Functional Calculus on BMO and related spaces

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Abstract: Let f be a Borel measurable function of the complex plane to itself. We consider the nonlinear operator T_f defined by $T_f[g] = f \circ g$, when g belongs to a certain subspace X of the space $BMO(\mathbb{R}^n)$ of functions with bounded mean oscillation on the Euclidean space. In particular, we investigate the case in which X is the whole of BMO, the case in which X is the space VMO of functions with vanishing mean oscillation, and the case in which X is the closure in BMO of the smooth functions with compact support. We characterize those f's for which T_f maps X to itself, those f's for which T_f is continuous from X to itself, and those f's for which T_f is differentiable in X.

1 Introduction and main results.

In this paper, we characterize those Borel measurable functions f of the complex plane $\mathbb{C}$ to itself such that the nonlinear superposition operator T_f defined by

$$T_f[g] := f \circ g$$

takes $BMO(\mathbb{R}^n)$ and several spaces related to $BMO(\mathbb{R}^n)$ to themselves. Also continuity and differentiability of T_f will be discussed.

This paper may be considered as a continuation of the investigations of Fominykh [6], of Chevalier [3], and of Brezis and Nirenberg [2]. Whereas Fominykh and Chevalier have characterized all functions f such that $T_f(BMO) \subseteq BMO$ in cases $n = 1$, and $n \geq 1$, respectively, Brezis and Nirenberg have shown that the uniform continuity of f suffices to ensure that T_f acts in $VMO(\mathbb{R}^n)$.

We are going to consider T_f in $BMO(\mathbb{R}^n)$, in $VMO(\mathbb{R}^n)$, in $CMO(\mathbb{R}^n)$ and in their respective inhomogeneous counterparts $bmo(\mathbb{R}^n)$, $vmo(\mathbb{R}^n)$ and $cmo(\mathbb{R}^n)$. For the definition of these spaces, we refer to Section 2. (The reader should be aware of the fact that the symbols VMO and CMO are used with different meanings at different places in the literature.) It turns out that the behaviour of T_f can differ strongly on these various classes.

We start by analyzing the acting condition of T_f.

Here and in the sequel we require, without further reference, the validity of the following

Assumption *f is a Borel measurable function of $\mathbb{C}$ to itself.*

We first introduce the following more general form of Fominykh-Chevalier Theorem.

Theorem 1 *The following properties are equivalent.*

(i) $\sup_{x,y\in\mathbb{C}} (1 + |x - y|)^{-1}|f(x) - f(y)| < +\infty.$

(ii) $T_f[BMO(\mathbb{R}^n)] \subseteq BMO(\mathbb{R}^n)$.

(iii) $T_f[bmo(\mathbb{R}^n)] \subseteq bmo(\mathbb{R}^n)$.

(iv) $T_f[cmo(\mathbb{R}^n)] \subseteq BMO(\mathbb{R}^n)$.

Furthermore, if any of the above properties is satisfied, then T_f maps bounded subsets of $BMO(\mathbb{R}^n)$ to bounded subsets of $BMO(\mathbb{R}^n)$, and bounded subsets of $bmo(\mathbb{R}^n)$ to bounded subsets of $bmo(\mathbb{R}^n)$.

Next we extend the result of Brezis and Nirenberg which we mentioned before by establishing the necessity of the uniform continuity in case of VMO.

Theorem 2 *The following properties are equivalent.*

(a) f *is uniformly continuous.*

(b) $T_f[VMO(\mathbb{R}^n)] \subseteq VMO(\mathbb{R}^n)$.

(c) $T_f[vmo(\mathbb{R}^n)] \subseteq vmo(\mathbb{R}^n)$.

(d) $T_f[cmo(\mathbb{R}^n)] \subseteq VMO(\mathbb{R}^n)$.

Furthermore, if any of the above properties is satisfied, then T_f maps bounded subsets of $VMO(\mathbb{R}^n)$ to bounded subsets of $VMO(\mathbb{R}^n)$, and bounded subsets of $vmo(\mathbb{R}^n)$ to bounded subsets of $vmo(\mathbb{R}^n)$.

In cases of *cmo* and CMO, we have the following nice conclusion, which can be deduced from Theorem 2 and from a continuity result for T_f (cf. Proposition 2 of Section 5.)

Corollary 1 *The following two statements hold.*

- *We have $T_f[cmo(\mathbb{R}^n)] \subseteq cmo(\mathbb{R}^n)$ if and only f is uniformly continuous and $f(0) = 0$.*
- *We have $T_f[CMO(\mathbb{R}^n)] \subseteq CMO(\mathbb{R}^n)$ if and only f is uniformly continuous.*

We now turn to discuss the continuity of the operator T_f. Brezis and Nirenberg [2, Lem. A.8, p. 238] have proved that if f is a uniformly continuous function, and if $\mathcal{M}$ is a compact Riemann manifold, then T_f is continuous from $BMO(\mathcal{M})$ to itself at all points of $VMO(\mathcal{M})$. By exploiting the same arguments, we can prove that T_f is continuous from $bmo(\mathbb{R}^n)$ to itself at all points of $vmo(\mathbb{R}^n)$, and that T_f is continuous from $BMO(\mathbb{R}^n)$ to itself at all points of $CMO(\mathbb{R}^n)$ (cf. Proposition 2 of Section 5.) With this respect, we observe that when $\mathcal{M}$ is compact, there is no difference between $CMO(\mathcal{M})$ and $VMO(\mathcal{M})$. Instead, $CMO(\mathbb{R}^n) \neq VMO(\mathbb{R}^n)$ and, as we shall see in Theorem 4, the uniform continuity of f does not suffice to guarantee the continuity of T_f at the points of $VMO(\mathbb{R}^n)$. By combining such continuity result with Theorem 2 and with Corollary 1, we obtain the following characterization.

Theorem 3 *The following two statements hold.*

(J) T_f *is continuous from $vmo(\mathbb{R}^n)$ to itself or from $CMO(\mathbb{R}^n)$ to itself if and only if f is uniformly continuous.*

(JJ) T_f *is continuous from $cmo(\mathbb{R}^n)$ to itself if and only if f is uniformly continuous and $f(0) = 0$.*

By Theorem 2, by Corollary 1, and by Theorem 3, we can immediately deduce the following characterization, inspired by the famous corresponding result for superposition operators acting in first order Sobolev spaces of Marcus and Mizel [9].

Corollary 2 *Let X be either $vmo(\mathbb{R}^n)$, or $cmo(\mathbb{R}^n)$, or $CMO(\mathbb{R}^n)$. Then the following properties are equivalent.*

(1) $T_f[X] \subseteq X$, i.e., T_f acts in X.

(2) T_f maps bounded subsets of X to bounded subsets of X.

(3) T_f is continuous from X to itself.

Very different instead, are the cases of $bmo(\mathbb{R}^n)$, $BMO(\mathbb{R}^n)$ and $VMO(\mathbb{R}^n)$. Brezis and Nirenberg [2, p. 240] have proved that even the Lipschitz continuous function $\max\{0,t\}$ does not generate a continuous superposition operator on $bmo(\mathbb{R}^n)$. A more complete picture is given by the following degeneracy result.

Theorem 4 *Let X be either $BMO(\mathbb{R}^n)$, or $VMO(\mathbb{R}^n)$, or $bmo(\mathbb{R}^n)$. Then T_f is continuous from X to $BMO(\mathbb{R}^n)$ if and only if f is $\mathbb{R}$-affine.*

We now turn to consider the differentiability of the operator T_f, and we present the following degeneracy result.

