ANALYTICITY OF A NONLINEAR OPERATOR ASSOCIATED TO THE CONFORMAL REPRESENTATION OF A DOUBLY CONNECTED DOMAIN IN SCHAUDER SPACES

Massimo Lanza de Cristoforis
Dipartimento di Matematica Pura ed Applicata, Università di Padova
Via Belzoni 7, 35131 Padova, Italy.

Sergei V. Rogosin
Department of Mathematics and Mechanics, Belarusian State University
F. Skaryny av. 4, 220050 Minsk, Belarus.

We consider a suitably normalized Riemann map $g_{[\boldsymbol{\zeta}]}$ of the plane annulus $\mathbb{A}_{r[\boldsymbol{\zeta}]} \equiv \{z \in \mathbb{C} : r[\boldsymbol{\zeta}] < |z| < 1\}$, with $r[\boldsymbol{\zeta}] < 1$, to the plane doubly connected domain $\mathbb{A}[\boldsymbol{\zeta}]$ enclosed by the pair of Jordan curves $\boldsymbol{\zeta} \equiv (\zeta^i, \zeta^o)$, and we present a nonlinear singular integral equation approach to prove that the nonlinear operator which takes the pair of functions (ζ^i, ζ^o) to the triple of functions $\left(r^{-1}[\boldsymbol{\zeta}]g^{(-1)}_{[\boldsymbol{\zeta}]} \circ \zeta^i, g^{(-1)}_{[\boldsymbol{\zeta}]} \circ \zeta^o, r[\boldsymbol{\zeta}]\right)$ is real analytic in Schauder spaces.

Keywords: Conformal representation of doubly connected domains; Nonlinear singular integral equations; Integrals of Cauchy type; Nonlinear operators; Schauder spaces.

Classification Categories: AMS No. 30C20, 30E20, 45G05, 47H30

1. Introduction

As it is well-known (cf. *e.g.* Goluzin [10]), given a doubly connected domain in the complex plane, bounded by an inner simple closed curve ζ^i and by an outer simple closed curve ζ^o, there exist a unique $r[\boldsymbol{\zeta}] \in]0,1[$ depending on $\boldsymbol{\zeta} \equiv (\zeta^i, \zeta^o)$ and a unique holomorphic homeomorphism $g_{[\boldsymbol{\zeta}]}$ of the annulus

$$\mathbb{A}_{r[\boldsymbol{\zeta}]} \equiv \{z \in \mathbb{C} : r[\boldsymbol{\zeta}] < |z| < 1\}$$

onto the doubly connected annular domain $\mathbb{A}[\boldsymbol{\zeta}]$ enclosed by ζ^i, ζ^o, which satisfies a suitable normalizing condition which we specify later.

We prove that the nonlinear operator which takes the pair $\boldsymbol{\zeta} \equiv (\zeta^i, \zeta^o)$ to the triple of maps $\left(r^{-1}[\boldsymbol{\zeta}]g^{(-1)}_{[\boldsymbol{\zeta}]} \circ \zeta^i, g^{(-1)}_{[\boldsymbol{\zeta}]} \circ \zeta^o, r[\boldsymbol{\zeta}]\right) \equiv (h^i[\boldsymbol{\zeta}], h^o[\boldsymbol{\zeta}], r[\boldsymbol{\zeta}]) \equiv (\mathbf{h}[\boldsymbol{\zeta}], r[\boldsymbol{\zeta}])$ is real analytic in Schauder spaces. This problem is strictly related to that of the dependence of $g_{[\boldsymbol{\zeta}]}$ upon the pair of curves $\boldsymbol{\zeta}$, and arises in questions of hydrodynamics, of aerodynamics, and of composite materials. At the end of the paper, we briefly outline some potential applications of our analyticity theorem.

In connection to our present work on doubly connected domains, we mention the classical result of Radó [26], which asserts the continuity of the Riemann Map of a simply connected Jordan domain upon the boundary curve in the topology of the uniform convergence. For extensive references to the contributions of different authors to this question, we refer the reader to the monographs of Gaier [7], of Kantorovich and Krylov [13], and of Goluzin [10]. More recently, Coifman and Meyer [4] have proved the analyticity of an operator related to that considered in this paper for unbounded simply connected domains with boundary assigned with an arc-length parametrized curve with direction of the tangent vector prescribed by a function of class BMO. Wu [31], with the advice of Coifman, and with the ideas of Coifman and Meyer [4], has obtained two analyticity statements for arc-length parametrized Jordan domains which have certain symmetries and for curves which are close to a circle. Lanza [17] has shown, in the frame of Schauder spaces, the analyticity of an operator related to that of this paper for Jordan domains by exploiting a PDE approach, which views the simply connected Jordan domain as parametrized by a function of the unit disk of the complex plane to the complex plane. Lanza and Rogosin [20] have presented an integral equation approach to prove the analyticity of an operator related to that considered in Lanza [17]. To the best of the authors' knowledge however, the analytic dependence of operators as $\left(r^{-1}[\boldsymbol{\zeta}]g^{(-1)}_{[\boldsymbol{\zeta}]}\circ\zeta^{i}, g^{(-1)}_{[\boldsymbol{\zeta}]}\circ\zeta^{o}, r[\boldsymbol{\zeta}]\right)$ upon $\boldsymbol{\zeta}$ for *doubly* connected domains has not been treated.

We are developing here the approach proposed by Lanza and Rogosin [20] for the case of simply connected Jordan domains. While for simply connected domains, the problem can be reduced to study a nonlinear integral equation with singular kernel, for a doubly connected domain, we are led to derive and study a *system* of integral equations which contains both singular and nonsingular integrals. As a first step, we derive a system of integral equations involving $\boldsymbol{\zeta}$, $\mathbf{h}[\boldsymbol{\zeta}]$, $r[\boldsymbol{\zeta}]$, which generalizes to doubly connected domains the equation proposed in Lanza [16], Lanza and Rogosin [20], and we show that the set of solutions $(\boldsymbol{\zeta}, \mathbf{h}[\boldsymbol{\zeta}], r[\boldsymbol{\zeta}])$ of such system coincides with the graph of the function $\boldsymbol{\zeta} \mapsto (\mathbf{h}[\boldsymbol{\zeta}], r[\boldsymbol{\zeta}])$. At this point, it would be natural to try to deduce the smoothness of the solution set by applying the Implicit Function Theorem, but we observe that the corresponding linearized problem is not well-posed. Thus we introduce a modified system of equations, which we show to have the same solutions of the original system, and we deduce the analyticity of $\mathbf{h}[\cdot]$ and of $r[\cdot]$ by applying the Implicit Function Theorem to the modified system of equations.

Technically, we encounter mainly three difficulties. The first is inherent with obtaining a few results for conformal maps of annuli to doubly connected domains bounded by curves in Schauder spaces, which are known to hold for simply connected domains. The second difficulty is inherent with the regularity of the nonlinear operators involved in the modified system of equations, and we overcome it by employing a theorem of Lanza and Preciso [19], which can be considered as a Schauder space version for Jordan domains of the known result of Coifman and Meyer [4]. The third difficulty is connected with the unique solvability of the linearized problem for the modified system of equations. To overcome this difficulty, we use the theory of linear singular integral equations developped in Muskhelishvili [22], Gakhov [8], and in Gohberg and Krupnik [9], as well as the related theory of boundary value problems for holomorphic functions (cf. Gakhov [8].) In particular, we mention that the well-posedness of the linearized system associated to the modified system of equations has been established by a two step argument. As a first step, we prove the existence of a unique solution with a certain regularity and we represent such solution explicitly in terms of certain integral operators. Then by using such representation formulas and by exploiting certain properties of the involved integral operators, we show that the solution has in fact the required regularity.

2. Notation and Auxiliary Results

The inverse function of a function f is denoted by $f^{(-1)}$ as opposed to the reciprocal of a complex-valued function g, which is denoted by g^{-1}. We denote by $\mathbb{D}$ the open unit disk in $\mathbb{C}$, by $\mathbb{T}$ the boundary of $\mathbb{D}$, and by $\mathrm{cl}\,\mathbb{D}$ the closure on $\mathbb{D}$. For all $r \in]0,+\infty[$, $r\mathbb{T}$ denotes the circle $\{z \in \mathbb{C} : |z| = r\}$, and $r\mathbb{D}$ denotes the disk $\{z \in \mathbb{C} : |z| < r\}$. If $r \in]0,1[$, then $\mathbb{A}_r$ denotes the annulus $\{z \in \mathbb{C} : r < |z| < 1\}$. We denote by $\mathrm{id}_{\mathbb{T}}$ the identity map in $\mathbb{T}$. As customary, $\Re z$ and $\Im z$ stand for the real and imaginary part of $z \in \mathbb{C}$, respectively. For all open subsets $\mathbb{S}$ of $\mathbb{C}$, we denote by $H(\mathbb{S})$ the space of all holomorphic functions in $\mathbb{S}$. By $\int_{\mathbb{T}} f(s)\,ds$ we understand the line integral of the function f of $\mathbb{T}$ to $\mathbb{C}$ computed with respect to the parametrization $\theta \mapsto e^{i\theta}$, $\theta \in [0,2\pi]$ of $\mathbb{T}$. Let $\mathbb{N}$ be the set of nonnegative integers including 0. Let $m \in \mathbb{N}$, $r \in]0,+\infty[$. Then $C^m(r\mathbb{T},\mathbb{C})$ denotes the space of m times continuously differentiable functions from $r\mathbb{T}$ to $\mathbb{C}$. $C^{m,\alpha}(r\mathbb{T},\mathbb{C})$ denotes the subspace of $C^m(r\mathbb{T},\mathbb{C})$ of those functions, which have m-th order derivatives that are Hölder continuous with exponent $\alpha \in]0,1]$. Let $B \subseteq \mathbb{C}$. We set $C^{m,\alpha}(r\mathbb{T},B) \equiv \{f \in C^{m,\alpha}(r\mathbb{T},\mathbb{C}) : f(r\mathbb{T}) \subseteq B\}$. It is well-known that the space $C^{m,\alpha}(r\mathbb{T},\mathbb{C})$ endowed with the norm

$$\|f\|_{m,\alpha} \equiv \sum_{j=0}^{m} \sup_{t\in r\mathbb{T}} |D^j f(t)| + \sup\left\{ \frac{|D^m f(s) - D^m f(t)|}{|s-t|^\alpha} : s,t \in r\mathbb{T}, s \neq t \right\}$$

is a Banach space. Similarly, if $\mathbb{S}$ is an open bounded subset of $\mathbb{C}$, we define $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S},\mathbb{R})$ to be the space of m-times continuously differentiable real-valued functions in $\mathbb{S}$ such that all the partial derivatives up to order m admit a continuous extension to $\mathrm{cl}\,\mathbb{S}$, and such that the partial derivatives of order m are α-Hölder continuous. By $C^{m,\alpha}(\mathrm{cl}\,\mathbb{S},\mathbb{R}^2)$ we understand $(C^{m,\alpha}(\mathrm{cl}\,\mathbb{S},\mathbb{R}))^2$, and we take as norm of a pair of functions the sum of the norms of the components. It can be readily verified that for any fixed $r > 0$ the restriction to the boundary of a function of class $C^{m,\alpha}(\mathrm{cl}\,r\mathbb{D},\mathbb{C})$ is of class $C^{m,\alpha}(r\mathbb{T},\mathbb{C})$. Similarly, if $r \in]0,1[$ and $f \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{A}_r,\mathbb{C})$, then $f_{|\mathbb{T}} \in C^{m,\alpha}(\mathbb{T},\mathbb{C})$, and $f_{|r\mathbb{T}} \in C^{m,\alpha}(r\mathbb{T},\mathbb{C})$. For standard definitions of calculus in normed spaces, we refer *e.g.* to Berger [3] and to Prodi and Ambrosetti [25].

The following Theorem collects known facts related to singular integrals with Cauchy kernels and to Cauchy type integrals.

Theorem 2.1. *Let $\alpha \in]0,1[$, $m \in \mathbb{N}$, $r \in]0,+\infty[$. Then the following statements hold.*

(i) *For all* $f \in C^{m,\alpha}(r\mathbb{T},\mathbb{C})$*, the singular integral*

$$\mathbf{S}_r[f](\tau) \equiv \frac{1}{\pi i}\int_{r\mathbb{T}} \frac{f(\sigma)}{\sigma-\tau}\,d\sigma, \qquad \forall \tau \in r\mathbb{T}, \tag{2.1}$$

exists in the sense of the principal value, and $\mathbf{S}_r[f](\cdot) \in C^{m,\alpha}(r\mathbb{T},\mathbb{C})$*. The operator* $\mathbf{S}_r$ *defined by* (2.1) *is linear and continuous from* $C^{m,\alpha}(r\mathbb{T},\mathbb{C})$ *to* $C^{m,\alpha}(r\mathbb{T},\mathbb{C})$*.*

(ii) *For all* $f \in C^{m,\alpha}(r\mathbb{T},\mathbb{C})$*, the function* $\mathbf{C}_r[f]$ *of* $\mathbb{C}\setminus\{r\mathbb{T}\}$ *to* $\mathbb{C}$ *defined by*

$$\mathbf{C}_r[f](z) \equiv \frac{1}{2\pi i}\int_{r\mathbb{T}} \frac{f(\sigma)}{\sigma-z}\,d\sigma, \qquad \forall z \in \mathbb{C}\setminus\{r\mathbb{T}\},$$

is holomorphic. The function $\mathbf{C}_r[f]_{|r\mathbb{D}}$ *admits a continuous extension to* $\mathrm{cl}\, r\mathbb{D}$*, which we denote by* $\mathbf{C}_r^+[f]$*, and the function* $\mathbf{C}_r[f]_{|\mathbb{C}\setminus \mathrm{cl}\, r\mathbb{D}}$ *admits a continuous extension to* $\mathbb{C}\setminus r\mathbb{D}$*, which we denote by* $\mathbf{C}_r^-[f]$*. Then we have* $\mathbf{C}_r^+[f] \in C^{m,\alpha}(\mathrm{cl}\, r\mathbb{D},\mathbb{C}) \cap H(r\mathbb{D})$*,* $\mathbf{C}_r^-[f] \in C^0(\mathbb{C}\setminus r\mathbb{D}) \cap C^{m,\alpha}(r\mathbb{T},\mathbb{C}) \cap H(\mathbb{C}\setminus \mathrm{cl}\, r\mathbb{D})$*. Furthermore,* $\lim_{z\to\infty}\mathbf{C}_r[f](z) = 0$*, and the Sokhotsky-Plemelj formulas*

$$\mathbf{C}_r^{\pm}[f](t) = \pm\frac{1}{2}f(t) + \frac{1}{2}\mathbf{S}_r[f](t), \qquad \forall t \in r\mathbb{T}.$$

hold.

