m-1,\alpha}\left(\mathrm{cl}\,\Phi_1\left(\mathbf{A}_\delta^+\right),\mathbf{R}^n\right)$, and $b^-\in C^{m-1,\alpha}\left(\mathrm{cl}\,\Phi_1\left(\mathbf{A}_\delta^-\right),\mathbf{R}^n\right)$, and $g\circ\Phi_1^{(-1)}\in C^{m,\alpha}\left(\phi_1\left(\partial\mathbf{B}_n\right)\right)$, and $g_1\circ\Phi_1^{(-1)}\in C^{m-1,\alpha}\left(\phi_1\left(\partial\mathbf{B}_n\right)\right)$, and $h^+\circ\Phi_1^{(-1)}\in C^{m,\alpha}\left(\Phi_1\left((1-\delta)\partial\mathbf{B}_n\right)\right)$, and $h^-\circ\Phi_1^{(-1)}\in C^{m,\alpha}\left(\Phi_1\left((1+\delta)\partial\mathbf{B}_n\right)\right)$. Thus, by Lemma 3.10, there exist elements a^- of $C^{m,\alpha}\left(\mathrm{cl}\Phi_1\left(\mathbf{A}_\delta^-\right)\right)$ such that

$$\begin{cases} \Delta a^-=\sum_{j=1}^n \partial b_j^-/\partial\xi_j & \text{in } \mathcal{D}'\left(\Phi_1\left(\mathbf{A}_\delta^-\right)\right), \\ a^-=h^-\circ\Phi_1^{(-1)} & \text{on } \Phi_1\left((1+\delta)\partial\mathbf{B}_n\right), \end{cases}$$

and $a^+\in C^{m,\alpha}\left(\mathrm{cl}\,\Phi_1\left(\mathbf{A}_\delta^+\right)\right)$ such that

$$\begin{cases} \Delta a^+=\sum_{j=1}^n \partial b_j^+/\partial\xi_j & \text{in } \mathcal{D}'\left(\Phi_1\left(\mathbf{A}_\delta^+\right)\right), \\ a^+=a^-+g\circ\Phi_1^{(-1)} & \text{on } \Phi_1\left(\partial\mathbf{B}_n\right)=\phi_1\left(\partial\mathbf{B}_n\right), \\ a^+=h^+\circ\Phi_1^{(-1)} & \text{on } \Phi_1\left((1-\delta)\partial\mathbf{B}_n\right), \end{cases}$$

Obviously, $g_1 \circ \Phi_1^{(-1)} - Da^+ \cdot \nu_{\phi_1} + Da^- \cdot \nu_{\phi_1} \in C^{m-1,\alpha}\left(\phi_1\left(\partial \mathbf{B}_n\right)\right)$, and Theorem 3.2 implies that

$$
\begin{aligned}
& v^+\left[\phi_1, g_1 - \left(Da^+ \cdot \nu_{\phi_1} - Da^- \cdot \nu_{\phi_1}\right) \circ \phi_1\right] \in C^{m,\alpha}\left(\operatorname{cl} \Phi_1\left(\mathbf{A}_\delta^+\right)\right), \\
& v^-\left[\phi_1, g_1 - \left(Da^+ \cdot \nu_{\phi_1} - Da^- \cdot \nu_{\phi_1}\right) \circ \phi_1\right] \in C^{m,\alpha}\left(\operatorname{cl} \Phi_1\left(\mathbf{A}_\delta^-\right)\right).
\end{aligned}
$$

Clearly, system (3.18) admits a solution $(X^+, X^-) \in \mathcal{Y}$ if and only if system

(3.19)

$$
\begin{cases}
\Delta \tilde{X}^+ = 0 & \text{in } \mathcal{D}'\left(\Phi_1\left(\mathbf{A}_\delta^+\right)\right), \\
\Delta \tilde{X}^- = 0 & \text{in } \mathcal{D}'\left(\Phi_1\left(\mathbf{A}_\delta^-\right)\right), \\
\tilde{X}^+ - \tilde{X}^- = 0 & \text{on } \phi_1\left(\partial \mathbf{B}_n\right), \\
D\tilde{X}^+ \cdot \nu_{\phi_1} - D\tilde{X}^- \cdot \nu_{\phi_1} = 0 & \text{on } \phi_1\left(\partial \mathbf{B}_n\right), \\
\tilde{X}^+ = v^+\left[\phi_1, g_1 - \left(Da^+ \cdot \nu_{\phi_1} - Da^- \cdot \nu_{\phi_1}\right) \circ \phi_1\right] & \text{on } \Phi_1\left((1-\delta)\partial \mathbf{B}_n\right), \\
\tilde{X}^- = v^-\left[\phi_1, g_1 - \left(Da^+ \cdot \nu_{\phi_1} - Da^- \cdot \nu_{\phi_1}\right) \circ \phi_1\right] & \text{on } \Phi_1\left((1+\delta)\partial \mathbf{B}_n\right).
\end{cases}
$$