Theorem 5 *T_f is $\mathbb{R}$-differentiable from $\mathcal{D}(\mathbb{R}^n)$ endowed with the norm of $bmo(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$ if and only if f is $\mathbb{R}$-affine.*

This paper is organized as follows. In Section 2, we recall the definitions of BMO and of its subspaces. Sections 3 and 4 are devoted to the proofs of Theorems 1 and 2, respectively. Section 5 is devoted to the proof of the continuity statements and of Corollary 1, Section 6 is devoted to the proof of the statement concerning the differentiability. The last section is an Appendix, where we collect some technical facts, known in large part, which we exploit in the proofs.

2 Function spaces.

We recall that $BMO(\mathbb{R}^n)$ is the set of complex-valued locally integrable functions g on $\mathbb{R}^n$ such that

$$\|g\|_{BMO} := \sup_Q \fint_Q \left| g - \fint_Q g \right| < +\infty ,$$

where the supremum is taken on all cubes Q with sides parallel to the coordinate axes and where

$$\fint_Q g$$

denotes the mean value of the function g on Q. The quotient space of $BMO(\mathbb{R}^n)$ with the above seminorm over the constant functions is a Banach space. Since the operator T_f is clearly not defined on the quotient space, we prefer to consider $BMO(\mathbb{R}^n)$ as a Banach space of 'true' functions with the following norm:

$$\|g\|_* := \|g\|_{BMO} + \fint_{Q_0} |g| \qquad \forall g \in BMO(\mathbb{R}^n),$$

where Q_0 is the unit cube $[-1/2, +1/2]^n$. We denote by $bmo(\mathbb{R}^n)$ the linear subspace of $BMO(\mathbb{R}^n)$ consisting of those functions g which satisfy also the following condition

$$\sup_{|Q|\geq 1} \fint_Q |g| < +\infty ,$$

where $|Q|$ denotes the Lebesgue measure of Q or, equivalently,

$$\sup_{|Q|=1} \fint_Q |g| < +\infty$$

(cf. Lemma 7 of the Appendix.) It turns out that $bmo(\mathbb{R}^n)$ is a Banach space for the norm

$$\|g\|_{bmo} := \|g\|_{BMO} + \sup_{|Q|=1} \fint_Q |g| \qquad \forall g \in bmo(\mathbb{R}^n).$$

We denote by $cmo(\mathbb{R}^n)$ the closure of the set $\mathcal{D}(\mathbb{R}^n)$ of the C^∞ functions with compact support in $bmo(\mathbb{R}^n)$, and we endow $cmo(\mathbb{R}^n)$ with the norm of $bmo(\mathbb{R}^n)$. Similarly, we denote by $CMO(\mathbb{R}^n)$ the closure of $\mathcal{D}(\mathbb{R}^n)$ in $BMO(\mathbb{R}^n)$, and we endow $CMO(\mathbb{R}^n)$ with the norm of $BMO(\mathbb{R}^n)$.

According to Sarason [10], a function g of $BMO(\mathbb{R}^n)$ which satisfies the limiting condition

$$\lim_{a\to 0} \left(\sup_{|Q|\le a} \fint_Q \left| g - \fint_Q g \right| \right) = 0 \tag{1}$$

is said to be of vanishing mean oscillation. The subspace of $BMO(\mathbb{R}^n)$ consisting of the functions of vanishing mean oscillation is denoted $VMO(\mathbb{R}^n)$, and we endow $VMO(\mathbb{R}^n)$ with the norm of $BMO(\mathbb{R}^n)$. We note that the space $VMO(\mathbb{R}^n)$ considered by Coifman and Weiss [4] is different from that considered by Sarason, and it coincides with our $CMO(\mathbb{R}^n)$. As it is well known, $VMO(\mathbb{R}^n) \subsetneqq BMO(\mathbb{R}^n)$. For example, the function $\log|x|$ belongs to $BMO(\mathbb{R}^n)$, but not to $VMO(\mathbb{R}^n)$ (cf. *e.g.*, Stein [12, Ch. IV, §. I.1.2], and Brezis and Nirenberg [2, p. 211].) We set

$$vmo(\mathbb{R}^n) := VMO(\mathbb{R}^n) \cap bmo(\mathbb{R}^n),$$

and we endow the space $vmo(\mathbb{R}^n)$ with the norm of $bmo(\mathbb{R}^n)$.

For the convenience of the reader, we display all the subspaces of $BMO(\mathbb{R}^n)$ we have introduced in the following diagram:

$$\begin{array}{ccc} bmo(\mathbb{R}^n) & \subsetneqq & BMO(\mathbb{R}^n) \\ \cup \neq & & \cup \neq \\ vmo(\mathbb{R}^n) & \subsetneqq & VMO(\mathbb{R}^n) \\ \cup \neq & & \cup \neq \\ cmo(\mathbb{R}^n) & \subsetneqq & CMO(\mathbb{R}^n) \end{array}$$

where all inclusions are proper and continuous.

3 Proof of Theorem 1.

3.1 Alternative formulations of condition (i).

Proposition 1 *The condition (i) of Theorem 1 is equivalent to each of the following properties.*

- *(j) There exist two constants $\alpha > 0$ and $C > 0$ such that $|f(x) - f(y)| \le C$, for all complex numbers x, y satisfying inequality $|x - y| \le \alpha$.*
- *(k) f is the sum of a bounded Borel measurable function and of a Lipschitz continuous function.*

Proof. Obviously, condition (k) implies condition (i), and condition (i) implies condition (j). By a standard argument, condition (i) follows by condition (j). By Lemma 6 of the Appendix, condition (k) follows by condition (i).

3.2 Condition (i) implies conditions (ii), (iii) and (iv).

By Proposition 1, it suffices to consider separately, the case in which f is Lipschitz continuous, and the case in which f is bounded.

Assume first that f is Lipschitz continuous, with Lipschitz constant denoted $\mathrm{Lip}(f)$. Then we have

$$⨍_Q \left| f \circ g - f\left(⨍_Q g\right)\right| \leq \mathrm{Lip}(f)\|g\|_{BMO}$$

and

$$⨍_Q |f \circ g| \leq ⨍_Q |f \circ g - f(0)| + |f(0)| \leq \mathrm{Lip}(f)\left(⨍_Q |g|\right) + |f(0)|\,,$$

for all $g \in BMO(\mathbb{R}^n)$ and for all cubes Q. By inequality (21) of the Appendix, we obtain

$$\|f \circ g\|_{BMO} \leq 2\,\mathrm{Lip}(f)\|g\|_{BMO}\,, \tag{2}$$

$$\|f \circ g\|_* \leq 2\,\mathrm{Lip}(f)\|g\|_* + |f(0)|\,,$$

$$\|f \circ g\|_{bmo} \leq 2\,\mathrm{Lip}(f)\|g\|_{bmo} + |f(0)|\,.$$

Assume now that f is bounded. Then T_f takes $BMO(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$, a subspace of $bmo(\mathbb{R}^n)$.

3.3 Condition (iv) of Theorem 1 implies condition (j) of Proposition 1.

As customary in this type of problems (cf. *e.g.*, Katznelson [8, ch. VIII, § 8.3]), we first prove that the acting condition of T_f implies a property of local boundedness on bounded sets for T_f.