(iii) *Let* $\mathbf{I}$ *be the identity operator in* $C^{m,\alpha}(r\mathbb{T},\mathbb{C})$*. The function* $f \in C^{m,\alpha}(r\mathbb{T},\mathbb{C})$ *satisfies the equation*

$$(\mathbf{I}-\mathbf{S}_r)[f] = 0,$$

if and only if there exists a function $F \in C^{m,\alpha}(\mathrm{cl}\, r\mathbb{D},\mathbb{R}^2) \cap H(r\mathbb{D})$ *such that*

$$F(t) = f(t), \qquad \forall t \in r\mathbb{T}.$$

(iv) *Let* $r \in]0,1[$*. Let* $\mathbf{I}$ *be the identity operator in* $C^{m,\alpha}(s\mathbb{T},\mathbb{C})$*, for all* $s>0$*. The pair of functions* $\mathbf{f} \equiv (f^i, f^o) \in C^{m,\alpha}(r\mathbb{T},\mathbb{C}) \times C^{m,\alpha}(\mathbb{T},\mathbb{C})$ *satisfies the system*

$$\begin{cases} (\mathbf{I}+\mathbf{S}_r)\,[f^i] - 2\mathbf{C}_1[f^o] = 0 & \text{on} \quad r\mathbb{T}, \\ (\mathbf{I}-\mathbf{S}_1)\,[f^o] + 2\mathbf{C}_r[f^i] = 0 & \text{on} \quad \mathbb{T}, \end{cases}$$

if and only if there exists $F \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{A}_r) \cap H(\mathbb{A}_r)$ *such that* $F_{|r\mathbb{T}} = f^i$*,* $F_{|\mathbb{T}} = f^o$*. If such* F *exists, then*

$$F(z) = \mathbf{C}_1[f^o](z) - \mathbf{C}_r[f^i](z), \qquad \forall z \in \mathbb{A}_r.$$

(v) *If* $f \in C^{m,\alpha}(r\mathbb{T},\mathbb{R})$*, there exists a unique function* $\mathbf{H}_r f \in C^{m,\alpha}(r\mathbb{T},\mathbb{R})$ *such that* $f + i\mathbf{H}_r f$ *is the restriction to* $r\mathbb{T}$ *of a function* $F \in C^{m,\alpha}(\mathrm{cl}\, r\mathbb{D},\mathbb{R}^2) \cap H(r\mathbb{D})$ *which satisfies the condition* $\Im F(0) = 0$*. Furthermore*

$$i\mathbf{H}_r[f] = \mathbf{S}_r[f] - \frac{1}{2\pi i}\int_{r\mathbb{T}} \frac{f(\sigma)}{\sigma}\,d\sigma,$$
$$\mathbf{H}_r \circ \mathbf{H}_r[f] = -f + \frac{1}{2\pi i}\int_{r\mathbb{T}} \frac{f(\sigma)}{\sigma}\,d\sigma.$$

(vi) *If* $f \in C^{1,\alpha}(r\mathbb{T},\mathbb{C})$*, then* $D(\mathbf{S}_r[f]) = \mathbf{S}_r[Df]$*. If* $0 < s \neq r$*, then* $D(\mathbf{C}_r[f])(z) = \mathbf{C}_r[Df](z)$*, for all* $z \in s\mathbb{T}$*.*

(vii) *Let* $r \in]0,1[$. *Let* $(f^i, f^o) \in C^{m,\alpha}(r\mathbb{T},\mathbb{R}) \times C^{m,\alpha}(\mathbb{T},\mathbb{R})$ *be such that* $\int_{r\mathbb{T}} \frac{f^i(\sigma)}{\sigma}\, d\sigma = \int_{\mathbb{T}} \frac{f^o(\sigma)}{\sigma}\, d\sigma$. *Then there exists a unique element* F *of* $C^{m,\alpha}(\mathrm{cl}\,\mathbb{A}_r) \cap H(\mathbb{A}_r)$ *such that* $\Re F_{|r\mathbb{T}} = f^i$, $\Re F_{|\mathbb{T}} = f^o$, $\Im F(1) = 0$. *We denote such unique* F *by* $\mathbf{\Sigma}_r[f^i, f^o]$. *The operator* $\mathbf{\Sigma}_r$ *of*

$$\left\{ (f^i, f^o) \in C^{m,\alpha}(r\mathbb{T},\mathbb{R}) \times C^{m,\alpha}(\mathbb{T},\mathbb{R}) : \int_{r\mathbb{T}} \frac{f^i(\sigma)}{\sigma}\, d\sigma = \int_{\mathbb{T}} \frac{f^o(\sigma)}{\sigma}\, d\sigma \right\}$$

to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{A}_r) \cap H(\mathbb{A}_r)$, *which takes* (f^i, f^o) *to* $\mathbf{\Sigma}_r[f^i, f^o]$ *is linear.*

For statements (i), (ii) see Gakhov [8, p. 25] together with Vekua [28, pp. 21, 22]. For statement (iii) see Gakhov [8, p. 27] together with Vekua [28, p. 21]. Statement (iv) can be derived by the Cauchy formula and by statements (ii), (iii). For statement (v) see Gakhov [8, p. 45] and Wegert [30, p. 23] together with Vekua [28, p. 21]. For statement (vi), see Gakhov [8, p. 31]. For statement (vii), see Gaier [7, pp. 196–198] together with Theorem 2.1 (i), (ii).

We collect in the following Lemma a few known facts we need on the space $C^{m,\alpha}$.

Lemma 2.1. *Let* $m \in \mathbb{N}$, $\alpha \in]0,1[$, $r > 0$.

(i) *If* $m > 0$, $f \in C^{m,\alpha}(r\mathbb{T},\mathbb{C})$, $g \in C^{m,\alpha}(r\mathbb{T}, r\mathbb{T})$, *then* $f \circ g \in C^{m,\alpha}(r\mathbb{T},\mathbb{C})$.

(ii) *If* $m > 0$, $f \in C^{m,\alpha}(r\mathbb{T}, r\mathbb{T})$, f *is bijective and* $f'(t) \neq 0$, *for all* $t \in r\mathbb{T}$, *then the inverse function* $f^{(-1)}$ *belongs to* $C^{m,\alpha}(r\mathbb{T}, r\mathbb{T})$ *and* $D(f^{(-1)})(t) \neq 0$, *for all* $t \in r\mathbb{T}$.

(iii) *The pointwise product in* $C^{m,\alpha}(r\mathbb{T},\mathbb{C})$ *is bilinear and continuous.*

(iv) *The nonlinear operator* $\zeta \mapsto \zeta^{-1}$ *(the reciprocal of* ζ*) is real analytic from* $\{\zeta \in C^{m,\alpha}(r\mathbb{T},\mathbb{C}) : \zeta(t) \neq 0, \forall t \in r\mathbb{T}\}$ *to* $C^{m,\alpha}(r\mathbb{T},\mathbb{C})$.

(v) $C^{m+1}(r\mathbb{T},\mathbb{C})$ *is continuously imbedded in* $C^{m,\alpha}(r\mathbb{T},\mathbb{C})$. *If* $0 < \alpha < \beta < 1$, *then* $C^{m,\beta}(r\mathbb{T},\mathbb{C})$ *is compactly imbedded in* $C^{m,\alpha}(r\mathbb{T},\mathbb{C})$.

For appropriate references to a proof of Lemma 2.1, we refer to Lanza and Rogosin [20, Lemma 2.2].

3. Determination of a Nonlinear Integral Equation for the Conformal Representation

In this section, we begin by introducing the Riemann map for a doubly connected domain of class $C^{m,\alpha}$, and by presenting an argument that shows that such Riemann map is actually of class $C^{m,\alpha}$, as we need later in the paper (cf. Theorem 3.1.) Then we derive a system of integral equations for the boundary correspondence of the Riemann map (cf. Theorem 3.2.) Then we recast such system in the form of an abstract nonlinear operator equation $\mathbf{P}[\boldsymbol{\zeta}, \mathbf{h}, r] = 0$, involving $\boldsymbol{\zeta}$, the unknown radius r, and an unknown function $\mathbf{h}$, which determines the boundary values of the Riemann map. At this point, it would be natural to think of applying the Implicit Function Theorem to equation $\mathbf{P}[\boldsymbol{\zeta}, \mathbf{h}, r] = 0$ in order to deduce the regularity of the dependence of $\mathbf{h}$, r upon $\boldsymbol{\zeta}$. However, we discover that the corresponding linearized problem is not well-posed. Hence, we introduce a modified problem (cf. Theorem 3.5), whose linearized problem will reveal to be well-posed.

We now note that a regular curve is often defined as an equivalence class of regular parametrizations. However, we need to distinguish the different parametrizations. Thus we define a curve of class C^1 to be a map ζ of class C^1 from the boundary $\mathbb{T}$ of the unit disk $\mathbb{D}$ to $\mathbb{C}$. By a simple curve of class C^1, we understand an injective map of class C^1 from $\mathbb{T}$ to $\mathbb{C}$. Also, a curve ζ should not be confused with $\zeta(\mathbb{T})$.

If ζ is a simple closed curve of class C^1, we denote by $\mathbb{I}[\zeta]$ the bounded connected component of $\mathbb{C}\setminus\zeta(\mathbb{T})$ and by $\mathbb{E}[\zeta]$ the unbounded connected component of $\mathbb{C}\setminus\zeta(\mathbb{T})$.

Now we assume that ζ^i, ζ^o are two given C^1 simple closed curves, and that $\zeta^i(\mathbb{T})\subseteq \mathbb{I}[\zeta^o]$. Then we set $\boldsymbol{\zeta}\equiv(\zeta^i,\zeta^o)$, and

$$\mathbb{A}[\boldsymbol{\zeta}]\equiv\mathbb{I}[\zeta^o]\cap\mathbb{E}[\zeta^i].$$

By applying the Jordan Theorem to the curves ζ^i and ζ^o, it is easy to see that

$$\partial\mathbb{A}[\boldsymbol{\zeta}]=\zeta^i(\mathbb{T})\cup\zeta^o(\mathbb{T}). \tag{3.1}$$

By a simple contradiction argument, it can be readily verified that the following holds (cf. Lanza and Antman [18, p. 1201], Lanza [15, p. 124].)

Lemma 3.1. *The set*

$$\mathcal{Z}\equiv\left\{\zeta\in C^1(\mathbb{T},\mathbb{C}):\inf\left\{\left|\frac{\zeta(s)-\zeta(t)}{s-t}\right|:s,t\in\mathbb{T},s\neq t\right\}>0\right\}$$

coincides with the set of simple curves ζ of class $C^1(\mathbb{T},\mathbb{C})$ with nowhere vanishing ζ'. The set $\mathcal{Z}$ is open in $C^1(\mathbb{T},\mathbb{C})$, and the set

$$\boldsymbol{\mathcal{Z}}\equiv\left\{\boldsymbol{\zeta}\equiv(\zeta^i,\zeta^o)\in\mathcal{Z}^2:\zeta^i(\mathbb{T})\subseteq\mathbb{I}[\zeta^o]\right\}$$

is open in $\left(C^1(\mathbb{T},\mathbb{C})\right)^2$.

Let $\zeta\in\mathcal{Z}$. We denote by $w[\zeta]$ the winding number of the map $\theta\mapsto\zeta(e^{i\theta})$, $\theta\in[0,2\pi]$, with respect to any of the points of $\mathbb{I}[\zeta]$:

$$w[\zeta]\equiv\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{ds}{s-z}\qquad\forall z\in\mathbb{I}[\zeta].$$

The map $w[\cdot]$ is well-known to be constant on the open connected components of $\mathcal{Z}$. Since the curves of $\mathcal{Z}$ are simple, we have $w[\zeta]\in\{-1,1\}$, for all $\zeta\in\mathcal{Z}$. Then we have the following Theorem, which collects a few known facts on the Riemann map of a doubly connected domain.

Theorem 3.1. *Let $\alpha\in]0,1[$, $m\in\mathbb{N}\setminus\{0\}$. Let $\boldsymbol{\zeta}\equiv(\zeta^i,\zeta^o)\in\left(C^{m,\alpha}(\mathbb{T},\mathbb{C})\right)^2\cap\boldsymbol{\mathcal{Z}}$. Then there exist a unique $r[\boldsymbol{\zeta}]\in]0,1[$, and a unique holomorphic homeomorphism $g_{[\boldsymbol{\zeta}]}$ of the set*

$$\mathbb{A}_{r[\boldsymbol{\zeta}]}\equiv\{z\in\mathbb{C}:\ r[\boldsymbol{\zeta}]<|z|<1\}$$

onto $\mathbb{A}[\boldsymbol{\zeta}]$ such that $g_{[\boldsymbol{\zeta}]}$ admits a continuous extension of class $C^{m,\alpha}(\mathrm{cl}\,\mathbb{A}_{r[\boldsymbol{\zeta}]},\mathbb{R}^2)$, which we still denote by $g_{[\boldsymbol{\zeta}]}$, and such that

$$g_{[\boldsymbol{\zeta}]}(1)=\zeta^o(1).$$

Furthermore, $g_{[\boldsymbol{\zeta}]}$ is a homeomorphism of $\mathrm{cl}\,\mathbb{A}_{r[\boldsymbol{\zeta}]}$ onto $\mathrm{cl}\,\mathbb{A}[\boldsymbol{\zeta}]$, and $\frac{d}{dz}g_{[\boldsymbol{\zeta}]}(z)\neq0$, for all $z\in\mathrm{cl}\,\mathbb{A}_{r[\boldsymbol{\zeta}]}$.

