Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains

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Abstract

Using three different notions of generalized principal eigenvalue of linear second order elliptic operators in unbounded domains, we derive necessary and sufficient conditions for the validity of the maximum principle, as well as for the existence of positive eigenfunctions for the Dirichlet problem. Relations between these principal eigenvalues, their simplicity and several other properties are further discussed.

Contents

1 Definitions and main results

1.1 Introduction

What is the principal eigenvalue of a general linear second order elliptic operator in an unbounded domain associated with Dirichlet conditions ? Under what conditions do such operators satisfy the maximum principle ? When do positive eigenfunctions exist ? These are some of the themes we discuss in this paper.

The Krein-Rutman theory provides the existence of the principal (or first) eigenvalue λ_Ω of an elliptic operator $-L$ in a bounded smooth domain Ω, under Dirichlet boundary condition. This eigenvalue is the bottom of the spectrum of $-L$, for the Dirichlet problem, it is simple and the associated eigenfunction is positive in Ω. The positivity of λ_Ω guarantees the existence of a unique solution to the inhomogeneous Dirichlet problem. These properties, together with several others, have been extended by H. Berestycki, L. Nirenberg and S. R. S. Varadhan [11] to the case of *bounded non-smooth* domains by introducing the notion of the *generalized principal eigenvalue*.

In the present paper, we consider the case of *unbounded domains*, continuing the study begun in [8], in collaboration with F. Hamel, and in [12]. Our aim is to emphasize the implications of unboundedness of the domain rather than lack of smoothness. For this reason, some of our results are stated for domains with smooth boundaries even though the techniques of [11] would allow one to extend them to non-smooth domains. Let us also say from the outset that rather than adopting a functional analytical point of view, the object of study here are the very partial differential equations (or inequalities) associated with the eigenvalue problem. We consider general linear second order not necessarily self-adjoint operators. As we show here, some of the basic properties of the principal eigenvalue fail in general in the unbounded case.

In [8], it has been pointed out that the generalized principal eigenvalue λ_1 of [11] is not suited for characterizing the existence of solutions for a class of semilinear problems in unbounded domains. It is further shown that another quantity - denoted by λ_1' - provides the right characterization.

Here, we introduce still another quantity, λ_1'', which turns out to provide a sufficient condition for the validity of the maximum principle in unbounded domains. The main object of this paper is to investigate the relations between the three quantities λ_1, λ_1', λ_1'' and their properties. The relations between λ_1 and λ_1' have been established in [8], [12], but only in low dimension. Here, we improve them to arbitrary dimension.

The forthcoming paper [10], in collaboration with G. Nadin, deals with extensions to parabolic operators of the notions introduced here. We also examine there the relationship of these notions with Lyapunov exponent type ideas. Applications to nonlinear problems will be further discussed in [10].

1.2 Motivations: semilinear problems, maximum principle and eigenfunctions

Our interest for the generalization of the notion of principal eigenvalue to unbounded domains originally stemmed from the study of the Fisher-KPP reaction-diffusion equation

$$\partial_t u - a_{ij}(x)\partial_{ij}u - b_i(x)\partial_i u = f(x,u), \quad t>0,\ x\in\mathbb{R}^N,$$

which arises for instance in some models in population dynamics. In such models, the large time behavior of the population - and in particular its persistence or extinction - is determined by the existence of a unique positive stationary solution. This, in turn, depends on the sign of the principal eigenvalue associated with the linearized operator about $u\equiv 0$:

$$\mathscr{L}u = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + f_s(x,0)u.$$

When the coefficients of the equation do not depend on x, the right notion of principal eigenvalue is the quantity λ_1 introduced in [11], whereas when the coefficients are periodic and $\mathscr{L}$ is self-adjoint, it is the periodic principal eigenvalue (see [7]). In the general case considered in [8], one needs to consider both λ_1 and λ_1', the latter being a kind of generalization of the periodic principal eigenvalue.

Furthermore, the study of asymptotic spreading speeds for general Fisher-KPP equations like the one above involves principal eigenvalues of families of associated linear operators. Building on the results and related notions to the ones presented here, properties about the asymptotic spreading speed for general non-homogeneous equations are established in [9].

Another motivation for our study comes from a very basic question: does the sign of the generalized principal eigenvalue λ_1 characterize the validity of the maximum principle for bounded solutions to linear equations in unbounded domains ? This is known to be the case for bounded domains. We show here that the answer is no. The necessary and sufficient conditions for the maximum principle will be shown here to hinge on λ_1' and on another generalization of the principal eigenvalue, denoted by λ_1''.

It is also a very natural question in itself to determine what are the eigenvalues associated with positive eigenfunctions for the Dirichlet condition, as well as their multiplicities, for general operators in unbounded domains.

1.3 Hypotheses and definitions

Throughout the paper, Ω denotes a domain in $\mathbb{R}^N$ (in general unbounded and possibly non-smooth) and L a general elliptic operator in non-divergence form:

$$Lu = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u$$

(the usual convention for summation from 1 to N on repeated indices is adopted). When we say that Ω is smooth we mean that it is of class $C^{1,1}$. We use the notation

$\underline{\alpha}(x)$, $\overline{\alpha}(x)$ to indicate respectively the smallest and the largest eigenvalues of the symmetric matrix $(a_{ij}(x))$, i. e.

$$\underline{\alpha}(x) := \min_{\substack{\xi\in\mathbb{R}^N\\|\xi|=1}} a_{ij}(x)\xi_i\xi_j, \qquad \overline{\alpha}(x) := \max_{\substack{\xi\in\mathbb{R}^N\\|\xi|=1}} a_{ij}(x)\xi_i\xi_j.$$

The basic assumptions on the coefficients of L are:

$$a_{ij}\in C^0(\overline{\Omega}), \qquad \forall\, x\in\overline{\Omega}, \quad \underline{\alpha}(x)>0, \qquad b_i,c\in L^\infty_{\mathrm{loc}}(\overline{\Omega}).$$

These hypotheses will always be understood, unless otherwise specified, since they are needed in most of our results. Note that we allow the ellipticity of (a_{ij}) to degenerate at infinity. Also, $C^0(\overline{\Omega})$ denotes the space of functions which are continuous on $\overline{\Omega}$, but not necessarily bounded. Additional hypotheses will be explicitly required in some of the statements below. The operator L is said to be uniformly elliptic if $\inf_\Omega \underline{\alpha}>0$ and is termed self-adjoint if it can be written in the form

$$Lu = \partial_i(a_{ij}(x)\partial_j u) + c(x)u.$$

It is well known that if the domain Ω is bounded and smooth then the Krein-Rutman theory (see [20]) implies the existence of a unique real number $\lambda=\lambda_\Omega$ such that the problem

$$\begin{cases} -L\varphi = \lambda\varphi & \text{a.e. in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases}$$

admits a positive solution $\varphi\in W^{2,p}(\Omega)$, $\forall\, p<\infty$. The quantity λ_Ω and the associated eigenfunction φ (which is unique up to a multiplicative constant) are respectively called Dirichlet principal eigenvalue and eigenfunction of $-L$ in Ω. Henceforth, we keep the notation λ_Ω for this Dirichlet principal eigenvalue.

The Krein-Rutman theory cannot be applied if Ω is non-smooth or unbounded (except for problems in periodic settings), because the resolvent of $-L$ is not compact. However, the fundamental properties of the Dirichlet principal eigenvalue have been extended in [11] to the case of non-smooth bounded domains considering the following notion:

$$(1.1)\quad \lambda_1(-L,\Omega) := \sup\{\lambda \ :\ \exists\,\phi\in W^{2,N}_{\mathrm{loc}}(\Omega),\ \phi>0,\ (L+\lambda)\phi\le 0 \text{ a.e. in } \Omega\}.$$

If Ω is bounded and smooth, then $\lambda_1(-L,\Omega)$ coincides with the classical Dirichlet principal eigenvalue λ_Ω. An equivalent definition was previously given by S. Agmon in [1] in the case of operators in divergence form defined on Riemannian manifolds and, for general operators, by R. D. Nussbaum and Y. Pinchover [22], building on a result by M. H. Protter and H. F. Weinberger [27].

The quantity defined by (1.1) is our first notion of a generalized principal eigenvalue in an unbounded domain. We also consider here two other generalizations.

Definition 1.1. For given Ω and L, we set

(1.2)
$$\lambda_1'(-L,\Omega) := \inf\{\lambda \ : \ \exists\, \phi \in W^{2,N}_{\mathrm{loc}}(\Omega)\cap L^\infty(\Omega),\ \phi > 0,\ (L+\lambda)\phi \geq 0 \text{ a.e. in } \Omega,\ \forall\, \xi \in \partial\Omega,\ \lim_{x\to\xi}\phi(x) = 0\};$$

(1.3) $$\lambda_1''(-L,\Omega) := \sup\{\lambda \ : \ \exists\, \phi \in W^{2,N}_{\mathrm{loc}}(\Omega),\ \inf_\Omega \phi > 0,\ (L+\lambda)\phi \leq 0 \text{ a.e. in } \Omega\}.$$

The quantity λ_1' has been introduced in [7], [8] and it also coincides with λ_Ω if Ω is bounded and smooth. However, in contradistinction with λ_1, it is equal to the periodic principal eigenvalue when Ω and L are periodic. Later on we will show that these two properties are fulfilled by λ_1'' as well.

If Ω is smooth then the three quantities $\lambda_1(-L,\Omega)$, $\lambda_1'(-L,\Omega)$, $\lambda_1''(-L,\Omega)$ - if finite- are eigenvalues for $-L$ in Ω under Dirichlet boundary conditions. This follows from Theorems 1.4 and 1.7 part (ii) below and the obvious inequality $\lambda_1''(-L,\Omega) \leq \lambda_1(-L,\Omega)$. But, as shown in Section 8, the principal eigenvalues λ_1', λ_1'' do not have in general admissible eigenfunctions, i.e. eigenfunctions satisfying the additional requirements of being bounded from above or having positive infimum far from $\partial\Omega$ respectively.

It may occur that the sets in the definitions (1.1), (1.2) or (1.3) are empty (see Section 2.3). In such cases, we set $\lambda_1(-L,\Omega) := -\infty$, $\lambda_1'(-L,\Omega) := +\infty$, $\lambda_1''(-L,\Omega) := -\infty$ respectively. A sufficient (yet not necessary) condition for the sets in (1.1), (1.3) to be nonempty is: $\sup_\Omega c < \infty$, as is immediately seen by taking $\phi \equiv 1$ in the formulas. We will find that $\lambda_1' < +\infty$ when Ω is smooth as a consequence of a comparison result between λ_1 and λ_1', Theorem 1.7 part (ii). If Ω is non-smooth then the boundary condition in (1.2) is too strong a requirement, and one should relax it in the sense of [11]. However, we do not stress the non-smooth aspect in the present paper. In Section 2.1, we show that, if Ω is uniformly smooth and L is uniformly elliptic and has bounded coefficients, then the definition (1.3) of λ_1'' does not change if the condition $\inf\phi > 0$ is only required in any subset of Ω having positive distance from $\partial\Omega$. This condition is more natural because it is satisfied by the classical Dirichlet principal eigenfunction when Ω is bounded and smooth.

The admissible functions for λ_1' and λ_1'' are bounded respectively from above and from below by (a positive constant times) the function $\beta \equiv 1$. Considering instead an arbitrary barrier β yields further extensions of these definitions.

Definition 1.2. For given Ω, L and positive function $\beta : \Omega \to \mathbb{R}$, we set
$$\lambda_\beta'(-L,\Omega) := \inf\{\lambda \ : \ \exists\, \phi \in W^{2,N}_{\mathrm{loc}}(\Omega),\ 0 < \phi \leq \beta,\ (L+\lambda)\phi \geq 0 \text{ a.e. in } \Omega,\ \forall\, \xi \in \partial\Omega,\ \lim_{x\to\xi}\phi(x) = 0\};$$
$$\lambda_\beta''(-L,\Omega) := \sup\{\lambda \ : \ \exists\, \phi \in W^{2,N}_{\mathrm{loc}}(\Omega),\ \phi \geq \beta,\ (L+\lambda)\phi \leq 0 \text{ a.e. in } \Omega\}.$$

If λ_1', λ_1'' arise in the study of the existence and uniqueness of positive *bounded* solutions for the Dirichlet problem, λ_β', λ_β'' come into play when considering solutions with prescribed maximal (or minimal) growth β. We will mainly focus here on λ_1' and λ_1'', but we also derive properties with λ_β', λ_β'' along the way.

1.4 Statement of the main results

We start with investigating the existence of eigenvalues associated with positive eigenfunctions satisfying Dirichlet boundary conditions. These are given by the problem

$$\begin{cases} -L\varphi = \lambda\varphi & \text{a.e. in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \ \ (\text{if } \Omega \neq \mathbb{R}^N) \end{cases} \tag{1.4}$$

Definition 1.3. We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of $-L$ in Ω (associated with positive eigenfunction), under Dirichlet boundary condition, if the problem (1.4) admits a positive solution $\varphi \in W^{2,p}_{\text{loc}}(\overline{\Omega})$, $\forall\, p < \infty$. Such a solution is called (positive) *eigenfunction* and the set of all eigenvalues is denoted by $\mathscr{E}$.

In the following, since we only deal with positive eigenfunctions, we omit to mention it. If Ω is bounded and smooth then it is well known that $\mathscr{E} = \{\lambda_\Omega\}$ and that this eigenvalue is simple. This property is improved in [11] to non-smooth domains, by replacing λ_Ω with $\lambda_1(-L,\Omega)$ and imposing the boundary conditions on a suitable subset of $\partial\Omega$. The picture changes drastically in the case of unbounded domains. Indeed, in Section 3 below, we derive the following characterization.

Theorem 1.4. *If Ω is unbounded and smooth then $\mathscr{E} = (-\infty, \lambda_1(-L,\Omega)]$.*

Theorem 1.4 improves the property already known that the set of eigenvalues associated with eigenfunctions *without prescribed conditions* on $\partial\Omega$ coincides with $(-\infty, \lambda_1(-L,\Omega)]$ (see, e.g., [1]). Note that in the case of bounded smooth domains this property still holds if one prescribes the Dirichlet condition on a proper subset of the boundary. This example might lead one to believe that the reason why $\mathscr{E}$ does not reduce to a singleton when Ω is unbounded is that no Dirichlet condition is imposed at infinity. Counter-example 3.3 in Section 3 shows that this is not the case, even if one imposes an exponential decay.

Recently, we came across the work [17] by Y. Furusho and Y. Ogura (1981) that does not seem to be very well known. In that paper, they prove Theorem 1.4 but only in the case where Ω is an exterior smooth domain (and L has smooth coefficients). The main difficulty when dealing with general unbounded domain is that, in order to construct a solution, one needs to control the behavior near the boundary of a family of solutions in bounded domains. We achieve this by use of an appropriate version of the *boundary Harnack inequality* (also known as Carleson estimate) due to H. Berestycki, L. Caffarelli and L. Nirenberg [6].

Let us further point out that, if L has Hölder continuous coefficients, the problem of the existence of eigenfunctions (vanishing on $\partial\Omega$) can also be approached by using the Green function and the Martin boundary theory (see, e.g., [24]). However, as far as we know, the result of Theorem 1.4 was not previously derived in the generality in which we state it here.

Next, we derive a necessary and a sufficient condition, expressed in terms of λ_1' and of λ_1'' respectively, for the validity of the maximum principle in unbounded domains. With maximum principle we mean the following:

Definition 1.5. We say that the operator L satisfies the *maximum principle* (*MP* for short) in Ω if every function $u \in W^{2,N}_{\text{loc}}(\Omega)$ such that

$$Lu \geq 0 \ \text{ a.e. in } \Omega, \qquad \sup_{\Omega} u < \infty, \qquad \forall\, \xi \in \partial\Omega,\ \limsup_{x\to\xi} u(x) \leq 0,$$

satisfies $u \leq 0$ in Ω.

Note that no conditions are imposed at infinity, except for the boundedness from above. This condition is redundant if Ω is bounded. In the case of bounded smooth domains, it is well known that the MP holds iff $\lambda_\Omega > 0$. This result is improved in [11] to bounded non-smooth domains by replacing λ_Ω with $\lambda_1(-L,\Omega)$ and considering a refined version of the maximum principle. The extensions and results in the general theory of [11] are recalled in the Appendix A here. For unbounded domains, we will show that the validity of the MP is not related to the sign of λ_1 (even if one restricts Definition 1.5 to subsolutions decaying exponentially to 0, see Counter-example 3.3), but rather to those of λ_1' and λ_1''.

Theorem 1.6. *The operator L satisfies the MP in Ω*

(i) if $\lambda_1''(-L,\Omega) > 0$ and the coefficients of L satisfy

$$\sup_{\Omega} c < \infty, \qquad \limsup_{\substack{x\in\Omega\\ |x|\to\infty}} \frac{|a_{ij}(x)|}{|x|^2} < \infty, \qquad \limsup_{\substack{x\in\Omega\\ |x|\to\infty}} \frac{b(x)\cdot x}{|x|^2} < \infty; \tag{1.5}$$

(ii) only if $\lambda_1'(-L,\Omega) \geq 0$.

Condition (1.5) is specific to the metric of $\mathbb{R}^N$. Actually, many of the results here can be extended to more general Remaniann manifolds. For this purpose, this condition (1.5) should be modified by involving the corresponding metric.

Y. Pinchover pointed out to us that the hypothesis on the a_{ij} in (1.5) is sharp for statement (i) to hold. Indeed, it can be proved that the operator $Lu = (1+|x|)^{2+\varepsilon}\Delta u - u$ in $\mathbb{R}^N$, with $\varepsilon > 0$ and $N \geq 3$, satisfies $\lambda_1''(-L,\mathbb{R}^N) \geq 1$ but the equation $Lu = 0$ in $\mathbb{R}^N$ admits positive bounded solutions (actually, one can show that $\lambda_1'(-L,\mathbb{R}^N) = -\infty$). Moreover, it is easy to construct operators with $b_i(x) = O(|x|^{1+\varepsilon})$ for which $\lambda_1'' > 0$ and the MP does not hold. Let us mention that the continuity of (a_{ij}) is not used in the proof of Theorem 1.6, and that the ellipticity is only required to hold locally uniformly in Ω. Theorem 1.6 is a particular case of Theorem 4.2 below, which asserts that the MP holds for subsolutions satisfying $\sup u/\beta < \infty$ instead of $\sup u < \infty$ - for a given barrier function β growing at most exponentially - if $\lambda''_\beta(-L,\Omega) > 0$ and only if $\lambda'_\beta(-L,\Omega) \geq 0$, where λ'_β, λ''_β are given by Definition 1.2. In the case of operators with bounded coefficients and subsolutions bounded from above, the implication $\lambda''_1 > 0 \Rightarrow$ MP is implicitly contained in Lemma 2.1 of [8]. Note that if Ω is bounded (possibly non-smooth) then Proposition 6.1 of [11] yields $\lambda''_1(-L,\Omega) = \lambda_1(-L,\Omega)$. This is why, in that case, $\lambda_1(-L,\Omega) > 0 \Rightarrow$ MP. In the limiting case where λ'_1 and λ''_1 are equal to 0, the MP might or might not hold (see Remark 4.3 below).

Next, we derive some relations between the generalized principal eigenvalues λ_1, λ'_1 and λ''_1.

Theorem 1.7. *Let Ω be smooth. Then, the following properties hold:*

(i) if L is self-adjoint and the a_{ij} are bounded then $\lambda_1(-L,\Omega) = \lambda'_1(-L,\Omega)$;
(ii) for general L it holds that $\lambda'_1(-L,\Omega) \leq \lambda_1(-L,\Omega)$;
(iii) under the growth condition (1.5) *it holds that $\lambda''_1(-L,\Omega) \leq \lambda'_1(-L,\Omega)$.*

From the above result and the definitions of λ_1 and λ'_1 it follows that, if a self-adjoint operator L with $a_{ij} \in L^\infty(\Omega)$ admits a *bounded* (positive) eigenfunction associated with an eigenvalue $\lambda \in \mathbb{R}$, then necessarily $\lambda = \lambda_1(-L,\Omega) = \lambda'_1(-L,\Omega)$.

We actually prove Theorem 1.7 part (i) with $\lambda'_1(-L,\Omega)$ replaced by $\lambda'_\beta(-L,\Omega)$, for any barrier β with subexponential growth.

In the case of uniformly elliptic operators with bounded smooth coefficients, the inequality $\lambda'_1 \leq \lambda_1$ was proved in [12] in dimension 1, together with the inequality $\lambda'_1 \geq \lambda_1$ for self-adjoint operators in dimension less than 4 (subsequently improved to non-smooth operators in [25]). The question in arbitrary dimension was stated as an open problem. With the results here, it is now completely solved. Instead, the relations between λ'_1 and λ''_1 are not fully understood; Theorem 1.7 part (iii) gives only a partial information. Indeed, we do not know any example of operators for which $\lambda''_1 < \lambda'_1$. We leave it as an open problem to prove the following

Conjecture 1.8. *If Ω is smooth and L has bounded coefficients then $\lambda'_1(-L,\Omega) = \lambda''_1(-L,\Omega)$.*

We are able to prove Conjecture 1.8 in some particular cases, where we actually show that all three notions of generalized principal eigenvalues coincide.