Lemma 1 *If conditions $T_f[cmo(\mathbb{R}^n)] \subseteq BMO(\mathbb{R}^n)$ and $f(0) = 0$ hold, then there exist a cube Q and two constants C_1, $C_2 > 0$ such that $\|f \circ g\|_* \leq C_2$ for any $g \in cmo(\mathbb{R}^n)$ with $\operatorname{supp} g \subseteq Q$ and $\|g\|_{bmo} \leq C_1$.*

Proof. We argue by contradiction. We assume that for any cube Q and for any positive numbers C_1, C_2, there exists $g \in cmo(\mathbb{R}^n)$ with $\operatorname{supp} g \subseteq Q$, $\|g\|_{bmo} \leq C_1$ and $\|f \circ g\|_* > C_2$. Let $(Q_j)_{j \geq 1}$ be a sequence of disjoint cubes. Let $\widetilde{Q}_j$ be the cube with the same center as that of Q_j, and with sidelength equal to one half of that of Q_j. Let $\phi_j \in \mathcal{D}(\mathbb{R}^n)$ be such that $\phi_j(x) = 1$ on $\widetilde{Q}_j$ and $\phi_j(x) = 0$ out of Q_j. According to Lemma 11 of the Appendix, there exists $\gamma_j > 0$ such that

$$\|g\phi_j\|_* \leq \gamma_j \|g\|_*\,, \tag{3}$$

for all $g \in BMO(\mathbb{R}^n)$. By the contradiction assumption, there exist functions $g_j \in cmo(\mathbb{R}^n)$ such that

$$\operatorname{supp} g_j \subseteq \widetilde{Q}_j\,, \quad \|g_j\|_{bmo} \leq 2^{-j}\,, \quad \|f \circ g_j\|_* > j\gamma_j\,.$$

Now we set $g := \sum_{j=1}^{\infty} g_j$. Then $g \in cmo(\mathbb{R}^n)$. Moreover, since

$$\sum_{j=1}^{\infty} \int_Q |g_j| \leq \sum_{j=1}^{\infty} \|g_j\|_{bmo} < \infty\,,$$

for all unit cubes Q of $\mathbb{R}^n$, then

$$\sum_{j=1}^{\infty} |g_j(x)| < +\infty \quad \text{a.e. in } \mathbb{R}^n\,.$$

Thus we also have

$$g(x) = \sum_{j=1}^{\infty} g_j(x) \quad \text{a.e. in } \mathbb{R}^n\,.$$

Then by condition $f(0) = 0$, we deduce that

$$(f \circ g)\phi_j = f \circ g_j \quad \text{a.e. in } \mathbb{R}^n\,.$$

By assumption, we have $f \circ g \in BMO(\mathbb{R}^n)$. Then inequality (3) implies that

$$j\gamma_j \leq \gamma_j \|f \circ g\|_* \quad \forall j \geq 1\,,$$

a contradiction. ■

We now prove the following Lemma, which we also employ in the rest of the paper, and which is inspired by an argument of Bourdaud [1].

Lemma 2 *Assume that there exist constants* $c_1 > 0$, $c_2 > 0$, $c_3 \geq 0$, *and a cube* K *such that*

$$\sup_{|Q|<c_2} ⨍_Q \left| f \circ g - \left(⨍_Q f \circ g \right) \right| \leq c_3, \tag{4}$$

whenever $g \in \mathcal{D}(\mathbb{R}^n)$ *and* $\|g\|_{bmo} \leq c_1$, $\operatorname{supp} g \subseteq K$, *then there exists a constant* $k > 0$ *depending only on the cube* K *such that*

$$\sup \left\{ |f(a) - f(b)| : \, a, b \in \mathbb{C}, \ |a - b| \leq kc_1 \right\} \leq 4^{n+1} c_3\,. \tag{5}$$

Proof. By translation invariance of the norm in $bmo(\mathbb{R}^n)$ and of the supremum in (4), and by Lemma 12 of the Appendix, and by replacing c_1 and c_2 by $\alpha_1 c_1$ and $\alpha_2 c_2$, for some strictly positive constants α_1 and α_2 depending only on K, we can assume that $K = Q_0$. Then we take $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that $\phi = 1$ on $\frac{1}{2}Q_0$ and $\operatorname{supp}\phi \subseteq Q_0$, $0 \leq \phi \leq 1$. Let a, b be two complex numbers such that

$$|a - b| \leq \frac{\alpha_1 c_1}{6}. \tag{6}$$

According to Lemma 8 of the Appendix, there exist a function $\theta \in \mathcal{D}(\mathbb{R}^n)$ and an integer $j \geq 1$ such that $\operatorname{supp}\theta \subseteq Q_0$, $\theta = 1$ on the cube $2^{-j}Q_0$, $2^{-nj} \leq \alpha_2 c_2$, and

$$|a| \|\theta\|_{bmo} \leq \frac{\alpha_1 c_1}{2}. \tag{7}$$

Now we set

$$g(x) = (b - a)\phi(2^{j+1}x) + a\theta(x) \qquad \forall x \in \mathbb{R}^n.$$

Clearly, $g \in \mathcal{D}(\mathbb{R}^n)$ and $\operatorname{supp} g \subseteq Q_0$. Then by the inequalities (6) and (7), by the boundedness of ϕ, and by inequality $\|\cdot\|_{bmo} \leq 3\|\cdot\|_\infty$, we have

$$\|g\|_{bmo} \leq \alpha_1 c_1.$$

Thus by our assumption, we have

$$⨍_{2^{-j}Q_0} \left| f \circ g - \left(⨍_{2^{-j}Q_0} f \circ g \right) \right| \leq c_3.$$

Clearly, $f(g(x)) = f(b)$ on $2^{-j-2}Q_0$, and $f(g(x)) = f(a)$ on $2^{-j}Q_0 \setminus 2^{-j-1}Q_0$. Thus we obtain

$$|f(b) - f(a)| \leq \\ \left| f(b) - \left(⨍_{2^{-j}Q_0} f \circ g \right) \right| + \left| f(a) - \left(⨍_{2^{-j}Q_0} f \circ g \right) \right| \leq c_3 4^{n+1},$$

and we can take $k = \alpha_1/6$. ■

Next we assume that $T_f[cmo(\mathbb{R}^n)] \subseteq BMO(\mathbb{R}^n)$. By possibly subtracting $f(0)$, we can assume that $f(0) = 0$. Then condition (j) holds by Lemma 1 and by Lemma 2.

4 Proof of Theorem 2 .

Brezis and Nirenberg [2, Lem. A.7, p. 238] have proved that condition (b) follows by condition (a). Solely for the sake of completeness, we report here their proof.

We say that a function ω of $[0,\infty[$ to itself is *a modulus of continuity* for the function f provided that

$$|f(x) - f(y)| \leq \omega(|x - y|) \qquad \forall x, y \in \mathbb{C}, \qquad \lim_{t\to 0} \omega(t) = 0\,, \tag{8}$$

Now let f be a uniformly continuous function. As it is well known, there exists a concave increasing modulus of continuity ω for f (cf. *e.g.*, DeVore and Lorentz [5, Lem. 6.1, p. 43].) Thus by Jensen's inequality and by inequality (22) of the Appendix, we have

$$\begin{aligned} &⨍_Q \left| f \circ g - \left(⨍_Q f \circ g\right)\right| \leq \\ &\leq \omega\left(⨍_Q ⨍_Q |g(x) - g(y)|\, dxdy\right) \leq \omega\left(2⨍_Q \left|g - \left(⨍_Q g\right)\right|\right) \end{aligned} \tag{9}$$

for all cubes Q, and for all $g \in BMO(\mathbb{R}^n)$. Inequality (9) implies the validity of condition (b). Since condition (b) implies condition (iv) of Theorem 1, then, by Theorem 1, condition (b) implies condition (c). Since condition (d) clearly follows by condition (c), it remains to prove that condition (d) implies the uniform continuity of f.