Proof. We first consider the existence. There is clearly no loss of generality in assuming that $0 \in \mathbb{I}[\zeta^i]$. The existence is well-known to hold if ζ^i and ζ^o are real analytic. Thus we now reduce the proof to the real analytic case. For all simple closed curves $\zeta \in C^{m,\alpha}(\mathbb{T},\mathbb{C})$ with $\zeta'(t) \neq 0$ for all $t \in \mathbb{T}$, $0 \in \mathbb{I}[\zeta]$, we denote by f_ζ the unique holomorphic homeomorphism of $\mathbb{D}$ onto $\mathbb{I}[\zeta]$ such that $f_\zeta(0) = 0$, $f'_\zeta(0) > 0$. It is well-known that f_ζ extends to a homeomorphism of $\mathrm{cl}\,\mathbb{D}$ onto $\mathrm{cl}\,\mathbb{I}[\zeta]$, and that $f_\zeta \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{R}^2)$, $f'_\zeta(z) \neq 0$ for all $z \in \mathrm{cl}\,\mathbb{D}$ (cf. *e.g.* Pommerenke [24, Thms. 3.5, 3.6, pp. 48–49].) Now we set $q_1(z) \equiv \left[f_{1/\zeta^i}(1/z)\right]^{-1}$, for all $z \in \mathbb{C} \setminus \mathbb{D}$. Clearly, q_1 is the Riemann map of $\mathbb{E}[\mathrm{id}_\mathbb{T}]$ onto $\mathbb{E}[\zeta^i]$, normalized by $q_1(\infty) = \infty$, $q'_1(\infty) > 0$. Furthermore, $q_{1|\mathbb{T}} \in C^{m,\alpha}(\mathbb{T},\mathbb{C})$. Since $\zeta^o \in C^{m,\alpha}(\mathbb{T},\mathbb{C})$, and $q_1^{(-1)}$ is holomorphic on a neighborhood of $\zeta^o(\mathbb{T})$, we have $q_1^{(-1)} \circ \zeta^o \in C^{m,\alpha}(\mathbb{T},\mathbb{C})$, and $q_1^{(-1)} \circ \zeta^o(\mathbb{T}) \subseteq \mathbb{E}[\mathrm{id}_\mathbb{T}]$. Thus the map $q_2 \equiv f_{q_1^{(-1)}\circ\zeta^o}$ belongs to $C^{m,\alpha}(\mathrm{cl}\,\mathbb{D},\mathbb{R}^2)$. Now we set $\tilde{\boldsymbol{\zeta}} \equiv (\tilde{\zeta}^i,\tilde{\zeta}^o) \equiv \left(q_2^{(-1)} \circ q_1^{(-1)} \circ \zeta^i, q_2^{(-1)} \circ q_1^{(-1)} \circ \zeta^o\right)$. Clearly, $q_2^{(-1)} \circ q_1^{(-1)}$ is a holomorphic homeomorphism of $\mathbb{A}[\boldsymbol{\zeta}]$ onto $\mathbb{A}[\tilde{\boldsymbol{\zeta}}]$, which extends to a homeomorphism of $\mathrm{cl}\,\mathbb{A}[\boldsymbol{\zeta}]$ onto $\mathrm{cl}\,\mathbb{A}[\tilde{\boldsymbol{\zeta}}]$. Now we note that $\tilde{\zeta}^i(\mathbb{T}) = q_2^{(-1)} \circ q_1^{(-1)} \circ \zeta^i(\mathbb{T}) = q_2^{(-1)}(\mathbb{T})$, and that $\tilde{\zeta}^o(\mathbb{T}) = q_2^{(-1)} \circ q_1^{(-1)} \circ \zeta^o(\mathbb{T}) = \mathbb{T}$. Thus we have $\mathbb{A}[\tilde{\boldsymbol{\zeta}}] = \mathbb{A}\left[\left(q_2^{(-1)}, \mathrm{id}_\mathbb{T}\right)\right]$. Since $q_2^{(-1)}{}_{|\mathbb{T}}$, and $\mathrm{id}_\mathbb{T}$ are restrictions to $\partial\mathbb{D}$ of holomorphic functions, then it is well-known (cf. *e.g.* Goluzin [10, Thms. 1, 2, p. 208]), that there exists a unique $r \in]0,1[$, and a unique holomorphic homeomorphism q of $\mathbb{A}_r$ onto $\mathbb{A}\left[\left(q_2^{(-1)}, \mathrm{id}_\mathbb{T}\right)\right]$, such that q extends to a homeomorphism of $\mathrm{cl}\,\mathbb{A}_r$ onto $\mathrm{cl}\,\mathbb{A}\left[\left(q_2^{(-1)}, \mathrm{id}_\mathbb{T}\right)\right]$, and such that $q(1) = q_2^{(-1)} \circ q_1^{(-1)} \circ \zeta^o(1)$. A standard argument implies that q can be extended holomorphically to an open neighborhood of $\mathrm{cl}\,\mathbb{A}_r$, and that $q'(z) \neq 0$, for all $z \in \mathrm{cl}\,\mathbb{A}_r$ (cf. *e.g.* Ahlfors [1, pp. 233, 234].) Now we set $g_{[\tilde{\boldsymbol{\zeta}}]} \equiv q_1 \circ q_2 \circ q$. It can be readily verified that $g_{[\tilde{\boldsymbol{\zeta}}]}$ satisfies the properties of the statement. If $g^\#_{[\tilde{\boldsymbol{\zeta}}]}$ is another map which satisfies the properties of the statement, then the map $q_2^{(-1)} \circ q_1^{(-1)} \circ g^\#_{[\tilde{\boldsymbol{\zeta}}]}$ satisfies the same properties of q, and thus $q = q_2^{(-1)} \circ q_1^{(-1)} \circ g^\#_{[\tilde{\boldsymbol{\zeta}}]}$ by uniqueness of the map q. □

Then we can consider the nonlinear operator which takes a pair $\boldsymbol{\zeta} \equiv (\zeta^i,\zeta^o) \in \boldsymbol{\mathcal{Z}}$ to the triple of maps $(\mathbf{h}[\boldsymbol{\zeta}], r[\boldsymbol{\zeta}]) \equiv (h^i[\boldsymbol{\zeta}], h^o[\boldsymbol{\zeta}], r[\boldsymbol{\zeta}])$, where

$$\mathbf{h}[\boldsymbol{\zeta}] \equiv \left(h^i[\boldsymbol{\zeta}], h^o[\boldsymbol{\zeta}]\right) \equiv \left(r^{-1}[\boldsymbol{\zeta}] g^{(-1)}_{[\boldsymbol{\zeta}]} \circ \zeta^i, g^{(-1)}_{[\boldsymbol{\zeta}]} \circ \zeta^o\right). \tag{3.2}$$

By definition, $h^i[\boldsymbol{\zeta}], h^o[\boldsymbol{\zeta}]$ are both self maps of $\mathbb{T}$. We now prove the following technical fact.

Lemma 3.2. *Let* $\alpha \in]0,1[, m \in \mathbb{N} \setminus \{0\}, r \in]0,1[$. *Let* $\mathbf{f} \equiv (f^i, f^o) \in C^{m,\alpha}(r\mathbb{T},\mathbb{C}) \times C^{m,\alpha}(\mathbb{T},\mathbb{C})$. *Then*

$$\begin{cases} (\mathbf{I} + \mathbf{S}_r)[f^i] - 2\mathbf{C}_1[f^o] = 0 & \text{on} \quad r\mathbb{T}, \\ (\mathbf{I} - \mathbf{S}_1)[f^o] + 2\mathbf{C}_r[f^i] = 0 & \text{on} \quad \mathbb{T}, \end{cases} \tag{3.3}$$

holds if and only if

$$
(3.4) \qquad \begin{cases} \Re \left\{(\mathbf{I}+\mathbf{S}_r)[f^i] - 2\mathbf{C}_1[f^o]\right\} = 0 & \text{on} \quad r\mathbb{T}, \\ \Re \left\{(\mathbf{I}-\mathbf{S}_1)[f^o] + 2\mathbf{C}_r[f^i]\right\} = 0 & \text{on} \quad \mathbb{T}, \\ \Im \left\{\mathbf{C}_1[f^o](0) - \mathbf{C}_r[f^i](0)\right\} = 0. \end{cases}
$$

Proof. Obviously, if (3.3) holds, then the first two equations of (3.4) hold. Furthermore, by Theorem 2.1 (iv), there exists $F \in C^0(\mathrm{cl}\,\mathbb{A}_r) \cap H(\mathbb{A}_r)$ such that $F_{|r\mathbb{T}} = f^i$, $F_{|\mathbb{T}} = f^o$. Since $\frac{F(z)}{z}$ is holomorphic in $\mathbb{A}_r$, and continuous on $\mathrm{cl}\,\mathbb{A}_r$, we have $\frac{1}{2\pi i}\int_{r\mathbb{T}} \frac{F(s)}{s} ds = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{F(s)}{s} ds$, and thus the third equation of (3.4) holds.

Conversely, let $\mathbf{f} \equiv (f^i, f^o)$ satisfy (3.4). Then by Sokhotsky-Plemelj formulas, we have $\Re \left\{-2\mathbf{C}_1^-[f^o] + 2\mathbf{C}_r[f^i]\right\} = 0$ on $\mathbb{T}$. Since $-2\mathbf{C}_1^-[f^o] + 2\mathbf{C}_r[f^i]$ belongs to $H(\mathbb{C} \setminus \mathrm{cl}\,\mathbb{D}) \cap C^0(\mathbb{C} \setminus \mathbb{D})$ and vanishes at infinity, we have $-2\mathbf{C}_1^-[f^o] + 2\mathbf{C}_r[f^i] = 0$ on $\mathbb{C} \setminus \mathbb{D}$. Then by Sokhotsky-Plemelj formulas we obtain the second equation of (3.3). We now prove the first equation of (3.3). By Theorem 2.1, the function $2\mathbf{C}_r^+[f^i] - 2\mathbf{C}_1[f^o]$ belongs to $C^0(\mathrm{cl}\,r\mathbb{D}, \mathbb{R}^2) \cap H(r\mathbb{D})$. By Sokhotsky-Plemelj formulas, and by the first equation of (3.4), we obtain $\Re\left\{2\mathbf{C}_r^+[f^i] - 2\mathbf{C}_1[f^o]\right\} = 0$ on $r\mathbb{T}$. Therefore $2\mathbf{C}_r^+[f^i] - 2\mathbf{C}_1[f^o]$ equals a purely imaginary constant on $\mathrm{cl}\,r\mathbb{D}$, and thus by the third equation of (3.4) we obtain $-2\mathbf{C}_1[f^o](z) + 2\mathbf{C}_r^+[f^i](z) = i\Im\left\{-2\mathbf{C}_1[f^o](0) + 2\mathbf{C}_r^+[f^i](0)\right\} = 0$, for all $z \in \mathrm{cl}\,r\mathbb{D}$. □

A simple topological argument based on connectivity of $\mathbb{T}$ shows immediately the validity of the following.

Remark 3.1. *If h is in $\mathcal{Z}$ and if $h(\mathbb{T}) \subseteq \mathbb{T}$, then h is a bijection of $\mathbb{T}$ onto $\mathbb{T}$.*

Theorem 3.2. *Let $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{A}$ be the set defined by*

$$
\mathcal{A} \equiv \Big\{ (\boldsymbol{\zeta}, \mathbf{h}, r) \in (C^{m,\alpha}(\mathbb{T}, \mathbb{C}))^4 \times]0,1[: \boldsymbol{\zeta} \equiv (\zeta^i, \zeta^o) \in \boldsymbol{\mathcal{Z}}, \mathbf{h} \equiv (h^i, h^o) \in (\mathcal{Z})^2, \\ rh^i(\mathbb{T}) \cap h^o(\mathbb{T}) = \emptyset, 0 \in \mathbb{I}[h^i], 0 \in \mathbb{I}[h^o], \ \Re\{h^o(1)\} > 0 \Big\}.
$$

Let $\mathbf{P}$ be the nonlinear operator of $\mathcal{A}$ to $(C^{m,\alpha}(\mathbb{T},\mathbb{R}))^4 \times \mathbb{R}^2$ defined by

$$\begin{aligned}
&\mathbf{P}[\boldsymbol{\zeta},\mathbf{h},r](\tau) \equiv (\mathbf{P}_j[\boldsymbol{\zeta},\mathbf{h},r](\tau))_{j=1,\dots,6} \equiv \\
&\Bigg(\Re\left\{\zeta^i(\tau) + \frac{w[h^i]}{\pi i}\int_{\mathbb{T}}\frac{\zeta^i(\sigma)h^{i\prime}(\sigma)}{h^i(\sigma)-h^i(\tau)}d\sigma - \frac{w[h^o]}{\pi i}\int_{\mathbb{T}}\frac{\zeta^o(\sigma)h^{o\prime}(\sigma)}{h^o(\sigma)-rh^i(\tau)}d\sigma\right\}, \\
&\Re\left\{\zeta^o(\tau) - \frac{w[h^o]}{\pi i}\int_{\mathbb{T}}\frac{\zeta^o(\sigma)h^{o\prime}(\sigma)}{h^o(\sigma)-h^o(\tau)}d\sigma + \frac{w[h^i]}{\pi i}\int_{\mathbb{T}}\frac{\zeta^i(\sigma)h^{i\prime}(\sigma)}{h^i(\sigma)-r^{-1}h^o(\tau)}d\sigma\right\}, \\
&h^i(\tau)\cdot\overline{h^i(\tau)} - 1, \\
&h^o(\tau)\cdot\overline{h^o(\tau)} - 1, \\
&\Im\left\{\frac{w[h^i]}{2\pi i}\int_{\mathbb{T}}\frac{\zeta^i(\sigma)h^{i\prime}(\sigma)}{h^i(\sigma)}d\sigma - \frac{w[h^o]}{2\pi i}\int_{\mathbb{T}}\frac{\zeta^o(\sigma)h^{o\prime}(\sigma)}{h^o(\sigma)}d\sigma\right\}, \\
&\Im\left\{h^o(1)\right\}\Bigg).
\end{aligned}$$

If $\boldsymbol{\zeta} \in (C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \cap \boldsymbol{\mathcal{Z}}$, then the function $\mathbf{h}[\boldsymbol{\zeta}]$ defined in (3.2) *and the radius $r[\boldsymbol{\zeta}]$ defined in Theorem 3.1 satisfy $(\boldsymbol{\zeta},\mathbf{h}[\boldsymbol{\zeta}],r[\boldsymbol{\zeta}]) \in \mathcal{A}$ and $\mathbf{P}[\boldsymbol{\zeta},\mathbf{h}[\boldsymbol{\zeta}],r[\boldsymbol{\zeta}]] = 0$. Conversely, if $(\boldsymbol{\zeta},\mathbf{h},r) \in \mathcal{A}$, and if $\mathbf{P}[\boldsymbol{\zeta},\mathbf{h},r] = 0$, then r equals $r[\boldsymbol{\zeta}]$, $\mathbf{h} \equiv (h^i,h^o)$ equals $\mathbf{h}[\boldsymbol{\zeta}]$, and in particular, both functions h^i and h^o are bijections of $\mathbb{T}$ onto $\mathbb{T}$. Finally, the domain $\mathcal{A}$ of $\mathbf{P}$ is open in the real Banach space $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^4 \times \mathbb{R}$.*

Proof. We first assume that $\boldsymbol{\zeta} \in (C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \cap \boldsymbol{\mathcal{Z}}$. By Theorem 3.1, by (3.2), and by Lemma 2.1 (i), (ii), we can conclude that $\mathbf{h}[\boldsymbol{\zeta}] \in (C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2$, that both $h^i[\boldsymbol{\zeta}]$ and $h^o[\boldsymbol{\zeta}]$ are bijections of $\mathbb{T}$ onto $\mathbb{T}$, that $0 \in \mathbb{I}[h^i[\boldsymbol{\zeta}]] \cap \mathbb{I}[h^o[\boldsymbol{\zeta}]]$, that $h^o[\boldsymbol{\zeta}](1) = 1$, and that $\frac{d}{d\tau}h^i[\boldsymbol{\zeta}](\tau) \neq 0$, $\frac{d}{d\tau}h^i[\boldsymbol{\zeta}](\tau) \neq 0$ for all $\tau \in \mathbb{T}$. Then $\mathbf{h}[\boldsymbol{\zeta}] \in \mathcal{Z}^2$ by Lemma 3.1. In particular, $(\boldsymbol{\zeta},\mathbf{h}[\boldsymbol{\zeta}],r[\boldsymbol{\zeta}]) \in \mathcal{A}$. By equality $h^i[\boldsymbol{\zeta}](\mathbb{T}) = \mathbb{T} = h^o[\boldsymbol{\zeta}](\mathbb{T})$ we conclude that $\mathbf{P}_3[\boldsymbol{\zeta},\mathbf{h}[\boldsymbol{\zeta}],r[\boldsymbol{\zeta}]] = \mathbf{P}_4[\boldsymbol{\zeta},\mathbf{h}[\boldsymbol{\zeta}],r[\boldsymbol{\zeta}]] = 0$. Since $g_{[\boldsymbol{\zeta}]}(1) = \zeta^o(1)$, we obtain equality $\mathbf{P}_6[\boldsymbol{\zeta},\mathbf{h}[\boldsymbol{\zeta}],r[\boldsymbol{\zeta}]] = 0$. By applying Theorem 2.1 (iv) to the function $g_{[\boldsymbol{\zeta}]}$, by (3.2), and by changing the variables in the improper integrals, we conclude that $\mathbf{P}_j[\boldsymbol{\zeta},\mathbf{h}[\boldsymbol{\zeta}],r[\boldsymbol{\zeta}]] = 0, j = 1,2$. Since $\frac{g_{[\boldsymbol{\zeta}]}(z)}{z}$ is holomorphic in $\mathbb{A}_{r[\boldsymbol{\zeta}]}$, and continuous in $\mathrm{cl}\,\mathbb{A}_{r[\boldsymbol{\zeta}]}$ we deduce the validity of $\mathbf{P}_5[\boldsymbol{\zeta},\mathbf{h}[\boldsymbol{\zeta}],r[\boldsymbol{\zeta}]] = 0$.