Theorem 1.9. *Let Ω be unbounded and smooth. Then $\lambda_1(-L,\Omega) = \lambda''_1(-L,\Omega)$ ($= \lambda'_1(-L,\Omega)$ if* (1.5) *holds) in each of the following cases:*

1) L is a self-adjoint, uniformly elliptic operator with bounded coefficients and either $N = 1$ or $\Omega = \mathbb{R}^N$ and L is radially symmetric;
2) $L = \tilde{L} + \gamma(x)$, where $\tilde{L}$ is an elliptic operator such that $\lambda_1(-\tilde{L},\Omega) = \lambda''_1(-\tilde{L},\Omega)$ and $\gamma \in L^\infty(\Omega)$ is nonnegative and satisfies $\displaystyle\lim_{\substack{x\in\Omega \\ |x|\to\infty}} \gamma(x) = 0$;

3)
$$\lambda_1(-L,\Omega) \leq -\limsup_{\substack{x\in\Omega\\|x|\to\infty}} c(x);$$

4) the a_{ij} are bounded, L is uniformly elliptic and it is either self-adjoint or in non-divergence form with $\lim\limits_{\substack{x\in\Omega\\|x|\to\infty}} b(x) = 0$, and
$$\forall\, r>0,\ \forall\, \beta < \limsup_{\substack{x\in\Omega\\|x|\to\infty}} c(x), \quad \exists\, B_r(x_0)\subset\Omega \ \ s.\ t. \ \inf_{B_r(x_0)} c > \beta.$$

We remark that the hypothesis on c in the case 4 of Theorem 1.9 is fulfilled if $\Omega = \mathbb{R}^N$ and $c(x) \to \gamma(x/|x|)$ as $|x|\to\infty$, with γ lower semicontinuous. Cases 2-4 will be derived from a general result - Theorem 7.6 below - which provides a useful characterization for λ_1''. One of the tools used in its proof is an extension of the boundary Harnack inequality to inhomogeneous Dirichlet problems. Another tool is a continuity property of $\lambda_1(-L,\Omega)$ with respect to perturbations of the domain Ω. In particular, we derive the following continuity property with respect to *exterior* perturbations, which is of independent interest.

Theorem 1.10. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a family of domains such that $\Omega_1\backslash\Omega$ is bounded, $\partial\Omega$ is smooth in a neighborhood of $\overline{\Omega_1\backslash\Omega}$ and*
$$\forall\, n\in\mathbb{N}, \quad \Omega_n \supset \Omega_{n+1} \supset \Omega, \qquad \bigcap_{n\in\mathbb{N}} \overline{\Omega}_n = \overline{\Omega}.$$
Then $\lambda_1(-L,\Omega_n) \nearrow \lambda_1(-L,\Omega)$ as $n\to\infty$.

In the above statement, it is understood that the coefficients of L satisfy the hypotheses of Section 1.3 in Ω_1 and not only in Ω. Theorem 1.10 is an important feature of the principal eigenvalue λ_1. Contrary to interior convergence of domains (cf. Proposition 2.3 part (iv) below), continuity with respect to exterior perturbations is a subtle issue and it may possibly fail (see, e.g., [4], [15] for the case of bounded domains). We discuss several aspects of this property in Section 7.2.

In Section 8, we discuss the existence of admissible eigenfunctions for λ_1', λ_1'', as well as the simplicity of λ_1. A sufficient condition for the latter is derived by using the notion of solution of *minimal growth at infinity*. This is in the spirit but a slightly different version of the notion introduced by S. Agmon in his pioneering and important paper [1]. Combining this condition with Theorems 1.4, 1.6 and the characterization of λ_1'' given in Theorem 7.6, we are able to extend the basic properties of the classical Dirichlet principal eigenvalue to the case of unbounded domains, provided that c is negative at infinity.

Proposition 1.11. *Let Ω be unbounded and smooth and let*
$$\xi := \limsup_{\substack{x\in\Omega\\|x|\to\infty}} c(x).$$

The following properties hold:

(i) if $\xi < 0$ and (1.5) *holds then L satisfies the MP in Ω iff $\lambda_1(-L,\Omega) > 0$;*

(ii) if $\lambda_1(-L,\Omega) < -\xi$ then any positive function $v \in W^{2,N}_{loc}(\Omega)$ satisfying $(L + \lambda_1(-L,\Omega))v \leq 0$ a.e. in Ω coincides, up to a scalar multiple, with the eigenfunction φ_1 associated with $\lambda_1(-L,\Omega)$. Moreover, φ_1 is bounded and, if the coefficients of L are bounded, it decays exponentially to 0.

Actually, hypothesis (1.5) is not required in the "only if" implication of statement (i). If $\xi = 0$ then $\lambda_1(-L,\Omega) > 0$ does not imply the MP, even if $c < 0$ everywhere (see Remark 8.8). Statement (ii) was announced and used in our previous paper [13]. There, we dealt with Neumann problems in smooth infinite cylinders. This led us to define a notion of generalized principal eigenvalue which incorporates the Neumann boundary condition. However, the proof presented below works exactly at the same way in that case. The last statement of Proposition 1.11 part (ii) follows from a general result about the exponential decay of subsolutions of the Dirichlet problem - Proposition 8.7 below.

We conclude by investigating the continuity of λ_1 with respect to the coefficients, as well as its behavior as the size of the zero and the second order coefficients blows up or the ellipticity degenerates.

Let us point out that some of the results concerning λ_1 and λ_1'' still hold if Ω is not connected. This is seen by noticing that $\lambda_1(-L,\Omega)$ and $\lambda_1''(-L,\Omega)$ coincide with the infimum of the λ_1 and λ_1'' in the connected components of Ω. Exceptions are: the results about the existence of eigenfunctions, such as Theorem 1.4, the implication MP$\Rightarrow \lambda_1 > 0$ in Proposition 1.11 part (i) (unless Ω has a finite number of connected components). Note that λ_1' is equal to the supremum of the λ_1' in the connected components of Ω. We further remark that, if Ω is connected, the definition (1.1) of λ_1 does not change if one replaces $\phi > 0$ with $\phi \geq 0$, $\phi \not\equiv 0$. This is no longer true if Ω is not connected.

As was already mentioned above, most of the results of this paper can be extended to the case of linear elliptic equations on noncompact manifolds. There are only few points, such as condition (1.5), where the volume growth of balls and other properties of $\mathbb{R}^N$ are used and need to be adapted to this more general setting.

2 Preliminary considerations on the definitions and assumptions

2.1 Exploring other possible definitions

To start with, we address the question of what happens if one enlarges the class of admissible functions in definition (1.3). For $\varepsilon > 0$, we set

$$\Omega^\varepsilon := \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}.$$

Proposition 2.1. *Let Ω be uniformly of class $C^{2,1}$ and L be a uniformly elliptic operator with a_{ij}, b_i bounded and c bounded from above. Then, the quantity*

$\lambda_1''(-L,\Omega)$ *defined by* (1.3) *satisfies*

$$\lambda_1''(-L,\Omega) = \sup\{\lambda\ :\ \exists\,\phi \in W^{2,N}_{\mathrm{loc}}(\Omega), \forall\,\varepsilon>0,\ \inf_{\Omega^\varepsilon}\phi>0,\ (L+\lambda)\phi \le 0 \text{ a.e. in } \Omega\}.$$

Proof. To prove the statement it is sufficient to show that if $\lambda \in \mathbb{R}$, $\phi \in W^{2,N}_{\mathrm{loc}}(\Omega)$ satisfy

$$\forall\,\varepsilon>0,\quad \inf_{\Omega^\varepsilon}\phi>0,\qquad (L+\lambda)\phi\le 0 \text{ a.e. in } \Omega,$$

then every $\tilde\lambda<\lambda$ belongs to the set in (1.3). We can assume without loss of generality that $\lambda=0$, so that $\tilde\lambda<0$, and that $\Omega\neq\mathbb{R}^N$. For $x\in\overline{\Omega}$, set $d(x):=\mathrm{dist}(x,\Omega)$. Since Ω is uniformly of class $C^{2,1}$, we know from [21] that, for $\varepsilon>0$ small enough, the distance function d belongs to $W^{2,\infty}(\Omega\backslash\Omega^\varepsilon)$. Furthermore, $|\nabla d|=1$ in $\Omega\backslash\Omega^\varepsilon$. Define the function $v(x):=\cos(kd(x))$, where k is a positive constant that will be chosen later. For a.e. $x\in\Omega\backslash\Omega^\varepsilon$ it holds that

$$\begin{aligned} Lv &= -ka_{ij}(x)[kv\partial_i d\partial_j d+\sin(kd)\partial_{ij}d]-k\sin(kd)b_i(x)\partial_i d+c(x)v\\ &\le(-k^2\underline{\alpha}(x)+c(x))v+(Ck+k|b(x)|)|\sin(kd)|,\end{aligned}$$

where $C=\sum_{i,j}\|a_{ij}\partial_{ij}d\|_\infty$. Hence, since $v\ge|\sin(kd)|$ in $\Omega\backslash\Omega^{\frac{\pi}{4k}}$, setting $\delta:=\min(\varepsilon,\frac{\pi}{4k})$ we get

$$Lv\le(-k^2\inf_\Omega\underline{\alpha}+\sup_\Omega c+Ck+k\sup_\Omega|b|)v\quad\text{a.e. in }\Omega\backslash\Omega^\delta.$$

It is then possible to choose $k>0$ in such a way that $Lv\le0$ a.e. in $\Omega\backslash\Omega^\delta$. Let $\chi:\mathbb{R}\to[0,+\infty)$ be a smooth cutoff function satisfying

$$\chi=1\quad\text{in }[0,1/2],\qquad \chi=0\quad\text{in }[1,+\infty).$$

Then, for $x\in\Omega$, define $w(x):=v(x)\chi(\frac{1}{\delta}d(x))$. The function w is nonnegative, smooth, belongs to $W^{2,\infty}(\Omega)$, vanishes on $\overline{\Omega}^\delta$ and it satisfies

$$\inf_{\Omega\backslash\Omega^{\delta/2}}w>0,\qquad Lw\le0\quad\text{a.e. in }\Omega\backslash\Omega^{\delta/2}.$$

We finally set $\tilde\phi(x):=h\phi(x)+w(x)$, for some positive constant h. This function satisfies

$$\inf_\Omega\tilde\phi\ge\min\left(h\inf_{\Omega^{\delta/2}}\phi,\ \inf_{\Omega\backslash\Omega^{\delta/2}}w\right)>0,\qquad L\tilde\phi\le0\quad\text{a.e. in }\Omega\backslash\Omega^{\delta/2}.$$

Moreover, for a.e. $x\in\Omega^{\delta/2}$,

$$(L+\tilde\lambda)\tilde\phi\le h\tilde\lambda\phi+(L+\tilde\lambda)w.$$

Therefore, for h large enough, we have that $(L+\tilde\lambda)\tilde\phi\le0$ a.e. in Ω. This shows that $\tilde\lambda$ belongs to the set in (1.3). □

The above proof leads us to formulate the following.

Open problem 2.2. *Does the result of Proposition 2.1 hold true if one drops the uniform ellipticity and boundedness of the coefficients of L ?*

Starting from definition (1.1), one could define several quantities by replacing "sup" with "inf", $(L+\lambda)\phi \leq 0$ with $(L+\lambda)\phi \geq 0$ as well as by adding the conditions $\sup\phi < \infty$ or $\inf\phi > 0$ (or, more generally, $\sup\frac{\phi}{\beta} < \infty$ or $\inf\frac{\phi}{\beta} > 0$ for a given barrier function β). Let us explain why we focus on the ones in Definition 1.1 and their extensions with a barrier function β.

First of all, it is clear that if $c \in L^\infty(\Omega)$ then replacing sup with inf in definition (1.1) gives $-\infty$, whereas taking $(L+\lambda)\phi \geq 0$ instead of $(L+\lambda)\phi \leq 0$ gives $+\infty$. This is true even if one adds the conditions $\sup\phi < \infty$ or $\inf\phi > 0$.

Two other possibilities are thus left.

$$\tilde{\lambda}_1'(-L,\Omega) := \sup\{\lambda \ : \ \exists\, \phi \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega),\ \phi > 0,\ (L+\lambda)\phi \leq 0 \text{ a.e. in } \Omega\},$$

$$\tilde{\lambda}_1''(-L,\Omega) := \inf\{\lambda \ : \ \exists\, \phi \in W^{2,N}_{\mathrm{loc}}(\Omega),\ \inf_{\Omega}\phi > 0,\ (L+\lambda)\phi \geq 0 \text{ a.e. in } \Omega\}.$$

One can show that, if L is a uniformly elliptic operator with bounded coefficients, then $\tilde{\lambda}_1''(-L,\Omega) = -\infty$. Instead, if c is bounded from above, the quantity $\tilde{\lambda}_1'(-L,\Omega)$ is a well defined real number satisfying $-\sup_\Omega c \leq \tilde{\lambda}_1'(-L,\Omega) \leq \lambda_1(-L,\Omega)$. However, its sign is not related to the validity of the MP. This is seen, by means of Theorem 1.6, considering the operator L defined in Counter-example 3.3, that satisfies

$$\lambda_1''(-L,\mathbb{R}) \leq \lambda_1'(-L,\mathbb{R}) < 0 < \lambda_1(-L,\mathbb{R}) = \tilde{\lambda}_1'(-L,\mathbb{R}).$$

2.2 Previously known properties of λ_1 and λ_1'

In this section, we present some known properties of λ_1 and λ_1'. We recall that, for a bounded smooth domain Ω, λ_Ω denotes the classical principal eigenvalue of $-L$ in Ω under Dirichlet boundary conditions.

Proposition 2.3. *The generalized principal eigenvalue $\lambda_1(-L,\Omega)$ defined by* (1.1) *satisfies the following properties:*

(i) if Ω is bounded and smooth then $\lambda_1(-L,\Omega) = \lambda_\Omega$;

(ii)

$$-\sup_{\Omega} c \leq \lambda_1(-L,\Omega) \leq Cr^{-2},$$

where $0 < r \leq 1$ is the radius of some ball B contained in Ω and $C > 0$ only depends on N, $\inf_B \underline{\alpha}$ and the $L^\infty(B)$ norms of a_{ij}, b_i, c;

(iii) if $\Omega' \subset \Omega$ then $\lambda_1(-L,\Omega') \geq \lambda_1(-L,\Omega)$, with strict inequality if Ω' is bounded and $|\Omega \backslash \Omega'| > 0$;

(iv) if $(\Omega_n)_{n\in\mathbb{N}}$ is a family of nonempty domains such that

$$\Omega_n \subset \Omega_{n+1}, \qquad \bigcup_{n\in\mathbb{N}} \Omega_n = \Omega,$$

then $\lambda_1(-L,\Omega_n) \searrow \lambda_1(-L,\Omega)$ as $n \to \infty$;

(v) if $\lambda_1(-L,\Omega) > -\infty$ then there exists a positive function $\varphi \in W^{2,p}_{\mathrm{loc}}(\Omega)$, $\forall\, p<\infty$, satisfying

$$-L\varphi = \lambda_1(-L,\Omega)\varphi \quad \textit{a.e. in } \Omega; \tag{2.1}$$

(vi) if L is self-adjoint then

$$\lambda_1(-L,\Omega) = \inf_{\substack{\phi\in C^1_c(\Omega)\\ \phi\not\equiv 0}} \frac{\int_\Omega (a_{ij}(x)\partial_i\phi\,\partial_j\phi - c(x)\phi^2)}{\int_\Omega \phi^2}, \tag{2.2}$$

where $C^1_c(\Omega)$ denotes the space of compactly supported, C^1 functions in Ω. In particular, $\lambda_1(-L,\Omega)$ is nondecreasing with respect to the matrix (a_{ij});

(vii) in its dependence on c, $\lambda_1(-L,\Omega)$ is nonincreasing (i. e. $h\geq 0$ in Ω implies $\lambda_1(-(L+h),\Omega)\leq\lambda_1(-L,\Omega)$), concave and Lipschitz-continuous (using the L^∞ norm) with Lipschitz constant 1;

(viii) for uniformly elliptic operators with bounded coefficients, $\lambda_1(-L,\Omega)$ is locally Lipschitz-continuous with respect to the b_i, with Lipschitz constant depending only on N, Ω, the ellipticity constants and the L^∞ norm of c.

The above properties, in particular (i), motivate the terming of "generalized principal eigenvalue". Property (i) can be deduced from a mini-max formula in [28]. The upper bound in (ii) is Lemma 1.1 of [11]. The lower bound follows immediately from the definition, as does the inequality $\geq$ in (iii). The strict inequality in the case of bounded Ω' is given by Theorem 2.4 of [11] (and it actually holds in greater generality, cf. Remark 7.4 below). The proofs of (iv), (v) can be found in [1] in the case of operators with smooth coefficients (see also [11] for general operators in bounded, non-smooth domains), but the same arguments apply to the general case. We point out that if Ω is smooth then Theorem 1.4 above is a much stronger result than Proposition 2.3 part (v), providing in particular a function φ satisfying in addition the Dirichlet boundary condition. Property (vi) follows from (i), (iv) and the Rayleigh-Ritz variational formula for the classical Dirichlet principal eigenvalue, as shown in [1]. Properties (vii) and (viii) are respectively Propositions 2.1 and 5.1 in [11].

Remark 2.4. The monotonicity of λ_1 with respect to c is strict if Ω is bounded, whereas it might not be the case if Ω is unbounded (see the proof of Proposition 8.1 below for an example). Likewise, the decreasing monotonicity with respect to the domain given by Proposition 2.3 part (iii) might not be strict in the case of unbounded domains. For example, in dimension 1, the operator $Lu=u''$ clearly satisfies $\lambda_1(-L,\mathbb{R})=\lambda_1(-L,\mathbb{R}_+)=0$.

The generalized principal eigenvalue $\lambda_1'(-L,\Omega)$ also coincides with the Dirichlet principal eigenvalue λ_Ω if Ω is bounded and smooth. Moreover, it coincides with the periodic principal eigenvalue λ_p under Dirichlet boundary conditions if Ω is smooth and Ω and L are periodic with the same period. We say that Ω is periodic, with period $(l_1,\dots,l_N)\in\mathbb{R}^N_+$, if $\Omega+\{l_ie_i\}=\Omega$ for $i=1,\dots,N$, where

$\{e_1,\dots,e_N\}$ is the canonical basis of $\mathbb{R}^N$; the operator L is said to be periodic, with period $(l_1,\dots,l_N)$, if its coefficients are periodic with the same period $(l_1,\dots,l_N)$. We recall that λ_p is the unique real number λ such that the problem (1.4) admits a positive periodic solution. Such a solution, which is unique up to a multiplicative constant, is called periodic principal eigenfunction.

Proposition 2.5. *The generalized principal eigenvalue $\lambda_1'(-L,\Omega)$ defined by* (1.2) *satisfies the following properties:*

(i) if Ω is bounded and smooth then $\lambda_1'(-L,\Omega)=\lambda_\Omega$;
(ii) if Ω is smooth then $\lambda_1'(-L,\Omega)<+\infty$ and, if in addition (1.5) *holds, then $\lambda_1'(-L,\Omega)\in\mathbb{R}$;*
(iii) if Ω is smooth and Ω, L are periodic, with the same period, then $\lambda_1'(-L,\Omega)=\lambda_p$.

The fact that the set of "admissible functions" in (1.2) could be empty was not discussed in previous papers, where, essentially, only the case $\Omega=\mathbb{R}^N$ and L with bounded coefficients was treated. Statements (i) and (iii) are proved in [8] (for operators with smooth coefficients) and [29], requiring, in both cases, the additional condition that the functions ϕ in (1.2) are uniformly Lipschitz-continuous. In the general case where this extra condition is not imposed, properties (i), (ii) and the inequality $\lambda_1'\geq\lambda_p$ in (iii) can be deduced from the properties of λ_1 and λ_1'' - Propositions 2.3, 5.1 - by means of Theorem 1.7 here. The inequality $\lambda_1'\leq\lambda_p$ is immediately obtained by taking ϕ equal to the periodic principal eigenfunction in (1.2).

2.3 Finiteness of λ_1

By Proposition 2.3 part (ii), we know that $\lambda_1\in\mathbb{R}$ if c is bounded above. Otherwise, it could be equal to $-\infty$, that is, the set of admissible functions in (1.1) could be empty.

Proposition 2.6. *Let Ω be a smooth domain and L be a uniformly elliptic operator with a_{ij}, b_i bounded and c such that there exists a positive constant δ and a sequence $(x_n)_{n\in\mathbb{N}}$ satisfying*

$$\forall\, n\in\mathbb{N},\quad B_\delta(x_n)\subset\Omega,\qquad \lim_{n\to\infty}\inf_{B_\delta(x_n)} c=+\infty. \tag{2.3}$$

Then, $\lambda_1(-L,\Omega)=-\infty$ and the MP, as stated in Definition 1.5, does not hold for L in Ω.