4.1 Condition (d) implies condition (a).

We need the following technical lemma.

Lemma 3 *If conditions $T_f[cmo(\mathbb{R}^n)] \subseteq VMO(\mathbb{R}^n)$ and $f(0) = 0$ hold, then for every $\varepsilon > 0$, there exist a cube K contained in the cube Q_0, and two constants $c_1 > 0$, $c_2 > 0$ such that*

$$⨍_Q \left| f \circ g - \left(⨍_Q f \circ g\right)\right| \leq \varepsilon,$$

for all $g \in cmo(\mathbb{R}^n)$ with $\operatorname{supp} g \subseteq K$, $\|g\|_{bmo} \leq c_1$, and for all cubes Q with $|Q| \leq c_2$.

Proof. By contradiction, we assume that there exists $\bar\varepsilon > 0$ such that for any cube K contained in $K_0 := Q_0$, and for all positive numbers $c_1 > 0$, $c_2 > 0$, there exist $g \in cmo(\mathbb{R}^n)$ with support in K, $\|g\|_{bmo} \leq c_1$, and $|Q| \leq c_2$ such that

$$\bar\varepsilon \leq ⨍_Q \left| f \circ g - \left(⨍_Q f \circ g\right)\right|.$$

We now define a family of disjoint cubes contained in K_0. Namely, we take

$$K_j := 2^{-1}(j+1)^{-2} K_0 + j^{-1}\mathbf{e}_1\,,$$

for j natural, $j \geq 3$, $\mathbf{e}_1 := (1, 0, \ldots, 0)$. Now let $\phi \in \mathcal{D}(\mathbb{R}^n)$, with $\phi = 1$ on $\frac{1}{2}K_0$, and with $\operatorname{supp}\phi \subseteq K_0$, $\phi_j(x) := \phi\left(2(j+1)^2(x - j^{-1}\mathbf{e}_1)\right)$. Clearly, $\|\nabla\phi_j\|_\infty = 2(j+1)^2\|\nabla\phi\|_\infty$. By our contradiction assumption, there exist functions $g_j \in cmo(\mathbb{R}^n)$ and cubes Q_j such that $\operatorname{supp} g_j \subseteq K'_j := 2^{-2}(j+1)^{-2}K_0 + j^{-1}\mathbf{e}_1$, $\|g_j\|_{bmo} \leq 2^{-j}$, $|Q_j| \leq 2^{-jn}$,

$$\bar{\varepsilon} \leq ⨍_{Q_j} \left| f \circ g_j - \left(⨍_{Q_j} f \circ g_j \right) \right| .$$

Since g_j vanishes outside K_j and $f(0) = 0$, we have $Q_j \cap K_j \neq \emptyset$, and thus $Q_j \subseteq K_0$ for $j \geq 3$. Now, we set $g := \sum_{j=3}^{\infty} g_j$. Then $g \in cmo(\mathbb{R}^n)$. Moreover, as in the proof of Lemma 1, we have $(f \circ g)\phi_j = f \circ g_j$. By assumption, we have $f \circ g \in VMO(\mathbb{R}^n)$. Thus by our contradiction assumption, by inequality $j \leq 2|\log |Q_j||$ and by Lemma 10 of the Appendix, we obtain

$$\bar{\varepsilon} \leq 2 \left[⨍_{Q_j} \left| f \circ g - \left(⨍_{Q_j} f \circ g \right) \right| \right] + \\ +2\sqrt{n}|Q_j|^{1/n} \left(1 + 2|\log |Q_j||\right)^2 \|\nabla \phi\|_\infty \left[C\|f \circ g\|_{BMO} \left(1 + |\log |Q_j||\right) + \left| ⨍_{K_0} f \circ g \right| \right] ,$$

for all $j \geq 3$. Then by letting j tend to infinity and by observing that $f \circ g \in VMO(\mathbb{R}^n)$, we obtain a contradiction. ■

Next we assume that $T_f[cmo(\mathbb{R}^n)] \subseteq VMO(\mathbb{R}^n)$. By possibly subtracting $f(0)$, we can assume that $f(0) = 0$. Then by Lemma 3 and by Lemma 2, the function f is uniformly continuous.

5 Proof of the continuity statements for T_f.

We first introduce a continuity statement for T_f, which we prove by an argument of Brezis and Nirenberg.

Proposition 2 *Let f be uniformly continuous. If $g \in vmo(\mathbb{R}^n)$, then T_f is continuous at g as a map of $bmo(\mathbb{R}^n)$ to itself. If $g \in CMO(\mathbb{R}^n)$, then T_f is continuous at g as a map of $BMO(\mathbb{R}^n)$ to itself.*

Proof. The proof is based on an inequality which we present in the following Lemma.

Lemma 4 *If f has a concave increasing modulus of continuity ω as in (8), then we have*

$$⨍_Q |f \circ (g+v) - f \circ g - ⨍_Q (f \circ (g+v) - f \circ g)| \\ \leq \min \left(2\omega(2⨍_Q |g - ⨍_Q g|) + \omega(2⨍_Q |v - ⨍_Q v|), 2\omega(⨍_Q |v|) \right) ,$$

for all locally integrable functions g and v on $\mathbb{R}^n$, and for all cubes Q.

Proof. The left hand side of the above inequality is less than or equal to

$$I := ⨍_Q ⨍_Q |f(g(x) + v(x)) - f(g(x)) - f(g(y) + v(y)) + f(g(y))| \, dx \, dy .$$

Then we have

$$I \leq ⨍_Q ⨍_Q (|f(g(x) + v(x)) - f(g(x) + v(y))| + |f(g(x)) - f(g(y))| + \\ +|f(g(x) + v(y)) - f(g(y) + v(y))|) \, dx \, dy \leq \\ \leq \omega \left(⨍_Q ⨍_Q |v(x) - v(y)| \, dx \, dy \right) + 2\omega \left(⨍_Q ⨍_Q |g(x) - g(y)| \, dx \, dy \right) \leq$$

$$\leq \omega\left(2⨍_Q|v-⨍_Q v|\right)+2\omega\left(2⨍_Q|g-⨍_Q g|\right).$$

On the other hand

$$I \leq ⨍_Q ⨍_Q \left(|f(g(x)+v(x))-f(g(x))|+|f(g(y)+v(y))-f(g(y))|\right) dx\,dy \leq$$

$$\leq 2\omega\left(⨍_Q ⨍_Q |v(x)|\,dx\,dy\right)=2\omega\left(⨍_Q|v|\right).$$

Thus the proof of the Lemma is complete. ■

We now return to the proof of Proposition 2. We find it convenient to introduce some notation. If Q is a cube with center a, and sidelength $r>0$, then we set $\tau(Q):=|a|+r$,

$$W_R := \sup_{\tau(Q)\geq R} ⨍_Q \left|g-⨍_Q g\right| \quad \text{and} \quad M_c := \sup_{|Q|\leq c} ⨍_Q \left|g-⨍_Q g\right|.$$

Furthermore, for any function $v\in BMO(\mathbb{R}^n)$, we set

$$I_Q(v) := ⨍_Q \left| f\circ(g+v)-f\circ g-⨍_Q\left(f\circ(g+v)-f\circ g\right)\right|.$$

Let ω be a concave increasing modulus of continuity for f.