Conversely, assume that $(\boldsymbol{\zeta},\mathbf{h},r)$ is an element of $\mathcal{A}$ and that $\mathbf{P}[\boldsymbol{\zeta},\mathbf{h},r] = 0$. By Remark 3.1, we have $h^i(\mathbb{T}) = \mathbb{T}$ and $h^o(\mathbb{T}) = \mathbb{T}$. Thus by Lemma 2.1, the pair of functions defined by $(g^i(t),g^o(t)) \equiv \left(\zeta^i \circ h^{i(-1)}(r^{-1}t), \zeta^o \circ h^{o(-1)}(t)\right)$ belongs to $C^{m,\alpha}(r\mathbb{T},\mathbb{C}) \times C^{m,\alpha}(\mathbb{T},\mathbb{C})$. By equality $\mathbf{P}[\boldsymbol{\zeta},\mathbf{h},r] = 0$, and by the formula of change of variables in improper integrals we obtain

$$\begin{cases}
\Re\left\{(\mathbf{I}+\mathbf{S}_r)[g^i] - 2\mathbf{C}_1[g^o]\right\} = 0 & \text{on} \quad r\mathbb{T}, \\
\Re\left\{(\mathbf{I}-\mathbf{S}_1)[g^o] + 2\mathbf{C}_r[g^i]\right\} = 0 & \text{on} \quad \mathbb{T}, \\
\Im\left\{\mathbf{C}_1[g^o](0) - \mathbf{C}_r[g^i](0)\right\} = 0.
\end{cases}$$

By Theorem 2.1 (iv), and by Lemma 3.2, there exists $g \in C^{m,\alpha}(\mathrm{cl}\,\mathbb{A}_r, \mathbb{R}^2) \cap H(\mathbb{A}_r)$ such that $g_{|r\mathbb{T}}(t) = g^i(t)$, for all $t \in r\mathbb{T}$, $g_{|\mathbb{T}}(t) = g^o(t)$, for all $t \in \mathbb{T}$. Since g is injective on the boundary $r\mathbb{T} \cup \mathbb{T}$ of $\mathbb{A}_r$, g is injective on $\mathbb{A}_r$ by the Argument Principle. By the Open Mapping Theorem (cf. *e.g.* Ahlfors [1, Cor. 1, p. 132]) g maps open subsets of $\mathbb{A}_r$ to open subsets of $\mathbb{C}$. Accordingly, $g(\mathbb{A}_r)$ is an open bounded connected subset of $\mathbb{C}$. We now prove that $\partial g(\mathbb{A}_r) \subseteq g(r\mathbb{T} \cup \mathbb{T})$. If $p \in \partial g(\mathbb{A}_r)$, then there exists a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ in $\mathbb{A}_r$ such that $\lim_{n\to\infty} g(\xi_n) = p$. By possibly selecting a convergent subsequence, we can assume that the sequence $\{\xi_n\}_{n\in\mathbb{N}}$ converges to some $\xi \in \mathrm{cl}\,\mathbb{A}_r$. Since g is open in $\mathbb{A}_r$, we must have $\xi \in r\mathbb{T} \cup \mathbb{T}$. Thus $p = g(\xi) \in g(r\mathbb{T} \cup \mathbb{T})$. Then a simple topological argument based on the connectivity of $\mathbb{E}[g(r\mathbb{T})], \mathbb{I}[g(\mathbb{T})], \mathbb{A}[\boldsymbol{\zeta}]$, and on the fact that $\mathbb{A}[\boldsymbol{\zeta}]$ is doubly connected and bounded, shows that $g(\mathbb{A}_r) = \mathbb{A}[\boldsymbol{\zeta}]$. In particular g is a holomorphic homeomorphism of $\mathbb{A}_r$ onto $\mathbb{A}[\boldsymbol{\zeta}]$. Since $h^o(\mathbb{T}) \subseteq \mathbb{T}$ and $\Re\{h^o(1)\} > 0$, condition $\mathbf{P}_6[\boldsymbol{\zeta}, \mathbf{h}, r] = 0$ implies that $h^o(1) = 1$. Thus $g^o(1) = \zeta^o(1)$. Hence by the uniqueness inferred by Theorem 3.1, we conclude that $g = g_{[\boldsymbol{\zeta}]}$ and that $r = r[\boldsymbol{\zeta}]$. Since the norm of $C^{m,\alpha}(\mathbb{T}, \mathbb{C})$ is stronger than that of the uniform convergence, we can invoke Lemma 3.1 to conclude that the set $\mathcal{A}$ is open. □

We note that the appearance of the winding numbers $w[h_0^i]$, $w[h_0^o]$ in the statement of Theorem 3.2 is associated to the application of the rule of change of variables in the line integrals, performed in order to obtain the equation $\mathbf{P}[\boldsymbol{\zeta}, \mathbf{h}, r] = 0$, and with the fact that our curves ζ^i, ζ^o may have all possible orientations.

In view of the validity of Theorem 3.2, it is natural to think of proving a regularity theorem for the nonlinear operator $(\mathbf{h}[\cdot], r[\cdot])$ by applying the Implicit Function Theorem to the equation $\mathbf{P}[\boldsymbol{\zeta}, \mathbf{h}, r] = 0$ in $\mathcal{A}$. To do so, we need to prove that $\mathbf{P}$ is regular. Thus we state the following result of Lanza and Preciso [19], which may be regarded as an extension of the corresponding result of Coifman and Meyer [4].

Theorem 3.3. *Let $\alpha \in]0,1[, m \in \mathbb{N} \setminus \{0\}$. Then the (nonlinear) operator which takes (ζ, h) to the function $\frac{1}{\pi i}\int_{\mathbb{T}} \frac{\zeta(\sigma)h'(\sigma)}{h(\sigma)-h(\tau)} d\sigma$ is real analytic from $(C^{m,\alpha}(\mathbb{T}, \mathbb{C}) \cap \mathcal{Z}) \times \{h \in C^{m,\alpha}(\mathbb{T}, \mathbb{C}) \cap \mathcal{Z} : 0 \in \mathbb{I}[h]\}$ to $C^{m,\alpha}(\mathbb{T}, \mathbb{C})$. Furthermore, if (ζ_0, h_0) belongs to $(C^{m,\alpha}(\mathbb{T}, \mathbb{C}) \cap \mathcal{Z}) \times \{h \in C^{m,\alpha}(\mathbb{T}, \mathbb{C}) \cap \mathcal{Z} : 0 \in \mathbb{I}[h]\}$, then the real differential at h_0 of the map $h \mapsto \frac{1}{\pi i}\int_{\mathbb{T}} \frac{\zeta_0(\sigma)h'(\sigma)}{h(\sigma)-h(\tau)} d\sigma$ is delivered by the map*

$$\mu \mapsto \frac{1}{\pi i}\int_{\mathbb{T}} \frac{\mu(\tau) - \mu(\sigma)}{h_0(\sigma) - h_0(\tau)} \zeta_0'(\sigma) d\sigma,$$

for all $\mu \in C^{m,\alpha}(\mathbb{T}, \mathbb{C})$.

Then by the previous theorem, by Theorem 2.1 (i), by Lemma 2.1 (iii), (iv), by standard calculus in normed spaces, and by integration by parts, we deduce the following.

Theorem 3.4. *Let $\alpha \in]0,1[, m \in \mathbb{N} \setminus \{0\}$. Then the operator $\mathbf{P}$ is real analytic on $\mathcal{A}$. Furthermore, for all $(\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0) \in \mathcal{A}$ the real differential $\partial_{(\mathbf{h},r)}\mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0]$ of $\mathbf{P}$*

with respect to $(\mathbf{h}, r) \equiv (h^i, h^o, r)$ *at the point* (h_0^i, h_0^o, r_0) *is delivered by the formula*

$$\begin{aligned}
&\partial_{(\mathbf{h},r)}\mathbf{P}[\boldsymbol{\zeta}_0,\mathbf{h}_0,r_0](\mu^i,\mu^o,d) = \\
&= \Bigg(\Re\left\{\frac{w[h_0^i]}{\pi i}\int_{\mathbb{T}}\frac{\mu^i(\tau)-\mu^i(\sigma)}{h_0^i(\sigma)-h_0^i(\tau)}\zeta_0^{i\prime}(\sigma)d\sigma + \frac{w[h_0^o]}{\pi i}\int_{\mathbb{T}}\frac{\mu^o(\sigma)\zeta_0^{o\prime}(\sigma)}{h_0^o(\sigma)-r_0h_0^i(\tau)}d\sigma\right.\\
&\qquad\left. -\frac{\mu^i(\tau)r_0w[h_0^o]}{\pi i}\int_{\mathbb{T}}\frac{\zeta_0^{o\prime}(\sigma)d\sigma}{h_0^o(\sigma)-r_0h_0^i(\tau)} - \frac{h_0^i(\tau)w[h_0^o]d}{\pi i}\int_{\mathbb{T}}\frac{\zeta_0^{o\prime}(\sigma)d\sigma}{h_0^o(\sigma)-r_0h_0^i(\tau)}\right\},\\
&\qquad\Re\left\{-\frac{w[h_0^o]}{\pi i}\int_{\mathbb{T}}\frac{\mu^o(\tau)-\mu^o(\sigma)}{h_0^o(\sigma)-h_0^o(\tau)}\zeta_0^{o\prime}(\sigma)d\sigma - \frac{w[h_0^i]}{\pi i}\int_{\mathbb{T}}\frac{\mu^i(\sigma)\zeta_0^{i\prime}(\sigma)}{h_0^i(\sigma)-r_0^{-1}h_0^o(\tau)}d\sigma\right.\\
&\qquad\left. +\frac{\mu^o(\tau)w[h_0^i]}{r_0\pi i}\int_{\mathbb{T}}\frac{\zeta_0^{i\prime}(\sigma)d\sigma}{h_0^i(\sigma)-r_0^{-1}h_0^o(\tau)} - \frac{h_0^o(\tau)w[h_0^i]d}{r_0^2\pi i}\int_{\mathbb{T}}\frac{\zeta_0^{i\prime}(\sigma)d\sigma}{h_0^i(\sigma)-r_0^{-1}h_0^o(\tau)}\right\},\\
&\qquad h_0^i(\tau)\cdot\overline{\mu^i(\tau)} + \mu^i(\tau)\cdot\overline{h_0^i(\tau)},\\
&\qquad h_0^o(\tau)\cdot\overline{\mu^o(\tau)} + \mu^o(\tau)\cdot\overline{h_0^o(\tau)},\\
&\qquad -\Im\left\{\frac{w[h_0^i]}{2\pi i}\int_{\mathbb{T}}\frac{\mu^i(\sigma)}{h_0^i(\sigma)}\zeta_0^{i\prime}(\sigma)d\sigma - \frac{w[h_0^o]}{2\pi i}\int_{\mathbb{T}}\frac{\mu^o(\sigma)}{h_0^o(\sigma)}\zeta_0^{o\prime}(\sigma)d\sigma\right\},\\
&\qquad \Im\{\mu^o(1)\}\Bigg).
\end{aligned}$$

By the previous Theorem, we deduce the following.

Proposition 3.1. *Let* $\alpha \in]0,1[, m \in \mathbb{N}\setminus\{0\}$. *Let* $(\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0) \in \mathcal{A}$. *If* $\mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0] = 0$, *then*

$$\frac{1}{2\pi i}\int_{\mathbb{T}}\partial_{(\mathbf{h},r)}\mathbf{P}_2[\boldsymbol{\zeta}_0,\mathbf{h}_0,r_0](\mu^i,\mu^o,d)\frac{h_0^{o\prime}(\sigma)}{h_0^o(\sigma)}d\sigma = 0, \tag{3.5}$$

for all $(\mu^i, \mu^o, d) \in (C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \times \mathbb{R}$.