admits a solution $(\tilde{X}^+, \tilde{X}^-) \in C^{m,\alpha}\left(\operatorname{cl} \Phi_1\left(\mathbf{A}_\delta^+\right)\right) \times C^{m,\alpha}\left(\operatorname{cl} \Phi_1\left(\mathbf{A}_\delta^-\right)\right)$, and in case of existence, we have

$$
\begin{aligned}
& \tilde{X}^+ = X^+ \circ \Phi_1^{(-1)} - a^+ + v^+\left[\phi_1, g_1 - \left(Da^+ \cdot \nu_{\phi_1} - Da^- \cdot \nu_{\phi_1}\right) \circ \phi_1\right] \\
& \tilde{X}^- = X^- \circ \Phi_1^{(-1)} - a^- + v^-\left[\phi_1, g_1 - \left(Da^+ \cdot \nu_{\phi_1} - Da^- \cdot \nu_{\phi_1}\right) \circ \phi_1\right]
\end{aligned}
$$

Thus we now show existence for system (3.19). By the third and fourth equation of (3.19), and by a standard argument based on the Divergence Theorem, system (3.19) is equivalent to the following system for $\tilde{X} \in C^{m,\alpha}\left(\operatorname{cl} \Phi_1\left(\mathbf{A}_\delta\right)\right)$

(3.20)

$$
\begin{cases}
\Delta \tilde{X} = 0 & \text{in } \mathcal{D}'\left(\Phi_1\left(\mathbf{A}_\delta\right)\right), \\
\tilde{X} = v^+\left[\phi_1, g_1 - \left(Da^+ \cdot \nu_{\phi_1} - Da^- \cdot \nu_{\phi_1}\right) \circ \phi_1\right] & \text{on } \Phi_1\left((1-\delta)\partial \mathbf{B}_n\right), \\
\tilde{X} = v^-\left[\phi_1, g_1 - \left(Da^+ \cdot \nu_{\phi_1} - Da^- \cdot \nu_{\phi_1}\right) \circ \phi_1\right] & \text{on } \Phi_1\left((1+\delta)\partial \mathbf{B}_n\right).
\end{cases}
$$

By Lemma 3.10, such a system has a unique solution $\tilde{X}$.

We now show uniqueness for system (3.17). If $(F^+, F^-, g, g_1, h^+, h^-) = 0$ for some $(X^+, X^-) \in \mathcal{Y}$, then system (3.18) holds with $b^+ = 0$, $b^- = 0$. Then we set $\tilde{X} \equiv X^+ \circ \Phi_1^{(-1)}$ on $\operatorname{cl} \Phi_1\left(\mathbf{A}_\delta^+\right)$, $\tilde{X} \equiv X^- \circ \Phi_1^{(-1)}$

on $\mathrm{cl}\,\Phi_1\left(\mathbf{A}_\delta^-\right)$. The function $\tilde{X}$ is harmonic in $\Phi_1\left(\mathbf{A}_\delta^+\right) \cup \Phi_1\left(\mathbf{A}_\delta^-\right)$, and continuous on $\mathrm{cl}\,\Phi_1\left(\mathbf{A}_\delta\right)$. Thus by exploiting the third and the fourth equation of (3.18) with $g = g_1 = 0$ and by a standard argument based on the Divergence Theorem, $\tilde{X}$ can be shown to be harmonic on $\Phi_1(\mathbf{A}_\delta)$. By the fifth and the sixth equation of (3.18) with $h^\pm = 0$, $\tilde{X}$ vanishes on $\partial\Phi_1(\mathbf{A}_\delta)$. Hence, $\tilde{X} = 0$, and (X^+, X^-) must be zero.

We now turn to the proof of statement (ii). We first consider the map $W^+[\cdot,\cdot]$. We observe that the linear map Γ of $C^{m,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+\right)$ to $C^{m-1,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+\right) \times C^{m-1,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+, \mathbf{R}^n\right)$ defined by

$$\Gamma[g] \equiv (g, \partial_{x_1} g, \dots, \partial_{x_n} g) \quad \forall\, g \in C^{m,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+\right),$$

is a linear homeomorphism of $C^{m,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+\right)$ onto the image space $\mathrm{Im}\,\Gamma$, a subspace of $C^{m-1,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+\right) \times C^{m-1,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+, \mathbf{R}^n\right)$. Thus it suffices to show that the nonlinear maps $W^+[\cdot,\cdot]$ and $(\partial/\partial x_i)W^+[\cdot,\cdot]$ for $i = 1, \dots, n$ are real analytic from $\left(C^{m,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n\right) \cap \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}\right) \times C^{m,\alpha}\left(\partial\mathbf{B}_n\right)$ to $C^{m-1,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+\right)$. Now let $R > 1 + \delta$. By Lemma 2.3 (ii), (iii), there exists a linear and continuous extension operator $\mathbf{F}$ of $C^{m,\alpha}\left(\partial\mathbf{B}_n\right)$ to $C^{m,\alpha}\left(\mathrm{cl}\,\mathbf{B}_n(0,R)\right)$ such that $\mathbf{F}[f]_{|\partial\mathbf{B}_n} = f$, for all $f \in C^{m,\alpha}\left(\partial\mathbf{B}_n\right)$. By Theorem 3.1, and by Lemma 3.3, we have the following identities