Let $g\in vmo(\mathbb{R}^n)$ and $\varepsilon>0$. By definition of $vmo(\mathbb{R}^n)$, there exists $0<c\leq 2^{-1}$ such that

$$\omega(2M_c)\leq\varepsilon. \tag{10}$$

Then we can take $\eta>0$ such that

$$\omega(\eta/c)\leq\varepsilon.$$

Now let $v\in bmo(\mathbb{R}^n)$ with $\|v\|_{bmo}\leq\eta$. Let Q be a cube. If $|Q|\leq c$, then by Lemma 4 and by (10), we have

$$I_Q(v)\leq 2\varepsilon+\omega(2\|v\|_{bmo})\leq 3\varepsilon.$$

If $c<|Q|\leq 1$, we have

$$⨍_Q|v|\leq c^{-1}\|v\|_{bmo}$$

and thus

$$I_Q(v)\leq 2\omega\left(c^{-1}\|v\|_{bmo}\right)\leq 2\varepsilon.$$

Moreover, if $|Q|=1$, then

$$⨍_Q|f\circ(g+v)-f\circ g|\leq\omega\left(⨍_Q|v|\right)\leq\omega\left(\|v\|_{bmo}\right).$$

Finally, we obtain

$$\sup_{|Q|\leq 1} I_Q(v)+\sup_{|Q|=1}⨍_Q|f\circ(g+v)-f\circ g|\leq 4\varepsilon,$$

for all $\|v\|_{bmo}\leq\eta$. Then by Lemma 7 of the Appendix, the operator T_f is continuous from $bmo(\mathbb{R}^n)$ to itself at g.

Now we assume that $g\in CMO(\mathbb{R}^n)$. Again, we choose $0<c\leq 2^{-1}$ such that (10) holds. By Lemma 15 of the Appendix, there exists some $R\geq 1$ such that

$$\omega(2W_R)\leq\varepsilon. \tag{11}$$

By applying Lemma 9 of the Appendix to $|v|$, there exists a constant $C(n,c,R)\geq 2$, such that

$$⨍_Q|v|\leq C(n,c,R)\|v\|_*,$$

for all $v \in BMO(\mathbb{R}^n)$, and for all cubes Q such that $|Q| > c$ and $\tau(Q) < R$. Then we choose $\eta > 0$ such that $\omega(\eta C(n, c, R)) \leq \varepsilon$ and $\omega(\eta/c) \leq \epsilon$.

Now let $v \in BMO(\mathbb{R}^n)$ such that $\|v\|_* \leq \eta$. If $|Q| \leq c$ or if $\tau(Q) \geq R$, then by (10), and by (11), and by Lemma 4, we have $I_Q(v) \leq 3\varepsilon$. If $|Q| > c$ and $\tau(Q) < R$, we have $I_Q(v) \leq 2\omega(\|v\|_* C(n, c, R)) \leq 2\varepsilon$. We conclude that $\sup_Q I_Q(v) \leq 3\varepsilon$. Moreover,

$$⨍_{Q_0} |f \circ (g + v) - f \circ g| \leq \omega \left(⨍_{Q_0} |v| \right) \leq \varepsilon,$$

and thus the proof of Proposition 2 is complete. ■

5.1 Proof of Corollary 1.

If T_f acts in $cmo(\mathbb{R}^n)$ or in $CMO(\mathbb{R}^n)$, then $T_f[cmo(\mathbb{R}^n)] \subseteq VMO(\mathbb{R}^n)$ and, by Theorem 2, f is uniformly continuous. If $T_f[cmo(\mathbb{R}^n)] \subseteq cmo(\mathbb{R}^n)$, then the constant function $f(0) = T_f[0]$ belongs to $cmo(\mathbb{R}^n)$. Then by Lemma 13 of the Appendix, we have $f(0) = 0$.

Now assume that f is uniformly continuous. By Theorem 2 and by Proposition 2, we know that T_f is continuous from $CMO(\mathbb{R}^n)$ to $VMO(\mathbb{R}^n)$, and from $cmo(\mathbb{R}^n)$ to $vmo(\mathbb{R}^n)$. Thus, it suffices to prove the following two inclusions.

$$T_f[\mathcal{D}(\mathbb{R}^n)] \subseteq CMO(\mathbb{R}^n)\,, \tag{12}$$

$$T_f[\mathcal{D}(\mathbb{R}^n)] \subseteq cmo(\mathbb{R}^n) \quad \text{if } f(0) = 0\,. \tag{13}$$

If $f(0) = 0$, then $T_f[\mathcal{D}(\mathbb{R}^n)]$ is included in the space $C_c(\mathbb{R}^n)$ of continuous functions with compact support. Since any such function is a uniform limit of functions of $\mathcal{D}(\mathbb{R}^n)$, we obtain $C_c(\mathbb{R}^n) \subseteq cmo(\mathbb{R}^n)$. Thus the proof of (13) is complete. If $f(0) \neq 0$, we apply (13) to the function $f - f(0)$. Then, for all $g \in \mathcal{D}(\mathbb{R}^n)$, we have $f \circ g - f(0) \in cmo(\mathbb{R}^n)$. By Lemma 14 of the Appendix, all constant functions belong to $CMO(\mathbb{R}^n)$. Thus we obtain $f \circ g \in CMO(\mathbb{R}^n)$, for all $g \in \mathcal{D}(\mathbb{R}^n)$. ■

5.2 Proof of Theorem 3.

Statement (J) is an immediate consequence of Theorem 2, of Corollary 1 and of Proposition 2. By definition of $cmo(\mathbb{R}^n)$, statement (JJ) is an immediate consequence of statement (J), and of Corollary 1. ■

5.3 Proof of Theorem 4.

We first introduce the following preliminary Lemma.

Lemma 5 *If the superposition operator T_f of the space $\mathcal{D}(\mathbb{R}^n)$ endowed with the norm $\|\cdot\|_{bmo}$ to $BMO(\mathbb{R}^n)$ is continuous at the constant function 0, then f is uniformly continuous.*

Proof. By possibly subtracting $f(0)$ from f, we can assume that $f(0) = 0$. Accordingly, $T_f[0] = 0$. Let $\varepsilon > 0$ be arbitrary. By continuity of T_f at 0, there exists $r > 0$ such that $\|f \circ g\|_* \leq \varepsilon$ if $g \in \mathcal{D}(\mathbb{R}^n)$ and if $\|g\|_{bmo} \leq r$. Then by Lemma 2, we conclude that f is uniformly continuous. ■

We are now ready to prove Theorem 4. As usual, we can assume that $f(0) = 0$. Let α, β be two arbitrary complex numbers.

First we assume that T_f is continuous from $bmo(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$. By Lemma 8 of the Appendix, there exists a sequence $(\theta_j)_{j\geq 1}$ of functions such that $\theta_j(x) = 1$ on the cube $K_j = [-j^{-1}, j^{-1}]^n$, and $\lim_{j\to\infty} \|\theta_j\|_{bmo} = 0$. Let γ denote the characteristic function of $[0,1]^n$. Clearly,

$$\begin{aligned} c_j &:= ⨍_{K_j}(f\circ(\beta\gamma+\alpha) - f\circ(\beta\gamma)) \\ &= 2^{-n}j^n\left[\int_{[0,j^{-1}]^n} (f(\beta\gamma(x)+\alpha) - f(\beta\gamma(x)))\,dx \right. \\ &\qquad \left. + \int_{K_j\setminus[0,j^{-1}]^n} (f(\beta\gamma(x)+\alpha) - f(\beta\gamma(x)))\,dx\right] \\ &= 2^{-n}(f(\beta+\alpha) - f(\beta)) + f(\alpha)(1-2^{-n})\,. \end{aligned}$$

Then we have

$$\begin{aligned} \|T_f[\beta\gamma+\alpha\theta_j] - T_f[\beta\gamma]\|_{BMO} &\geq 2^{-n}j^n\int_{[0,j^{-1}]^n} |f\circ(\beta\gamma+\alpha) - f\circ(\beta\gamma) - c_j| \\ &= 2^{-n}\,|f(\beta+\alpha) - f(\beta) - c_j| \\ &= 2^{-n}(1-2^{-n})\,|f(\beta+\alpha) - f(\beta) - f(\alpha)|\,. \end{aligned}$$

By taking the limit as j tends to infinity, we obtain

$$f(\alpha+\beta) = f(\alpha) + f(\beta) \qquad \forall \alpha\,,\beta\in\mathbb{C}\,.$$

Then by the continuity of f, which follows from Lemma 5, and by a classical argument, we can easily deduce that f is $\mathbb{R}$-linear.