Proof. By Theorem 3.2, we have $h_0^o(\mathbb{T}) = \mathbb{T}$ and $(\zeta_0^o \circ h_0^{o(-1)})' = g'_{\boldsymbol{\zeta}_0}$ on $\mathbb{T}$. By setting $k_0^o \equiv h_0^{o(-1)}$ and $k_0^i \equiv h_0^{i\,(-1)}$ and by applying Theorem 2.1 (iv), we have

$$(\zeta_0^o \circ k_0^o(t))' - \frac{1}{\pi i}\int_{\mathbb{T}}\frac{(\zeta_0^o \circ k_0^o(s))'}{s-t}ds = -\frac{1}{r_0\pi i}\int_{\mathbb{T}}\frac{(\zeta_0^i \circ k_0^i(s))'}{s-r_0^{-1}t}ds \quad \forall t \in \mathbb{T}.$$

Then by changing the variable with the function $k_0^o \equiv h_0^{o(-1)}$ in (3.5), and by Sokhotsky-Plemelj formulas, we obtain that the integral in (3.5) is equal to

$$\begin{aligned}
&\frac{-w[h_0^o]}{2\pi i}\int_{\mathbb{T}}\Re\Big\{-2\mathbf{C}_1^-[(\mu^o\circ k_0^o)(\zeta_0^o\circ k_0^o)'](s)+\\
&\quad +2r_0\mathbf{C}_{r_0}[(\mu^i\circ k_0^i)(r_0^{-1}\cdot)((\zeta_0^i\circ k_0^i)(r_0^{-1}\cdot))'](s) + \frac{sd}{r_0}2\mathbf{C}_{r_0}[((\zeta_0^i\circ k_0^i)(r_0^{-1}\cdot))'](s)\Big\}\frac{ds}{s}.
\end{aligned}$$

Now the function F of $\mathbb{C} \setminus \mathbb{D}$ to $\mathbb{C}$ defined by

$$F(z) \equiv -2\mathbf{C}_1^-[(\mu^o \circ k_0^o)(\zeta_0^o \circ k_0^o)'](z) + 2r_0\mathbf{C}_{r_0}[(\mu^i \circ k_0^i)(r_0^{-1}\cdot)((\zeta_0^i \circ k_0^i)(r_0^{-1}\cdot))'](z) + \frac{zd}{r_0}2\mathbf{C}_{r_0}[((\zeta_0^i \circ k_0^i)(r_0^{-1}\cdot))'](z), \qquad \forall z \in \mathbb{C} \setminus \mathbb{D},$$

is holomorphic in $\mathbb{C} \setminus \mathrm{cl}\,\mathbb{D}$, continuous in $\mathbb{C} \setminus \mathbb{D}$ and vanishes at infinity. Then $\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{F(\frac{1}{\sigma})}{\sigma} d\sigma = 0$, and thus $\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{F(s)}{s} ds = 0$. Consequently $\frac{1}{2\pi}\int_0^{2\pi} \Re\{F(e^{i\theta})\}\, d\theta = 0$, and the validity of the statement follows. □

Remark 3.2. *By the previous proposition, we deduce that $\partial_{(\mathbf{h},r)}\mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0]$ cannot be surjective onto the target space $(C^{m,\alpha}(\mathbb{T},\mathbb{R}))^4 \times \mathbb{R}^2$ of $\mathbf{P}$. On the other hand we cannot impose the condition $\frac{1}{2\pi i}\int_{\mathbb{T}} f(\sigma)\frac{h^{o\prime}(\sigma)}{h^o(\sigma)} d\sigma = 0$ on the second component of the sestuples of the target space of $\mathbf{P}$ because such condition depends on h^o. Thus we cannot apply the Implicit Function Theorem directly to equation $\mathbf{P} = 0$ around a certain $(\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0) \in \mathcal{A}$ to deduce the regularity of $(\mathbf{h}[\cdot], r[\cdot])$.*

To circumvent the difficulty outlined in Remark 3.2, we employ an argument of Lanza and Rogosin [20] and we introduce a modified equation with the same solutions of equation $\mathbf{P}[\boldsymbol{\zeta}, \mathbf{h}, r] = 0$. To do so, we need the following two preliminary technical statements.

Remark 3.3. *For all $h \in \mathcal{Z}$ such that $0 \in \mathbb{I}[h]$, we have*

$$w[h] = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{h'(\sigma)}{h(\sigma)} d\sigma \in \{-1, 1\}. \tag{3.6}$$

Indeed, $w[h]$ equals the winding number of $\theta \mapsto h(e^{i\theta}), \theta \in [0, 2\pi]$ with respect to zero, and h is a simple closed curve of class C^1.

Lemma 3.3. *Let $\alpha \in]0,1[, m \in \mathbb{N} \setminus \{0\}$. If $(\boldsymbol{\zeta}, \mathbf{h}, r) \in \mathcal{A}$, and if $h^i(\mathbb{T}) = h^o(\mathbb{T}) = \mathbb{T}$, then*

$$\frac{w[h^o]}{2\pi i}\int_{\mathbb{T}} \Re\left\{\zeta^o(\tau) - \frac{w[h^o]}{\pi i}\int_{\mathbb{T}} \frac{\zeta^o(\sigma)h^{o\prime}(\sigma)}{h^o(\sigma) - h^o(\tau)} d\sigma + \frac{w[h^i]}{\pi i}\int_{\mathbb{T}} \frac{\zeta^i(\sigma)h^{i\prime}(\sigma)}{h^i(\sigma) - r^{-1}h^o(\tau)} d\sigma\right\} \frac{h^{o\prime}(\tau)}{h^o(\tau)} d\tau = 0. \tag{3.7}$$

Proof. By assumption, h^o is a diffeomorphism of $\mathbb{T}$. Thus by changing variable in all the integrals of (3.7) by means of the functions $k^o \equiv h^{o(-1)}$, $k^i \equiv h^{i(-1)}$ (cf. Gakhov [8, p. 17]), we can show that the left hand side of (3.7) equals

$$\frac{w[h^o]}{2\pi i}\int_{\mathbb{T}} \Re\left\{-2\mathbf{C}_1^-[\zeta^o \circ k^o](t) + 2\mathbf{C}_r[\zeta^i \circ k^i(r^{-1}\cdot)](t)\right\} \frac{dt}{t}.$$

Now, the function F of $\mathbb{C} \setminus \mathbb{D}$ to $\mathbb{C}$ defined by

$$F(z) \equiv -2\mathbf{C}_1^-[\zeta^o \circ k^o](z) + 2\mathbf{C}_r[\zeta^i \circ k^i(r^{-1}\cdot)](z) \quad \forall z \in \mathbb{C} \setminus \mathbb{D},$$

is holomorphic in $\mathbb{C} \setminus \mathrm{cl}\,\mathbb{D}$, continuous in $\mathbb{C} \setminus \mathbb{D}$ and vanishes at infinity. Then $\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{F(\frac{1}{\sigma})}{\sigma} d\sigma = 0$, and thus $\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{F(s)}{s} ds = 0$ and (3.7) follows. □

We are now ready to introduce the modified equation mentioned above.

Theorem 3.5. *Let* $\alpha \in]0,1[, m \in \mathbb{N} \setminus \{0\}$*. Let* $(\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0) \in \mathcal{A}$*, and* $h_0^o(\mathbb{T}) = \mathbb{T}$*. Let* $\Pi_{h_0^o}$ *be the map of* $(C^{m,\alpha}(\mathbb{T},\mathbb{R}))^4 \times \mathbb{R}^2$ *to itself defined by*

$$\Pi_{h_0^o}[\beta^i, \beta^o, \gamma^i, \gamma^o, a, b] \equiv \left(\beta^i, \beta^o - w^{-1}[h_0^o]\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\beta^o(\sigma) h_0^{o\prime}(\sigma)}{h_0^o(\sigma)} d\sigma, \gamma^i, \gamma^o, a, b\right).$$

The map $\Pi_{h_0^o}$ *is linear and continuous and maps its domain onto the space*

$$V_{h_0^o}^{m,\alpha} \equiv \left\{(\beta^i, \beta^o, \gamma^i, \gamma^o, a, b) \in (C^{m,\alpha}(\mathbb{T},\mathbb{R}))^4 \times \mathbb{R}^2 : \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\beta^o(\sigma) h_0^{o\prime}(\sigma)}{h_0^o(\sigma)} d\sigma = 0\right\}.$$

The restriction of $\Pi_{h_0^o}$ *to* $V_{h_0^o}^{m,\alpha}$ *is the identity map. An element* $(\boldsymbol{\zeta}, \mathbf{h}, r) \in \mathcal{A}$ *satisfies the equation*

$$\mathbf{P}[\boldsymbol{\zeta}, \mathbf{h}, r] = 0, \tag{3.8}$$

if and only if it satisfies the equation

$$\Pi_{h_0^o}\mathbf{P}[\boldsymbol{\zeta}, \mathbf{h}, r] = 0. \tag{3.9}$$

Proof. Since $h_0^o \in \mathcal{Z}$ and $h_0^o(\mathbb{T}) = \mathbb{T}$, h_0^o is a diffeomorphism of $\mathbb{T}$ to itself and $\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\beta^o(\sigma) h_0^{o\prime}(\sigma)}{h_0^o(\sigma)} d\sigma$ is real when β^o is real-valued. Then by calculating the integral in the definition of $V_{h_0^o}^{m,\alpha}$, we can easily verify that $\Pi_{h_0^o}$ maps its domain to $V_{h_0^o}^{m,\alpha}$. Obviously $\Pi_{h_0^o}$ restricts the identity on $V_{h_0^o}^{m,\alpha}$ and is linear and continuous. In particular, $\Pi_{h_0^o}$ is surjective. It is also clear that the equality (3.8) implies (3.9). Assume now conversely that (3.9) holds. Then we have

$$\mathbf{P}_2[\boldsymbol{\zeta}, \mathbf{h}, r](\tau) - w^{-1}[h_0^o]\frac{1}{2\pi i}\int_{\mathbb{T}} \mathbf{P}_2[\boldsymbol{\zeta}, \mathbf{h}, r](\sigma)\frac{h_0^{o\prime}(\sigma)}{h_0^o(\sigma)} d\sigma = 0, \tag{3.10}$$

and $h^i(\mathbb{T}) \subseteq \mathbb{T}$, $h^o(\mathbb{T}) \subseteq \mathbb{T}$. We now multiply (3.10) by $\frac{1}{2\pi i}\frac{h^{o\prime}(\tau)}{h^o(\tau)}$ and integrate in $\tau \in \mathbb{T}$. By Remark 3.1 and by (3.6), (3.7), we obtain

$$w[h^o]w^{-1}[h_0^o]\frac{1}{2\pi i}\int_{\mathbb{T}} \mathbf{P}_2[\boldsymbol{\zeta}, \mathbf{h}, r](\sigma)\frac{h_0^{o\prime}(\sigma)}{h_0^o(\sigma)} d\sigma = 0. \tag{3.11}$$

By (3.6) we have $w[h^o] \neq 0$. Then by definition of $\Pi_{h_0^o}$ and of $\mathbf{P}$, we have $\mathbf{P}[\boldsymbol{\zeta}, \mathbf{h}, r] = \Pi_{h_0^o}\mathbf{P}[\boldsymbol{\zeta}, \mathbf{h}, r]$. □

We now investigate the regularity of the nonlinear operator $\mathbf{h}$ from $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \cap \boldsymbol{\mathcal{Z}}$ to $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2$ defined in formula (3.2), and of the nonlinear operator $r[\cdot]$ from $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \cap \boldsymbol{\mathcal{Z}}$ to $]0,1[$ defined in Theorem 3.1, by means of an application of the Implicit Function Theorem to the equation

$$\Pi_{h_0^o}\mathbf{P}[\boldsymbol{\zeta}, \mathbf{h}, r] = 0, \tag{3.12}$$

around $(\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0) \in \mathcal{A}$. In the next section we turn our attention to the linearized problem.

4. A Preliminary Existence and Uniqueness Theorem for the Linearized Problem

In this section, we first rewrite the linearization of the modified equation delivered at the end of the previous section (cf. (3.12)) in a suitable form (cf. Lemma 4.1.) Then we show that such linearized problem has a unique solution of class $C^{m-1,\alpha}$, when the data are of class $C^{m,\alpha}$. Also, we exibit a representation formula for the solution of the linearized problem (cf. (4.11), (4.13)), which involves certain integral operators, such as the Schwarz operator. We now turn to study the linear operator $\partial_{(\mathbf{h},r)}\left(\Pi_{h_0^o}\circ\mathbf{P}\right)$. It clearly suffices to consider the linear operator $\partial_{(\mathbf{h},r)}\mathbf{P}$. Thus we introduce the following.

Lemma 4.1. *Let $\alpha\in]0,1[, m\in\mathbb{N}\setminus\{0\}$. Let $(\boldsymbol{\zeta}_0,\mathbf{h}_0,r_0)\in\mathcal{A}$, and $\mathbf{P}[\boldsymbol{\zeta}_0,\mathbf{h}_0,r_0]=0$. Let $k_0^i\equiv(h_0^i)^{(-1)}$, $k_0^o\equiv(h_0^o)^{(-1)}$. Let $(\boldsymbol{\beta},\boldsymbol{\gamma},a,b)\equiv(\beta^i,\beta^o,\gamma^i,\gamma^o,a,b)$ be an element of $(C^{m,\alpha}(\mathbb{T},\mathbb{R}))^4\times\mathbb{R}^2$. Then the triple $(\boldsymbol{\mu},d)\equiv(\mu^i,\mu^o,d)\in(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2\times\mathbb{R}$ satisfies*

$$\partial_{(\mathbf{h},r)}\mathbf{P}[\boldsymbol{\zeta}_0,\mathbf{h}_0,r_0](\boldsymbol{\mu},d)=(\boldsymbol{\beta},\boldsymbol{\gamma},a,b),\tag{4.1}$$

if and only if the triple $(\boldsymbol{\nu},d)\equiv(\nu^i,\nu^o,d)\equiv(\mu^i\circ k_0^i,\mu^o\circ k_0^o,d)\in(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2\times\mathbb{R}$ satisfies the following system

$$\begin{cases}\Re\Big\{-r_0 g'_{[\boldsymbol{\zeta}_0]}(t)\nu^i(r_0^{-1}t)-r_0\mathbf{S}_{r_0}[g'_{[\boldsymbol{\zeta}_0]}(\cdot)\nu^i(r_0^{-1}\cdot)](t)+2\mathbf{C}_1^+[g'_{[\boldsymbol{\zeta}_0]}(\cdot)\nu^o(\cdot)](t)\\ \qquad -\dfrac{d\cdot t}{r_0}2\mathbf{C}_1^+[g'_{[\boldsymbol{\zeta}_0]}(\cdot)](t)\Big\}=(\beta^i\circ k_0^i)(r_0^{-1}t)\quad\forall t\in r_0\mathbb{T},\\ \Re\Big\{-g'_{[\boldsymbol{\zeta}_0]}(t)\nu^o(t)+\mathbf{S}_1[g'_{[\boldsymbol{\zeta}_0]}(\cdot)\nu^o(\cdot)](t)-2r_0\mathbf{C}_{r_0}^-[g'_{[\boldsymbol{\zeta}_0]}(\cdot)\nu^i(r_0^{-1}\cdot)](t)\\ \qquad -\dfrac{d\cdot t}{r_0}2\mathbf{C}_{r_0}^-[g'_{[\boldsymbol{\zeta}_0]}(\cdot)](t)\Big\}=(\beta^o\circ k_0^o)(t)\quad\forall t\in\mathbb{T},\\ 2\Re\left\{\dfrac{r_0}{t}\nu^i(r_0^{-1}t)\right\}=(\gamma^i\circ k_0^i)(r_0^{-1}t)\quad\forall t\in r_0\mathbb{T},\\ 2\Re\left\{\dfrac{\nu^o(t)}{t}\right\}=(\gamma^o\circ k_0^o)(t)\quad\forall t\in\mathbb{T},\\ -\Im\left\{r_0\mathbf{C}_{r_0}[g'_{[\boldsymbol{\zeta}_0]}(\cdot)\nu^i(r_0^{-1}\cdot)](0)-\mathbf{C}_1[g'_{[\boldsymbol{\zeta}_0]}(\cdot)\nu^o(\cdot)](0)\right\}=a,\\ \Im\left\{\nu^0(1)\right\}=b.\end{cases}\tag{4.2}$$

Proof. By Remark 3.1, both functions h_0^i and h_0^o are bijections of $\mathbb{T}$ onto $\mathbb{T}$. Hence we can change the variables of integration in the first, second, and fifth equation of the formula for the differential of $\mathbf{P}$ of Theorem 3.4 by setting $\sigma=k_0^i(r_0^{-1}s)$, where σ is in the argument of $(\zeta_0^i)'(\cdot)$, and by setting $\sigma=k_0^o(s)$, where σ is in the argument of $(\zeta_0^o)'(\cdot)$. Then we set $\tau=k_0^i(r_0^{-1}t)$ in the first and in the third equation of (4.1), and $\tau=k_0^o(t)$ in the second and fourth equation of (4.1). Furthermore, we note that

$$g'_{[\boldsymbol{\zeta}_0]}(t)=\begin{cases}\frac{d}{dt}\left((\zeta_0^i\circ k_0^i)(r_0^{-1}t)\right) & \forall t\in r_0\mathbb{T},\\ \frac{d}{dt}\left((\zeta_0^o\circ k_0^o)(t)\right) & \forall t\in\mathbb{T},\end{cases}$$

and that $g'_{[\boldsymbol{\zeta}_0]} \in C^{m-1,\alpha}(\mathrm{cl}\,\mathbb{A}_{r_0},\mathbb{R}^2) \cap H(\mathbb{A}_{r_0})$. Then we can apply Theorem 2.1 to deduce (4.2). By performing the inverse change of variables, we obtain (4.1) from (4.2). □

We are now in position to prove our preliminary result on existence and uniqueness for the system (4.2).