Proof. Since, for $\lambda\in\mathbb{R}$, $L-\lambda$ satisfies the same condition (2.3) as L and, by definition,

$$\lambda_1(-L,\Omega)=\lambda_1(-(L-\lambda),\Omega)-\lambda,$$

to prove that $\lambda_1(-L,\Omega)=-\infty$ it is sufficient to show that $\lambda_1(-L,\Omega)\leq 0$. Thus, owing to Proposition 2.3 part (iii), it is enough to show that $\lambda_1(-L,B_\delta(x_n))\leq 0$ for some $n \in \mathbb{N}$. Consider a function $\vartheta \in C^2([0,\delta])$ satisfying

$$\vartheta > 0 \text{ in } [0,\delta), \qquad \vartheta'(0) = 0, \qquad \vartheta(\delta) = \vartheta'(\delta) = 0, \qquad \vartheta'' > 0 \text{ in } [\frac{\delta}{2}, \delta]$$

(for instance, $\vartheta(r) := \cos(\frac{\pi}{\delta} r) + 1$). The functions $(\theta_n)_{n\in\mathbb{N}}$ defined by $\theta_n(x) := \vartheta(|x - x_n|)$ satisfy, a.e. in $B_\delta(x_n) \backslash B_{\frac{\delta}{2}}(x_n)$,

$$a_{ij}(x)\partial_{ij}\theta_n + b_i(x)\partial_i\theta_n \geq \left(\inf_\Omega \underline{\alpha}\right) \vartheta''(|x - x_n|) - k|\vartheta'(|x - x_n|) + \vartheta(|x - x_n|)|,$$

for some k independent of n. There exists then $\rho \in (0,\delta)$, independent of n, such that $a_{ij}(x)\partial_{ij}\theta_n + b_i(x)\partial_i\theta_n > 0$ a.e. in $B_\delta(x_n) \backslash B_\rho(x_n)$. On the other hand,

$$L\theta_n \geq -k' + \left(\inf_{B_\delta(x_n)} c\right)\left(\min_{[0,\rho]} \vartheta\right) \quad \text{a.e. in } B_\rho(x_n),$$

where k' is another positive constant independent of n. As a consequence, using the hypothesis on c, we can find $n \in \mathbb{N}$ such that $L\theta_n > 0$ a.e. in $B_\delta(x_n)$. Taking $\phi = \theta_n$ in (1.2) we obtain $\lambda_1'(-L, B_\delta(x_n)) \leq 0$. Eventually, statement (i) of Propositions 2.3 and 2.5 yield

$$0 \geq \lambda_1'(-L, B_\delta(x_n)) = \lambda_{B_\delta(x_n)} = \lambda_1(-L, B_\delta(x_n)).$$

That the MP does not hold in this case (in fact, as soon as $-\infty \leq \lambda_1 < 0$) follows from Theorems 1.6 and 1.7 part (ii), proved in Sections 4 and 6 respectively. $\square$

The hypothesis on c in the previous statement cannot be weakened by $\sup c = +\infty$. One can see this by considering, in dimension 1, the operator $Lu := u'' + c(x)u$, with $c(x) := v'(x) - v^2(x)$ and $v \in C^1(\mathbb{R})$ such that $\sup(v' - v^2) = +\infty$. We leave to the reader to check that such a function v exists. Since the function $\phi(x) = e^{-\int_0^x v(t)dt}$ satisfies $L\phi = 0$, it follows that $\lambda_1(-L,\mathbb{R}) \geq 0$. We now show that if the b_i are unbounded then it may happen that $\lambda_1(-L,\Omega) \in \mathbb{R}$ even though c satisfies (2.3).

Proposition 2.7. *If the a_{ij} are bounded, $b(x) \cdot x$ does not change sign for $|x|$ large and it holds that*

$$\limsup_{\substack{x\in\Omega \\ |x|\to\infty}} \frac{c(x)}{\frac{|b(x)\cdot x|}{|x|} + 1} < +\infty,$$

then $\lambda_1(-L,\Omega)$ is finite.

Proof. Let $\phi \in C^2(\mathbb{R}^N)$ be a positive function satisfying, for $|x| \geq 1$, $\phi(x) = e^{\pm\sigma|x|}$, where the $\pm$ is in agreement with the sign of $-b \cdot x$ at infinity, and $\sigma > 0$ will be chosen later. Direct computation shows that

$$L\phi = \left[\frac{a_{ij}x_ix_j}{|x|^2}\sigma^2 \pm \left(\frac{\mathrm{Tr}(a_{ij})}{|x|} - \frac{a_{ij}x_ix_j}{|x|^3} + \frac{b\cdot x}{|x|}\right)\sigma + c\right]\phi \quad \text{a.e. in } \Omega\backslash B_1.$$

Using the hypotheses, we can then choose σ large enough and $\lambda \in \mathbb{R}$ such that $(L+\lambda)\phi < 0$ a.e. in Ω. Hence, $\lambda_1(-L,\Omega) \geq \lambda$. On the other hand, $\lambda_1(-L,\Omega) <$

$+\infty$ by Proposition 2.3 part (ii). This proof also shows that $\lambda_1''(-L,\Omega)\in\mathbb{R}$, under the same conditions, when $b(x)\cdot x<0$ at infinity. □

3 Existence of positive eigenfunctions vanishing on $\partial\Omega$

We now prove the characterization of the set of eigenvalues $\mathscr{E}$. One of the main tools we require is the following boundary Harnack inequality, quoted from [6], which extends the previous versions of [14], [5]. We recall that, for $\delta>0$, Ω^δ denotes the set $\{x\in\Omega\ :\ \mathrm{dist}(x,\partial\Omega)>\delta\}$.

Theorem 3.1 ([6]). *Let Ω be a bounded domain and Ω' be an open subset of Ω such that $T:=\partial\Omega\cap(\Omega'+B_\eta)$ is of class $C^{1,1}$, for some $\eta>0$. Then, any nonnegative solution $u\in W^{2,N}_{loc}(\Omega)\cap C^0(\Omega\cup T)$ of*

$$\begin{cases} Lu=0 & \text{a.e. in } \Omega\\ u=0 & \text{on } T,\end{cases}$$

satisfies

$$\sup_{\Omega'} u\le C\inf_{\Omega^\delta} u,$$

for all $\delta>0$ such that $\Omega^\delta\neq\emptyset$, with C depending on N, Ω, δ, η, $\inf\underline{\alpha}$ and the L^∞ norms of a_{ij}, b_i, c.

Proof of Theorem 1.4. From the definition (1.1) of $\lambda_1(-L,\Omega)$ it follows that $\mathscr{E}\subset(-\infty,\lambda_1(-L,\Omega)]$. This concludes the proof if $\lambda_1(-L,\Omega)=-\infty$. Let us prove the reverse inclusion when $\lambda_1(-L,\Omega)>-\infty$. We can assume, without loss of generality, that $0\in\Omega$. Since Ω is smooth, a compactness argument (that we leave to the reader) shows that, for any $n\in\mathbb{N}$, there exists $r(n)\ge n$ such that $\Omega\cap B_n$ is contained in a single connected component of $\Omega\cap B_{r(n)}$. Let Ω_n denote this connected component. It is not restrictive to assume that $\Omega_n\subset\Omega_{n+1}$ for $n\in\mathbb{N}$. Hence, Proposition 2.3 part (iv) yields

$$\lim_{n\to\infty}\lambda_1(-L,\Omega_n)=\lambda_1(-L,\Omega).$$

We first show the existence of an eigenfunction associated with $\lambda_1(-L,\Omega)$, and then of one associated with λ, for any given $\lambda<\lambda_1(-L,\Omega)$.

Step 1: $\lambda_1(-L,\Omega)\in\mathscr{E}$.

For $n\in\mathbb{N}$, let φ^n be the generalized principal eigenfunction of $-L$ in Ω_n, normalized by $\varphi^n(0)=1$. This eigenfunction is obtained in the work of Berestycki, Nirenberg and Varadhan [11] (note that Ω_n, in general, is not smooth). For the reader's ease, some of the main results of that paper are described in Appendix A here. In particular, the existence of φ^n is provided by Property A.1. Fix $m\in\mathbb{N}$. Since for $n>m$, φ^n belongs to $W^{2,p}(\Omega\cap B_m)$, $\forall\, p<\infty$, and vanishes on $\partial\Omega\cap B_m$, applying the boundary Harnack inequality - Theorem 3.1 - with $\Omega=\Omega_{m+1}$, $\Omega'=\Omega\cap B_m$, $\eta=1$ and $\delta<\mathrm{dist}(0,\partial\Omega)$, we find a constant C_m such that

$$\forall\, n>m,\quad \sup_{\Omega\cap B_m}\varphi^n\le C_m.$$

Thus, the elliptic local boundary estimate of Agmon, Douglis and Nirenberg [2] (see also Theorem 9.13 of [18]) implies that the $(\varphi^n)_{n>m}$ are uniformly bounded in $W^{2,p}(\Omega\cap B_{m-\frac{1}{2}})$ (since $\overline{B}_{m-\frac{1}{2}}\cap\partial\Omega$ is contained in a smooth boundary portion of $B_m\cap\Omega$). Consequently they converge, up to subsequences, weakly in $W^{2,p}(\Omega\cap B_{m-\frac{1}{2}})$ and, by Morrey's inequality (see, e.g., Theorem 7.26 part (ii) of [18]), strongly in $C^1(\overline{\Omega}\cap B_{m-1})$ to a nonnegative solution ϕ^m of

$$\begin{cases} -L\phi^m=\lambda_1(-L,\Omega)\phi^m & \text{a.e. in } \Omega\cap B_{m-1} \\ \phi^m=0 & \text{on } \partial\Omega\cap B_{m-1}. \end{cases}$$

In particular, $\phi^m(0)=1$ and then ϕ^m is positive in $\Omega\cap B_{m-1}$ by the strong maximum principle. Therefore, using a diagonal method, we can extract a subsequence of $(\varphi^n)_{n\in\mathbb{N}}$ converging to a positive function ϕ which is a solution of the above problem for all $m>1$. That is, $\lambda_1(-L,\Omega)\in\mathscr{E}$.

Step 2: $(-\infty,\lambda_1(-L,\Omega))\subset\mathscr{E}$.

Take $\lambda<\lambda_1(-L,\Omega)$. Since Ω is unbounded and connected, $\Omega_n\backslash\overline{B}_{n-1}\neq\emptyset$ for all $n\in\mathbb{N}$. Let $(f_n)_{n\in\mathbb{N}}$ be a family of continuous, nonpositive and not identically equal to zero functions such that

$$\forall\, n\in\mathbb{N},\quad \mathrm{supp}f_n\subset\Omega_n\backslash\overline{B}_{n-1}.$$

Since for $n\in\mathbb{N}$, $\lambda_1(-L,\Omega_n)>\lambda_1(-L,\Omega)>\lambda$ by Proposition 2.3 part (iii), we have $\lambda_1(-(L+\lambda),\Omega_n)>0$. Hence, Property A.5 provides a bounded solution $u^n\in W^{2,N}_{loc}(\Omega_n\cup(B_n\cap\partial\Omega))$ of

$$\begin{cases} (L+\lambda)u^n=f_n & \text{a.e. in } \Omega_n \\ u^n\overset{u_0}{=}0 & \text{on } \partial\Omega_n. \end{cases}$$

The meaning of the relaxed boundary condition $u^n\overset{u_0}{=}0$ is recalled in Appendix A. However, we only use here the fact that it implies $u^n=0$ in the classical sense on the smooth portion $B_n\cap\partial\Omega$. Note that u^n is nonnegative by Property A.2, and then it is strictly positive in Ω_n by the strong maximum principle. Moreover, Lemma 9.16 in [18] yields $u^n\in W^{2,p}_{loc}(\Omega_n\cup(B_n\cap\partial\Omega))$, $\forall\, p<\infty$. For $n\in\mathbb{N}$, the function v^n defined by

$$v^n(x):=\frac{u^n(x)}{u^n(0)},$$

belongs to $W^{2,p}(\Omega\cap B_{n-1})$, it is positive and satisfies: $v^n(0)=1$,

$$\begin{cases} -Lv^n=\lambda v^n & \text{a.e. in } \Omega\cap B_{n-2} \\ v^n=0 & \text{on } \partial\Omega\cap B_n. \end{cases}$$

We can thereby proceed exactly as in step 1, with $(v^n)_{n\in\mathbb{N}}$ in place of $(\varphi^n)_{n\in\mathbb{N}}$, and infer that $\lambda\in\mathscr{E}$. □

Remark 3.2. Actually, the arguments in the proof of Theorem 1.4 yield a more general statement. Namely, if Ω is unbounded and has a smooth boundary portion

T then $\mathscr{E}_T = (-\infty, \lambda_1(-L,\Omega)]$, where

$$\mathscr{E}_T := \{\lambda \in \mathbb{R} \,:\, \exists\, \phi \in W^{2,p}_{\mathrm{loc}}(\Omega \cup T),\ \forall\, p < \infty,\ \phi > 0,\ -L\phi = \lambda\phi \text{ a.e. in } \Omega,\\ \phi = 0 \text{ on } T\}.$$

We now exhibit an example where the set of eigenvalues does not reduce to $\{\lambda_1(-L,\Omega)\}$ even if one restricts to (positive) eigenfunctions decaying to 0 at infinity. This example also shows that $\lambda_1 > 0$ does not imply the validity of the MP for subsolutions which are nonpositive also at infinity, and not only on $\partial\Omega$.

Counter-example 3.3. *There exists an operator L in $\mathbb{R}$ such that $\lambda_1(-L,\mathbb{R}) > 0$ and, for all $\lambda \in [0, \lambda_1(-L,\mathbb{R})]$, there is a positive function $\phi \in W^{2,p}(\mathbb{R})$, $\forall\, p < \infty$, satisfying*

$$-L\phi = \lambda\phi \quad \text{a.e. in } \mathbb{R}, \qquad \limsup_{|x|\to\infty} \phi(x) e^{|x|} \le 1.$$

Proof. Consider the operator L defined by

$$Lu(x) := \begin{cases} u''(x) - 4u'(x) + 3u(x) & \text{if } x < -\frac{\pi}{4} \\ u''(x) + u(x) & \text{if } -\frac{\pi}{4} \le x \le \frac{\pi}{4} \\ u''(x) + 4u'(x) + 3u(x) & \text{if } x > \frac{\pi}{4}. \end{cases}$$

In order to show that $\lambda_1(-L,\mathbb{R}) > 0$, we explicitly construct a function $v \in W^{2,\infty}(\mathbb{R})$ such that $(L+\lambda)v \le 0$, for some $\lambda > 0$. We set

$$v(x) := \begin{cases} ke^{2x} & \text{if } x < -\frac{\pi}{4} \\ \cos(\gamma x) & \text{if } -\frac{\pi}{4} \le x \le \frac{\pi}{4} \\ ke^{-2x} & \text{if } x > \frac{\pi}{4}, \end{cases}$$

where $k = e^{\frac{\pi}{2}} \cos(\frac{\pi}{4}\gamma)$ and γ is the solution in $(1,2)$ of the equation

$$\gamma \tan\left(\frac{\pi}{4}\gamma\right) - 2 = 0.$$

We leave to the reader to check that $v \in W^{2,\infty}(\mathbb{R})$. We see that $Lv = -v$ for $|x| > \pi/4$. For $|x| < \pi/4$, we find $Lv = (1-\gamma^2)v$. Hence, $(L+\lambda)v \le 0$ a.e. in $\mathbb{R}$, with $\lambda = \min(1, \gamma^2 - 1) > 0$. Now, direct computation shows that the function

$$\underline{u}(x) := \begin{cases} e^x & \text{if } x < -\frac{\pi}{4} \\ \sqrt{2} e^{-\frac{\pi}{4}} \cos(x) & \text{if } -\frac{\pi}{4} \le x \le \frac{\pi}{4} \\ e^{-x} & \text{if } x > \frac{\pi}{4} \end{cases}$$

belongs to $W^{2,\infty}(\mathbb{R})$ and satisfies $L\underline{u} = 0$ in $\mathbb{R}\backslash\{\pm\pi/4\}$. For $\lambda \in [0, \lambda_1(-L,\mathbb{R})]$, let ϕ be the associated positive eigenfunction constructed as in the proof of Theorem 1.4, with $\Omega_n = B_n$. It is clear that, when $\lambda < \lambda_1(-L,\mathbb{R})$, it is possible to take an even function f_n in that construction. Hence, the symmetry of L implies that ϕ is even. Normalize ϕ in such a way that $\phi(0) < \underline{u}(0)$. By property (iv) of Proposition 2.3, $\lambda_1(-L, B_r) > 0$ for r large enough. Thus, if $\underline{u}(\pm r) \le \phi(\pm r)$ for such values of r, the MP yields a contradiction. This shows that $\phi(x) < \underline{u}(x)$ for $|x|$ large enough, which concludes the proof.

We remark that $\lambda_1'(-L,\mathbb{R}) \leq -1$, as is seen by taking $\phi \equiv 1$ in (1.2). This is in agreement with Theorem 1.6. $\square$

4 Maximum principle

We derive Theorem 1.6 as a particular case of a result concerning subsolutions bounded from above by (constant times) a barrier β. The function β is positive and satisfies either

$$\exists\, \sigma > 0, \quad \limsup_{\substack{x\in\Omega \\ |x|\to\infty}} \beta(x)|x|^{-\sigma} = 0, \tag{4.1}$$

if the coefficients of L satisfy (1.5), or

$$\exists\, \sigma > 0, \quad \limsup_{\substack{x\in\Omega \\ |x|\to\infty}} \beta(x)e^{-\sigma|x|} = 0, \tag{4.2}$$

if they satisfy the stronger hypothesis

$$\sup_\Omega c < \infty, \qquad \sup_\Omega a_{ij} < \infty, \qquad \sup_{x\in\Omega} \frac{b(x)\cdot x}{|x|} < \infty. \tag{4.3}$$

Definition 4.1. Let β be a positive function on Ω. We say that the operator L satisfies the β*-MP* in Ω if every function $u \in W^{2,N}_{\mathrm{loc}}(\Omega)$ such that

$$Lu \geq 0 \ \text{ a.e. in } \Omega, \qquad \sup_\Omega \frac{u}{\beta} < \infty, \qquad \forall\, \xi \in \partial\Omega, \ \limsup_{x\to\xi} u(x) \leq 0,$$

satisfies $u \leq 0$ in Ω.

Theorem 1.6 represents the particular case $\beta \equiv 1$ of the following statement.

Theorem 4.2. *The operator L satisfies the β-MP in Ω*

(i) if $\lambda''_\beta(-L,\Omega) > 0$ and either (1.5)*,* (4.1) *or* (4.3)*,* (4.2) *hold;*

(ii) only if $\lambda'_\beta(-L,\Omega) \geq 0$.

Proof. Statement (ii) is an immediate consequence of Definition 1.2. Indeed, if $\lambda'_\beta(-L,\Omega) < 0$ then there are $\lambda < 0$ and a positive function $\phi \in W^{2,N}_{\mathrm{loc}}(\Omega)$ such that

$$\phi \leq \beta, \quad L\phi \geq -\lambda\phi \ \text{ a.e. in } \Omega, \qquad \forall\, \xi \in \partial\Omega, \ \lim_{x\to\xi} \phi(x) \leq 0.$$

Hence, ϕ violates the β-MP.

Let us prove (i). Assume by contradiction that there exists a function $u \in W^{2,N}_{\mathrm{loc}}(\Omega)$ which is positive somewhere in Ω and satisfies

$$Lu \geq 0 \ \text{ a.e. in } \Omega, \qquad \sup_\Omega \frac{u}{\beta} < \infty, \qquad \forall\, \xi \in \partial\Omega, \ \limsup_{x\to\xi} u(x) \leq 0.$$

Since $\lambda''_\beta(-L,\Omega) > 0$, by Definition 1.2 there exists $\lambda > 0$ and a function $\phi \in W^{2,N}_{\mathrm{loc}}(\Omega)$ such that

$$\phi \geq \beta, \qquad (L+\lambda)\phi \leq 0, \quad \text{a.e. in } \Omega.$$

In particular, up to renormalization, we can assume that $\phi \geq u$ in Ω. We want to modify ϕ in order to obtain a function that grows faster than u at infinity and is still a supersolution in a suitable subset of Ω. To this aim, we consider a positive smooth function $\chi : \mathbb{R}^N \to \mathbb{R}$ such that, for $|x| > 1$, $\chi(x) = |x|^\sigma$ if β satisfies (4.1) or $\chi(x) = e^{\sigma|x|}$ if β satisfies (4.2). For $n \in \mathbb{N}$, we set

$$\phi_n(x) := \phi(x) + \frac{1}{n}\chi(x), \qquad k_n := \sup_\Omega \frac{u}{\phi_n}.$$

Note that the sequence $(k_n)_{n\in\mathbb{N}}$ is positive, nondecreasing and bounded from above by 1. Thus, it is convergent. Moreover, since

$$\limsup_{\substack{x\in\Omega\\|x|\to\infty}} \frac{u(x)}{\phi_n(x)} \leq n\left(\sup_\Omega \frac{u}{\beta}\right) \limsup_{\substack{x\in\Omega\\|x|\to\infty}} \frac{\beta(x)}{\chi(x)} = 0, \qquad \forall\, \xi \in \partial\Omega, \quad \limsup_{x\to\xi} \frac{u(x)}{\phi_n(x)} = 0,$$

there exists $x_n \in \Omega$ such that $k_n = \frac{u(x_n)}{\phi_n(x_n)}$. We claim that, for n large enough, $L\phi_n < 0$ in a neighborhood of x_n. The operator L acts on a radial function $\theta(x) = \vartheta(|x|)$ in the following way:

$$L\theta(x) = A(x)\vartheta''(|x|) + B(x)\vartheta'(|x|) + c(x)\vartheta(|x|),$$

where

$$A(x) := \frac{a_{ij}(x)x_ix_j}{|x|^2}, \qquad B(x) := \frac{b(x)\cdot x}{|x|} + \frac{\mathrm{Tr}(a_{ij}(x))}{|x|} - \frac{a_{ij}(x)x_ix_j}{|x|^3}. \tag{4.4}$$

Hence, for a.e. $x \in \Omega\backslash B_1$, in the case where β satisfies (4.1) we get

$$\begin{aligned} L\chi &= \left(\sigma(\sigma-1)\frac{A(x)}{|x|^2} + \sigma\frac{B(x)}{|x|} + c(x)\right)\chi \\ &\leq \left(\sigma(N+\sigma-2)\frac{\overline{\alpha}(x)}{|x|^2} + \sigma\frac{b(x)\cdot x}{|x|^2} + c(x)\right)\chi, \end{aligned}$$

while, in the case of condition (4.2), we get

$$\begin{aligned} L\chi &= (\sigma^2 A(x) + \sigma B(x) + c(x))\chi \\ &\leq \left[\sigma\left(\sigma + \frac{N-1}{|x|}\right)\overline{\alpha}(x) + \sigma\frac{b(x)\cdot x}{|x|} + c(x)\right]\chi. \end{aligned}$$

Therefore, in both cases, there exists a positive constant C such that $L\chi \leq C\chi$ a.e. in Ω. Let us estimate the "penalization" term $\frac{1}{n}\chi(x_n)$. For $n \in \mathbb{N}$, we find that

$$\frac{1}{k_{2n}} \leq \frac{\phi_{2n}(x_n)}{u(x_n)} = \frac{\phi(x_n) + \frac{1}{2n}\chi(x_n)}{u(x_n)} = \frac{1}{k_n} - \frac{\chi(x_n)}{2nu(x_n)},$$

and then that

$$\frac{\chi(x_n)}{n} \leq 2\left(\frac{1}{k_n} - \frac{1}{k_{2n}}\right)u(x_n).$$

As a consequence, for $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that, for a.e. $x \in B_{\delta_n}(x_n)$,

$$\frac{1}{n} L\chi(x) \le C\frac{\chi(x)}{n} \le 3C\left(\frac{1}{k_n} - \frac{1}{k_{2n}}\right) u(x) \le 3C\left(\frac{1}{k_n} - \frac{1}{k_{2n}}\right) \phi(x).$$

Thus,

$$L\phi_n \le \left[-\lambda + 3C\left(\frac{1}{k_n} - \frac{1}{k_{2n}}\right)\right]\phi \quad \text{a.e. in } B_{\delta_n}(x_n).$$

Since the sequence $(k_n)_{n\in\mathbb{N}}$ is convergent, we can then find $n \in \mathbb{N}$ such that $L\phi_n < 0$ a.e. in $B_{\delta_n}(x_n)$. Whence we infer that the nonnegative function $w_n := k_n\phi_n - u$ satisfies $Lw_n < 0$ a.e. in $B_{\delta_n}(x_n)$ and vanishes at x_n. This contradicts the strong maximum principle. □

Remark 4.3. If $\lambda_1'(-L,\Omega) = \lambda_1''(-L,\Omega) = 0$ then the MP might or might not hold. Indeed, if L and Ω are periodic then $\lambda_1'(-L,\Omega)$ and $\lambda_1''(-L,\Omega)$ coincide with λ_p. Hence, if $\lambda_p = 0$, the periodic principal eigenfunction violates the MP. On the other hand, the operator L introduced at the beginning of the proof of Proposition 8.1 below satisfies the MP and $\lambda_1'(-L,\mathbb{R}) = \lambda_1''(-L,\mathbb{R}) = 0$.