We now assume that T_f is continuous from $VMO(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$. Again, Lemma 5 implies the continuity of f. Let M be a sufficiently large positive constant. Let K_j, K_j', K_j'' be the cubes of center $a_j = 2M4^j\mathbf{e}_1$ and halfsidelength $2^j, 2^j+1$, and 2^{j+1}, respectively. We note that

$$|K_j'\setminus K_j| = O(2^{j(n-1)}) \quad \text{as} \quad j\to+\infty \tag{15}$$

and that the cubes K_j'' are pairwise disjoint. Let $(\phi_j)_{j\geq 1}$ be a sequence of functions of $\mathcal{D}(\mathbb{R}^n)$ such that

$$\phi_j(x) = 1 \quad \text{for}\, x\in[-1,1]^n\,, \quad \phi_j(x) = 0 \quad \text{for}\, x\notin[-1-2^{-j}, 1+2^{-j}]^n$$

and

$$|\phi_j| \leq 2, \qquad \sup_{j\geq 1} 2^{-j}\|\nabla\phi_j\|_\infty < +\infty\,. \tag{16}$$

We define the function g by setting

$$g(x) = \phi_j\left(\frac{x-a_j}{2^j}\right) \qquad \text{if } x\in K_j'' \text{ for some } j\geq 1,$$

and $g(x) = 0$ elsewhere. From (16) we deduce that g and ∇g are bounded. Hence $g\in VMO(\mathbb{R}^n)$. Let $(\psi_j)_{j\geq 1}$ be the sequence of functions introduced in Lemma 8 of the

Appendix. Let $u_j(x) := \psi_j\left(M^{-1}(x-a_j)\right)$. Then $u_j \in \mathcal{D}(\mathbb{R}^n)$, $\|u_j\|_{BMO} = \|\psi_j\|_{BMO}$ and $u_j(x) = 0$ on Q_0, for j sufficiently large. Thus we have

$$\lim_{j\to+\infty} \|u_j\|_* = 0$$

and $u_j(x) = 1$ on the cube K''_j. We now set

$$c_j := ⨍_{K''_j} (f\circ(\beta g+\alpha u_j) - f\circ(\beta g))\,.$$

Clearly,

$$c_j = \frac{1}{|K''_j|}\left(|K_j|(f(\beta+\alpha)-f(\beta)) + |K''_j\setminus K'_j| f(\alpha) + A_j\right)\ ,$$

where $A_j = \int_{K'_j\setminus K_j}(f\circ(\beta g+\alpha u_j) - f\circ(\beta g))$. By (15) and by the uniform continuity of f, we deduce that $A_j = O(2^{j(n-1)})$. Moreover,

$$|K''_j\setminus K'_j| = (2^n-1)|K_j| - |K'_j\setminus K_j|\,.$$

Hence

$$c_j = 2^{-n}(f(\beta+\alpha)-f(\beta)) + (1-2^{-n})f(\alpha) + \varepsilon_j\ ,$$

with $\lim_{j\to\infty}\varepsilon_j = 0$. Then we have

$$\|T_f[\beta g+\alpha u_j] - T_f[\beta g]\|_{BMO} \geq \frac{1}{|K''_j|}\int_{K_j} |f\circ(\beta g+\alpha u_j) - f\circ(\beta g) - c_j| =$$

$$= 2^{-n}\left|(1-2^{-n})(f(\beta+\alpha)-f(\beta)-f(\alpha)) - \varepsilon_j\right|\ .$$

Thus by taking the limit as $j\to+\infty$, we obtain $f(\beta+\alpha) = f(\beta)+f(\alpha)$. ■

5.4 Open questions.

We end this section by mentioning some open problems concerning the continuity of T_f.

1. By Theorem 4, there are no nonlinear uniformly continuous function f for which T_f is continuous from the *whole* of $BMO(\mathbb{R}^n)$, or of $VMO(\mathbb{R}^n)$, or of $bmo(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$. However, we did not characterize the points of continuity of T_f.

2. Are there nonlinear functions f for which T_f is locally Hölder continuous on $vmo(\mathbb{R}^n)$, $cmo(\mathbb{R}^n)$ or $CMO(\mathbb{R}^n)$?

6 Proof of Theorem 5.

A function f of $\mathbb{C}$ to itself can be viewed as a function of two real variables, say y_1, y_2. As a first step, we prove that $\frac{\partial f}{\partial y_1}$ and $\frac{\partial f}{\partial y_2}$ exist. We consider for example $\frac{\partial f}{\partial y_1}$. Let $\phi\in\mathcal{D}(\mathbb{R}^n)$ be real valued and equal to one on Q_0. Since T_f is differentiable at $c\phi$ for all $c\in\mathbb{C}$, we have

$$\lim_{t\to 0} t^{-1}\left\{T_f[c\phi+t\phi] - T_f[c\phi]\right\} = dT_f[c\phi](\phi) \qquad \text{in } BMO(\mathbb{R}^n). \tag{17}$$

Since $BMO(\mathbb{R}^n)$ is continuously imbedded in the space of locally summable functions, we deduce that there exists a sequence $(j_k)_{k\geq 1}$ in $\mathbb{N}$ such that $\lim_{k\to\infty} j_k = \infty$ and

$$\lim_{k\to\infty} j_k\left\{f\circ(c\phi+j_k^{-1}\phi) - f\circ(c\phi)\right\} = dT_f[c\phi](\phi) \qquad \text{a.e. in } \mathbb{R}^n. \tag{18}$$

Since the argument of the limit in (18) is constant on Q_0 for each k, such limit must exist and have a constant value β_c for all $x \in Q_0$. Now let $(t_l)_{l\geq 1}$ be an arbitrary sequence in $\mathbb{R} \setminus \{0\}$ converging to 0. We show that an arbitrary subsequence of $(t_l)_{l\geq 1}$ has a subsequence $(t_{l_k})_{k\geq 1}$ such that $\lim_{k\to\infty} t_{l_k}^{-1} \left\{ f(c + t_{l_k}^{-1}) - f(c) \right\} = \beta_c$. Then the existence of $\frac{\partial f}{\partial y_1}(c) = \beta_c$ will follow by a standard argument. By (17), there exists a subsequence $(t_{l_k})_{k\geq 1}$ such that

$$\lim_{k\to\infty} t_{l_k}^{-1} \left\{ f \circ (c\phi + t_{l_k}\phi) - f \circ (c\phi) \right\} = dT_f[c\phi](\phi) \qquad \text{a.e. in } \mathbb{R}^n. \tag{19}$$