$$\begin{aligned} (3.21) \quad & W^+[\Phi, f] \\ &= -\sum_{j=1}^n \sum_{l=1}^n \frac{\partial}{\partial x_l}\left(V^+\left[\Phi, \left(\frac{(D\Phi)^{-t} \cdot \nu_{\mathbf{B}_n}}{\left|(D\Phi)^{-t} \cdot \nu_{\mathbf{B}_n}\right|}\right)_j f\right]\right)\left((D\Phi)^{-1}\right)_{lj}, \end{aligned}$$

and

$$\begin{aligned} (3.22) \quad & \frac{\partial}{\partial x_i}\left(W^+[\Phi, f]\right) \\ &= \sum_{j,l,r=1}^n \frac{\partial \Phi_r}{\partial x_i} \frac{\partial}{\partial x_l}\left(V^+\left[\Phi, M_{rj}[f, \Phi]\right]\right)\left((D\Phi)^{-1}\right)_{lj}, \end{aligned}$$

for all $i = 1, \dots, n$, where

$$M_{rj}[f,\Phi] = \left|(D\Phi)^{-t} \cdot \nu_{\mathbf{B}_n}\right|^{-1} \cdot \Bigg\{ \Bigg[\sum_{l=1}^{n} \Big((D\Phi)^{-1} \Big)_{lr} (\nu_{\mathbf{B}_n})_l \Bigg] \Bigg[\sum_{l=1}^{n} \frac{\partial(\mathbf{F}[f])}{\partial x_l} \Big((D\Phi)^{-1} \Big)_{lj} \Bigg] - \Bigg[\sum_{l=1}^{n} \Big((D\Phi)^{-1} \Big)_{lj} (\nu_{\mathbf{B}_n})_l \Bigg] \Bigg[\sum_{l=1}^{n} \frac{\partial(\mathbf{F}[f])}{\partial x_l} \Big((D\Phi)^{-1} \Big)_{lr} \Bigg] \Bigg\}.$$

By the real analyticity of $\mathbf{F}[\cdot]$, and of the trace operator, by Lemma 2.1, by the real analyticity of $V^+[\cdot,\cdot]$, by equations (3.21) and (3.22), we conclude that $W^+[\cdot,\cdot]$, $(\partial/\partial x_i)W^+[\cdot,\cdot]$ are real analytic from $(C^{m,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta, \mathbf{R}^n) \cap \mathcal{A}'_{\mathrm{cl}\,\mathbf{A}_\delta}) \times C^{m,\alpha}(\partial\mathbf{B}_n)$ to $C^{m-1,\alpha}(\mathrm{cl}\,\mathbf{A}^+_\delta)$. Similarly, we can show that $W^-[\cdot,\cdot]$ depends real analytically on (Φ, f). □

We are now ready to prove our main result.

Theorem 3.12. *Let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in]0,1[$. Then the following statements hold.*

(i) *The map $V[\cdot,\cdot]$ of $(C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}) \times C^{m-1,\alpha}(\partial\mathbf{B}_n)$ to the space $C^{m,\alpha}(\partial\mathbf{B}_n)$ defined by* (1.1) *is real analytic.*

(ii) *The map $W[\cdot,\cdot]$ of $(C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}) \times C^{m,\alpha}(\partial\mathbf{B}_n)$ to the space $C^{m,\alpha}(\partial\mathbf{B}_n)$ defined by* (1.2) *is real analytic.*

Proof. We first consider statement (i). It clearly suffices to show that if $(\phi_0, f_0) \in (C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}) \times C^{m-1,\alpha}(\partial\mathbf{B}_n)$, then $V[\cdot,\cdot]$ is real analytic in a neighborhood of (ϕ_0, f_0). Now let $\mathcal{W}_0$, δ, $\mathbf{E}_0$ be as in Proposition 2.8. By Theorem 3.1 and (3.15), we have $V[\phi,f] = v^+[\phi,f]\circ\phi = V^+[\mathbf{E}_0[\phi], f]$ on $\partial\mathbf{B}_n$ for all $(\phi, f) \in \mathcal{W}_0 \times C^{m-1,\alpha}(\partial\mathbf{B}_n)$. Thus it suffices to prove that the pair $(v^+[\phi,f]\circ\phi, v^-[\phi,f]\circ\phi)$ depends real analytically on (ϕ, f) around (ϕ_0, f_0), a fact which is an immediate consequence of Propositions 2.8 and 3.11. We now turn to the proof of statement (ii). By Theorem 3.1 (ii), we know that $w^+[\phi,f]$ defined as in (3.16) satisfies the following equality

$$W[\phi,f] = w^+[\phi,f]\circ\phi - \frac{1}{2}f = W^+[\mathbf{E}_0[\phi], f] - \frac{1}{2}f \quad \text{on} \quad \partial\mathbf{B}_n. \tag{3.23}$$

Then we can conclude by invoking Propositions 2.8 and 3.11 as in the proof of statement (i). $\square$