5 Properties of λ_1''

Proposition 5.1. *The quantity $\lambda_1''(-L,\Omega)$ defined by* (1.3) *satisfies the following properties:*

(i) if Ω is bounded and smooth then $\lambda_1''(-L,\Omega) = \lambda_\Omega$;

(ii)

$$-\sup_\Omega c \le \lambda_1''(-L,\Omega) \le \lambda_1(-L,\Omega);$$

(iii) if $\Omega' \subset \Omega$ then $\lambda_1''(-L,\Omega') \ge \lambda_1''(-L,\Omega)$;

(iv) if Ω is smooth and Ω, L are periodic, with the same period, then $\lambda_1''(-L,\Omega) = \lambda_p$;

(v) in its dependence on c, $\lambda_1''(-L,\Omega)$ is nonincreasing, concave and Lipschitz-continuous (using the L^∞ norm) with Lipschitz constant 1;

(vi) for uniformly elliptic operators with bounded coefficients, $\lambda_1''(-L,\Omega)$ is locally Lipschitz-continuous with respect to the b_i, with Lipschitz constant depending only on N, Ω, the ellipticity constants and the L^∞ norm of c.

Proof. The first inequality in property (ii) follows by taking $\phi \equiv 1$ in (1.3). The second inequality in (ii), as well as property (iii), are immediate consequences of the definition.

(i) From (ii) and Proposition 2.3 part (i) it follows that $\lambda_1''(-L,\Omega) \le \lambda_\Omega$. The reverse inequality is a consequence of Lemma 7.7. Note that if Ω is of class $C^{2,1}$ then this inequality also follows from the characterization of Proposition 2.1, taking ϕ equal to the Dirichlet principal eigenfunction of $-L$ in Ω.

(iv) We consider the periodic principal eigenfunction φ of $-L$ in Ω, under Dirichlet boundary conditions. Taking $\phi = \varphi$ in the characterization of Proposition

2.1 yields $\lambda_1''(-L,\Omega) \geq \lambda_p$. Assume now by contradiction that $\lambda_1''(-L,\Omega) > \lambda_p$. Thus, by Theorem 1.6, the operator $(L+\lambda_p)$ satisfies the MP in Ω. This is in contradiction with the existence of the periodic principal eigenfunction.

Properties (v), (vi) follow from the same arguments used to prove the analogous properties for λ_1 (cf. Propositions 2.1, 5.1 of [11]). □

We now derive a result about the admissible functions ϕ in (1.3). It will be used in the sequel to obtain the sufficient conditions for the equivalence of λ_1, λ_1', λ_1''.

Proposition 5.2. *If Ω has a $C^{1,1}$ boundary portion $T \subset \partial\Omega$ then the definition* (1.3) *of $\lambda_1''(-L,\Omega)$ does not change if one further requires $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega \cup T)$, $\forall\, p < \infty$.*

Proof. We prove the statement by showing that, if for some $\lambda \in \mathbb{R}$ there exists a function $\phi \in W^{2,N}_{\mathrm{loc}}(\Omega)$ satisfying

$$\inf_{\Omega} \phi > 0, \qquad (L+\lambda)\phi \leq 0 \quad \text{a.e. in } \Omega,$$

then we can find a function $u \in W^{2,p}_{\mathrm{loc}}(\Omega \cup T)$, $\forall\, p < \infty$, with the same properties. The function u will be obtained as a solution of a suitable nonlinear problem.

First, by renormalizing ϕ and replacing c with $c+\lambda$, the problem is reduced to the case where $\inf_{\Omega} \phi = 2$ and $\lambda = 0$. Consider the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by $f(x,s) := |c(x)|g(s)$, where

$$g(s) = \begin{cases} -1 & \text{for } s \leq 1 \\ s-2 & \text{for } s \in (1,2) \\ 0 & \text{for } s \geq 2. \end{cases}$$

Setting $\underline{u} \equiv 1$, we find that, a.e. in Ω,

$$L\underline{u} \geq f(x,\underline{u}), \qquad L\phi \leq f(x,\phi), \qquad \underline{u} < \phi.$$

Standard arguments provide a function $u \in W^{2,p}_{loc}(\Omega)$ satisfying $1 \leq u \leq \phi$ and $Lu = f(x,u)$ a.e. in Ω. More precisely, one constructs solutions of problems in bounded domains invading Ω by an iterative method and then uses a diagonal extraction procedure. However, getting the improved regularity $u \in W^{2,p}_{loc}(\Omega \cup T)$ is delicate, especially because ϕ may blow up at T. Moreover, in order to pass to the unbounded domain, one needs a version of the boundary Harnack inequality for solutions of inhomogeneous problems. This is the object of Appendix B. Let us now describe the method in detail.

The first step consists in solving semilinear problems in bounded domains with Dirichlet conditions on smooth portions of the boundary. Namely, we derive the following

Lemma 5.3. *Let Ω be a bounded domain with a $C^{1,1}$ boundary portion T and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that $f(\cdot,0) \in L^N(\Omega)$ and $f(x,\cdot)$ is uniformly Lipschitz continuous in $\mathbb{R}$, uniformly with respect to $x \in \Omega$. Assume further that the problem*

$$\begin{cases} Lu = f(x,u) & \text{a.e. in } \Omega \\ u \overset{u_0}{=} 0 & \text{on } \partial\Omega \end{cases}$$

has a subsolution $\underline{u} \in W^{2,N}_{loc}(\Omega) \cap L^\infty(\Omega)$ *and a supersolution* $\overline{u} \in W^{2,N}_{loc}(\Omega)$ *such that* $\underline{u} \le 0 \le \overline{u}$ *in* Ω*. Then, there exists a function* $u \in W^{2,N}_{loc}(\Omega \cup T) \cap L^\infty(\Omega)$ *satisfying*

$$\begin{cases} Lu = f(x,u) & \textit{a.e. in } \Omega \\ u = 0 & \textit{on } T \\ \underline{u} \le u \le \overline{u} & \textit{in } \Omega. \end{cases}$$

Note that, in the above statement, one can replace the 0 boundary conditions with a more general datum $\psi \in W^{2,N}(\Omega)$, provided that $\underline{u}$, $\overline{u}$ satisfy $\underline{u} \le \psi \le \overline{u}$. Let us postpone the proof of this Lemma until we complete the argument to prove Proposition 5.2. We assume that $0 \in \Omega$. For $n \in \mathbb{N}$, let Ω_n denote the connected component of $\Omega \cap B_n$ containing 0 and let T_n be its portion of the boundary of class at least $C^{1,1}$. T_n is open in the topology of $\partial\Omega_n$ and is nonempty for n large enough. Consider the functions $u^n \in W^{2,N}_{loc}(\Omega_n \cup T_n) \cap L^\infty(\Omega_n)$ provided by Lemma 5.3 satisfying

$$\begin{cases} Lu^n = f(x,u^n) & \text{a.e. in } \Omega_n \\ u^n = 1 & \text{on } T_n \\ 1 \le u^n \le \phi & \text{in } \Omega_n. \end{cases}$$

Since $|f(x,u^n)| \le |c(x)|$ and $1 \le u^n \le \phi$, using interior estimates and a diagonal argument, we find that the sequence $(u^n)_{n\in\mathbb{N}}$ converges (up to subsequences) locally uniformly in Ω to a function $u \in W^{2,N}_{loc}(\Omega)$ satisfying

$$Lu = f(x,u) \le 0 \quad \text{a.e. in } \Omega, \qquad 1 \le u \le \phi \quad \text{in } \Omega.$$

It remains to show that $u \in W^{2,p}_{loc}(\Omega \cup T)$, $\forall\, p < \infty$. Let $K \subset\subset \Omega \cup T$. The smoothness of T implies that $K \subset\subset \Omega_m \cup T_m$, for m large enough. Take $\eta > 0$ such that $\partial\Omega_m \cap (K + B_{2\eta}) \subset T_m$. Applying the inhomogeneous boundary Harnack inequality given by Proposition B.1, with $\Omega = \Omega_m$ and $\Omega' = \Omega_m \cap (K + B_\eta)$, we find a constant C such that

$$\forall\, n \ge m, \qquad \sup_{\Omega_m \cap (K+B_\eta)} u^n \le C(u^n(0)+1) \le C(\phi(0)+1).$$

Hence, by the local boundary estimate, $(u^n)_{n\in\mathbb{N}}$ is bounded in $W^{2,p}(K)$ and then its limit u belongs to $W^{2,p}(K)$. □

Proof of Lemma 5.3. Replacing c with $c-k$ and $f(x,s)$ with $f(x,s)-ks$ if need be, with k greater than $\|c\|_{L^\infty(\Omega)}$ and the Lipschitz constant of $f(x,\cdot)$, it is not restrictive to assume that c is negative and that $f(x,\cdot)$ is decreasing. From Proposition 2.3 parts (ii) and (iii) it follows that $\lambda_1(-L,\mathscr{O}) > 0$ in any bounded domain $\mathscr{O}$. Hence, by Property A.5 in Appendix A, the problem

$$\begin{cases} Lu^1 = f(x,\underline{u}) & \text{a.e. in } \Omega \\ u^1 \overset{u_0}{=} 0 & \text{on } \partial\Omega \end{cases}$$

admits a unique bounded solution $u^1 \in W^{2,N}_{\text{loc}}(\Omega \cup T)$ (note that $f(x,\underline{u}) \in L^N(\Omega)$). The function $\underline{u}$ is a subsolution of this problem and $\overline{u}$ is a supersolution by the monotonicity of $f(x,\cdot)$. By the refined MP - Property A.2 - we get $\underline{u} \leq u^1$, but we cannot infer that $u^1 \leq \overline{u}$, because $\overline{u}$ may be unbounded. However, since the solution u^1 is obtained as the limit of solutions $(u_n^1)_{n\in\mathbb{N}}$ of the Dirichlet problem in a family of bounded smooth domains invading Ω (see the proof of Theorem 1.2 in [11]), the inequality $u^1 \leq \overline{u}$ follows by applying the refined MP to the functions $\overline{u} - u_n^1$. Proceeding as before, we construct by iteration a sequence $(u^j)_{j\in\mathbb{N}}$ in $W^{2,N}_{\mathrm{loc}}(\Omega \cup T) \cap L^\infty(\Omega)$ such that

$$\begin{cases} Lu^{j+1} = f(x,u^j) & \text{a.e. in } \Omega \\ u^{j+1} \overset{u_0}{=} 0 & \text{on } \partial\Omega \\ u^j \leq u^{j+1} \leq \overline{u} & \text{in } \Omega. \end{cases}$$

For $x \in \Omega$, let $u(x)$ be the limit of the nondecreasing sequence $(u^j(x))_{j\in\mathbb{N}}$. Let us show that the u^j are uniformly bounded in Ω. We write

$$Lu^j = f(x,0) + \zeta_j(x)u^j \quad \text{a.e. in } \Omega, \qquad \text{with } \zeta_j(x) := \frac{f(x,u^j) - f(x,0)}{u^j}.$$

Since the L^∞ norm of the ζ_j is less than or equal to the Lipschitz constant of $f(x,\cdot)$, applying the ABP estimate - Property A.6 - to u^j and $-u^j$ we infer that $(u^j)_{j\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Therefore, by the local boundary estimate, $(u^j)_{j\in\mathbb{N}}$ is bounded in $W^{2,N}(K)$, for any $K \subset\subset \Omega \cup T$. Whence, considering suitable subsequences of $(u^j)_{j\in\mathbb{N}}$ and applying the embedding theorem, we derive $u \in W^{2,N}_{\mathrm{loc}}(\Omega \cup T) \cap L^\infty(\Omega)$, $Lu = f(x,u)$ a.e. in Ω and $u = 0$ on T. This concludes the proof. □

6 Relations between λ_1, λ_1' and λ_1''

This section is devoted to the proof of Theorem 1.7. We will start from statement (ii). In our previous work [12], we proved it in dimension 1, using a direct argument, and we left the case of arbitrary dimension as an open problem. Here, we solve it by subtracting a quadratic penalization term that prevents solutions from being unbounded.

Proof of Theorem 1.7 part (ii). We prove the statement by showing that, for any given $\lambda > \lambda_1(-L,\Omega)$, $\lambda_1'(-L,\Omega) \leq \lambda$. We can assume without loss of generality that $\lambda = 0$. Since $\lambda_1(-L,\Omega) < 0$, by Proposition 2.3 part (iv) there exists a bounded smooth domain $\Omega' \subset \Omega$ such that $\lambda_{\Omega'} < 0$. Let φ' be the principal eigenfunction associated with $\lambda_{\Omega'}$, normalized by

$$\|\varphi'\|_{L^\infty(\Omega')} = \min\left(1, -\frac{\lambda_{\Omega'}}{\|c\|_{L^\infty(\Omega')}}\right).$$

Then, the functions $\underline{u}$, $\overline{u}$ defined by

$$\underline{u}(x) := \begin{cases} \varphi'(x) & \text{if } x \in \Omega' \\ 0 & \text{otherwise,} \end{cases} \qquad \overline{u}(x) := 1,$$

satisfy, a.e. in Ω,

$$L\underline{u} \geq c^+(x)\underline{u}^2, \qquad L\overline{u} \leq c^+(x)\overline{u}^2, \qquad \underline{u} \leq \overline{u},$$

where $c^+(x) = \max(c(x), 0)$. Thus, there exists a solution $u \in W^{2,p}_{loc}(\overline{\Omega})$, $\forall\, p < \infty$, of the problem

$$\begin{cases} Lu = c^+(x)u^2 & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $\underline{u} \leq u \leq \overline{u}$ in Ω (note that $\underline{u}$ is a "generalized subsolution" of the above problem because it is the supremum of two subsolutions). The existence of u follows from the same arguments as in the proof of Proposition 5.2, but here is actually simpler because the supersolution $\overline{u}$ is bounded. In particular, we see that $(L - c^+(x))u \leq 0$ a.e. in Ω and then the strong maximum principle yields $u > 0$ in Ω. Taking $\phi = u$ in (1.2) we eventually derive $\lambda_1'(-L, \Omega) \leq 0$. □

There is also a more direct, linear proof of Theorem 1.7 part (ii) [1]. The arguments, that we sketch now, make use of two independent results proved later on in this paper. As before, the aim is to show that $\lambda_1(-L, \Omega) < 0$ implies $\lambda_1'(-L, \Omega) \leq 0$. Let $(L_n)_{n\in\mathbb{N}}$ be the following family of operators:

$$L_n = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c_n(x), \qquad \text{with } c_n(x) := \begin{cases} c(x) & \text{if } |x| < n \\ \min(c(x), 0) & \text{otherwise.} \end{cases}$$

Note that $\lambda_1(-L_n, \Omega) > -\infty$ by Proposition 2.3 part (ii) and thus Theorem 1.4 implies that a principal eigenfunction φ_1^n of $-L_n$ in Ω (satisfying the Dirichlet boundary condition) does exist. Since $\lambda_1(-L, \Omega) < 0$, it follows from Proposition 9.2 part (i) that $\lambda_1(-L_n, \Omega) < 0$ for n large enough. Applying Proposition 1.11 part (ii) we deduce that φ_1^n is bounded for such values of n. Moreover, it satisfies

$$-L\varphi_1^n \leq -L_n\varphi_1^n = \lambda_1(-L_n, \Omega)\varphi_1^n < 0 \quad \text{a.e. in } \Omega.$$

We eventually infer that $\lambda_1'(-L, \Omega) \leq 0$.

Remark 6.1. As a byproduct of the above proofs of Theorem 1.7 part (ii), we have shown that the set in definition (1.2) is nonempty when Ω is smooth. If in addition c is bounded from above, the definition of $\lambda_1'(-L, \Omega)$ does not change if one restricts to subsolutions ϕ belonging to $W^{2,p}_{loc}(\overline{\Omega})$, $\forall\, p < \infty$. Indeed, if λ, ϕ satisfy the conditions in (1.2), then for any $\tilde{\lambda} > \lambda$ we can argue as in the first proof, with $\underline{u} = \varepsilon\phi$, ε small enough, and find a positive bounded subsolution of the Dirichlet problem for $L + \tilde{\lambda}$ in Ω satisfying the stronger regularity conditions.

Proof of Theorem 1.7 part (iii). Suppose that there exists $\lambda < \lambda_1''(-L, \Omega)$ and $\phi \in W^{2,N}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ such that

$$(L + \lambda)\phi \geq 0 \quad \text{a.e. in } \Omega, \qquad \forall\, \xi \in \partial\Omega, \; \limsup_{x\to\xi} \phi(x) \leq 0.$$

[1] The authors are grateful to a referee for suggesting this approach.

Since

$$\lambda_1''(-(L+\lambda),\Omega) = \lambda_1''(-L,\Omega) - \lambda > 0,$$

we know from Theorem 1.6 part (i) that the MP holds for the operator $(L+\lambda)$ in Ω. As a consequence, $\phi \le 0$ in Ω. This shows that $\lambda_1'(-L,\Omega) \ge \lambda_1''(-L,\Omega)$. □

Proof of Theorem 1.7 part (i). Owing to statement (ii), we only need to prove that $\lambda_1(-L,\Omega) \le \lambda_1'(-L,\Omega)$. That is, if $\lambda \in \mathbb{R}$ and $\phi \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ are such that

$$\phi > 0 \ \text{ in } \Omega, \qquad (L+\lambda)\phi \ge 0 \ \text{ a.e. in } \Omega, \qquad \forall\, \xi \in \partial\Omega, \ \lim_{x\to\xi} \phi(x) = 0,$$

then $\lambda_1(-L,\Omega) \le \lambda$. This will be achieved by the use of the variational formula (2.2). Clearly, the infimum in (2.2) can be taken over functions in $H^1_0(\Omega)$ with compact support in $\overline{\Omega}$. Note, however, that since no restriction is imposed on the behavior of c at infinity, one cannot consider the whole space $H^1_0(\Omega)$. Let $(\chi_r)_{r>1}$ be a family of cutoff functions uniformly bounded in $W^{1,\infty}(\mathbb{R}^N)$ and such that

$$\forall\, r > 1, \qquad \operatorname{supp}\chi_r \subset B_r, \qquad \chi_r = 1 \ \text{ in } B_{r-1}.$$

We can suppose that $\Omega \cap B_1 \ne \emptyset$. The functions $\phi\chi_r$ belong to $H^1_0(\Omega \cap B_r)$. Thus, for $r > 1$, we get

$$\begin{aligned}
\lambda_1(-L,\Omega) &\le \frac{\int_\Omega \left[a_{ij}(x)\partial_i(\phi\chi_r)\partial_j(\phi\chi_r) - c(x)\phi^2\chi_r^2\right]}{\int_\Omega \phi^2\chi_r^2} \\
&= \frac{\int_\Omega \left[a_{ij}(x)(\partial_i\phi)\chi_r\partial_j(\phi\chi_r) + a_{ij}(x)\phi(\partial_i\chi_r)\partial_j(\phi\chi_r) - c(x)\phi^2\chi_r^2\right]}{\int_\Omega \phi^2\chi_r^2}.
\end{aligned}$$

Integrating by parts the first term of the above sum yields

$$\begin{aligned}
\lambda_1(-L,\Omega) &\le \frac{\int_\Omega \left[(-L\phi)\phi\chi_r^2 - a_{ij}(x)(\partial_i\phi)(\partial_j\chi_r)\phi\chi_r + a_{ij}(x)\phi(\partial_i\chi_r)\partial_j(\phi\chi_r)\right]}{\int_\Omega \phi^2\chi_r^2} \\
&\le \lambda + \frac{\int_\Omega a_{ij}(x)(\partial_i\chi_r)(\partial_j\chi_r)\phi^2}{\int_\Omega \phi^2\chi_r^2}.
\end{aligned}$$

Since $\chi_r = 0$ outside B_r and $\chi_r = 1$ in B_{r-1}, we can then find a constant $k > 0$, only depending on $\sup_{r>1} \|\chi_r\|_{W^{1,\infty}(\mathbb{R}^N)}$ and $\|a_{ij}\|_{L^\infty(\Omega)}$, such that

$$\forall\, r > 1, \quad \lambda_1(-L,\Omega) \le \lambda + k\frac{\int_{\Omega\cap(B_r\setminus B_{r-1})} \phi^2}{\int_{\Omega\cap B_{r-1}} \phi^2}.$$

We obtain the desired inequality $\lambda_1(-L,\Omega) \le \lambda$ from the above formula by showing that

$$\liminf_{r\to\infty} \frac{\int_{\Omega\cap(B_r\setminus B_{r-1})} \phi^2}{\int_{\Omega\cap B_{r-1}} \phi^2} = 0.$$

Suppose by contradiction that there exists $\varepsilon > 0$ such that

$$\forall\, n > 2, \quad \frac{\int_{\Omega\cap(B_n\setminus B_{n-1})} \phi^2}{\int_{\Omega\cap B_{n-1}} \phi^2} \ge \varepsilon.$$

Hence, the sequence $j_n := \int_{\Omega\cap B_{n-1}} \phi^2$ satisfies $j_{n+1} - j_n \geq \varepsilon j_n$, that is, $j_n \geq j_2(1+\varepsilon)^{n-2}$. This is impossible because j_n grows at most at the rate n^N as $n \to \infty$. $\square$

Remark 6.2. The previous proof shows that Theorem 1.7 part (i) holds, more in general, with $\lambda_1'(-L,\Omega)$ replaced by $\lambda_\beta'(-L,\Omega)$ (given by Definition 1.2) provided that

$$\forall\, \sigma > 0, \quad \lim_{\substack{x\in\Omega \\ |x|\to\infty}} \beta(x) e^{-\sigma|x|} = 0.$$

On the other hand, if $\beta(x) = e^{\sigma|x|}$, $\sigma > 0$, then one can check that the operator $Lu = u''$ satisfies $\lambda_1(-L,\mathbb{R}) = 0 > -\sigma^2 = \lambda_\beta'(-L,\mathbb{R}) = \lambda_\beta''(-L,\mathbb{R})$.