By arguing as above, such limit exists at all points of Q_0, and has a constant value β'_c. Moreover, $\beta'_c = dT_f[c\phi](\phi)$ a.e. in Q_0. Then we have $\beta_c = \beta'_c$. Thus we can conclude that $\frac{\partial f}{\partial y_1}(c)$ exists for all $c \in \mathbb{C}$. Now let u, $v \in \mathcal{D}(\mathbb{R}^n)$, $v_1 := \mathrm{Re}\, v$, $v_2 := \mathrm{Im}\, v$. Clearly,

$$dT_f[u](v_1) = \lim_{t\to 0} t^{-1} \left\{ f \circ (u + tv_1) - f \circ u \right\} = \left(\frac{\partial f}{\partial y_1} \circ u \right) v_1 \qquad \text{in } BMO(\mathbb{R}^n),$$
$$dT_f[u](iv_2) = \lim_{t\to 0} t^{-1} \left\{ f \circ (u + tiv_2) - f \circ u \right\} = \left(\frac{\partial f}{\partial y_2} \circ u \right) v_2 \qquad \text{in } BMO(\mathbb{R}^n).$$

Thus by $\mathbb{R}$-linearity of the differential $dT_f[u]$, we have

$$dT_f[u](v_1 + iv_2) = \left(\frac{\partial f}{\partial y_1} \circ u \right) v_1 + \left(\frac{\partial f}{\partial y_2} \circ u \right) v_2 \,.$$

If T_f is $\mathbb{R}$-differentiable at $u = 0$, then so is the function that takes $u = u_1 + iu_2$ to $T_f[u] - u_1 \frac{\partial f}{\partial y_1}(0) - u_2 \frac{\partial f}{\partial y_2}(0) - f(0)$. Thus there is no loss of generality in assuming that $f(0) = \frac{\partial f}{\partial y_1}(0) = \frac{\partial f}{\partial y_2}(0) = 0$. Now we set

$$\sigma(t) := \sup \left\{ \frac{\|T_f[u]\|_{BMO}}{\|u\|_{bmo}} : u \in \mathcal{D}(\mathbb{R}^n),\ 0 < \|u\|_{bmo} \leq t \right\} \qquad \forall t > 0.$$

Then by conditions $T_f[0] = 0$ and $dT_f[0] = 0$, we have

$$\lim_{t\to 0} \sigma(t) = 0. \tag{20}$$

Clearly, $\|T_f[u]\|_{BMO} \leq t\sigma(t)$ whenever $u \in \mathcal{D}(\mathbb{R}^n)$ and $\|u\|_{bmo} \leq t$. Thus by applying Lemma 2 with $K = Q_0$, we conclude that

$$|f(a) - f(b)| \leq 4^{n+1} k^{-1} |a - b| \sigma(k^{-1}|a - b|) \,,$$

if $|a - b|$ is sufficiently small, where k is the constant of Lemma 2. Thus (20) implies that f is differentiable, and that its differential is identically zero. ■

Remark. By Theorem 5, there are no nonlinear uniformly continuous functions f for which T_f is differentiable from the *whole* of $vmo(\mathbb{R}^n)$ to $vmo(\mathbb{R}^n)$ or to $VMO(\mathbb{R}^n)$. However, we did not characterize the points of differentiability of T_f.

7 Appendix.

For the convenience of the reader, we collect in this Appendix some known results and some more or less elementary facts.

Lemma 6 *Let h be a measurable function of $\mathbb{R}^n$ to $\mathbb{C}$ such that*

$$\sup_{x,y\in\mathbb{R}^n} (1+|x-y|)^{-1}|h(x)-h(y)| < +\infty\,.$$

Then h is the sum of a bounded measurable function and of a continuously differentiable function with bounded first order derivatives.

Proof. Let μ be a Radon measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} (1+|y|)\, d|\mu|(y) < +\infty\,,$$

and $\mu(\mathbb{R}^n)=0$. By assumption, we have

$$|h*\mu(x)| = |\int_{\mathbb{R}^n} (h(x-y)-h(x))\; d\mu(y)| \le C\int_{\mathbb{R}^n} (1+|y|)\; d|\mu|(y)\,.$$

Thus $h*\mu$ is a bounded measurable function. Let $\phi\in\mathcal{D}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n}\phi = 1$. By taking μ equal to $\delta-\phi\,dx$ and to $\partial_j\phi\,dx$, for $j=1,\dots,n$, we deduce that $h-h*\phi$ and $h*\partial_j\phi$ are bounded and measurable. Then, by a classical argument, we see that $h*\phi$ is a function of class C^1 with bounded gradient. ■

We now turn to more specific properties of BMO functions. First we note that if g is a locally summable function in $\mathbb{R}^n$ and if Q is a cube, then

$$⨍_Q \left|g-\left(⨍_Q g\right)\right| \le 2⨍_Q|g-c| \qquad \forall c\in\mathbb{C}\,, \tag{21}$$

and

$$⨍_Q|g-⨍_Q g| \le ⨍_Q⨍_Q|g(x)-g(y)|\,dx\,dy \le 2⨍_Q|g-⨍_Q g|\,. \tag{22}$$

Lemma 7 *A locally integrable function g on $\mathbb{R}^n$ belongs to $bmo(\mathbb{R}^n)$ if and only if*

$$\sup_{|Q|\le 1}⨍_Q\left|g-⨍_Q g\right| + \sup_{|Q|=1}⨍_Q|g| < +\infty\,,$$

and the above expression defines an equivalent norm on $bmo(\mathbb{R}^n)$.

Proof. If the cube K has sidelength equal to an integer $N\ge 1$, then K is the union of N^n nonoverlapping cubes K_j of sidelength equal to 1. Hence

$$⨍_K|g| = \frac{1}{N^n}\sum_j ⨍_{K_j}|g| \le \sup_{|Q|=1}⨍_Q|g|\,.$$

If the cube K has a noninteger sidelength $r>1$, then $K\subset K'$, where the sidelength of K' is $[r]+1$. Then we have

$$⨍_K|g| \le \frac{|K'|}{|K|}⨍_{K'}|g| \le 2^n \sup_{|Q|=1}⨍_Q|g|\,.$$

Finally, for a cube such that $|Q|>1$, we have

$$⨍_Q\left|g-⨍_Q g\right| \le 2⨍_Q|g|\,.$$

■

Lemma 8 *There exist two sequences $(\theta_j)_{j\geq 1}$ and $(\psi_j)_{j\geq 1}$ of functions of $\mathcal{D}(\mathbb{R}^n)$ such that*

- $\theta_j(x) = 1$ *for* $|x| \leq 2^{-j}$, $\theta_j(x) = 0$ *for* $|x| \geq 1$, $0 \leq \theta_j \leq 1$, *for all* $j \geq 1$, *and* $\lim_{j\to\infty} \|\theta_j\|_{bmo} = 0$.
- $\psi_j(x) = 1$ *for* $|x| \leq 2^j$, $\psi_j(x) = 0$ *for* $|x| \geq 4^j$, $0 \leq \psi_j \leq 1$, *for all* $j \geq 1$, *and* $\lim_{j\to\infty} \|\psi_j\|_{BMO} = 0$.