We now compute the differentials of V and W by exploiting an argument of Lanza and Preciso [**13**, Section 4]. To do so, we first introduce some notation. If $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in]0,1]$, $\phi \in C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}$, then by combining Proposition 2.7 and Lemma 3.3, we can see that there exists a positive function $\tilde{\sigma}_n[\phi] \in C^{m-1,\alpha}(\partial\mathbf{B}_n)$ such that

$$\int_{\phi(\partial\mathbf{B}_n)} \omega(\eta)\, d\sigma_\eta = \int_{\partial\mathbf{B}_n} \omega \circ \phi(y)\tilde{\sigma}_n[\phi](y)\, d\sigma_y \quad \forall\, \omega \in L^1(\phi(\partial\mathbf{B}_n)).$$

Then we set

$$\tilde{\tau}_n[\phi] \equiv (\nu_\phi \circ \phi)\, \tilde{\sigma}_n[\phi],$$

where ν_ϕ denotes the unit exterior normal to $\mathcal{I}[\phi]$. Obviously,

$$\tilde{\sigma}_n[\phi] = |\tilde{\tau}_n[\phi]|.$$

By Proposition 2.8 and Lemma 3.3, we deduce that the following holds.

Proposition 3.13. *Let $m \in \mathbf{N} \setminus \{0\}$, $\alpha \in]0,1]$. Then the maps $\tilde{\tau}_n[\cdot]$ and $\tilde{\sigma}_n[\cdot]$ of $C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}$ to $C^{m-1,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n)$, and to $C^{m-1,\alpha}(\partial\mathbf{B}_n)$, which map ϕ to $\tilde{\tau}_n[\phi]$ and to $\tilde{\sigma}_n[\phi]$, respectively, are real analytic.*

We now compute the differentials of V, W in the following proposition. Since V and W are linear in the second variable, it suffices to consider the differentials with respect to the first variable ϕ.

Proposition 3.14. *Let $m, k \in \mathbf{N} \setminus \{0\}$, $\alpha \in]0,1[$. Let G_k denote the group of bijections of $\{1, \dots, k\}$ to itself. Let $d^k S_n(\eta)[\cdot]$ denote the kth order differential of S_n at $\eta \in \mathbf{R}^n \setminus \{0\}$. Then the following statements hold.*

(i) *If $(\phi_0, f_0) \in (C^{m,\alpha}(\partial\mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial\mathbf{B}_n}) \times C^{m-1,\alpha}(\partial\mathbf{B}_n)$, then the kth differential of $V[\cdot,\cdot]$ at (ϕ_0, f_0) with respect to the variable ϕ is* *delivered by the formula*

$$
\begin{aligned}
&\partial_\phi^k V[\phi_0, f_0][h_1, \dots, h_k](x) \\
&\quad = \int_{\partial \mathbf{B}_n} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} \\
&\qquad \cdot \sum_{\gamma \in G_k} d^j S_n\big(\phi_0(x) - \phi_0(y)\big)[h_{\gamma(1)}(x) - h_{\gamma(1)}(y), \\
&\qquad \dots, h_{\gamma(j)}(x) - h_{\gamma(j)}(y)] \\
&\qquad \cdot d^{k-j} \tilde{\sigma}_n[\phi_0][h_{\gamma(j+1)}, \dots, h_{\gamma(k)}] f_0(y)\, d\sigma_y,
\end{aligned} \tag{3.24}
$$

for all $x \in \partial \mathbf{B}_n$, *and for all* $(h_1, \dots, h_k) \in (C^{m,\alpha}(\partial \mathbf{B}_n, \mathbf{R}^n))^k$.

(ii) *If* $(\phi_0, f_0) \in (C^{m,\alpha}(\partial \mathbf{B}_n, \mathbf{R}^n) \cap \mathcal{A}_{\partial \mathbf{B}_n}) \times C^{m,\alpha}(\partial \mathbf{B}_n)$, *then the* k*th differential of* $W[\cdot,\cdot]$ *at* (ϕ_0, f_0) *with respect to the variable* ϕ *is delivered by the formula*

$$
\begin{aligned}
&\partial_\phi^k W[\phi_0, f_0][h_1, \dots, h_k](x) \\
&\quad = -\int_{\partial \mathbf{B}_n} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} \\
&\qquad \cdot \sum_{\gamma \in G_k} d^{j+1} S_n\left(\phi_0(x) - \phi_0(y)\right) \big[h_{\gamma(1)}(x) - h_{\gamma(1)}(y), \\
&\qquad \dots, h_{\gamma(j)}(x) - h_{\gamma(j)}(y), \\
&\qquad d^{k-j} \tilde{\tau}_n[\phi_0][h_{\gamma(j+1)}, \dots, h_{\gamma(k)}]\big] (f_0(y) - f_0(x))\, d\sigma_y,
\end{aligned} \tag{3.25}
$$

for all $x \in \partial \mathbf{B}_n$, *and for all* $(h_1, \dots, h_k) \in (C^{m,\alpha}(\partial \mathbf{B}_n, \mathbf{R}^n))^k$.