Theorem 4.1. *Let* $\alpha \in]0,1[$, $m \in \mathbb{N}\setminus\{0\}$. *Let* $(\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0) \in \mathcal{A}$, *and* $\mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0] = 0$. *If* $(\boldsymbol{\beta}, \boldsymbol{\gamma}, a, b) \in V^{m,\alpha}_{h^o_0}$, *then there exists a unique solution* $(\boldsymbol{\nu}, d) \in (C^{m-1,\alpha}(\mathbb{T},\mathbb{C}))^2 \times \mathbb{R}$ *of system* (4.2).

Proof. We adopt the following strategy. We first assume that for a given $(\boldsymbol{\beta}, \boldsymbol{\gamma}, a, b) \in V^{m,\alpha}_{h^o_0}$, system (4.2) has a solution $(\boldsymbol{\nu}, d) \in (C^{m-1,\alpha}(\mathbb{T},\mathbb{C}))^2 \times \mathbb{R}$. Then we show that $(\boldsymbol{\nu}, d)$ must necessarily be delivered by a certain formula which involves $(\boldsymbol{\beta}, \boldsymbol{\gamma}, a, b)$. Finally, we exploit such formula to deduce the existence and uniqueness of a solution $(\boldsymbol{\nu}, d) \in (C^{m-1,\alpha}(\mathbb{T},\mathbb{C}))^2 \times \mathbb{R}$.

Now let $(\boldsymbol{\beta}, \boldsymbol{\gamma}, a, b) \in V^{m,\alpha}_{h^o_0}$. To simplify the form of our formulas, we set

$$
\begin{cases}
\phi^i \equiv r_0 g'_{[\boldsymbol{\zeta}_0]}(t)\nu^i(r_0^{-1}t) & t \in r_0\mathbb{T}, \\
\phi^o \equiv g'_{[\boldsymbol{\zeta}_0]}(t)\nu^o(t) & t \in \mathbb{T}.
\end{cases}
\tag{4.3}
$$

Clearly $\phi^i \in C^{m-1,\alpha}(r_0\mathbb{T},\mathbb{C})$ and $\phi^o \in C^{m-1,\alpha}(\mathbb{T},\mathbb{C})$. Now we note that since the functions $\beta^i \circ k^i_0$, and $\beta^o \circ k^o_0$ are real-valued and since $\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\beta^o \circ k^o_0(s)}{s} ds = 0$, we can use Theorem 2.1 (ii), (v) and write

$$
\begin{cases}
(\beta^i \circ k^i_0)(r_0^{-1}t) = \Re\left\{2\mathbf{C}^+_{r_0}[\beta^i \circ k^i_0(r_0^{-1}\cdot)](t) - \mathbf{C}_{r_0}[\beta^i \circ k^i_0(r_0^{-1}\cdot)](0)\right\}, \\
(\beta^o \circ k^o_0)(t) = \Re\left\{-2\mathbf{C}^-_1[\beta^o \circ k^o_0(\cdot)](t)\right\}.
\end{cases}
$$

Then by Theorem 2.1 (ii), and by (4.2), we deduce the following system of equations

$$
(4.4)\quad \begin{cases}
\Re\Big\{2\mathbf{C}_1^+[\phi^o](t) - 2\mathbf{C}_{r_0}^+[\phi^i](t) - \dfrac{d\cdot t}{r_0} 2\mathbf{C}_1^+[g'_{\boldsymbol{\zeta}_0}](t) \\
\qquad - 2\mathbf{C}_{r_0}^+[\beta^i \circ k_0^i(r_0^{-1}\cdot)](t) + \mathbf{C}_{r_0}[\beta^i \circ k_0^i(r_0^{-1}\cdot)](0)\Big\} = 0 \quad \forall t \in r_0\mathbb{T}, \\
\Re\Big\{2\mathbf{C}_1^-[\phi^o](t) - 2\mathbf{C}_{r_0}^-[\phi^i](t) - \dfrac{d\cdot t}{r_0} 2\mathbf{C}_{r_0}^-[g'_{[\boldsymbol{\zeta}_0]}](t) \\
\qquad + 2\mathbf{C}_1^-[\beta^o \circ k_0^o(\cdot)](t)\Big\} = 0 \quad \forall t \in \mathbb{T}, \\
\Re\left\{\dfrac{\mathbf{C}_{r_0}^+[\phi^i](t) - \mathbf{C}_{r_0}^-[\phi^i](t)}{t g'_{[\boldsymbol{\zeta}_0]}(t)}\right\} = \dfrac{1}{2}(\gamma^i \circ k_0^i)(r_0^{-1}t) \quad \forall t \in r_0\mathbb{T}, \\
\Re\left\{\dfrac{\mathbf{C}_1^+[\phi^o](t) - \mathbf{C}_1^-[\phi^o](t)}{t g'_{[\boldsymbol{\zeta}_0]}(t)}\right\} = \dfrac{1}{2}(\gamma^o \circ k_0^o)(t) \quad \forall t \in \mathbb{T}, \\
-\Im\left\{\mathbf{C}_{r_0}[\phi^i](0) - \mathbf{C}_1[\phi^o](0)\right\} = a, \\
\Im\left\{\dfrac{\mathbf{C}_1^+[\phi^o](1) - \mathbf{C}_1^-[\phi^o](1)}{g'_{[\boldsymbol{\zeta}_0]}(1)}\right\} = b.
\end{cases}
$$

The term in braces in the first equation of (4.4) is the restriction to $r_0\mathbb{T}$ of a function which is holomorphic in $r_0\mathbb{D}$, continuous on $\mathrm{cl}\,(r_0\mathbb{D})$, and with zero real part on $r_0\mathbb{T}$. Thus such function is a purely imaginary constant, whose value is determined at 0 by the fifth equation of (4.4) and by the equality $\Im\left\{\mathbf{C}_{r_0}[\beta^i \circ k_0^i(r_0^{-1}\cdot)](0)\right\} = 0$. Similarly, the term in braces of the second equation of (4.4) is the restriction to $\mathbb{T}$ of a function which is holomorphic in $\mathbb{C}\setminus\mathrm{cl}\,\mathbb{D}$, continuous on $\mathbb{C}\setminus\mathbb{D}$, and which has zero real part on $\mathbb{T}$ and limiting value at infinity equal to $\frac{d}{r_0\pi i}\int_{r_0\mathbb{T}} g'_{[\boldsymbol{\zeta}_0]}(s)ds = 0$. Thus we conclude that such function is 0 in $\mathbb{C}\setminus\mathbb{D}$. Then we have the following.

$$
(4.5)\quad \begin{cases}
\mathbf{C}_{r_0}^+[\phi^i](z) = \mathbf{C}_1^+[\phi^o](z) - \dfrac{d\cdot z}{r_0}\mathbf{C}_1^+[g'_{\boldsymbol{\zeta}_0}](z) + \\
\qquad - \mathbf{C}_{r_0}^+[\beta^i \circ k_0^i(r_0^{-1}\cdot)](z) + \dfrac{1}{2}\mathbf{C}_{r_0}[\beta^i \circ k_0^i(r_0^{-1}\cdot)](0) - ia \quad \forall z \in r_0\mathbb{D}, \\
\mathbf{C}_1^-[\phi^o](z) = \mathbf{C}_{r_0}^-[\phi^i](z) + \dfrac{d\cdot z}{r_0}\mathbf{C}_{r_0}^-[g'_{[\boldsymbol{\zeta}_0]}](z) - \mathbf{C}_1^-[\beta^o \circ k_0^o(\cdot)](z) \quad \forall z \in \mathbb{C}\setminus\mathrm{cl}\,\mathbb{D}.
\end{cases}
$$

Now we set

$$
(4.6)\quad \Omega(z) \equiv \frac{\mathbf{C}_1^+[\phi^o](z) - \mathbf{C}_{r_0}^-[\phi^i](z)}{z g'_{[\boldsymbol{\zeta}_0]}(z)} - \frac{\mathbf{C}_1^+[g'_{[\boldsymbol{\zeta}_0]}](z) d}{r_0 g'_{[\boldsymbol{\zeta}_0]}(z)},
$$

and we note that condition $g'_{[\boldsymbol{\zeta}_0]} \in H(\mathbb{A}_{r_0}) \cap C^{m-1,\alpha}(\mathrm{cl}\,\mathbb{A}_{r_0}, \mathbb{R}^2)$, which holds by Theorem 3.1, and Theorem 2.1 (iv) imply that

$$\mathbf{C}_1^+[g'_{[\boldsymbol{\zeta}_0]}] - \mathbf{C}_{r_0}^-[g'_{[\boldsymbol{\zeta}_0]}] = g'_{[\boldsymbol{\zeta}_0]}. \tag{4.7}$$

Then by the third, the fourth, the sixth equation of (4.4), and by (4.5), the function Ω must satisfy the system

$$\begin{cases} \Re\,\Omega(t) = \Re\left\{\dfrac{\mathbf{C}_{r_0}^+[\beta^i \circ k_0^i(r_0^{-1}\cdot)](t) - \frac{1}{2}\mathbf{C}_{r_0}[\beta^i \circ k_0^i(r_0^{-1}\cdot)](0) + ia}{t g'_{[\boldsymbol{\zeta}_0]}(t)}\right\} + \\ \qquad + \dfrac{1}{2}(\gamma^i \circ k_0^i)(r_0^{-1}t) \equiv \delta^i(t) \quad \forall t \in r_0\mathbb{T}, \\ \Re\,\Omega(t) = \Re\left\{\dfrac{-\mathbf{C}_1^-[\beta^o \circ k_0^o(\cdot)](t)}{t g'_{[\boldsymbol{\zeta}_0]}(t)} - \dfrac{d}{r_0}\right\} + \dfrac{1}{2}(\gamma^o \circ k_0^o)(t) \equiv \delta^o(t) \quad \forall t \in \mathbb{T}, \\ \Im\,\Omega(1) = b - \Im\left\{\dfrac{\mathbf{C}_1^-[\beta^o \circ k_0^o(\cdot)](1)}{g'_{[\boldsymbol{\zeta}_0]}(1)}\right\}. \end{cases} \tag{4.8}$$

To shorten our notation, we find convenient to set

$$\begin{aligned} \omega^i(t) \equiv& \frac{\mathbf{C}_{r_0}^+[\beta^i \circ k_0^i(r_0^{-1}\cdot)](t) - \frac{1}{2}\mathbf{C}_{r_0}[\beta^i \circ k_0^i(r_0^{-1}\cdot)](0) + ia}{t} \quad \forall t \in r_0\mathbb{T}, \\ \omega^o(t) \equiv& \frac{-\mathbf{C}_1^-[\beta^o \circ k_0^o(\cdot)](t)}{t} \quad \forall t \in \mathbb{T}. \end{aligned} \tag{4.9}$$

By Theorem 2.1 (ii), the function $\frac{\Omega(z)}{z}$ is holomorphic in $\mathbb{A}_{r_0}$ and continuous on $\mathrm{cl}\,\mathbb{A}_{r_0}$. Then we must have

$$\frac{1}{2\pi i}\int_{r_0\mathbb{T}} \frac{\Omega(s)}{s}\,ds = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\Omega(s)}{s}\,ds. \tag{4.10}$$

Then by taking the real part in (4.10), we have

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \left\{ -\frac{d}{r_0} + \frac{1}{2}(\gamma^o \circ k_0^o)(t) - \Re\left\{\frac{\mathbf{C}_1^-[\beta^o \circ k_0^o(\cdot)](s)}{s g'_{[\boldsymbol{\zeta}_0]}(s)}\right\}\right\} \frac{ds}{s} = \frac{1}{2\pi i}\int_{r_0\mathbb{T}} \frac{\delta^i(s)}{s}\,ds,$$

and thus by solving for d, we obtain

$$d = \frac{r_0}{2\pi i}\int_{\mathbb{T}} \left\{ \frac{1}{2}(\gamma^o \circ k_0^o)(t) - \Re\left\{\frac{\mathbf{C}_1^-[\beta^o \circ k_0^o(\cdot)](s)}{s g'_{[\boldsymbol{\zeta}_0]}(s)}\right\}\right\} \frac{ds}{s} - \frac{r_0}{2\pi i}\int_{r_0\mathbb{T}} \frac{\delta^i(s)}{s}\,ds. \tag{4.11}$$