Proof. As we have pointed out in Section 2, the function $\log_2 |\cdot|$ belongs to $BMO(\mathbb{R}^n)$. Let α_n be its BMO-seminorm. Let $u \in C^\infty(\mathbb{R}^n)$ be such that $0 \leq u \leq 1$, and

$$u(t) = 1 \quad \text{for} \quad t \leq -1 \quad , u(t) = 0 \quad \text{for} \quad t \geq 0 \,.$$

Let θ_j and ψ_j be defined as follows.

$$\theta_j(x) = u\left(\frac{\log_2 |x|}{j}\right) \quad , \quad \psi_j(x) = u\left(\frac{\log_2 |x|}{j} - 2\right) .$$

By inequality (2), we have

$$\|\theta_j\|_{BMO} \leq 2j^{-1}\alpha_n \|u'\|_\infty \quad , \quad \|\psi_j\|_{BMO} \leq 2j^{-1}\alpha_n \|u'\|_\infty \,.$$

Moreover, if Q is a unit cube, we have

$$\int_Q |\theta_j(x)|\, dx \leq \int_{\mathbb{R}^n} u\left(\frac{\log_2 |x|}{j}\right) \leq j^{-1}\|u'\|_\infty \int_{|x|\leq 1} |\log_2 |x||\, dx \,.$$

Thus by Lemma 7, the sequences $(\theta_j)_{j\geq 1}$ and $(\psi_j)_{j\geq 1}$ have the required properties. ∎

Then we have the following Lemma, which can be proved as the corresponding statement for BMO functions on the unit circle (cf. *e.g.*, Stegenga [11].)

Lemma 9 *There exists a constant $C > 0$ depending only on n such that*

$$\left| ⨍_Q g - ⨍_{Q'} g \right| \leq C \left(1 + \left|\log \frac{|Q'|}{|Q|}\right|\right) \|g\|_{BMO} \,,$$

for all cubes Q, Q' with $Q \cap Q' \neq \emptyset$, and for all $g \in BMO(\mathbb{R}^n)$.

By Lemma 9, we can deduce the following.

Lemma 10 *There exists a constant $C > 0$, depending only on n, such that*

$$\begin{aligned} ⨍_Q \left| g\phi - \left(⨍_Q g\phi\right)\right| &\leq 2\|\phi\|_\infty \left(⨍_Q \left| g - \left(⨍_Q g\right)\right|\right) + \\ &+ \sqrt{n}|Q|^{1/n}\|\nabla\phi\|_\infty \left[C\|g\|_{BMO}\left(1 + \log\frac{|Q'|}{|Q|}\right) + \left|⨍_{Q'} g\right|\right] \end{aligned}$$

for all cubes Q, Q' with $Q \subseteq Q'$, for all $g \in BMO(\mathbb{R}^n)$, and for all bounded Lipschitz continuous functions ϕ of $\mathbb{R}^n$ to $\mathbb{C}$.

Proof. Let a be the center of the cube Q. By inequality (21), we have

$$\begin{aligned} ⨍_Q \left| g\phi - \left(⨍_Q g\phi\right)\right| &\leq 2⨍_Q \left| g\phi - \left(⨍_Q g\right)\phi(a)\right| \leq \\ &\leq 2\|\phi\|_\infty \left(⨍_Q \left| g - ⨍_Q g\right|\right) + \left|⨍_Q g\right| \sqrt{n}|Q|^{1/n}\|\nabla\phi\|_\infty. \end{aligned}$$

Then the statement follows by Lemma 9. ∎

Lemma 11 *For each $\phi \in \mathcal{D}(\mathbb{R}^n)$, there exists a constant $M(\phi) > 0$, depending only on ϕ and n, such that*

$$\|g\phi\|_{bmo} \leq M(\phi)\, \|g\|_* \,, \tag{23}$$

for all $g \in BMO(\mathbb{R}^n)$.

Proof. We denote by M a constant depending solely on n and ϕ whose value may change from equation to equation. Let $R > 0$ be such that $\operatorname{supp} \phi \subseteq [-R, R]^n$. Let Q be any cube such that $|Q| \leq 1$ and $Q \cap \operatorname{supp} \phi \neq \emptyset$. Then we have

$$Q \subseteq Q_1 := [-2 - R, 2 + R]^n.$$

By applying Lemma 9 to $|g|$, to the unit cube Q_0 and to Q_1, we obtain

$$⨍_{Q_1} |g| \leq M\|g\|_*.$$

Then by Lemma 10 , with $Q' = Q_1$, we have

$$⨍_Q \left| g\phi - \left(⨍_Q g\phi \right) \right| \leq M\|g\|_*.$$

Moreover, if $|Q| = 1$, then

$$⨍_Q |g\phi| \leq \|\phi\|_\infty |Q_1| ⨍_{Q_1} |g| \leq M\|g\|_*.$$

Hence,

$$\sup_{|Q|\leq 1} ⨍_Q \left| g\phi - \left(⨍_Q g\phi \right) \right| + \sup_{|Q|=1} ⨍_Q |g\phi| \leq M\|g\|_* \,,$$

and Lemma 7 yields the conclusion. ■

Remark. Inequality (23) does not follow immediately from the known characterizations of the multiplier spaces for BMO and bmo (cf. Janson [7], Stegenga [11]) because of the specific type of norms employed in both hand sides of inequality (23).

Lemma 12 *There exists $c > 0$ depending only on n such that*

$$\|g(\lambda(\cdot))\|_{bmo} \leq c\|g\|_{bmo} \,,$$

for all $\lambda \geq 1$ and for all $g \in bmo(\mathbb{R}^n)$.

Proof. Since the BMO seminorm is invariant by dilations, it suffices to estimate the means on the cubes with sidelength equal to 1. If K is such a cube, we obtain

$$⨍_K |g(\lambda(\cdot))| = ⨍_{\lambda K} |g| \leq \sup_{|Q|\geq 1} ⨍_Q |g| \,.$$

By Lemma 7, $\sup_{|Q|\geq 1} ⨍_Q |g|$ can be estimated in terms of a constant multiple of $\|g\|_{bmo}$, and thus the proof is complete. ■

Lemma 13 *If $g \in cmo(\mathbb{R}^n)$, then*

$$\lim_{a\to\infty} \int_{Q_a} |g| = 0,$$

where Q_a denotes the unit cube in $\mathbb{R}^n$ with center a. In particular, if g is constant, then g is zero.

Proof. The seminorm N on $bmo(\mathbb{R}^n)$ defined by $N(g) := \limsup_{a\to\infty} \fint_{Q_a} |g|$ is easily seen to be continuous. Moreover, N has value zero on $\mathcal{D}(\mathbb{R}^n)$. Thus $N(g) = 0$ for all elements g of $cmo(\mathbb{R}^n)$. ■

Lemma 14 *Any constant function belongs to $CMO(\mathbb{R}^n)$.*

Proof. Let ψ_j be the functions of Lemma 8. We have $\psi_j = 1$ on the unit cube Q_0. Hence $\|1 - \psi_j\|_* = \|\psi_j\|_{BMO}$, which tends to 0 as j tends to infinity. ■

Lemma 15 *If $g \in CMO(\mathbb{R}^n)$, then we have*

$$\lim_{R\to\infty} \left(\sup_{\tau(Q)\geq R} \fint_Q \left| g - \fint_Q g \right| \right) = 0,$$

where $\tau(Q)$ denotes the sum $|a| + r$ of the modulus $|a|$ of the center a of Q, and of $r := |Q|^{1/n}$.

Proof. The seminorm N on $BMO(\mathbb{R}^n)$ defined by

$$N(g) := \lim_{R\to\infty} \left(\sup_{\tau(Q)\geq R} \fint_Q \left| g - \fint_Q g \right| \right)$$

is easily seen to be continuous. Moreover, N has value zero on $\mathcal{D}(\mathbb{R}^n)$. Thus $N(g) = 0$ for all elements g of $CMO(\mathbb{R}^n)$. ■

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