Proof. We first consider statement (i). Since k-linear symmetric functions are uniquely determined by their values on the diagonal, we start by computing $\partial_\phi^k V[\phi_0, f_0][h, \dots, h]$ for $h \in C^{m,\alpha}(\partial \mathbf{B}_n, \mathbf{R}^n)$. To shorten our notation, we write h^k instead of $\overbrace{h, \dots, h}^{k \text{ terms}}$. By standard calculus in Banach spaces, and by (1.1) and by definition of $\tilde{\sigma}_n[\cdot]$, we have

(3.26)

$$\partial_\phi^k V[\phi_0, f_0][h^k](x) = \frac{d^k}{d\varepsilon^k}_{|\varepsilon=0} \Big\{ \int_{\partial\mathbf{B}_n} S_n((\phi_0(x) - \phi_0(y)) + \varepsilon(h(x) - h(y)))\tilde{\sigma}_n[\phi_0 + \varepsilon h](y) f_0(y)\, d\sigma_y \Big\} \quad \forall\, x \in \partial\mathbf{B}_n,$$

and now we would like to take the differentiation inside the integral. To justify such a step, one could exploit the local parametrizations of $\partial\mathbf{B}_n$ or extend the functions involved in the integrand in the vicinity of $\partial\mathbf{B}_n$. We choose the latter method. Thus we introduce $\mathcal{W}_0$, δ, $\mathbf{F}_0$, $\mathbf{E}_0$ of Proposition 2.8, and the trace operator R of $C^{m,\alpha}\left(\mathrm{cl}\,\mathbf{A}_\delta^+\right)$ to $C^{m,\alpha}\left(\partial\mathbf{B}_n\right)$. By Theorem 3.1 and by (3.15), we have

$$V[\phi, f] = v^+[\phi, f] \circ \phi = R\left[v^+[\phi, f] \circ \mathbf{E}_0[\phi]_{|\mathrm{cl}\,\mathbf{A}_\delta^+}\right]$$

for all $(\phi, f) \in \mathcal{W}_0 \times C^{m-1,\alpha}\left(\partial\mathbf{B}_n\right)$. By standard calculus in Banach space, and by Lemma 3.3, we obtain

$$\begin{aligned} &\partial_{|\phi=\phi_0}^k \left(v^+[\phi, f_0] \circ \mathbf{E}_0[\phi]_{|\mathrm{cl}\,\mathbf{A}_\delta^+}\right)[h^k](x) \\ &= \frac{d^k}{d\varepsilon^k}_{|\varepsilon=0} \Big\{ \int_{\partial\mathbf{B}_n} S_n(\mathbf{E}_0[\phi_0 + \varepsilon h](x) - \mathbf{E}_0[\phi_0 + \varepsilon h](y)) \\ &\qquad \cdot \sigma_n[\mathbf{E}_0[\phi_0 + \varepsilon h]](y)\mathbf{F}_0[f_0](y)\, d\sigma_y \Big\} \quad \forall\, x \in \mathrm{cl}\,\mathbf{A}_\delta^+. \end{aligned} \tag{3.27}$$

Obviously, the righthand side of (3.26) is just the restriction to $\partial\mathbf{B}_n$ of the righthand side of (3.27). Now we note that for $x \in \mathbf{A}_\delta^+$, the integrand of the righthand side of (3.27) is not singular. Then standard results on differentiation for integrals depending on a parameter imply that

(3.28)

$$\begin{aligned} &\partial_{|\phi=\phi_0}^k \left(v^+[\phi, f_0] \circ \mathbf{E}_0[\phi]_{|\mathrm{cl}\,\mathbf{A}_\delta^+}\right)[h^k](x) \\ &= \int_{\partial\mathbf{B}_n} \sum_{j=0}^k \binom{k}{j} \left\{ d^j S_n(\mathbf{E}_0[\phi_0](x) - \mathbf{E}_0[\phi_0](y)) \left[(\mathbf{F}_0[h](x) - \mathbf{F}_0[h](y))^j\right] \right\} \\ &\quad \cdot \frac{d^{k-j}}{d\varepsilon^{k-j}}_{|\varepsilon=0} \left\{\sigma_n[\mathbf{E}_0[\phi_0] + \varepsilon\mathbf{F}_0[h]](y)\right\} \mathbf{F}_0[f_0](y)\, d\sigma_y \quad \forall\, x \in \mathbf{A}_\delta^+. \end{aligned}$$

By Proposition 2.8, and by Lemma 3.3, the map $\sigma_n[\mathbf{E}_0[\cdot]]$ is real analytic from $\mathcal{W}_0$ to the space $C^{m-1,\alpha}(\mathrm{cl}\,\mathbf{A}_\delta)$, and thus the function $d^{k-j}\left\{\sigma_n[\mathbf{E}_0[\phi_0]+\varepsilon \mathbf{F}_0[h]](y)\right\}/d\varepsilon^{k-j}{}_{|\varepsilon=0}$ is continuous in $y \in \partial\mathbf{B}_n$. As is well known, cf., e.g., Gilbarg and Trudinger [**4**, p. 17], for all integers $j \geq 1$, $n \geq 2$, there exists $c(n,j) > 0$ such that

$$(3.29)\qquad \begin{gathered} |d^j S_n(y)[v_1,\ldots,v_j]| \leq c(n,j)|y|^{2-n-j}|v_1|\ldots|v_j| \\ \forall\, y \in \mathbf{R}^n \setminus \{0\},\ v_1,\ldots,v_j \in \mathbf{R}^n. \end{gathered}$$