7 Conditions for the equivalence of the three notions

7.1 Proof of Theorem 1.9, case 1

We start with a preliminary consideration.

Lemma 7.1. *Let Ω be bounded and L be self-adjoint. If $u \in H^1(\Omega)$, $\chi \in C^1(\overline{\Omega})$ satisfy $u^+\chi \in H_0^1(\Omega)$, $Lu \geq 0$ in Ω, then*

$$\frac{\lambda_1(-L,\Omega)}{\max_{\overline{\Omega}} \overline{\alpha}} \int_\Omega (u^+\chi)^2 \leq \int_\Omega (u^+|\nabla\chi|)^2.$$

Proof. The result follows from a well known inequality which is an immediate consequence of the divergence theorem (see, e.g., [16], [25]). Since $Lu \geq 0$, we derive

$$\begin{aligned}
0 &\geq \int_\Omega a_{ij}(x)\partial_j u \partial_i(u^+\chi^2) - c(x)uu^+\chi^2 \\
&= \int_\Omega a_{ij}(x)[\partial_j(u^+\chi)\partial_i(u^+\chi) - u^+\partial_j\chi\partial_i(u^+\chi) + u^+\chi\partial_j u^+\partial_i\chi] - c(x)(u^+\chi)^2 \\
&= \int_\Omega a_{ij}(x)\partial_j(u^+\chi)\partial_i(u^+\chi) - c(x)(u^+\chi)^2 - (u^+)^2 a_{ij}(x)\partial_i\chi\partial_j\chi \\
&\geq \lambda_1(-L,\Omega)\int_\Omega (u^+\chi)^2 - \left(\max_{\overline{\Omega}} \overline{\alpha}\right)\int_\Omega (u^+|\nabla\chi|)^2.
\end{aligned}$$

$\square$

Theorem 1.9 trivially holds if $\lambda_1(-L,\Omega) = -\infty$. Hence, the case 1 is a consequence of the following result.

Proposition 7.2. *Under the assumptions of Theorem 1.9 case 1, any $\lambda < \lambda_1(-L,\Omega)$ admits a (positive) eigenfunction in Ω with positive exponential growth.*

Proof. Case $N = 1$ and $\Omega = \mathbb{R}$.
For $n \in \mathbb{N}$, let v_n be the solution of $(L+\lambda)v_n = 0$ in $(-n,n)$ satisfying $v_n(-n) = M > 0$, $v_n(n) = 0$, with $M > 0$ such that $v_n(0) = 1$. Note that $\lambda_1(-L,(-n,n)) > \lambda$ and then v_n is positive by the maximum principle. By elliptic estimates and Harnack's inequality, $(v_n)_{n\in\mathbb{N}}$ converges (up to subsequences) locally uniformly to

a nonnegative solution v of $(L+\lambda)=0$ in $\mathbb{R}$. Since $v(0)=1$, the strong maximum principle implies that v is positive. We apply Lemma 7.1 to v_n, with $\chi=0$ in $(-\infty,0]$ and $\chi=1$ in $[1,+\infty)$. We derive

$$\int_1^n v_n^2 \le \int_0^n (v_n\chi)^2 \le \frac{\sup_{\mathbb{R}}\overline{\alpha}}{\lambda_1(-L,(-n,n))-\lambda}\int_0^n (v_n\chi')^2$$
$$\le \frac{\sup_{\mathbb{R}}\overline{\alpha}}{\lambda_1(-L,\mathbb{R})-\lambda}\int_0^1 (v_n\chi')^2.$$

Whence, letting $n\to\infty$, $\int_1^{+\infty} v^2<+\infty$. Next, consider a family $(\chi_n)_{n\in\mathbb{N}}$ of smooth functions satisfying

$$\chi_n(x)=0 \quad \text{for } |x|\ge n, \qquad \chi_n(x)=1 \quad \text{for } |x|\le n-1, \qquad |\chi_n'|\le 2 \quad \text{in } \mathbb{R}.$$

Lemma 7.1 yields

$$\int_{n-1\le|x|\le n} v^2 \ge \frac{1}{4}\int_{\mathbb{R}} (v\chi_n')^2 \ge \frac{\lambda_1(-L,\mathbb{R})-\lambda}{\overline{\alpha}}\int_{|x|\le n-1} v^2.$$

There exist then $k,\varepsilon>0$ such that

$$\forall n\in\mathbb{N}, \quad \int_{n-1\le|x|\le n} v^2 \ge k(1+\varepsilon)^n.$$

Since $\int_0^{+\infty} v^2<+\infty$, it follows that $\int_{-n\le x\le -n+1} v^2 \ge \frac{k}{2}(1+\varepsilon)^n$, for n large enough. Hence, we can find $-n<x_n<-n+1$ such that $v(x_n)\ge\sqrt{k}(1+\varepsilon)^{\frac{n}{2}}$. By Harnack's inequality we deduce that v has positive exponential growth at $-\infty$. Changing x in $-x$ in the coefficients of L and applying the above arguments yields the existence of a positive solution w of $L+\lambda=0$ in $\mathbb{R}$ which has positive exponential growth at $+\infty$. The function $v+w$ is an eigenfunction associated with λ with positive exponential growth.

Case $N=1$ and Ω is a half-line.

We can assume, without loss of generality, that $\Omega=(0,+\infty)$. Let $\lambda<\lambda_1(-L,\Omega)$ and let u be a positive solution of $(L+\lambda)=0$ in Ω satisfying $u(0)=0$. For $n\in\mathbb{N}$, applying Lemma 7.1 with $\chi=1$ in $[0,n-1]$, $\chi=0$ in $[n,+\infty)$ and $|\chi'|\le 2$ in $[n-1,n]$, we obtain

$$\int_0^{n-1} u^2 \le \varepsilon\int_{n-1}^n u^2,$$

for some ε independent of n. The same argument as above shows that u has exponential growth at $+\infty$.

Case $N>1$ and L is radially symmetric.

For $\lambda<\lambda_1(-L,\mathbb{R}^N)$, there exists a positive solution u of $(L+\lambda)=0$ in $\mathbb{R}^N$, which in addition is radially symmetric. Applying Lemma 7.1 with $\chi=1$ in B_{n-1}, $\chi=0$ outside B_n and $|\nabla\chi|\le 2$ in $B_n\backslash B_{n-1}$, we get

$$\int_{B_{n-1}} u^2 \le \varepsilon\int_{B_n\backslash B_{n-1}} u^2,$$

for some ε independent of n. As a consequence, $\int_{B_n\setminus B_{n-1}} u^2$ grows exponentially in n and then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with the same property and $n-1 < |x_n| < n$. The symmetry of u together with Harnack's inequality imply that u has exponential growth. □

7.2 Continuity of λ_1 with respect to decreasing sequences of domains

We know that λ_1 is continuous with respect to increasing sequences of domains (see statement (iv) of Proposition 2.3). We now derive the continuity property for sequences of sets approaching the domain from outside - Theorem 1.10.

Let us first sketch how one can derive the property in the case Ω bounded and smooth. Owing to the monotonicity of λ_1 with respect to the inclusion of domains, it is sufficient to prove the result in the case $\Omega_n = \bigcup_{x\in\Omega} B_{1/n}(x)$. To prove that $\lambda^* := \lim_{n\to\infty}\lambda_1(-L,\Omega_n) = \lambda_1(-L,\Omega)$, one considers the Dirichlet principal eigenfunction φ_1^n of $-L$ in Ω_n, normalized by $\|\varphi_1^n\|_{L^\infty(\Omega_n)} = 1$. By elliptic estimates, $(\varphi_1^n)_{n\in\mathbb{N}}$ converges (up to subsequences) to a solution $\tilde\varphi_1$ of $-L\tilde\varphi_1 = \lambda^*\tilde\varphi_1$ in Ω. Moreover, since the Ω_n are uniformly smooth for n large, the C^1 estimates up to the boundary yield $\tilde\varphi_1 = 0$ on $\partial\Omega$ and $\|\tilde\varphi_1\|_{L^\infty(\Omega)} = 1$. Hence, $\tilde\varphi_1$ is the Dirichlet principal eigenfunction of $-L$ in Ω, that is, $\lambda^* = \lambda_1(-L,\Omega)$.

Three types of difficulties arise in the general case. First, if Ω is not smooth one has to consider generalized principal eigenfunctions satisfying the Dirichlet boundary conditions in the relaxed sense of [11]. In particular, the C^1 boundary estimates are no longer available and then the passage to the limit in the boundary conditions is a subtle issue. Second, if Ω is unbounded then it might happen that $\lambda_1(-L,\Omega_n) = -\infty$ for all $n\in\mathbb{N}$. Third, in unbounded domains, the existence of a (positive) eigenfunction vanishing on the boundary does not characterize λ_1, as shown by Theorem 1.4.

Proof of Theorem 1.10. The proof is divided into three steps.

Step 1: reducing to domains with smooth boundary portions.
We want to replace $(\Omega_n)_{n\in\mathbb{N}}$ with a family of domains $(\mathscr{O}_r)$ having uniformly smooth boundaries in a neighborhood of $\overline{\Omega_1\setminus\Omega}$. Let U be a bounded neighborhood of $\overline{\Omega_1\setminus\Omega}$ such that $\overline{U}\cap\partial\Omega$ is smooth. Consider a nonnegative smooth function χ defined on $\overline{U}\cap\partial\Omega$ which is positive on $\overline{\Omega_1\setminus\Omega}\cap\partial\Omega$ and whose support is contained in U. Then, for $r>0$, define

$$\mathscr{O}_r := \Omega\cup\{\xi+\delta\nu(\xi)\ :\ \xi\in U\cap\partial\Omega,\ 0\le\delta<\frac{1}{r}\chi(\xi)\}.$$

where $\nu(\xi)$ stands for the outer normal to Ω at ξ. The smoothness of $U\cap\partial\Omega$ implies the existence of $r_0>0$ such that the $(\partial\mathscr{O}_r)_{r\ge r_0}$ are uniformly smooth in U. It is left to the reader to show that, for all $n\in\mathbb{N}$, there exists $k_n\in\mathbb{N}$ such that $\Omega_{k_n}\subset\mathscr{O}_n$. Hence, by Proposition 2.3, the sequences $(\lambda_1(-L,\Omega_n))_{n\in\mathbb{N}}$, $(\lambda_1(-L,\mathscr{O}_n))_{n\in\mathbb{N}}$ are nondecreasing and satisfy

$$\lambda^* := \lim_{n\to\infty}\lambda_1(-L,\mathscr{O}_n) \le \lim_{n\to\infty}\lambda_1(-L,\Omega_n) \le \lambda_1(-L,\Omega).$$

To prove the result it is then sufficient to show that $\lambda^* \geq \lambda_1(-L,\Omega)$. We argue by contradiction assuming that $\lambda^* < \lambda_1(-L,\Omega)$.

Step 2: the case Ω bounded.
Let φ_1^n be the generalized principal eigenfunction of $-L$ in $\mathscr{O}_n$, provided by Property A.1, normalized by $\|\varphi_1^n\|_{L^\infty(\mathscr{O}_n)} = 1$. For given $\tilde{\lambda} \in (\lambda^*, \lambda_1(-L,\Omega))$, we have that $\lambda_1(-(L+\tilde{\lambda}),\Omega) > 0$ and that $(L+\tilde{\lambda})\varphi_1^n > 0$ a.e. in Ω. Thus, the refined Alexandrov-Bakelman-Pucci estimate - Property A.6 - yields

$$\sup_{\Omega} \varphi_1^n \leq \sup_{\mathscr{O}_n \cap \partial\Omega} \varphi_1^n \left(1 + A(\sup_{\Omega} c^+ + \tilde{\lambda})|\Omega|^{1/N}\right),$$

for some A independent of n. On the other hand, for n large enough, every $x \in \mathscr{O}_n \backslash \Omega$ satisfies $\mathrm{dist}(x, \partial\mathscr{O}_n) \leq \frac{1}{n} \sup \chi$. Therefore, since φ_1^n vanishes on $\partial\mathscr{O}_n$, the local boundary estimate and the uniform smoothness of $(\partial\mathscr{O}_n)_{n \geq r_0}$ in U yield

$$\lim_{n\to\infty} \sup_{\mathscr{O}_n \backslash \Omega} \varphi_1^n = 0.$$

We eventually get $\lim_{n\to\infty} \|\varphi_1^n\|_{L^\infty(\mathscr{O}_n)} = 0$, which is a contradiction.

Step 3: the general case.
Suppose that $0 \in \Omega$. For $\rho > 0$, let $\mathscr{B}_\rho$ denote the connected component of $\Omega \cap B_\rho$ containing the origin. Since $\overline{U} \cap \partial\Omega$ is smooth, a compactness argument shows that there exists $\rho_0 > 0$ such that $(\Omega \cap U) \subset \mathscr{B}_{\rho_0}$. Thus, for all $\rho \geq \rho_0$ and $n \in \mathbb{N}$, the set $(\mathscr{O}_n \backslash \Omega) \cup \mathscr{B}_\rho$ is connected. Fix $\tilde{\lambda} \in (\lambda^*, \lambda_1(-L,\Omega))$. Proposition 2.3 part (iv) yields

$$\forall\, n \in \mathbb{N}, \quad \lim_{\rho\to\infty} \lambda_1(-L, (\mathscr{O}_n \backslash \Omega) \cup \mathscr{B}_\rho) = \lambda_1(-L, \mathscr{O}_n) \leq \lambda^* < \tilde{\lambda}.$$

It is then possible to choose $\rho_n > \rho_0$ in such a way that $\lambda_1(-L, (\mathscr{O}_n \backslash \Omega) \cup \mathscr{B}_{\rho_n}) < \tilde{\lambda}$. We can assume, without loss of generality, that the sequence $(\rho_n)_{n\in\mathbb{N}}$ is increasing and diverging. For fixed $n \in \mathbb{N}$, $n \geq r_0$, consider the following mapping:

$$\Theta(r) := \lambda_1(-L, (\mathscr{O}_r \backslash \Omega) \cup \mathscr{B}_{\rho_n}).$$

We know that $\Theta(n) < \tilde{\lambda}$ and, by step 2 and Proposition 2.3 part (iv), that Θ is continuous on $[n, +\infty)$ and satisfies

$$\lim_{r\to+\infty} \Theta(r) = \lambda_1(-L, \mathscr{B}_{\rho_n}) > \lambda_1(-L,\Omega) > \tilde{\lambda}.$$

Hence, there exists $r_n > n$ such that $\Theta(r_n) = \tilde{\lambda}$. We set $\tilde{\mathscr{O}}_n := (\mathscr{O}_{r_n} \backslash \Omega) \cup \mathscr{B}_{\rho_n}$. By Theorem 1.4 and Remark 3.2, there exists a function $\varphi_1 \in W^{2,p}_{loc}(\Omega \cup (U \cap \partial\Omega))$, $\forall\, p < \infty$, satisfying

$$\varphi_1 > 0 \quad \text{in } \Omega, \qquad -L\varphi_1 = \lambda_1(-L,\Omega)\varphi_1 \quad \text{a.e. in } \Omega, \qquad \varphi_1 = 0 \quad \text{on } U \cap \partial\Omega.$$

Let $\tilde{\varphi}_1^n$ be the generalized principal eigenfunction in $\tilde{\mathscr{O}}_n$, vanishing on $U \cap \partial\tilde{\mathscr{O}}_n$, given by Property A.1. We now use the following

Lemma 7.3. *Let $\mathscr{O}$ and K be an open and a compact subset of $\mathbb{R}^N$ such that $T := \partial\mathscr{O}\cap(K+B_\varepsilon)$ is smooth for some $\varepsilon>0$, and let $v\in W^{2,N}_{loc}(\mathscr{O})$ be positive and satisfy $Lv\leq 0$ a.e. in $\mathscr{O}$. Then, there exists a positive constant h such that*

$$\sup_{\mathscr{O}\cap K}\frac{u}{v}\leq h\|u\|_{W^{1,\infty}(\mathscr{O}\cap(K+B_\varepsilon))},$$

for all $u\in C^1(\mathscr{O}\cup T)$ satisfying $u\leq 0$ on T.

Let us postpone the proof of Lemma 7.3 and continue with the one of Theorem 1.10. Consider a neighborhood V of $\operatorname{supp}\chi$ such that $\overline{V}\subset U$. Applying Lemma 7.3 with $\mathscr{O}=\Omega\cap U$, $K=\overline{\Omega}\cap\partial V$, $u=\tilde{\varphi}_1^n$ and $v=\varphi_1$, we see that it is possible to normalize the $\tilde{\varphi}_1^n$ in such a way that

$$\forall\, n\geq r_0,\quad \inf_{\Omega\cap\partial V}\frac{\varphi_1}{\tilde{\varphi}_1^n}=1. \tag{7.1}$$

Note that the generalized principal eigenvalue λ_1 of $-(L+\tilde{\lambda})$ is positive in any connected component of $\Omega\cap\tilde{\mathscr{O}}_n\backslash\overline{V}$. Hence, owing to Property A.2, $\varphi_1\geq\tilde{\varphi}_1^n$ in $\Omega\cap\tilde{\mathscr{O}}_n\backslash\overline{V}$ by the refined MP. Moreover, since the $(\partial\tilde{\mathscr{O}}_n)_{n\geq r_0}$ are uniformly smooth in U, using the boundary Harnack inequality and the local boundary estimate we infer that, for any compact $K\subset\Omega\cup(U\cap\partial\Omega)$, the $\tilde{\varphi}_1^n$ are uniformly bounded in $W^{2,p}(K)$ for n large enough. Thus, by Morrey's inequality, they converge (up to subsequences) in $C^1_{loc}(\Omega\cup(U\cap\partial\Omega))$ to a nonnegative function $\tilde{\varphi}_1\in W^{2,p}_{loc}(\Omega\cup(U\cap\partial\Omega))$ satisfying

$$\begin{cases} -L\tilde{\varphi}_1=\tilde{\lambda}\,\tilde{\varphi}_1 & \text{a.e. in } \Omega\\ \tilde{\varphi}_1=0 & \text{on } U\cap\partial\Omega.\end{cases} \tag{7.2}$$

Furthermore, $\tilde{\varphi}_1\leq\varphi_1$ in $\Omega\backslash V$ and then in the whole Ω by the refined MP. Therefore, the difference $\varphi_1-\tilde{\varphi}_1$ is a nonnegative strict supersolution of (7.2). The strong maximum principle implies $\varphi_1-\tilde{\varphi}_1>0$ in Ω. Applying Lemma 7.3 with $u=\tilde{\varphi}_1$, $v=\varphi_1-\tilde{\varphi}_1$ and $L=L+\tilde{\lambda}$, we can find a positive constant h such that $\tilde{\varphi}_1\leq h(\varphi_1-\tilde{\varphi}_1)$ in $\overline{\Omega}\cap\partial V$, i.e., $\varphi_1\geq(1+h^{-1})\tilde{\varphi}_1$. Since $\tilde{\varphi}_1^n$ converges to $\tilde{\varphi}_1$ in $C^1_{loc}(\Omega\cup(U\cap\partial\Omega))$, using again Lemma 7.3 we can choose n large enough in such a way that $(2h+2)^{-1}\varphi_1\geq\tilde{\varphi}_1^n-\tilde{\varphi}_1$ in $\Omega\cap\partial V$. Gathering together these inequalities we derive

$$\varphi_1\geq(1+h^{-1})(\tilde{\varphi}_1^n-(2h+2)^{-1}\varphi_1)=(1+h^{-1})\tilde{\varphi}_1^n-(2h)^{-1}\varphi_1\quad\text{in }\Omega\cap\partial V.$$

This contradicts (7.1). □

It remains to prove Lemma 7.3. It is essentially a consequence of the Hopf lemma, even though the hypothesis on v does not allow one to apply it in its classical form.