By Theorem 2.1 (vii), we have

$$\Omega = \boldsymbol{\Sigma}_{r_0}\left[\delta^i, \delta^o\right] + i\left\{ b - \Im\left\{\frac{\mathbf{C}_1^-[\beta^o \circ k_0^o(\cdot)](1)}{g'_{[\boldsymbol{\zeta}_0]}(1)}\right\}\right\}, \tag{4.12}$$

with (δ^i, δ^o) defined in (4.8), and with d defined as in (4.11). Then by combining (4.3), (4.5), the definition (4.6) of Ω, equality (4.7), and (4.12), we obtain

$$\begin{aligned}
\nu^i(r_0^{-1}t) = & \frac{t}{r_0}\left\{ \mathbf{\Sigma}_{r_0}\left[\delta^i, \delta^o\right](t) - \frac{\omega^i(t)}{g'_{[\boldsymbol{\zeta}_0]}(t)} \right\} + \\
& \frac{it}{r_0}\left\{ b - \Im\left\{ \frac{\mathbf{C}_1^-[\beta^o \circ k_0^o(\cdot)](1)}{g'_{[\boldsymbol{\zeta}_0]}(1)} \right\}\right\} \qquad \forall t \in r_0\mathbb{T}, \\
\nu^o(t) = & t\left\{ \mathbf{\Sigma}_{r_0}\left[\delta^i, \delta^o\right](t) - \frac{\omega^o(t)}{g'_{[\boldsymbol{\zeta}_0]}(t)} + \frac{d}{r_0} \right\} + \\
& + it\left\{ b - \Im\left\{ \frac{\mathbf{C}_1^-[\beta^o \circ k_0^o(\cdot)](1)}{g'_{[\boldsymbol{\zeta}_0]}(1)} \right\}\right\} \qquad \forall t \in \mathbb{T},
\end{aligned} \tag{4.13}$$

where ω^i and ω^o have been defined in (4.9). By (4.11) and (4.13), we see that if $(\boldsymbol{\beta}, \boldsymbol{\gamma}, a, b) = 0$, then $(\boldsymbol{\nu}, d) = 0$, which implies the uniqueness for system (4.2). Finally, by using Theorem 2.1, it is easy to verify that if $(\boldsymbol{\beta}, \boldsymbol{\gamma}, a, b) \in V^{m,\alpha}_{h_0^o}$, then $(\boldsymbol{\nu}, d)$, defined by (4.11) and (4.13) is in $\left(C^{m-1,\alpha}(\mathbb{T}, \mathbb{C})\right)^2 \times \mathbb{R}$ and satisfies system (4.2). □

5. A Regularity Theorem for the Linearized Problem

The aim of this section is to show that the solution $\boldsymbol{\nu} \in \left(C^{m-1,\alpha}(\mathbb{T}, \mathbb{C})\right)^2$ of system (4.2), whose existence has been proved in Theorem 4.1, actually belongs to $\left(C^{m,\alpha}(\mathbb{T}, \mathbb{C})\right)^2$, when $(\boldsymbol{\beta}, \boldsymbol{\gamma}, a, b) \in V^{m,\alpha}_{h_0^o}$. It clearly suffices to show that the pair of functions $\boldsymbol{\nu}$ determined by formula (4.13) belongs to $\left(C^{m,\alpha}(\mathbb{T}, \mathbb{C})\right)^2$, when $(\boldsymbol{\beta}, \boldsymbol{\gamma}, a, b)$ belongs to $V^{m,\alpha}_{h_0^o}$. We note however that such fact is by no means obvious. Indeed, the function $g'_{[\boldsymbol{\zeta}_0]}$ which appears on the right-hand side of (4.13), is only of class $C^{m-1,\alpha}(\mathrm{cl}\,\mathbb{A}_{r_0}, \mathbb{R}^2)$.

We first introduce two technical Lemmas. The following is basically a restatement of Lanza and Rogosin [20, Lemma 5.1].

Lemma 5.1. *Let $\alpha \in]0,1[, r > 0$. Let $\psi \in C^{0,\alpha}(r\mathbb{T}, \mathbb{C}), \omega \in C^{1,\alpha}(r\mathbb{T}, \mathbb{C})$. Then for all $t \in r\mathbb{T}$, the integral*

$$Q(t) \equiv \frac{1}{\pi i}\int_{r\mathbb{T}} \psi(s)\frac{\omega(s) - \omega(t)}{s - t}ds$$

is convergent in the sense of an improper Riemann integral. Moreover, $Q(\cdot) \in C^{1,\alpha}(r\mathbb{T}, \mathbb{C})$.

Proof. By changing the variable with $s = r^{-1}\sigma$, we reduce the integral on $r\mathbb{T}$ to an integral on $\mathbb{T}$. Then the Lemma follows by Lanza and Rogosin [20, Lemma 5.1]. □

Lemma 5.2. *Let* $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$, $r \in]0,1[$. *If* $\psi \in C^{m-1,\alpha}(\mathrm{cl}\,\mathbb{A}_r, \mathbb{R}^2) \cap H(\mathbb{A}_r)$ *and if* $\omega \in C^{m,\alpha}(\partial\mathbb{A}_r, \mathbb{C})$, *then*

$$(\mathbf{I} + \mathbf{S}_r)[\omega\psi] \in C^{m,\alpha}(r\mathbb{T}, \mathbb{C}), \qquad (\mathbf{I} - \mathbf{S}_1)[\omega\psi] \in C^{m,\alpha}(\mathbb{T}, \mathbb{C}). \tag{5.1}$$

Proof. We first prove the second part of (5.1). Let $m = 1$. By Theorem 2.1 (iv), we have

$$\psi = \mathbf{S}_1[\psi] - 2\mathbf{C}_r[\psi] \qquad \text{on} \qquad \mathbb{T}.$$

Thus,

$$\omega(t)\psi(t) - \mathbf{S}_1[\omega\psi](t) = -\frac{1}{\pi i}\int_{\mathbb{T}} \psi(s)\frac{\omega(s) - \omega(t)}{s - t}\, ds - \omega(t)\frac{1}{\pi i}\int_{r\mathbb{T}} \frac{\psi(s)}{s - t}\, ds, \tag{5.2}$$

for all $t \in \mathbb{T}$. By Theorem 2.1 (ii), the last integral of (5.2) belongs to $C^\infty(\mathbb{T}, \mathbb{C})$. Thus Lemma 5.1 implies the validity of the second part of (5.1) for $m = 1$. To prove the statement for $m > 1$, we note that Theorem 2.1 implies that

$$D^{m-1}\{(\mathbf{I} - \mathbf{S}_1)[\omega\psi]\} = (\mathbf{I} - \mathbf{S}_1)\left[D^{m-1}(\omega\psi)\right] =$$
$$= \sum_{l=0}^{m-1} \binom{m-1}{l} (\mathbf{I} - \mathbf{S}_1)\left[D^{m-1-l}\omega D^l\psi\right] \qquad \text{on} \qquad \mathbb{T}.$$

Since $C^1(\mathbb{T}, \mathbb{C}) \subseteq C^{0,\alpha}(\mathbb{T}, \mathbb{C})$ and $C^1(\mathrm{cl}\,\mathbb{A}_r, \mathbb{R}^2) \subseteq C^{0,\alpha}(\mathrm{cl}\,\mathbb{A}_r, \mathbb{R}^2)$, our assumptions on ω, ψ imply that $D^{m-1-l}\omega \in C^{1,\alpha}(\mathbb{T}, \mathbb{C})$ and that $D^l\psi \in C^{0,\alpha}(\mathrm{cl}\,\mathbb{A}_r, \mathbb{R}^2) \cap H(\mathbb{A}_r)$. Thus we obtain the conclusion for $m > 1$ by exploiting the case $m = 1$. The proof of the first part of (5.1) is analogous. □

We are now in a position to prove that formulas (4.13) actually determine a solution of class $C^{m,\alpha}$.

Theorem 5.1. *Let* $\alpha \in]0,1[$, $m \in \mathbb{N} \setminus \{0\}$. *Let* $(\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0) \in \mathcal{A}$ *satisfy equation* $\mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0] = 0$. *For all* $(\boldsymbol{\beta}, \boldsymbol{\gamma}, a, b) \in V^{m,\alpha}_{h^o_0}$, *the pair of functions* $\boldsymbol{\nu} \equiv (\nu^i, \nu^o)$ *delivered by* (4.13) *belongs to* $(C^{m,\alpha}(\mathbb{T}, \mathbb{C}))^2$.

Proof. First we note that if we set $d_1 \equiv d - d_2$, with d defined by (4.11), and with d_2 defined by the equality

$$\frac{1}{2\pi i}\int_{r_0\mathbb{T}} \left\{\frac{1}{2}(\gamma^i \circ k^i_0)(r_0^{-1}s)\right\} \frac{ds}{s} = \frac{1}{2\pi i}\int_{\mathbb{T}} \left\{\frac{1}{2}(\gamma^o \circ k^o_0)(s) - \frac{d_2}{r_0}\right\} \frac{ds}{s},$$

then by (4.9) we have $\frac{1}{2\pi i}\int_{r_0\mathbb{T}} \Re\left(\frac{\omega^i}{g'_{[\boldsymbol{\zeta}_0]}}\right)\frac{ds}{s} = \frac{1}{2\pi i}\int_{\mathbb{T}}\left\{\Re\left(\frac{\omega^o}{g'_{[\boldsymbol{\zeta}_0]}}\right) - \frac{d_1}{r_0}\right\}\frac{ds}{s}$, and thus by Theorem 2.1 (vii) we can write

$$\boldsymbol{\Sigma}_{r_0}\left[\delta^i, \delta^o\right] = \boldsymbol{\Sigma}_{r_0}\left[\Re\left(\frac{\omega^i}{g'_{[\boldsymbol{\zeta}_0]}}\right), \Re\left(\frac{\omega^o}{g'_{[\boldsymbol{\zeta}_0]}}\right) - \frac{d_1}{r_0}\right] +$$
$$+ \frac{1}{2}\boldsymbol{\Sigma}_{r_0}\left[(\gamma^i \circ k^i_0)(r_0^{-1}\cdot), (\gamma^o \circ k^o_0)(\cdot) - \frac{2d_2}{r_0}\right].$$

Thus by (4.9), (4.13) it suffices to prove that if $\psi \in C^{m-1,\alpha}(\mathrm{cl}\,\mathbb{A}_{r_0}, \mathbb{R}^2) \cap H(\mathbb{A}_{r_0})$ and if $\omega = (\omega^i, \omega^o) \in C^{m,\alpha}(r_0\mathbb{T}, \mathbb{C}) \times C^{m,\alpha}(\mathbb{T}, \mathbb{C})$, and if

$$\frac{1}{2\pi i}\int_{r_0\mathbb{T}} \Re\left(\omega^i(s)\psi(s)\right)\frac{ds}{s} = \frac{1}{2\pi i}\int_{\mathbb{T}} \Re\left(\omega^o(s)\psi(s)\right) - \frac{d_1}{r_0}\frac{ds}{s},$$

then the following holds

$$\begin{cases} \mathbf{\Sigma}_{r_0}\left[\Re\left(\omega^i\psi\right), \Re\left(\omega^o\psi\right) - \dfrac{d_1}{r_0}\right] - \omega^i\psi \in C^{m,\alpha}(r_0\mathbb{T}, \mathbb{C}), \\ \mathbf{\Sigma}_{r_0}\left[\Re\left(\omega^i\psi\right), \Re\left(\omega^o\psi\right) - \dfrac{d_1}{r_0}\right] - \left(\omega^o\psi - \dfrac{d_1}{r_0}\right) \in C^{m,\alpha}(\mathbb{T}, \mathbb{C}). \end{cases} \tag{5.3}$$

By definition of the Schwarz operator for the annulus $\mathbf{\Sigma}_{r_0}[\cdot,\cdot]$ (cf. Theorem 2.1 (vii)), it suffices to show that the imaginary parts of the left hand sides of (5.3) belong to $C^{m,\alpha}(r_0\mathbb{T}, \mathbb{C})$, and to $C^{m,\alpha}(\mathbb{T}, \mathbb{C})$, respectively. By Theorem 2.1 (vii), the function $\mathbf{\Sigma}_{r_0}\left[\Re\left(\omega^i\psi\right), \Re\left(\omega^o\psi\right) - \frac{d_1}{r_0}\right]$ belongs to $C^{m-1,\alpha}(\mathrm{cl}\,\mathbb{A}_{r_0}, \mathbb{R}^2) \cap H(\mathbb{A}_{r_0})$. Thus we have

$$\begin{aligned} \xi^i \equiv \mathbf{\Sigma}_{r_0}\left[\Re\left(\omega^i\psi\right), \Re\left(\omega^o\psi\right) - \frac{d_1}{r_0}\right]_{|r_0\mathbb{T}} &\in \quad C^{m-1,\alpha}(r_0\mathbb{T}, \mathbb{C}), \\ \xi^o \equiv \mathbf{\Sigma}_{r_0}\left[\Re\left(\omega^i\psi\right), \Re\left(\omega^o\psi\right) - \frac{d_1}{r_0}\right]_{|\mathbb{T}} &\in \quad C^{m-1,\alpha}(\mathbb{T}, \mathbb{C}). \end{aligned}$$

By Theorem 2.1 (iv), (vii), the following holds

$$\begin{cases} \Im\left\{\xi^i(t)\right\} + \Im\left\{\mathbf{S}_{r_0}[\Re\left\{\xi^i\right\}](t)\right\} + \Im\left\{i\mathbf{S}_{r_0}[\Im\left\{\xi^i\right\}](t)\right\} = 2\Im\left\{\mathbf{C}_1[\xi^o](t)\right\} \quad \forall t \in r_0\mathbb{T}, \\ \Im\left\{\xi^o(t)\right\} - \Im\left\{\mathbf{S}_1[\Re\left\{\xi^o\right\}](t)\right\} - \Im\left\{i\mathbf{S}_1[\Im\left\{\xi^o\right\}](t)\right\} = -2\Im\left\{\mathbf{C}_{r_0}[\xi^i](t)\right\} \quad \forall t \in \mathbb{T}. \end{cases}$$

Then by Theorem 2.1 (v), we obtain

$$\begin{cases} \Im\left\{\xi^i(t)\right\} = -\mathbf{H}_{r_0}[\Re\left\{\xi^i(\cdot)\right\}](t) - \mathbf{C}_{r_0}[\Im\left\{\xi^i(\cdot)\right\}](0) + 2\Im\left\{\mathbf{C}_1[\xi^o(\cdot)](t), \ t \in r_0\mathbb{T}, \right. \\ \Im\left\{\xi^o(t)\right\} = \mathbf{H}_1[\Re\left\{\xi^o(\cdot)\right\}](t) + \mathbf{C}_1[\Im\left\{\xi^o(\cdot)\right\}](0) - 2\Im\left\{\mathbf{C}_{r_0}[\xi^i(\cdot)](t), \ t \in \mathbb{T}. \right. \end{cases}$$