By Lemma 2.1 (ii), $\mathbf{F}_0[h]$ is Lipschitz continuous on $\mathrm{cl}\,\mathbf{A}_\delta^+$, and by Proposition 2.8, $l_{\mathrm{cl}\,\mathbf{A}_\delta^+}[\mathbf{E}_0[\phi_0]] > 0$. Then by Vitali's convergence theorem, one can easily show that the righthand side of (3.28) depends continuously on $x \in \mathrm{cl}\,\mathbf{A}_\delta^+$. Since the lefthand side of (3.28) is continuous for $x \in \mathrm{cl}\,\mathbf{A}_\delta^+$, we deduce that

$$(3.30)\qquad \begin{aligned} &\partial^k_{|\phi=\phi_0} V[\phi_0,f_0][h^k](x) \\ &\quad = \int_{\partial\mathbf{B}_n} \sum_{j=0}^{k} \binom{k}{j} \left\{ d^j S_n\left(\phi_0(x)-\phi_0(y)\right)\left[(h(x)-h(y))^j\right]\right\} \\ &\qquad \cdot \frac{d^{k-j}}{d\varepsilon^{k-j}}_{|\varepsilon=0} \left\{\tilde{\sigma}_n[\phi_0+\varepsilon h](y)\right\} f_0(y)\, d\sigma_y \quad \forall\, x \in \partial\mathbf{B}_n. \end{aligned}$$

By inequality (3.29) and by the Lipschitz continuity of $h_1, \ldots, h_n$, and by inequality $l_{\partial\mathbf{B}_n}[\phi_0] > 0$, we deduce that the righthand side of equality (3.24), which we denote by $H[h_1,\ldots,h_n](x)$, defines a k-linear symmetric form on $\left(C^{m,\alpha}\left(\partial\mathbf{B}_n,\mathbf{R}^n\right)\right)^k$ for each $x \in \partial\mathbf{B}_n$. Obviously, $\partial^k_{|\phi=\phi_0} V[\phi_0,f_0][h^k] = H[h^k]$ for all $h \in C^{m,\alpha}\left(\partial\mathbf{B}_n,\mathbf{R}^n\right)$. Since k-linear symmetric forms are uniquely determined by their values on the diagonal, we can deduce the validity of statement (i).

We now prove statement (ii). As for statement (i), we first compute $\partial^k_\phi W[\phi_0,f_0][h^k](x)$ for $h \in C^{m,\alpha}\left(\partial\mathbf{B}_n,\mathbf{R}^n\right)$, and we face the problem of differentiating an integral similar to that of (3.26). To circumvent such a problem, we note that by Theorem 3.1 (ii) and (3.16), we have

$$\begin{aligned}W[\phi,f] &= w^+[\phi,f]\circ\phi - \frac{1}{2}f\\ &= R\left[w^+[\phi,f]\circ \mathbf{E}_0[\phi]_{|\mathrm{cl}\,\mathbf{A}_\delta^+} - \mathbf{F}_0[f]_{|\mathrm{cl}\,\mathbf{A}_\delta^+}\right] + \frac{1}{2}f\\ &= R\Big[w^+[\phi,f]\circ \mathbf{E}_0[\phi]_{|\mathrm{cl}\,\mathbf{A}_\delta^+}\\ &\qquad - \left(w^+[\phi,1]\circ \mathbf{E}_0[\phi]_{|\mathrm{cl}\,\mathbf{A}_\delta^+}\right)\mathbf{F}_0[f]_{|\mathrm{cl}\,\mathbf{A}_\delta^+}\Big] + \frac{1}{2}f\end{aligned}$$

for all $(\phi,f)\in\mathcal{W}_0\times C^{m,\alpha}(\partial\mathbf{B}_n)$. For the sake of brevity, we set

$$\tau_n[\phi] \equiv \left|\det\left(D\left(\mathbf{E}_0[\phi]\right)\right)\right| \left(\left(D\left(\mathbf{E}_0[\phi]\right)\right)^{-t}\cdot\nu_{\mathbf{B}_n}\right) \quad \forall\,\phi\in\mathcal{A}_{\partial\mathbf{B}_n}.$$

Now by the same argument of the proof of (i), we have

(3.31)

$$\begin{aligned}&\partial^k_{|\phi=\phi_0}\Big(w^+[\phi,f]\circ\mathbf{E}_0[\phi]_{|\mathrm{cl}\,\mathbf{A}_\delta^+}\\ &\qquad\qquad - (w^+[\phi,1]\circ\mathbf{E}_0[\phi]_{|\mathrm{cl}\,\mathbf{A}_\delta^+})\mathbf{F}_0[f]_{|\mathrm{cl}\,\mathbf{A}_\delta^+}\Big)[h^k](x)\\ &= -\int_{\partial\mathbf{B}_n}\sum_{j=0}^{k}\binom{k}{j}\Big\{d^{j+1}S_n\big(\mathbf{E}_0[\phi_0](x)-\mathbf{E}_0[\phi_0](y)\big)\\ &\qquad\qquad \Big[\big(\mathbf{F}_0[h](x)-\mathbf{F}_0[h](y)\big)^j, \frac{d^{k-j}}{d\varepsilon^{k-j}}{}_{|\varepsilon=0}\big(\tau_n[\phi_0+\varepsilon h](y)\big)\Big]\Big\}\\ &\qquad\qquad \cdot\left(\mathbf{F}_0[f_0](y)-\mathbf{F}_0[f_0](x)\right)d\sigma_y \quad \forall\,x\in\mathbf{A}_\delta^+.\end{aligned}$$