Proof of Lemma 7.3. Assume by contradiction that there exist a sequence of functions $(u^n)_{n\in\mathbb{N}}$ with $u^n\leq 0$ on T, $\|u^n\|_{W^{1,\infty}(\mathscr{O}\cap(K+B_\varepsilon))}=1$ and a sequence of points $(x_n)_{n\in\mathbb{N}}$ in $\mathscr{O}\cap K$ such that $u^n(x_n)>nv(x_n)$. Let ξ be the limit of (a subsequence of) $(x_n)_{n\in\mathbb{N}}$. It follows that $\xi\in\partial\mathscr{O}\cap K$. Let ξ_n be one of the projections of x_n on $\partial\mathscr{O}$. Clearly, $x_n,\xi_n\in K+B_\varepsilon$ for n large enough. Thus,

$$\limsup_{n\to\infty}\frac{v(x_n)}{|x_n-\xi_n|}\leq\limsup_{n\to\infty}\frac{u^n(x_n)-u^n(\xi_n)}{n|x_n-\xi_n|}=0. \tag{7.3}$$

On the other hand, since T is smooth, there exists $R>0$ such that $\mathscr{O}$ satisfies the interior sphere condition of radius R at the points ξ_n, for n large enough. That is, $x_n\in B_R(y_n)\subset\mathscr{O}$, where $y_n:=\xi_n-R\nu(\xi_n)$ and Fix $\rho\in(0,R)$. The existence of the positive supersolution v, together with Proposition 2.3 part (iii), imply that $0\leq\lambda_1(-L,\mathscr{O})<\lambda_1(-L,B_R(y_n)\backslash B_\rho(y_n))$. Therefore, owing to Property A.2 in Appendix A, one can follow the standard argument used to prove the Hopf lemma (see, e.g., [28] or Lemma 3.4 in [18]), comparing v with an exponential subsolution, and find a positive constant κ such that, for n large enough,

$$\forall\, x\in B_R(y_n)\backslash B_\rho(y_n),\qquad \frac{v(x)}{R-|x-y_n|}\geq\kappa\min_{\partial B_\rho(y_n)}v.$$

This contradicts (7.3). □

Theorem 1.10 does not hold in general if $\Omega_1\backslash\Omega$ is not bounded, as shown by Example 1.9 in [31]. The smoothness hypothesis on $\partial\Omega$ is also necessary, because it is possible to find two bounded domains $\Omega\subset\Omega'$ satisfying $\overline{\Omega}=\overline{\Omega'}$ and $\lambda_1(-\Delta,\Omega)>\lambda_1(-\Delta,\Omega')$ (see Remark 7.4 below). Hence, the sequence $(\Omega_n)_{n\in\mathbb{N}}$ identically equal to Ω' violates the convergence result.

Remark 7.4. If $\Omega\subset\Omega'$ are bounded and there exist $\xi\in\Omega'\cap\partial\Omega$ and $\delta>0$ such that $\Omega\cap B_\delta(\xi)$ has a connected component U satisfying the exterior cone condition at ξ (or, more generally, admitting a strong barrier at ξ) then $\lambda_1(-L,\Omega)>\lambda_1(-L,\Omega')$. To see this, consider the generalized principal eigenfunctions φ_1 and φ_1' of $-L$ in Ω and Ω' respectively, given by Property A.1. The function φ_1 can be obtained as the limit of the classical Dirichlet principal eigenfunctions of $-L$ in a family of smooth domains invading Ω, normalized by $\|\cdot\|_\infty=1$. As a consequence, the existence of the barrier function at ξ yields $\lim_{\substack{x\in U\\ x\to\xi}}\varphi_1(x)=0$. Since $\varphi_1'(\xi)>0$ by the strong maximum principle, we infer that φ_1 and φ_1' are linearly independent. Therefore, Property A.4 implies that $\lambda_1(-L,\Omega)>\lambda_1(-L,\Omega')$.

Note that Ω, Ω' fulfill the above property as soon as $\Omega'\backslash\Omega$ contains a $N-1$-dimensional Lipschitz manifold.

Remark 7.5. If Ω is bounded then the arguments in the proof of Theorem 1.10 work, with minor modifications, only assuming that $\partial\Omega$ is Lipschitz in a neighborhood of $\overline{\Omega_1\backslash\Omega}$. We do not know if the result holds for unbounded Lipschitz domains.

The first step of the proof of Theorem 1.10 consists in showing that the Ω_n approach Ω in the sense of the Hausdorff distance d_H [2]. Remark 7.4 shows that, in the

[2] For $A,B\subset\mathbb{R}^N$, $d_H(A,B):=\max(\rho(A,B),\rho(B,A))$, where $\rho(A,B):=\sup_{a\in A}\inf_{b\in B}|a-b|$.

non-smooth case, $\lambda_1(-L,\cdot)$ is not continuous with respect to d_H, and this is why the result of Theorem 1.10 may fail in that case. Note, however, that the domains Ω, Ω' in Remark 7.4 satisfy $d_H(\Omega^c,(\Omega')^c) > 0$. The Hausdorff distance between the complements is a better suited notion of distance for open sets (it implies for instance that if $d_H(\Omega_n^c,\Omega^c) \to 0$ then $\mathrm{int}\,(\bigcap_{n\in\mathbb{N}}\Omega_n) \subset \Omega \subset \bigcup_{n\in\mathbb{N}}\Omega_n$). A consequence of a γ-convergence result by Šverák [30] is that if $N = 2$, L is self-adjoint and $(\Omega_n)_{n\in\mathbb{N}}$ is a sequence of uniformly bounded domains, such that the number of connected components of Ω_n^c is uniformly bounded and $\lim_{n\to\infty} d_H(\Omega_n^c,\Omega^c) = 0$, then $\lim_{n\to\infty}\lambda_1(-L,\Omega_n) = \lambda_1(-L,\Omega)$. We refer to §2.3.3 in [19] for other continuity results for self-adjoint operators in bounded domains obtained via γ-convergence. Always in the case of bounded domains, A.-S. Sznitman proves in [31], Proposition 1.10, using a probabilistic approach, that the continuity of λ_1 with respect to decreasing sequences of domains $(\Omega_n)_{n\in\mathbb{N}}$ holds without any smoothness hypothesis on $\partial\Omega$, provided that $\bigcap_n \Omega_n = \Omega$. This hypothesis, which is stronger than $\lim_{n\to\infty} d_H(\Omega_n^c,\Omega^c) = 0$, is quite restrictive because, in general, $\bigcap_n \Omega_n$ is not an open set.

7.3 Proof of Theorem 1.9, cases 2-4

Below, we give a characterization of λ_1'' which provides a necessary and sufficient condition for the equivalence between λ_1, λ_1' and λ_1''. This characterization emphasizes that λ_1'' strongly reflects the properties of the operator at both finite distance and infinity.

Theorem 7.6. *If Ω is unbounded and smooth then*

$$\lambda_1''(-L,\Omega) = \min\left(\lambda_1(-L,\Omega), \lim_{r\to\infty}\lambda_1''(-L,\Omega\backslash\overline{B_r})\right).$$

As a consequence, $\lambda_1(-L,\Omega) = \lambda_1''(-L,\Omega)$ ($= \lambda_1'(-L,\Omega)$ if (1.5) *holds) iff*

$$\lim_{r\to\infty}\lambda_1''(-L,\Omega\backslash\overline{B_r}) \geq \lambda_1(-L,\Omega). \tag{7.4}$$

Proof. We first note that definitions (1.1) and (1.3) make good sense even if Ω is not connected, and that statements (ii), (iii) of Proposition 5.1 still hold in this case. Thus, the function $\lambda''(r) := \lambda_1''(-L,\Omega\backslash\overline{B_r})$ is nondecreasing with respect to r and satisfies

$$\lambda_1''(-L,\Omega) \leq \lim_{r\to\infty}\lambda''(r) \leq +\infty.$$

Hence, since $\lambda_1''(-L,\Omega) \leq \lambda_1(-L,\Omega)$ by definition, we find that

$$\lambda_1''(-L,\Omega) \leq \min\left(\lambda_1(-L,\Omega), \lim_{r\to\infty}\lambda''(r)\right).$$

To prove the reverse inequality, let us show that if there exists $\lambda \in \mathbb{R}$ satisfying

$$\lambda < \min\left(\lambda_1(-L,\Omega), \lim_{r\to\infty}\lambda''(r)\right),$$

then $\lambda_1''(-L,\Omega) \geq \lambda$. Take $R > 0$ such that $\lambda''(R) > \lambda$. We first prove the result in the case $\Omega = \mathbb{R}^N$. The proof in the general case is more involved and makes use of an auxiliary result - Lemma 7.7 below - derived from Theorem 1.10.

Since $\lambda''(R) > \lambda$, there exists $\phi \in W^{2,N}_{loc}(\mathbb{R}^N \backslash \overline{B}_R)$ with positive infimum and such that $(L+\lambda)\phi \leq 0$ a.e. in $\mathbb{R}^N \backslash \overline{B}_R$. By Proposition 5.2 and Morrey's inequality, we can assume without loss of generality that $\phi \in C^1(B^c_{R+1})$, where $B^c_{R+1} = \mathbb{R}^N \backslash B_{R+1}$. Let φ be an eigenfunction associated with $\lambda_1(-L,\mathbb{R}^N)$ (provided by statement (v) of Proposition 2.3) and $\chi \in C^2(\mathbb{R}^N)$ be nonnegative and satisfy

$$\chi = 0 \quad \text{in } B_{R+1}, \qquad \chi = 1 \quad \text{outside } B_{R+2}.$$

For $\varepsilon > 0$, define the function $u := \varphi + \varepsilon\chi\phi$. We see that $(L+\lambda)u \leq 0$ a.e. in $B_{R+1} \cup B^c_{R+2}$. On the other hand, for a.e. $x \in B_{R+2} \backslash B_{R+1}$,

$$\begin{aligned}(L+\lambda)u &\leq (L+\lambda)\varphi + \varepsilon[\chi(L+\lambda)\phi + 2a_{ij}\partial_i\chi\partial_j\phi + (a_{ij}\partial_{ij}\chi + b_i\partial_i\chi)\phi] \\ &\leq (\lambda - \lambda_1(-L,\mathbb{R}^N))\varphi + \varepsilon C,\end{aligned}$$

where C is a constant depending on N, the L^∞ norms of a_{ij}, b_i, the $W^{2,\infty}$ norm of χ and the $W^{1,\infty}$ norm of ϕ on $B_{R+2} \backslash B_{R+1}$. Therefore, for ε small enough the function u satisfies $(L+\lambda)u < 0$ a.e. in $B_{R+2} \backslash B_{R+1}$. Since u is an admissible function for λ_1'', we eventually obtain $\lambda_1''(-L,\mathbb{R}^N) \geq \lambda$.

Let us now turn to the case of a general smooth domain Ω. Assume that $\Omega \cap B_R \neq \emptyset$, otherwise we immediately get $\lambda_1''(-L,\Omega) = \lambda''(R) > \lambda$. The open set $\Omega \backslash \overline{B}_R$, being smooth in a neighborhood of ∂B_{R+1}, has a finite number of connected components $\mathscr{O}_1, \dots, \mathscr{O}_m$ intersecting ∂B_{R+1}. This is seen by a compactness argument that we leave to the reader. For $j \in \{1, \dots m\}$, we have $\lambda_1''(-L,\mathscr{O}_j) \geq \lambda_1''(R) > \lambda$. Since $\partial\mathscr{O}_j \backslash \partial B_R$ is smooth, by Proposition 5.2 there exists a function $\phi^j \in W^{2,p}_{\rm loc}(\overline{\mathscr{O}}_j \backslash \partial B_R)$, $\forall\, p < \infty$, satisfying

$$\inf_{\mathscr{O}_j} \phi^j > 0, \qquad (L+\lambda)\phi^j \leq 0 \quad \text{a.e. in } \mathscr{O}_j. \tag{7.5}$$

Define the function ϕ by setting $\phi(x) := \phi^j(x)$ if $x \in \mathscr{O}_j$. Note that $\Omega \backslash B_{R+1} \subset \bigcup_{j=1}^m \mathscr{O}_j$ because Ω is connected. Thus, $\phi \in W^{2,p}_{\rm loc}(\overline{\Omega} \backslash B_{R+1})$ satisfies (7.5) with $\mathscr{O}_j$ replaced by $\Omega \backslash \overline{B}_{R+1}$. We fix $\tilde\lambda \in (\lambda, \lambda_1(-L,\Omega))$ and consider a function $\tilde\varphi$ satisfying

$$-L\tilde\varphi = \tilde\lambda\tilde\varphi \quad \text{a.e. in } \Omega, \qquad \tilde\varphi > 0 \quad \text{in } \Omega \cup (\overline{B}_{R+2} \cap \partial\Omega).$$

The function $\tilde\varphi$ replaces the eigenfunction φ used in the case $\Omega = \mathbb{R}^N$. Its existence is given by the next lemma.

Lemma 7.7. *Assume that Ω has a $C^{1,1}$ boundary portion $T \subset \partial\Omega$ which is compact. Then, for any $\tilde\lambda < \lambda_1(-L,\Omega)$, there exists $\tilde\varphi \in W^{2,p}_{\rm loc}(\Omega \cup T)$, $\forall\, p < \infty$, such that*

$$-L\tilde\varphi = \tilde\lambda\tilde\varphi \quad \textit{a.e. in } \Omega, \qquad \tilde\varphi > 0 \quad \textit{in } \Omega \cup T.$$

Postponing the proof of Lemma 7.7 for a moment, let us complete the proof of Theorem 7.6. Consider the same function $\chi \in C^2(\mathbb{R}^N)$ as before. For $\varepsilon > 0$, the function $u := \tilde{\varphi} + \varepsilon\chi\phi$ satisfies $(L+\lambda)u \le 0$ a.e. in $\Omega \cap (B_{R+1} \cup B^c_{R+2})$. Moreover, since $\phi \in C^1(\overline{\Omega} \cap (\overline{B}_{R+2} \backslash B_{R+1}))$, the same computation as before shows that there exists C independent of ε such that

$$(L+\lambda)u \le (\lambda - \lambda_1(-L,\tilde{\Omega}))\tilde{\varphi} + \varepsilon C \quad \text{a.e. in } \Omega \cap (B_{R+2} \backslash B_{R+1}).$$

The latter quantity is negative for ε small enough because $\tilde{\varphi} > 0$ on $\overline{\Omega \cap B_{R+2}}$. Therefore, taking $\phi = u$ in (1.3) we get $\lambda_1''(-L,\Omega) \ge \lambda$.

The last statement of Theorem 7.6 follows immediately from Theorem 1.7. □

Proof of Lemma 7.7. Let U be a bounded neighborhood of T where $\partial\Omega$ is smooth. Consider an extension of the operator L - still denoted by L - to $\Omega \cup U$, satisfying the same hypotheses as L. As we have seen in the proof of Theorem 1.10, it is possible to construct a decreasing sequence of domains $(\mathscr{O}_n)_{n\in\mathbb{N}}$ satisfying

$$\overline{\mathscr{O}_1 \backslash \Omega} \subset U, \qquad \forall\, n \in \mathbb{N}, \quad \Omega \cup T \subset \mathscr{O}_n, \qquad \bigcap_{n\in\mathbb{N}} \overline{\mathscr{O}}_n = \overline{\Omega}.$$

Hence, by Theorem 1.10, $\lambda_1(-L,\mathscr{O}_n) > \tilde{\lambda}$ for n large enough. It then follows that there exists a positive function $\tilde{\varphi} \in W^{2,p}_{\mathrm{loc}}(\mathscr{O}_n)$, $\forall\, p < \infty$, satisfying $-L\tilde{\varphi} = \tilde{\lambda}\tilde{\varphi}$ a.e. in $\mathscr{O}_n$. In particular, $\tilde{\varphi} > 0$ on $\Omega \cup T \subset \mathscr{O}_n$. □

Conclusion of the proof of Theorem 1.9. Cases 2-4 are derived from Theorem 7.6, which is a powerful tool to understand when equality occurs. Thus, the aim is to prove (7.4).

Case 2) By the definition of λ_1'', it follows that

$$\begin{aligned}\lim_{r\to\infty} \lambda_1''(-L,\Omega\backslash\overline{B_r}) &\ge \lim_{r\to\infty} \left(\lambda_1''(-\tilde{L},\Omega\backslash\overline{B_r}) - \sup_{\Omega\backslash\overline{B_r}} \gamma \right) \\ &= \lim_{r\to\infty} \lambda_1''(-\tilde{L},\Omega\backslash\overline{B_r}).\end{aligned}$$

The last limit above is greater than or equal to $\lambda_1''(-\tilde{L},\Omega) = \lambda_1(-\tilde{L},\Omega)$. Since $\gamma \ge 0$, we see that $\lambda_1(-\tilde{L},\Omega) \ge \lambda_1(-L,\Omega)$. Hence, (7.4) holds.

Case 3) Proposition 5.1 part (ii) yields

$$\lim_{r\to\infty} \lambda_1''(-L,\Omega\backslash\overline{B_r}) \ge \lim_{r\to\infty} (-\sup_{\Omega\backslash\overline{B_r}} c) = -\limsup_{\substack{x\in\Omega \\ |x|\to\infty}} c(x) \ge \lambda_1(-L,\Omega).$$

Case 4) Owing to the case 3, it is sufficient to show that $\lambda_1(-L,\Omega) \le -\sigma$, for all $\sigma < \limsup_{\substack{x\in\Omega \\ |x|\to\infty}} c(x)$. Take such a σ. Consider first the case where L is self-adjoint. Let B be a ball contained in Ω. Proposition 2.3 part (iii) yields $\lambda_1(-L,\Omega) \le \lambda_1(-L,B)$. From the Rayleigh-Ritz formula, it then follows that

$$\lambda_1(-L,\Omega) \le \lambda_1(-L,B) \le \lambda_1(-\Delta,B) \sup_B \overline{\alpha} - \inf_B c.$$

Since, by hypothesis, we can find balls $B \subset \Omega$ with arbitrarily large radius such that $\inf_B c > \sigma$, we deduce that $\lambda_1(-L,\Omega) \leq -\sigma$. Consider now the case where L is not self-adjoint. By hypothesis, there exists $\delta > 0$ such that, for all $r > 0$, there is a ball B of radius r satisfying

$$\forall\, x \in B, \quad 4\underline{\alpha}(x)(c(x)-\sigma) \geq \delta.$$

Let B' be another ball of radius $r/4$ contained in the set $B \backslash B_{r/2}$. For large enough r, we find that

$$\forall\, x \in B', \quad 4\underline{\alpha}(x)(c(x)-\sigma) - |b(x)|^2 \geq \delta/2.$$

As shown in Lemma 3.1 of [8], if the radius of B' is large enough (depending on δ), the above condition ensures the existence of a C^2 function ϕ satisfying

$$(L-\sigma)\phi > 0 \text{ in } B', \qquad \phi > 0 \text{ in } B', \qquad \phi = 0 \text{ in } \partial B'.$$

As a consequence,

$$-\sigma \geq \lambda_1'(-L,B') = \lambda_{B'} = \lambda_1(-L,B') \geq \lambda_1(-L,\Omega).$$

□

Remark 7.8. If the function γ in the case 2 of Theorem 1.9 is compactly supported in Ω, then $\lambda_1(-L,\Omega) = \lambda_1''(-L,\Omega)$ holds true even for Ω non-smooth.

8 Existence and uniqueness of the principal eigenfunctions

We now investigate the simplicity of λ_1. Another natural question is to know whether the generalized principal eigenvalues λ_1', λ_1'' have corresponding eigenvalues that satisfy the additional requirements of their definitions. This section is devoted to these questions.

We say that an eigenfunction φ is admissible for λ_1' (resp. λ_1'') if it satisfies

$$\sup_{\Omega} \varphi < \infty \qquad (\text{resp. } \forall\, \varepsilon > 0, \quad \inf_{\Omega_\varepsilon} \varphi > 0),$$

where Ω_ε is defined in Section 2.1. Throughout this section, we assume that $\lambda_1, \lambda_1', \lambda_1'' \in \mathbb{R}$ (which is for instance the case if $\sup c < +\infty$).

From Theorem 1.4 we know that if Ω is smooth then there always exist eigenfunctions with eigenvalues λ_1, λ_1', λ_1'' respectively. But, as we show below, λ_1' and λ_1'' may not have admissible eigenfunctions. Moreover, λ_1, λ_1', λ_1'' are generally not simple.

Proposition 8.1. *There exist operators L for which there are several linearly independent eigenfunctions associated with the eigenvalues $\lambda_1(-L,\Omega)$, $\lambda_1'(-L,\Omega)$, $\lambda_1''(-L,\Omega)$. There are also operators such that λ_1' or λ_1'' have several linearly independent admissible eigenfunctions and others for which they do not have any.*

Proof. Let $Lu = u'' + c(x)u$ in $\mathbb{R}$, with $c < 0$ in $(-1,1)$ and $c = 0$ outside. We show that λ_1' has no admissible eigenfunctions and that $\lambda_1(-L,\mathbb{R}) = \lambda_1'(-L,\mathbb{R}) = \lambda_1''(-L,\mathbb{R}) = 0$ is not simple, even in the class of admissible eigenfunctions for λ_1''. Let φ_- and φ_+ be the solutions to $Lu = 0$ in $\mathbb{R}$ satisfying $\varphi_\pm(\pm1) = 1$, $\varphi_\pm'(\pm1) = 0$. By ODE arguments we find that φ_- and φ_+ are positive and satisfy

$$\varphi_- = 1 \text{ in } (-\infty,-1], \qquad \varphi_+ = 1 \text{ in } [1,+\infty), \qquad \lim_{x\to\mp\infty} \varphi_\pm(x) = +\infty.$$

Consequently, they are linearly independent and thus they generate the space of solutions to $Lu = 0$ in $\mathbb{R}$. Taking $\phi = \varphi_-$ in (1.3) and using Theorem 1.7 we derive $\lambda_1''(-L,\mathbb{R}) = \lambda_1'(-L,\mathbb{R}) = \lambda_1(-L,\mathbb{R}) = 0$.