By Theorem 2.1 (ii), the last two terms on the right-hand sides of the last two equations are of class C^∞. Then, it suffices to show that

$$\begin{cases} \Im\left\{(\omega^i\psi)(\cdot)\right\} + \mathbf{H}_{r_0}\left[\Re\left\{\xi^i(\cdot)\right\}\right](\cdot) \in C^{m,\alpha}(r_0\mathbb{T}, \mathbb{C}), \\ \Im\left\{(\omega^o\psi)(\cdot) - d_1 r_0^{-1}\right\} - \mathbf{H}_1\left[\Re\left\{(\xi^o(\cdot)\right\}\right](\cdot) \in C^{m,\alpha}(\mathbb{T}, \mathbb{C}). \end{cases} \tag{5.4}$$

We consider the second part of (5.4), the first part can be proved similarly . By Theorem 2.1 (ii) it follows that

$$\begin{aligned} (\mathbf{I} - \mathbf{S}_1)[\omega^o\psi](t) = (\mathbf{I} - \mathbf{S}_1)[\omega^o\psi - d_1 r_0^{-1}](t) = \\ = \left\{\Re\left(\omega^o\psi\right)(t) - d_1 r_0^{-1} + \mathbf{H}_1[\Im\left\{\omega^o\psi\right\}](t)\right\} + \\ + i\left\{\Im\left(\omega^o\psi\right)(t) - \mathbf{H}_1\left[\Re\left\{\omega^o\psi\right\} - d_1 r_0^{-1}\right](t)\right\} + \\ - \mathbf{C}_1\left[\Re\left\{\omega^o\psi\right\} - d_1 r_0^{-1}\right](0) - i\mathbf{C}_1\left[\Im\left\{\omega^o\psi\right\}\right](0) \qquad \text{on } \mathbb{T}. \end{aligned}$$

Since $\Re\left\{\omega^o\psi\right\} - d_1 r_0^{-1} = \Re\,\xi^o$ on $\mathbb{T}$, then the second part of (5.4) follows immediately from Lemma 5.2. □

Corollary 5.1. *Let* $\alpha \in]0,1[, m \in \mathbb{N}\setminus\{0\}$*. Let* $(\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0) \in \mathcal{A}$*, be such that* $\mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0] = 0$*. For all* $(\boldsymbol{\beta}, \boldsymbol{\gamma}, a, b) \in V^{m,\alpha}_{h_0^o}$*, there exists a unique solution* $(\boldsymbol{\nu}, d) \in (C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \times \mathbb{R}$ *of system* (4.2)*.*

Proof. By Theorem 4.1, there exists a unique solution $(\boldsymbol{\nu}, d) \in (C^{m-1,\alpha}(\mathbb{T},\mathbb{C}))^2 \times \mathbb{R}$ given by formulas (4.11), (4.13). Moreover, Theorem 5.1 shows that the functions ν^i, ν^o determined by formulas (4.13) are in fact of class $C^{m,\alpha}(\mathbb{T},\mathbb{C})$. □

6. Analiticity of the Nonlinear Operators $\mathbf{h}[\cdot]$, and $r[\cdot]$

We are ready to prove our main result.

Theorem 6.1. *Let* $\alpha \in]0,1[, m \in \mathbb{N}\setminus\{0\}$*. Then the set* $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \cap \boldsymbol{\mathcal{Z}}$ *is open in* $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2$ *and the (nonlinear) operator* $(\mathbf{h}[\cdot], r[\cdot])$ *of* $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \cap \boldsymbol{\mathcal{Z}}$ *to* $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \times]0,1[$ *defined by* $(\mathbf{h}[\boldsymbol{\zeta}], r[\boldsymbol{\zeta}]) \equiv \left(r^{-1}[\boldsymbol{\zeta}] g^{(-1)}_{[\boldsymbol{\zeta}]} \circ \zeta^i, g^{(-1)}_{[\boldsymbol{\zeta}]} \circ \zeta^o, r[\boldsymbol{\zeta}]\right)$ *is real analytic.*

Proof. By Theorem 3.2 the graph of $(\mathbf{h}[\cdot], r[\cdot])$ coincides with the set of zeros of $\mathbf{P}$ in $\mathcal{A}$. Then it suffices to show that the set of zeros of $\mathbf{P}$ is (locally around each of its points) the graph of a real analytic operator. We now fix $(\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0) \in \mathcal{A}$ with $\mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0] = 0$ and prove that the set of zeros of $\mathbf{P}$ around $(\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0)$ is the graph of a real analytic operator. By Theorem 3.2 we have $h_0^i(\mathbb{T}) = \mathbb{T}$, and $h_0^o(\mathbb{T}) = \mathbb{T}$. Thus by Theorem 3.5, the set of zeros of the operator $\mathbf{P}$ coincides with the set of zeros of operator $\Pi_{h_0^o} \circ \mathbf{P}$, which maps $\mathcal{A}$ to $V^{m,\alpha}_{h_0^o}$. We now check that the assumptions of the Implicit Function Theorem for real analytic maps (cf. *e.g.* Prodi and Ambrosetti [25, Thm. 11.6, p. 101]) are fulfilled. The analiticity of $\Pi_{h_0^o} \circ \mathbf{P}$ follows from that of $\mathbf{P}$ (cf. Theorem 3.4), and by the linearity and continuity of $\Pi_{h_0^o}$ (cf. Theorem 3.5). Thus it suffices to show that $\partial_{(\mathbf{h},r)} \left(\Pi_{h_0^o} \circ \mathbf{P}\right)[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0](\cdot)$ is a linear homeomorphism of $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \times \mathbb{R}$ onto $V^{m,\alpha}_{h_0^o}$. By the linearity of $\Pi_{h_0^o}$ we have

$$\partial_{(\mathbf{h},r)} \left(\Pi_{h_0^o} \circ \mathbf{P}\right)[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0] = \Pi_{h_0^o} \circ \partial_{(\mathbf{h},r)} \mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0].$$

Since the restriction of $\Pi_{h_0^o}$ to $V^{m,\alpha}_{h_0^o}$ is the identity map, it suffices to show that the linear operator $\partial_{(\mathbf{h},r)} \mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0]$ is a homeomorphism of $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \times \mathbb{R}$ onto $V^{m,\alpha}_{h_0^o}$. By Corollary 5.1, the operator $\partial_{(\mathbf{h},r)} \mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0](\cdot)$ is a bijection of $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \times \mathbb{R}$ onto $V^{m,\alpha}_{h_0^o}$. Since $\partial_{(\mathbf{h},r)} \mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0](\cdot)$ is continuous, the Open Mapping Theorem implies that the map $\partial_{(\mathbf{h},r)} \mathbf{P}[\boldsymbol{\zeta}_0, \mathbf{h}_0, r_0](\cdot)$ is a homeomorphism of $(C^{m,\alpha}(\mathbb{T},\mathbb{C}))^2 \times \mathbb{R}$ onto $V^{m,\alpha}_{h_0^o}$, and thus the proof is complete. □

We now briefly indicate a few potential applications of our final Theorem 6.1. First of all, we remark that the number $r[\boldsymbol{\zeta}]$ is the reciprocal of the so-called conformal modulus of the domain $\mathbb{A}[\boldsymbol{\zeta}]$ and depends on the geometry of the domain. Thus, as a corollary of our analyticity statement, we obtain that the conformal modulus of the domain $\mathbb{A}[\boldsymbol{\zeta}]$ (cf. *e.g.*, Gaier [7, p. 185]) depends real analytically upon the domain itself. In particular, since $r[\boldsymbol{\zeta}]$ depends analytically on $\boldsymbol{\zeta}$, the inner radius of the annulus of definition of the Riemann Mapping of a doubly connected domain varies analytically as the domain itself is perturbed analytically. This fact may be of interest in the applications. For example, in the study of the Hele-Shaw moving boundary value problem (cf. Gustaffson [11]), the region occupied at the instant t by a fluid sucked from a narrow channel is represented as the image of a certain univalent function $f(\cdot,t)$, which is assumed to satisfy a certain initial boundary value problem, which for brevity we do not state here. Now, if the region occupied by the fluid is simply connected, the domain of the function $f(\cdot,t)$ is taken to be the unit disk $\mathbb{D}$, and a local result for small positive t has been proved by Reissig and Rogosin [27], under a certain regularization condition of the given problem, which is ill-posed. However, if one considers the case in which the fluid may occupy a doubly connected domain (cf. Gustaffson [12]), then the problem becomes more difficult and the domain of $f(\cdot,t)$ should be chosen as an annulus of inner radius r depending on the domain, and thus on t. Thus Theorem 6.1 may be employed to study such dependence, as well as the dependence of the boundary values of the Riemann map upon the boundary curves of the domain.

Finally, one could consider two disjoint Jordan domains bounded by two plane curves ζ_1, ζ_2, and consider them as sections of electrically conducting cylinders in the presence of an electrostatic field uniform at infinity in $\mathbb{C} \setminus \{\mathbb{I}[\zeta_1] \cup \mathbb{I}[\zeta_2]\}$. As it is well-known, the complex potential of the electric field is determined by a conformal map of $\mathbb{C} \setminus \{\mathbb{I}[\zeta_1] \cup \mathbb{I}[\zeta_2]\}$ onto some doubly connected domain conformally equivalent to an annulus. Thus our Theorem 6.1 could be employed to study the dependence of the electric field on the boundary curves ζ_1, ζ_2, as ζ_1, ζ_2 are varied. Similarly, if $\mathbb{I}[\zeta_1]$ and $\mathbb{I}[\zeta_2]$ are thought as sections of two impermeable cylinders in the presence of an incompressible, inviscid, irrotational potential flow with constant velocity at infinity in $\mathbb{C} \setminus \{\mathbb{I}[\zeta_1] \cup \mathbb{I}[\zeta_2]\}$, one could study the dependence of the velocity field and of the pressure field on the boundary curves ζ_1 and ζ_2, as ζ_1, ζ_2 are varied.

Acknowledgement The authors are thankful to Prof. Christian Pommerenke for a substantial hint in the proof of Theorem 3.1.

The authors are indebted to the 'Gruppo Nazionale per l'Analisi Funzionale e le sue Applicazioni' of the Italian 'Consiglio Nazionale delle Ricerche' and to 'The Belarusian Fund for Fundamental Scientific Research' for their financial support. The second author would like to express his deep gratitude to Prof. M. Lanza de Cristoforis for the warm hospitality during his visit to the University of Padova in July-August 1998.

References

[1] L. Ahlfors, *Complex Analysis*, third edition, McGraw-Hill, New York, etc., 1979.

[2] N. I. Akhieser, *Elements of the Theory of Elliptic Functions*, Amer. Math. Soc., Providence, Rhode Island, 1990.

[3] M. S. Berger, *Nonlinearity and Functional Analysis*, Academic Press, New York, 1977.

[4] R. R. Coifman and Y. Meyer, Lavrentiev's Curves and Conformal Mappings, Institut Mittag-Leffler, Report No. 5, 1983.

[5] M. A. Efendiev and W. L. Wendland, Nonlinear Riemann-Hilbert problems for multiply connected domains, *Nonlinear Analysis. Theory, methods & applications*, **27**, No. 1 (1996), 37–58.

[6] M. A. Efendiev and W. L. Wendland, Nonlinear Riemann-Hilbert problems without transversality, *Math. Nachr.*, **183** (1997), 73–89.

[7] D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer Verlag, Berlin, 1964.

[8] F. D. Gakhov, *Boundary Value Problems*, Pergamon Press, Oxford-London etc., 1966.

[9] I. Gohberg and N. Krupnik, *Introduction to the Theory of One-Dimensional Singular Integral Equations*, vol. I, II, Birkhäuser Verlag, Basel-Boston-Berlin, 1992.

[10] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Amer. Math. Soc., Providence, Rhode Island, 1969.

[11] B. Gustafsson, Applications of variational inequalities to a moving boundary problem for Hele-Shaw flows, *SIAM Jour. Math. Anal.*, **16** (1985), 279–300.

[12] B. Gustafsson, An ill-posed moving boundary problem for doubly connected domains, *Arkiv för Math.*, **25** (1987), 231–253.

[13] L. V. Kantorovich and V. I. Krylov, *Approximate Methods of Higher Analysis, Interscience Publ.*, New York, 1964.

[14] M. V. Keldysh, On the problem of conformal mapping of multiply connected domains onto canonical domains, *Uspekhi Math. Nauk*, **6** (1939), 90–119, (Russian).

[15] M. Lanza de Cristoforis, A note on conformal representation in Sobolev spaces, *Complex Variables*, **20** (1992), 121–133.

[16] M. Lanza de Cristoforis, Conformal representation and domain dependence, *Boundary Value Problems, Special Functions and Fractional Calculus* (I.V. Gaishun and A.A. Kilbas (eds.)), Proceedings of International Conference, Minsk, Belarus, February 16-20, 1996, 195–203.

[17] M. Lanza de Cristoforis, A functional decomposition theorem for the conformal representation, *J. of Math. Soc. of Japan*, **49** (1997), 759–780.

[18] M. Lanza de Cristoforis and S. S. Antman, The large deformation of nonlinearly elastic tubes in two-dimensional flows, *SIAM J. Math. Anal.*, **22** (1991), 1193–1221.

[19] M. Lanza de Cristoforis and L. Preciso, On the Analiticity of the Cauchy Integral in Schauder Spaces, *Journal of Integral Equations and Applications*, **11** (1999), 363–391.

[20] M. Lanza de Cristoforis and S. V. Rogosin, Analiticity of a nonlinear operator associated to the conformal representation in Schauder spaces. An integral equation approach, 1997, (to appear).

[21] S. G. Mikhlin, *Integral Equations and their applications to certain problems in mechanics, mathematical physics and technology*, Pergamon Press, Oxford-London etc., 1964.

[22] N. I. Muskhelishvili, *Singular Integral Equations*, P. Noordhoff N.V., Groningen, 1953.

[23] W. Pogorzelski, *Integral Equations and their Applications*, vol. I, Pergamon Press, Oxford-London etc., 1966.

[24] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer Verlag, Berlin etc., 1992.

[25] G. Prodi and A. Ambrosetti, *Analisi non lineare*, Editrice Tecnico Scientifica, Pisa, 1973.

[26] T. Radó, Sur la representation conforme de domaines variables, *Acta Univ.Szeged*, **1** (1923), 180–186.

[27] M. Reissig and S. V. Rogosin, with an appendix of F. Huebner, Analytical and numerical treatment of a complex model for Hele-Shaw moving boundary value problems with kinetic undercooling regularization, *Euro J. Appl. Math.*, **10** (1998), 561–579.

[28] I. N. Vekua, *Generalized Analytic Functions*, Pergamon Press, Oxford-London etc., 1962.

[29] S. E. Warschawski, On the higher derivatives at the boundary in conformal mapping, *Trans. Amer. Math. Soc.*, **38** (1935), 310–340.

[30] E. Wegert, *Nonlinear Boundary Value Problems for Holomorphic Functions and Singular Integral Equations*, Akademie Verlag, Berlin, 1992.

[31] S. Wu, Analytic dependence of Riemann mappings for bounded domains and minimal surfaces, *Comm. on Pure and Appl. Math.*, **XLVI** (1993), 1303–1326.