Since the map of $\mathcal{W}_0$ to $C^{m-1,\alpha}(\partial\mathbf{B}_n,\mathbf{R}^n)$ which takes ϕ to $\tau_n[\phi]$ is real analytic, then the function $d^{k-j}\left(\tau_n[\phi_0+\varepsilon h]\right)/d\varepsilon^{k-j}{}_{|\varepsilon=0}$ is continuous on $\partial\mathbf{B}_n$. By Lemma 2.1 (ii), (iii), $\mathbf{F}_0[h]$ is Lipschitz continuous on $\mathrm{cl}\,\mathbf{A}_\delta^+$, and $\mathbf{F}_0[f_0]$ is Hölder continuous with exponent α on $\mathrm{cl}\,\mathbf{A}_\delta^+$, and by Lemma 2.8, $l_{\mathrm{cl}\,\mathbf{A}_\delta^+}[\mathbf{E}_0[\phi_0]]>0$ and $\left(D\left(\mathbf{E}_0[\phi_0]\right)\right)^{-t}\cdot\nu_{\mathbf{B}_n}$ is continuous and bounded. Then by (3.29), and by Vitali's convergence theorem, one can easily show that the righthand side of (3.31) depends continuously on $x\in\mathrm{cl}\,\mathbf{A}_\delta^+$. Since the lefthand side of (3.31) is continuous for $x\in\mathrm{cl}\,\mathbf{A}_\delta^+$, we deduce that (3.31) holds also for $x\in\partial\mathbf{B}_n$. The same argument also implies that the righthand side of (3.25) defines a k-linear symmetric form on $\left(C^{m,\alpha}\left(\partial\mathbf{B}_n,\mathbf{R}^n\right)\right)^k$ for each $x\in\partial\mathbf{B}_n$. Thus we can conclude as in statement (i). □

As we have seen in Proposition 3.14, in order to compute the differentials of V, W, one needs to know the differentials of the functions $\tilde{\sigma}_n[\cdot]$, $\tilde{\tau}_n[\cdot]$. Thus we now present a different way of writing $\tilde{\sigma}_n[\cdot]$, $\tilde{\tau}_n[\cdot]$ which makes it easy to compute such differentials explicitly.

We denote by $\wedge$ the standard vector product of $n-1$ vectors in $\mathbf{R}^n$, cf., e.g., Schwartz [**22**, p. 250]. If $\phi \in \mathcal{A}_{\partial \mathbf{B}_n}$, then we can consider $\phi(\partial \mathbf{B}_n)$ as oriented by the exterior normal field ν_ϕ, and $\partial \mathbf{B}_n$ as oriented by $\nu_{\partial \mathbf{B}_n}$. Then we shall say that ϕ has index 1 if ϕ is orientation preserving, and that ϕ has index -1 if ϕ is orientation reversing. We formalize the known properties of such an index in the following lemma.

Lemma 3.15. *Let $\{e_1, \dots, e_n\}$ denote the canonical basis of $\mathbf{R}^n$. Let $v_1(y)$, ... , $v_{n-1}(y)$ be an orthonormal basis of the tangent space $T_y \partial \mathbf{B}_n$ to $\partial \mathbf{B}_n$ at y such that*

$$\begin{aligned} &\nu_{\mathbf{B}_n}(y) = (v_1(y) \cdot e_2)e_1 - (v_1(y) \cdot e_1)e_2 && \text{if } n = 2, \\ &\nu_{\mathbf{B}_n}(y) = v_1(y) \wedge \cdots \wedge v_{n-1}(y) && \text{if } n > 2, \end{aligned} \tag{3.32}$$

for all $y \in \partial \mathbf{B}_n$. If $\phi \in \mathcal{A}_{\partial \mathbf{B}_n}$, then there exists $\operatorname{ind}[\phi] \in \{-1, 1\}$ *such that*

(3.33)

$$\operatorname{ind}[\phi] = \operatorname{sgn} \{\nu_\phi(\phi(y)) \cdot ((d\phi(y)[v_1(y)] \cdot e_2)e_1 - (d\phi(y)[v_1(y)] \cdot e_1)e_2)\} \qquad \text{if } n = 2,$$

$$\operatorname{ind}[\phi] = \operatorname{sgn} \{\nu_\phi(\phi(y)) \cdot (d\phi(y)[v_1(y)] \wedge \cdots \wedge d\phi(y)[v_{n-1}(y)])\} \qquad \text{if } n > 2,$$

for all $y \in \partial \mathbf{B}_n$, where we set $\operatorname{sgn}(t) = 1$ *for $t > 0$,* $\operatorname{sgn}(t) = -1$ *for $t < 0$. Moreover, the function* $\operatorname{ind}[\cdot]$ *of $\mathcal{A}_{\partial \mathbf{B}_n}$ to $\{-1, 1\}$ which takes ϕ to* $\operatorname{ind}[\phi]$ *is continuous.*