To exhibit an example of non-existence of admissible eigenfunctions for λ_1'', we will make use of Proposition 1.11, proved at the end of this section. Consider the operator $Lu = u'' + c(x)u$ in $\mathbb{R}$, with $c = 0$ in $(-\pi,\pi)$, $c = -1$ outside $(-\pi,\pi)$. By Proposition 2.3 part (iii) we see that $\lambda_1(-L,\mathbb{R}) < \lambda_1(-L,(-\pi,\pi)) = 1/4$. Thus, Theorem 1.9 yields $\lambda_1''(-L,\mathbb{R}) = \lambda_1(-L,\mathbb{R})$. But Proposition 1.11 implies that the eigenfunction associated with $\lambda_1(-L,\mathbb{R})$ is unique (up to a scalar multiple) and vanishes at infinity.

Lastly, an example of non-uniqueness of admissible eigenfunctions for λ_1' is given by the operator

$$Lu := u'' + \frac{2x}{1+x^2}u' \text{ in } \mathbb{R}.$$

In fact, the functions $u_1 \equiv 1$ and $u_2(x) = \arctan(x) + \pi$ satisfy $Lu = 0$ in $\mathbb{R}$. Taking $\phi = u_1$ in the definition of λ_1' and λ_1'' we get $\lambda_1'(-L,\mathbb{R}) \leq 0 \leq \lambda_1''(-L,\mathbb{R})$. Hence, $\lambda_1''(-L,\mathbb{R}) = \lambda_1'(-L,\mathbb{R}) = 0$ by statement (iii) of Theorem 1.7 and, as a consequence, λ_1' is not simple. $\square$

Let us mention two other examples of non non-existence of admissible eigenfunctions for λ_1' and λ_1'' respectively, this time in higher dimension, that can be exhibited using the theory of critical operators (see, e.g.,[26]). The first one is $L = \Delta + c(x)$ in $\mathbb{R}^2$, where

$$c(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{n\in\mathbb{N}} B_{r_n}(x_n) \\ 0 & \text{otherwise,} \end{cases}$$

with $(x_n)_{n\in\mathbb{N}}$, $(r_n)_{n\in\mathbb{N}}$ such that the $B_{r_n}(x_n)$ are disjoint and $|x_n|, r_n \to \infty$. Clearly, $\lambda_1(-L,\mathbb{R}^2) = \lambda_1'(-L,\mathbb{R}^2) = \lambda_1(-L,\mathbb{R}^2) = 0$, but one can show that the equation $L = 0$ does not admit positive bounded solutions in $\mathbb{R}^2$ (see [25]). An example where no admissible eigenfunctions exist for λ_1'' is $L = \Delta + c(x)$ in $\mathbb{R}^N$, $N \geq 3$, with $c \leq 0$ chosen in such a way that L is *critical*. Then $\lambda_1(-L,\mathbb{R}^N) = \lambda_1''(-L,\mathbb{R}^N) = 0$, and the (unique up to a scalar multiple) positive solution of $L = 0$ behaves at infinity like $|x|^{2-N}$.

In order to derive a sufficient condition for the simplicity of λ_1, we introduce the notion of "minimal growth at infinity". This notion slightly differs from the one of S. Agmon [1] (see Remark 8.3 below). In the case of smooth domains, the sufficient condition we obtain - Theorem 8.5 - is more general than Theorem 5.5 in [1] (whose proof can be found in [23], see Lemma 4.6 and Remark 4.8 therein). Let us mention that another sufficient condition for the simplicity of λ_1 can be expressed in terms of the criticality property of the operator (see, e.g., §4 in [26] and the references therein).

Definition 8.2. Let Ω be unbounded. A positive function $u \in W^{2,N}_{loc}(\Omega)$ satisfying

$$Lu = 0 \quad \text{a.e. in } \Omega, \tag{8.1}$$

is said to be a solution of (8.1) of *minimal growth at infinity* if for any $\rho > 0$ and any positive function $v \in W^{2,N}_{loc}(\Omega\backslash B_\rho)$ satisfying $Lv \leq 0$ a.e. in $\Omega\backslash B_\rho$, there exist $R \geq \rho$ and $k > 0$ such that $ku \leq v$ in $\Omega\backslash B_R$.

Remark 8.3. Our Definition 8.2 of minimal growth at infinity differs from the original one of Agmon [1]. There, B_ρ and B_R are replaced by two compact sets $K \subset K' \subset \Omega$. Thus, Agmon's definition regards minimal growth both at infinity and at the boundary, whereas ours only deals with behavior at infinity. Indeed, Agmon calls it "minimal growth at infinity in Ω". Using the refined maximum principle in bounded domains, one readily sees that solutions of minimal growth at infinity vanishing on $\partial\Omega$ fulfill Agmon's definition. Hence, owing to Theorem 5.5 in [1], they are unique up to a scalar multiple. This fact is expressed in the next statement, whose simple proof is included here for the sake of completeness. Another difference with Agmon's approach is that he also considers positive solutions in proper subsets $\Omega\backslash E$, without imposing condition on ∂E. Such solutions can always be constructed, no matter what the sign of $\lambda_1(-L,\Omega)$ is, satisfying in addition the minimal growth condition. When E reduces to a single point, this type of solutions is used to investigate the removability of singularities.

Proposition 8.4 ([1]). *Let Ω be unbounded and $u \in W^{2,N}_{loc}(\Omega) \cap C^0(\overline{\Omega})$ be a solution of* (8.1) *of minimal growth at infinity vanishing on $\partial\Omega$. Then, for any positive function $v \in W^{2,N}_{loc}(\Omega)$ satisfying $Lv \leq 0$ a.e. in Ω, there exists $\kappa > 0$ such that $v \equiv \kappa u$ in Ω. In particular, $\lambda_1(-L,\Omega) = 0$.*

Proof. Taking $\phi = u$ in (1.1) yields $\lambda_1(-L,\Omega) \geq 0$. Consider a function v as in the statement. The quantity

$$\kappa := \inf_{\Omega} \frac{v}{u}$$

is a nonnegative real number. Suppose by way of contradiction that $v - \kappa u > 0$ in Ω. Applying Definition 8.2 with $v - \kappa u$ in place of v, we can find $R, h > 0$ such that $hu \leq v - \kappa u$ in $\Omega\backslash B_R$. By Property A.2, we know that the refined MP holds in any connected component $\mathscr{O}$ of $\Omega \cap B_R$, because $\lambda_1(-L,\mathscr{O}) > 0$ by Proposition 2.3 part (iii). As a consequence, $hu \leq v - \kappa u$ in the whole Ω. This contradicts the definition of κ. Therefore, $v - \kappa u$ vanishes somewhere in Ω, and then everywhere by the strong maximum principle. □

From Proposition 8.4 it follows in particular that λ_1 is simple, in the class of positive functions, as soon as it admits an eigenfunction having minimal growth at infinity (we recall that eigenfunctions are assumed to vanish on $\partial\Omega$). Here we derive a sufficient condition for this to hold.

Theorem 8.5. *If Ω is unbounded and smooth and $\lambda_1(-L,\Omega)$ satisfies*

$$\lambda_1(-L,\Omega) < \lim_{r\to\infty} \lambda_1(-L,\Omega\backslash\overline{B}_r),$$

then the associated eigenfunction is a solution of (2.1) *of minimal growth at infinity and, therefore, $\lambda_1(-L,\Omega)$ is simple in the class of positive functions.*

Proof. It is not restrictive to assume that $\lambda_1(-L,\Omega)=0$. Consider the same family of bounded domains $(\Omega_n)_{n\in\mathbb{N}}$ as in the proof of Theorem 1.4, i.e.,

$$\forall\, n\in\mathbb{N},\quad \Omega\cap B_n\subset\Omega_n\subset\Omega_{n+1}\subset\Omega.$$

As we have seen there, the generalized principal eigenfunctions φ^n of $-L$ in Ω_n - provided by Property A.1 - normalized by $\varphi^n(x_0)=1$, for a given $x_0\in\Omega$, converge (up to subsequences) in $C^1_{loc}(\overline{\Omega})$ to an eigenfunction φ^* with eigenvalue $\lambda_1(-L,\Omega)$. We claim that φ^* is a solution of (2.1) of minimal growth at infinity. By hypothesis, there exists $R>0$ such that $\lambda_1(-L,\Omega\backslash\overline{B}_R)>0$. Let $\mathscr{O}_1,\dots,\mathscr{O}_m$ be the connected components of $\Omega\backslash\overline{B}_R$ intersecting ∂B_{R+1} (which are finite due to the smoothness of Ω). It follows from Proposition 2.3 that there is $n_0\in\mathbb{N}$ such that

$$\forall\, j\in\{1,\dots m\},\ n\geq n_0,\quad \lambda_1(-L,\mathscr{O}_j)\geq\lambda_1(-L,\Omega\backslash\overline{B}_R)>\lambda_1(-L,\Omega_n)>0.$$

Let $\phi^j>0$ satisfy $-L\phi^j=\lambda_1(-L,\mathscr{O}_j)\phi^j$ a.e. in $\mathscr{O}_j$ (see statement (v) of Proposition 2.3). Since $\varphi^n\to\varphi^*$ in $C^1_{loc}(\overline{\Omega})$, by Lemma 7.3 it is possible to normalize ϕ^j in such a way that

$$\forall\, n\in\mathbb{N},\quad \phi^j\geq\varphi^n\quad\text{on } \Omega_n\cap\mathscr{O}_j\cap\partial B_{R+1}.$$

Hence, for $n\geq n_0$, applying the refined MP in every connected component of $\Omega_n\cap\mathscr{O}_j\backslash\overline{B}_{R+1}$ - which holds due to Property A.2 - we get $\varphi^n\leq\phi^j$ in $\Omega_n\cap\mathscr{O}_j\backslash B_{R+1}$. It follows that, for given $\varepsilon>0$, the function $\varphi^n-\varepsilon\phi^j$ satisfies

$$L(\varphi^n-\varepsilon\phi^j)\geq[-\lambda_1(-L,\Omega_n)+\varepsilon\lambda_1(-L,\mathscr{O}_j)]\varphi^n\quad\text{a.e. in } \Omega_n\cap\mathscr{O}_j\backslash B_{R+1}.$$

Therefore, since $(\lambda_1(-L,\Omega_n))_{n\in\mathbb{N}}$ converges to 0, there exists $n_1\in\mathbb{N}$ such that $L(\varphi^n-\varepsilon\phi^j)>0$ a.e. in $\Omega_n\cap\mathscr{O}_j\backslash B_{R+1}$ for $n\geq n_1$. Consider now a function v as in Definition 8.2. Let $R'>\max(\rho,R+1)$. By Lemma 7.3, there exists $h>0$ such that

$$\forall\, n\in\mathbb{N},\quad hv\geq\varphi^n\quad\text{on } \Omega_n\cap\partial B_{R'}.$$

For $n\geq n_1$, applying once again the refined MP we then obtain $\varphi^n-\varepsilon\phi^j\leq hv$ in $\Omega_n\cap\mathscr{O}_j\backslash B_{R'}$. Letting $n\to\infty$ we finally derive $\varphi^*-\varepsilon\phi^j\leq hv$ in $\mathscr{O}_j\backslash B_{R'}$. Since the latter holds for all $j\in\{1,\dots m\}$ and $\varepsilon>0$, we eventually infer that $\varphi^*\leq hv$ in $\Omega\backslash B_{R'}$. This concludes the proof. $\square$

Corollary 8.6. *If Ω is unbounded and smooth, the a_{ij} are bounded and the b_i, c satisfy*

$$\lim_{\substack{x\in\Omega\\|x|\to\infty}} \frac{b(x)\cdot x}{|x|} = \pm\infty, \qquad \sup_{\Omega} c < \infty, \tag{8.2}$$

then the eigenfunction associated with $\lambda_1(-L,\Omega)$ is a solution of (2.1) *of minimal growth at infinity, and it satisfies*

$$\forall\,\sigma>0, \quad \lim_{\substack{x\in\Omega\\|x|\to\infty}} \varphi(x)e^{\pm\sigma|x|} = 0,$$

where the $\pm$ is in agreement with the $\pm$ in (8.2).

Proof. For $\sigma>0$, define the function ϕ by $\phi(x) := e^{\mp\sigma|x|}$, where the $\mp$ is in agreement with the $\pm$ in (8.2). The same computation as in the proof of Proposition 2.7 shows that $(L+\lambda_1(-L,\Omega)+1)\phi\le 0$ a.e. in $\Omega\backslash B_r$, for r large enough. Therefore, $\lambda_1(-L,\Omega\backslash B_r)\ge\lambda_1(-L,\Omega)+1$. The result then follows from Theorem 8.5. □

We now derive a result about the exponential decay of subsolutions of the Dirichlet problem. This will be used to prove the last statement of Proposition 1.11.

Proposition 8.7. *Let Ω be unbounded and smooth, L be an elliptic operator with bounded coefficients such that*

$$\limsup_{\substack{x\in\Omega\\|x|\to\infty}} c(x) < 0,$$

and A, B be the functions in (4.4). *Set*

$$\Gamma_- := \limsup_{\substack{x\in\Omega\\|x|\to\infty}} \frac{B(x)-\sqrt{B^2(x)-4A(x)c(x)}}{2A(x)},$$

$$\Gamma_+ := \liminf_{\substack{x\in\Omega\\|x|\to\infty}} \frac{B(x)+\sqrt{B^2(x)-4A(x)c(x)}}{2A(x)}.$$

Then, for any function $u\in W^{2,N}_{\mathrm{loc}}(\Omega)$ satisfying

$$Lu\ge 0 \text{ a.e. in } \Omega, \qquad \forall\,\xi\in\partial\Omega,\ \limsup_{x\to\xi} u(x)\le 0$$

and such that

$$\exists\,\gamma\in[0,-\Gamma_-), \quad \limsup_{\substack{x\in\Omega\\|x|\to\infty}} u(x)e^{-\gamma|x|}\le 0,$$

it holds that

$$\forall\,\eta\in(0,\Gamma_+), \quad \limsup_{\substack{x\in\Omega\\|x|\to\infty}} u(x)e^{\eta|x|}\le 0.$$

Proof. Let $\eta \in (0, \Gamma_+)$. Consider two numbers $\underline{\sigma} \in (\Gamma_-, -\gamma)$ and $\overline{\sigma} \in (\eta, \Gamma_+)$. By hypothesis, there exists $R > 0$ such that, for a.e. $x \in \Omega \backslash B_{R-1}$, $u(x) \le e^{\gamma|x|}$, $c(x) < 0$ and

$$\underline{\sigma} > \frac{B(x) - \sqrt{B^2(x) - 4A(x)c(x)}}{2A(x)}, \qquad \overline{\sigma} < \frac{B(x) + \sqrt{B^2(x) - 4A(x)c(x)}}{2A(x)}.$$

For any $n \in \mathbb{N}$, define the function

$$\overline{u}_n(x) := e^{R(\gamma+\overline{\sigma})-\overline{\sigma}|x|} + e^{(R+n)(\gamma+\underline{\sigma})-\underline{\sigma}|x|}.$$

Since for $\sigma \in \mathbb{R}$ we have $Le^{-\sigma|x|} = (A(x)\sigma^2 - B(x)\sigma + c(x))e^{\sigma|x|}$, we infer that $L\overline{u}_n \le 0$ a.e. in $x \in \Omega \backslash \overline{B}_{R-1}$. Moreover, $\overline{u}_n \ge u$ on $\Omega \cap (\partial B_{R+n} \cup \partial B_R)$. Consequently, applying the maximum principle in any connected component of $\Omega \cap (B_{R+n} \backslash \overline{B}_R)$ (where $c < 0$) we get

$$\forall\, n \in \mathbb{N},\ x \in \Omega \cap (B_{R+n} \backslash B_R), \quad u(x) \le e^{R(\gamma+\overline{\sigma})-\overline{\sigma}|x|} + e^{(R+n)(\gamma+\underline{\sigma})-\underline{\sigma}|x|}.$$

Letting n go to infinity in the above inequality yields

$$\forall\, x \in \Omega \backslash B_R, \quad u(x) \le e^{R(\gamma+\overline{\sigma})-\overline{\sigma}|x|},$$

which concludes the proof. □

It is not hard to see that the upper bounds for γ and η are optimal.

Proof of Proposition 1.11. Proposition 5.1 part (ii) yields

$$\lim_{r\to\infty} \lambda_1(-L, \Omega \backslash \overline{B_r}) \ge \lim_{r\to\infty} \lambda_1''(-L, \Omega \backslash \overline{B_r}) \ge \lim_{r\to\infty} (-\sup_{\Omega \backslash \overline{B_r}} c) = -\xi.$$

Hence, if $\xi < 0$ and $\lambda_1(-L, \Omega) > 0$ we find that $\lambda_1''(-L, \Omega) > 0$ by Theorem 7.6. Then the MP holds due to Theorem 1.6. Suppose now that $\lambda_1(-L, \Omega) < -\xi$ (which is the case if $\xi < 0$ and $\lambda_1(-L, \Omega) \le 0$). Theorem 8.5 implies that the eigenfunction φ_1 associated with $\lambda_1(-L, \Omega)$ has minimal growth at infinity. Since $v \equiv 1$ satisfies $(L + \lambda_1(-L, \Omega))v < 0$ a.e. in $\Omega \backslash B_\rho$, for ρ large enough, Definition 8.2 implies that φ_1 is bounded. This concludes the proof of statement (i) and, owing to Propositions 8.4 and 8.7, statement (ii) also follows. □

Remark 8.8. The hypothesis $\xi < 0$ in Proposition 1.11 part (i) is sharp. Indeed, we can construct an operator L in $\mathbb{R}$, with a negative zero-order term vanishing at $\pm\infty$, for which $\lambda_1(-L, \mathbb{R}) > 0$ but the MP does not hold. To this aim, consider a nondecreasing odd function $b \in C^0(\mathbb{R})$ such that $b = 2$ in $(1/\sqrt{3}, +\infty)$. Direct computation shows that the function $u(x) := 2 - (x^2+1)^{-1}$ satisfies

$$\forall\, x \in \mathbb{R}, \quad c(x) := -\frac{u'' + b(x)u'}{u} < 0, \qquad \lim_{x\to\pm\infty} c(x) = 0.$$

Defining the operator L by $Lv := v'' + b(x)v' + c(x)v$, we get $Lu = 0$ in $\mathbb{R}$. It is easily seen that the function ϕ defined by $\phi(x) := e^{-|x|}$ for $|x| \ge 1/\sqrt{3}$ can be extended to the whole line as a positive smooth function satisfying $\phi'' + b(x)\phi' + \varepsilon\phi < 0$ in $\mathbb{R}$, for some $\varepsilon > 0$. As a consequence, $\lambda_1(-L, \mathbb{R}) \ge \varepsilon$. Note that if instead of $\mathbb{R}$ we consider the half line $\mathbb{R}^+$, we still have $\lambda_1(-L,\mathbb{R}^+) \geq \varepsilon$ and $u-1$ violates the MP there.

9 Continuous dependence of λ_1 with respect to the coefficients

We know from statements (vii), (viii) of Proposition 2.3 that λ_1 is Lipschitz-continuous (using the L^∞ norm) in its dependence on the coefficients b_i and c. Let us show that, if $\Omega = \mathbb{R}^N$ and the coefficients are Hölder continuous, Schauder's estimates and Harnack's inequality imply the Lipschitz-continuity with respect to the a_{ij} too. We point out that it is possible to use $\sup_{x\in\Omega} \|\cdot\|_{L^p(B_1(x))}$, $p > 1$, instead of the L^∞ norm and to deal with discontinuous b_i, c. This was shown by A. Ancona in Theorem 2' of [3] using much more involved arguments than the simple observation presented below.

Proposition 9.1. *Let $L_k = a^k_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$, $k = 1,2$, be two uniformly elliptic operators with coefficients in $C^{0,\delta}(\mathbb{R}^N)$, $\delta \in (0,1)$. Then,*

$$|\lambda_1(-L_1,\mathbb{R}^N) - \lambda_1(-L_2,\mathbb{R}^N)| \leq C \sum_{i,j=1}^N \|a^1_{ij} - a^2_{ij}\|_{L^\infty(\mathbb{R}^N)},$$

where C depends on N, the ellipticity constants of the operators and the Hölder norms of the coefficients.

Proof. For $k \in \{1,2\}$, let φ_k be an eigenfunction of $-L_k$ in $\mathbb{R}^N$ associated with $\lambda_1(-L_k,\mathbb{R}^N)$, provided by Proposition 2.3 part (v). We know that $\varphi_k \in C^{2,\delta}(\mathbb{R}^N)$. It holds that

$$(L_2 + \lambda_1(-L_1,\mathbb{R}^N))\varphi_1 = (a^2_{ij} - a^1_{ij})\partial_{ij}\varphi_1 \quad \text{in } \mathbb{R}^N.$$

By Schauder's interior estimates (see, e.g., Theorem 6.2 in [18]) there exists $h > 0$, only depending on N, the ellipticity constants and the Hölder norms of the coefficients of L_1, such that, for $x \in \mathbb{R}^N$, $\|\varphi_1\|_{C^2(B_1(x))} \leq h\|\varphi_1\|_{L^\infty(B_2(x))}$. Hence, Harnack's inequality yields

$$\forall\, x \in \mathbb{R}^N, \quad \|\varphi_1\|_{C^2(B_1(x))} \leq C \inf_{B_2(x)} \varphi_1 \leq C\varphi_1(x),$$

for some positive constant C. As a consequence, $(L_2 + \lambda)\varphi_1 \leq 0$ in $\mathbb{R}^N$, with

$$\lambda = \lambda_1(-L_1,\mathbb{R}^N) - C \sum_{i,j=1}^N \|a^1_{ij} - a^2_{ij}\|_{L^\infty(\mathbb{R}^N)}.$$

Taking $\phi = \varphi_1$ in the definition of $\lambda_1(-L_2,\mathbb{R}^N)$, we then derive

$$\lambda_1(-L_2,\mathbb{R}^N) \geq \lambda_1(-L_1,\mathbb{R}^N) - C \sum_{i,j=1}^N \|a^1_{ij} - a^2_{ij}\|_{L^\infty(\mathbb{R}^N)}.$$

Exchanging the roles of L_1 and L_2, one gets the two-sided inequality. $\square$

Next, we derive a semicontinuity property under some weak convergence hypotheses on the coefficients, as well as a continuity result when $\Omega = \mathbb{R}^N$ and the limit operator has continuous coefficients.