Proof. Let $\phi_0 \in \mathcal{A}_{\partial \mathbf{B}_n}$. Let $\mathcal{W}_0$, δ, $\mathbf{E}_0$ be as in Proposition 2.8. Let $n > 2$. By linear algebra, and by (3.32), we have

$$\begin{aligned} d\phi(y)&[v_1(y)] \wedge \cdots \wedge d\phi(y)[v_{n-1}(y)] \\ &= \det\left(D(\mathbf{E}_0[\phi])(y)\right) D(\mathbf{E}_0[\phi])^{-t}(y) \nu_{\partial \mathbf{B}_n}(y) \\ &= \left|D(\mathbf{E}_0[\phi])^{-t}(y) \nu_{\partial \mathbf{B}_n}(y)\right| \det\left(D(\mathbf{E}_0[\phi])(y)\right) \nu_\phi \circ \phi(y), \end{aligned}$$

and thus the righthand side of (3.33) equals sgn $(\det (D(\mathbf{E}_0[\phi])(y)))$ for all $y \in \partial\mathbf{B}_n$ and $\phi \in \mathcal{W}_0$. Similarly, we can argue for $n = 2$. Then by Proposition 2.8 and by the continuity of sgn $(\cdot)$ from $\mathbf{R} \setminus \{0\}$ to $\{-1, 1\}$, we conclude that the righthand side of (3.33) is independent of $y \in \partial\mathbf{B}_n$, and that ind $[\cdot]$ defines a continuous function of $\mathcal{A}_{\partial\mathbf{B}_n}$ to $\{-1, 1\}$. □

Then we have the following.

Proposition 3.16. *Let $e_1, \dots, e_n$, and $v_1(y), \dots, v_{n-1}(y)$ for all $y \in \partial\mathbf{B}_n$ be as in Lemma 3.15. Then we have*

$$(3.34)\quad \begin{aligned} \tilde{\tau}_n[\phi](y) &= \mathrm{ind}\,[\phi]\,\{(d\phi(y)[v_1(y)] \cdot e_2)e_1 - (d\phi(y)[v_1(y)] \cdot e_1)e_2\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n = 2, \\ \tilde{\tau}_n[\phi](y) &= \mathrm{ind}\,[\phi]\,\{d\phi(y)[v_1(y)] \wedge \cdots \wedge d\phi(y)[v_{n-1}(y)]\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n > 2, \end{aligned}$$

for all $y \in \partial\mathbf{B}_n$, and for all $\phi \in \mathcal{A}_{\partial\mathbf{B}_n}$.

Proof. Let δ, Φ be as in Proposition 2.7. As in the previous proof, one can easily check that $(\det D\Phi(y))\,(D\Phi)^{-t}(y) \cdot \nu_{\mathbf{B}_n}(y)$ equals the term in braces of (3.34). By definition of $\tilde{\tau}_n[\phi]$, and by Lemma 3.3, we can see that $\tilde{\tau}_n[\phi](y)$ equals $|\det D\Phi(y)|\,(D\Phi)^{-t}(y) \cdot \nu_{\mathbf{B}_n}(y)$. As pointed out in the proof of Lemma 3.15, ind $[\phi]$ equals sgn $(\det D\Phi(y))$ for all $y \in \partial\mathbf{B}_n$. Then we conclude that (3.34) holds. □

We note that the differentials of $\tilde{\tau}_n$ of order $k \geq n$ vanish identically and that in order to obtain the differentials of order less than k one can use Leibnitz rule. In particular, we note that, for $k = 1$, we obtain

$$\begin{aligned} d\tilde{\tau}_2[\phi][h](y) &= \mathrm{ind}\,[\phi]\,\{(dh(y)[v_1(y)] \cdot e_2)e_1 - (dh(y)[v_1(y)] \cdot e_1)e_2\} \\ d\tilde{\tau}_n[\phi][h](y) &= \mathrm{ind}\,[\phi]\,\{dh(y)[v_1(y)] \wedge d\phi(y)[v_2(y)] \\ &\quad \wedge \cdots \wedge d\phi(y)[v_{n-1}(y)] \\ &\quad + \cdots + d\phi(y)[v_1(y)] \wedge \dots \\ &\quad \wedge d\phi(y)[v_{n-2}(y)] \wedge dh(y)[v_{n-1}(y)]\} \quad \text{if } n > 2, \end{aligned}$$

for all $h \in C^{m,\alpha}(\partial \mathbf{B}_n, \mathbf{R}^n)$. Finally, we note that

$$d\tilde{\sigma}_n[\phi][h] = (\nu_\phi \circ \phi) \cdot d\tilde{\tau}_n[\phi][h]$$

for all $h \in C^{m,\alpha}(\partial \mathbf{B}_n, \mathbf{R}^n)$.

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