Proposition 9.2. *Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of operators in Ω of the type*

$$L_n u = a_{ij}^n(x)\partial_{ij}u + b_i^n(x)\partial_i u + c^n(x)u.$$

The following properties hold true:

(i) if for any $r > 0$, the sequences $(a_{ij}^n)_{n\in\mathbb{N}}$, $(b_i^n)_{n\in\mathbb{N}}$, $(c^n)_{n\in\mathbb{N}}$ are bounded in $L^\infty(\Omega\cap B_r)$, the (a_{ij}^n) are in $C(\overline{\Omega})$ with smallest eigenvalues $\underline{\alpha}^n$ satisfying $\inf_{n\in\mathbb{N}}\inf_{\Omega\cap B_r}\underline{\alpha}^n > 0$, and there is $p > 1$ such that $a_{ij}^n \to a_{ij}$ in $L^p_{\mathrm{loc}}(\Omega)$ and $b_i^n \rightharpoonup b_i$, $c^n \rightharpoonup c$ in $L^1_{\mathrm{loc}}(\Omega)$, then

$$\lambda_1(-L,\Omega) \geq \limsup_{n\to\infty} \lambda_1(-L_n,\Omega);$$

(ii) if $\Omega = \mathbb{R}^N$, L is uniformly elliptic, $a_{ij} \in C^{0,\delta}(\mathbb{R}^N)$, the b_i, c are bounded and uniformly continuous and $a_{ij}^n \to a_{ij}$, $b_i^n \to b_i$, $c^n \to c$ in $L^\infty(\mathbb{R}^N)$, then

$$\lambda_1(-L,\mathbb{R}^N) = \lim_{n\to\infty} \lambda_1(-L_n,\mathbb{R}^N).$$

Proof. We write for short $\lambda_1 := \lambda_1(-L,\Omega)$ and $\lambda_1^n := \lambda_1(-L_n,\Omega)$. By hypothesis, in both cases (i) and (ii), the sequence $(\lambda_1^n)_{n\in\mathbb{N}}$ is bounded from above due to Proposition 2.3 part (ii).

(i) Consider a subsequence of $(\lambda_1^n)_{n\in\mathbb{N}}$ (that we still call $(\lambda_1^n)_{n\in\mathbb{N}}$) tending to $\lambda^* := \limsup_{n\in\mathbb{N}} \lambda_1^n$. We know that $\lambda^* < +\infty$. Let us suppose that $\lambda^* > -\infty$, because otherwise there is nothing to prove. For $n \in \mathbb{N}$, let φ^n be a generalized principal eigenfunction associated with λ_1^n, normalized by $\varphi^n(x_0) = 1$, where x_0 is a given point in Ω. By usual arguments, the φ^n converge (up to subsequences) in $C^1_{\mathrm{loc}}(\Omega)$ and weakly in $W^{2,q}_{\mathrm{loc}}(\Omega)$, $\forall\, q < \infty$, to a nonnegative function $\varphi \in W^{2,q}_{\mathrm{loc}}(\Omega)$ satisfying $\varphi(x_0) = 1$. Then, it easily follows from the hypotheses that $L_n\varphi^n$ converges to $L\varphi$ in the sense of $\mathscr{D}'(\Omega)$. Therefore, $(L+\lambda^*)\varphi = 0$ in $\mathscr{D}'(\Omega)$ and thus, as $\varphi \in W^{2,q}_{\mathrm{loc}}(\Omega)$, also a.e. in Ω. The strong maximum principle then yields $\varphi > 0$ in Ω. Consequently, taking $\phi = \varphi$ in (1.1) we derive $\lambda_1 \geq \lambda^*$.

(ii) Suppose first that the b_i, c are uniformly Hölder continuous. Arguing as in the proof of Proposition 9.1 and then using Proposition 2.3 parts (vii), (viii), we can find a positive constant C such that, for $n \in \mathbb{N}$,

$$\lambda_1^n \geq \lambda_1 - C\left(\sum_{i,j=1}^N \|a_{ij}^n - a_{ij}\|_{L^\infty(\mathbb{R}^N)} + \sum_{i=1}^N \|b_i^n - b_i\|_{L^\infty(\mathbb{R}^N)}\right) - \|c^n - c\|_{L^\infty(\mathbb{R}^N)}.$$

The result follows from the above inequality and statement (i).

In order to deal with b_i, c uniformly continuous, for any fixed $\varepsilon > 0$ consider some smooth functions b_i^ε, c^ε satisfying

$$\|b_i - b_i^\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq \varepsilon, \qquad \|c - c^\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq \varepsilon,$$

obtained for instance by convolution with a mollifier (this is where the uniform continuity of b_i, c is required). Then, define the operators

$$L^\varepsilon := a_{ij}(x)\partial_{ij} + b_i^\varepsilon(x)\partial_i + c^\varepsilon(x),$$
$$L_n^\varepsilon := a_{ij}^n(x)\partial_{ij} + (b_i^n(x) - b_i(x) + b_i^\varepsilon(x))\partial_i + (c^n(x) - c(x) + c^\varepsilon(x)),$$

and call $\lambda_1^\varepsilon := \lambda_1(-L^\varepsilon, \mathbb{R}^N)$, $\lambda_1^{n,\varepsilon} := \lambda_1(-L_n^\varepsilon, \mathbb{R}^N)$. Since L^ε has Hölder continuous coefficients, we know that there exists $n_\varepsilon \in \mathbb{N}$ such that, for $n \geq n_\varepsilon$, $|\lambda_1^{n,\varepsilon} - \lambda_1^\varepsilon| \leq \varepsilon$. Hence, by statements (vii), (viii) of Proposition 2.3 there exists a positive constant C', independent of n and ε, such that

$$\forall\, n \geq n_\varepsilon, \quad |\lambda_1^n - \lambda_1| \leq |\lambda_1^n - \lambda_1^{n,\varepsilon}| + |\lambda_1^{n,\varepsilon} - \lambda_1^\varepsilon| + |\lambda_1 - \lambda_1^\varepsilon| \leq (2C'+1)\varepsilon.$$

□

In the last part of this section, we investigate the behavior of λ_1 as the zero and the second order terms blow up as well as when the ellipticity degenerates.

For $\gamma \in \mathbb{R}$, consider the operator

$$L_\gamma^c u := a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + \gamma c(x)u.$$

We set $\lambda_1^c(\gamma) := \lambda_1(-L_\gamma^c, \Omega)$.

Theorem 9.3. *The function $\lambda_1^c : \mathbb{R} \to [-\infty, +\infty)$ is concave and satisfies the following properties:*

(i) $\lambda_1^c(0) \geq 0$*;*

(ii) if c is lower semicontinuous then

$$\lim_{\gamma\to+\infty} \frac{\lambda_1^c(\gamma)}{\gamma} = -\sup_\Omega c;$$

(iii) if c is upper semicontinuous then

$$\lim_{\gamma\to-\infty} \frac{\lambda_1^c(\gamma)}{\gamma} = -\inf_\Omega c.$$

Moreover, if c is bounded then λ_1^c is uniformly Lipschitz-continuous with Lipschitz constant $\|c\|_{L^\infty(\Omega)}$.

Proof. The concavity and the Lipschitz-continuity follow from Proposition 2.3 part (vii). Statement (i) is an immediate consequence of definition (1.1). Let us prove (ii). Proposition 2.3 part (ii) implies that, for $\gamma > 0$, $\lambda_1^c(\gamma) \geq -\gamma \sup c$. Hence, to prove the statement it is sufficient to show that $\limsup_{\gamma\to+\infty} \lambda_1^c(\gamma)/\gamma \leq -\sup_\Omega c$. The lower semicontinuity of c implies that, for any given $\varepsilon > 0$, there exists a ball $B \subset \Omega$ such that $c > \sup_\Omega c - \varepsilon$ in B. Let λ_B and φ denote the Dirichlet principal eigenvalue and eigenfunction of the operator $-a_{ij}(x)\partial_{ij} - b_i(x)\partial_i$ in B. For $\gamma > 0$, the function φ satisfies, a.e. in B,

$$(L_\gamma^c + \gamma(-\sup_\Omega c + 2\varepsilon))\varphi = [-\lambda_B + \gamma(c(x) - \sup_\Omega c + 2\varepsilon)]\varphi > (\varepsilon\gamma - \lambda_B)\varphi.$$

Therefore, for $\gamma \geq \lambda_B/\varepsilon$, taking $\phi = \varphi$ in (1.2) we get $\lambda_1'(-L_\gamma^c, B) \leq \gamma(-\sup_\Omega c + 2\varepsilon)$. Since $\lambda_1'(-L_\gamma^c, B) = \lambda_1(-L_\gamma^c, B)$, Proposition 2.3 part (iii) yields

$$-\sup_\Omega c + 2\varepsilon \geq \limsup_{\gamma\to+\infty} \frac{\lambda_1(-L_\gamma^c, B)}{\gamma} \geq \limsup_{\gamma\to+\infty} \frac{\lambda_1^c(\gamma)}{\gamma}.$$

The proof of (ii) is thereby achieved due to arbitrariness of ε. Statement (iii) follows from (ii) by replacing the operator L with $a_{ij}(x)\partial_{ij} + b_i(x)\partial_i - c(x)$. $\square$

Remark 9.4. In the proof of Theorem 9.3, we have shown that

$$\lim_{\gamma\to+\infty} \frac{\lambda_1^c(\gamma)}{\gamma} \leq -\sup\{k \in \mathbb{R} \,:\, \exists \text{ a ball } B \subset \Omega \text{ such that } c(x) \geq k \text{ in } B\}.$$

Clearly, if c is lower semicontinuous then the right-hand side of the above inequality coincides with $-\sup_\Omega c$.

For $\alpha > 0$, we define

$$L_\alpha^a u := \alpha a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u.$$

We set for brief $\lambda_1^a(\alpha) := \lambda_1(-L_\alpha^a, \Omega)$.

Theorem 9.5. *The function $\lambda_1^a : \mathbb{R}_+ \to [-\infty, +\infty)$ satisfies the following properties:*

(i) if L has bounded coefficients then λ_1^a is locally Lipschitz-continuous on $\mathbb{R}_+$;

(ii) if Ω contains balls of arbitrarily large radius and L is uniformly elliptic with bounded coefficients, then

$$\liminf_{\alpha\to+\infty} \lambda_1^a(\alpha) \geq -\limsup_{\substack{x\in\Omega\\|x|\to\infty}} c(x), \qquad \limsup_{\alpha\to+\infty} \lambda_1^a(\alpha) \leq -\liminf_{\substack{x\in\Omega\\|x|\to\infty}} c(x);$$

(iii) if the L_α^a are self-adjoint then λ_1^a is concave and nondecreasing. If in addition c is lower semicontinuous then

$$\lim_{\alpha\to0^+} \lambda_1^a(\alpha) = -\sup_\Omega c.$$

Proof. (i) For $\alpha > 0$, we can write $L_\alpha^a = \alpha L_{1/\alpha}^{b,c}$, with

$$L_{1/\alpha}^{b,c} := a_{ij}(x)\partial_{ij} + \frac{1}{\alpha} b_i(x)\partial_i + \frac{1}{\alpha} c(x).$$

Therefore, $\lambda_1^a(\alpha) = \alpha\lambda_1(-L_{1/\alpha}^{b,c}, \Omega)$. The statement then follows from statements (vii), (viii) of Proposition 2.3.

(ii) We make use of the estimate (4.3) in [8]. It implies that

$$\lambda_1^a(\alpha) \leq -\liminf_{\substack{x\in\Omega\\|x|\to\infty}} \left(c(x) - \frac{|b(x)|^2}{4\alpha \inf \underline{\alpha}} \right).$$

Consequently,

$$\limsup_{\alpha\to+\infty} \lambda_1^a(\alpha) \leq -\liminf_{\substack{x\in\Omega\\|x|\to\infty}} c(x).$$

In order to prove that

$$\liminf_{\alpha\to+\infty}\lambda_1^a(\alpha) \geq -\limsup_{\substack{x\in\Omega\\|x|\to\infty}} c(x), \tag{9.1}$$

we define the function $\phi(x) := \vartheta(\alpha^{-1/8}|x|)$, with

$$\vartheta(\rho) := (e^\rho + e^{-\rho})^{-\alpha^{-1/2}}.$$

As $\vartheta'(\rho) \leq 0$ for $\rho \geq 0$, it follows that, for a.e. $x \in \Omega$,

$$\begin{aligned} a_{ij}\partial_{ij}\phi(x) &= A(x)\alpha^{-1/4}\vartheta''(\alpha^{-1/8}|x|) + \alpha^{-1/8}\vartheta'(\alpha^{-1/8}|x|)\frac{\sum_{i=1}^N a_{ii}(x) - A(x)}{|x|} \\ &\leq A(x)\alpha^{-1/4}\vartheta''(\alpha^{-1/8}|x|), \end{aligned}$$

where $A(x) = \frac{a_{ij}(x)x_ix_j}{|x|^2} \geq \underline{\alpha}(x)$. Thus, direct computation yields

$$L_\alpha^a\phi \leq \left[A(x)\alpha^{1/4}\left((1+\alpha^{-1/2})g(\alpha^{-1/8}|x|) - 1\right) + \|b\|_\infty\alpha^{-5/8} + c(x)\right]\phi,$$

with

$$g(\rho) := \left(\frac{e^\rho - e^{-\rho}}{e^\rho + e^{-\rho}}\right)^2.$$

For given $\varepsilon > 0$, let $R > 0$ be such that $c \leq \limsup_{|x|\to\infty} c(x) + \varepsilon$ a.e. in $\Omega\backslash B_R$. For α large enough and for a.e. $x \in \Omega \cap B_R$ it holds true that $g(\alpha^{-1/8}|x|) \leq 1/2$, and then that

$$L_\alpha^a\phi \leq \left(\frac{1}{2}A(x)\alpha^{1/4}(-1+\alpha^{-1/2}) + \|b\|_\infty\alpha^{-5/8} + c(x)\right)\phi.$$

On the other hand, for a.e. $x \in \Omega\backslash B_R$ we find

$$L_\alpha^a\phi \leq \left(A(x)\alpha^{-1/4} + \|b\|_\infty\alpha^{-5/8} + \limsup_{|x|\to\infty} c(x) + \varepsilon\right)\phi.$$

Consequently, $L_\alpha^a\phi \leq (\limsup_{|x|\to\infty} c(x) + 2\varepsilon)\phi$ a.e. in $\mathbb{R}^N$ for α large enough. Therefore, by definition (1.1) we obtain

$$\liminf_{\alpha\to+\infty}\lambda_1^a(\alpha) \geq -\limsup_{|x|\to\infty} c(x) - 2\varepsilon,$$

which concludes the proof due to the arbitrariness of ε.

(iii) Proposition 2.3 part (vi) implies that the function λ_1^a is concave and nondecreasing. Since $L_\alpha^a = \alpha L_{1/\alpha}^c$, it holds that $\lambda_1^a(\alpha) = \alpha\lambda_1^c(1/\alpha)$. The last statement then follows by applying Theorem 9.3 part (ii). □

Appendix A: Known results in bounded non-smooth domains

Even though in the present paper we are only interested in the case Ω smooth, in some of the proofs we deal with intersections of smooth domains, which are no longer smooth. This is why we require some of the tools developed in [11] to treat the non-smooth case. When Ω is non-smooth, the Dirichlet boundary condition has to be relaxed to a weaker sense:

$$u \overset{u_0}{=} 0 \quad (\text{resp. } u \overset{u_0}{\leq} 0) \quad \text{on } \partial\Omega,$$

which means that, if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in Ω converging to a point of $\partial\Omega$ such that $\lim_{n\to\infty} u_0(x_n) = 0$, then

$$\lim_{n\to\infty} u(x_n) = 0 \quad (\text{resp. } \limsup_{n\to\infty} u(x_n) \leq 0),$$

where u_0 is the "boundary function" associated with the problem (see [11]). We do not need to define the function u_0 here since, in the proofs, we only use the information $u \overset{u_0}{=} 0$ on smooth portions of $\partial\Omega$. It suffices to know that, there, it coincides with the standard Dirichlet condition. Indeed, it turns out that if $u \overset{u_0}{=} 0$ on $\partial\Omega$ then it can be extended as a continuous function to every $\xi \in \partial\Omega$ admitting a so called "strong barrier" by setting $u(\xi) = 0$. Since any point $\xi \in \partial\Omega$ satisfying the exterior cone condition admits a strong barrier, it follows that u vanishes continuously on smooth boundary portions of $\partial\Omega$.

We now assume that L is uniformly elliptic and that

$$a_{ij} \in C^0(\overline{\Omega}), \qquad b_i, c \in L^\infty(\Omega).$$

Definition A.1. We say that the operator L satisfies the *refined MP* in Ω if every function $u \in W^{2,N}_{\text{loc}}(\Omega)$ such that

$$Lu \geq 0 \text{ a.e. in } \Omega, \qquad \sup_\Omega u < \infty, \qquad u \overset{u_0}{\leq} 0 \text{ on } \partial\Omega,$$

satisfies $u \leq 0$ in Ω.

Properties A ([11])**.** Let Ω be a general bounded domain. Then, the following properties hold:

A.1 There exists a positive bounded function $\varphi_1 \in W^{2,p}_{loc}(\Omega)$, $\forall\, p < \infty$, called generalized principal eigenfunction of $-L$ in Ω, satisfying

$$\begin{cases} -L\varphi_1 = \lambda_1(-L,\Omega)\varphi_1 & \text{a.e. in } \Omega \\ \varphi_1 \overset{u_0}{=} 0 & \text{on } \partial\Omega; \end{cases}$$

moreover, if Ω has a $C^{1,1}$ boundary portion $T \subset \partial\Omega$, then $\varphi_1 \in W^{2,p}_{loc}(\Omega \cup T)$ and $\varphi_1 = 0$ on T;

A.2 If $\lambda_1(-L,\Omega) > 0$ then L satisfies the refined MP in Ω;

A.3 If $\phi \in W^{2,N}_{loc}(\Omega)$ is bounded from above and satisfies

$$-L\phi \leq \lambda_1(-L,\Omega)\phi \quad \text{a.e. in } \Omega, \qquad \phi \overset{u_0}{\leq} 0 \quad \text{on } \partial\Omega,$$

then ϕ is a constant multiple of the generalized principal eigenfunction φ_1;

A.4 If there exists a positive function $\phi \in W^{2,N}_{loc}(\Omega)$ satisfying

$$L\phi \leq 0 \quad \text{a.e. in } \Omega,$$

then either $\lambda_1(-L,\Omega) > 0$ or $\lambda_1(-L,\Omega) = 0$ and ϕ is a constant multiple of φ_1;

A.5 If $\lambda_1(-L,\Omega) > 0$ then, given $f \in L^N(\Omega)$, there is a unique bounded solution $u \in W^{2,N}_{loc}(\Omega)$ satisfying

$$\begin{cases} Lu = f & \text{a.e. in } \Omega \\ u \overset{u_0}{=} 0 & \text{on } \partial\Omega; \end{cases}$$

moreover, if Ω has a $C^{1,1}$ boundary portion $T \subset \partial\Omega$, then $u \in W^{2,N}_{loc}(\Omega \cup T)$ and $u = 0$ on T;

A.6 If $\lambda_1(-L,\Omega) > 0$ and $u \in W^{2,N}_{loc}(\Omega)$ is bounded above and satisfies

$$Lu \geq f \quad \text{a.e. in } \Omega, \qquad u \overset{u_0}{\leq} \beta \quad \text{on } \partial\Omega,$$

for some nonpositive function $f \in L^N(\Omega)$ and nonnegative constant β, then

$$\sup_{\Omega} u \leq \beta + A\left(\|f\|_{L^N(\Omega)} + \beta \sup c^+ |\Omega|^{1/N}\right),$$

where A only depends on Ω, $\lambda_1(-L,\Omega)$, $\inf \underline{\alpha}$ and the L^∞ norms of a_{ij}, b_i, c.

Property A.1 is Theorem 2.1 in [11], except for the improved regularity of φ_1 near the smooth boundary portion T. The latter follows from the standard local boundary estimate, even though a technical difficulty arises because φ_1 does not belong to $W^{2,p}(\Omega)$. However, it can be overcome using the same approximation argument as in the proof of Lemma 6.18 in [18]. The same is true for the last statement of A.5. The other properties refer to the following results of [11]: A.2 is Theorem 1.1, A.3 is Corollary 2.2, A.4 is Corollary 2.1, A.5 is Theorem 1.2, A.6 is Theorem 1.3.

Appendix B: The inhomogeneous boundary Harnack inequality

Using the refined Alexandrov-Bakelman-Pucci estimate, we extend the boundary Harnack inequality - Theorem 3.1 - to solutions of inhomogeneous Dirichlet problems.

Proposition B.1. *Let Ω be a bounded domain and Ω' be an open subset of Ω such that $T := \partial\Omega \cap (\Omega' + B_\eta)$ is of class $C^{1,1}$, for some $\eta > 0$. Then, any nonnegative function $u \in W^{2,N}_{loc}(\Omega \cup T)$ such that*

$$L^N(\Omega) \ni Lu \le 0 \quad a.e.\ in\ \Omega,$$

satisfies

$$\sup_{\Omega'} u \le \sup_T u + C\left(\inf_{\Omega^\delta} u + \|Lu\|_{L^N(\Omega)} + \sup_\Omega c^+ \sup_T u\right),$$

for all $\delta > 0$ such that $\Omega^\delta \neq \emptyset$, with C depending on N, Ω, δ, η, $\inf \underline{\alpha}$, the L^∞ norms of a_{ij}, b_i, c and $\lambda_1(-L,\Omega)$.

Proof. Suppose first that $\lambda_1(-L,\Omega) \le 0$. If u vanishes somewhere in Ω then $u \equiv 0$ by the strong maximum principle and the statement trivially holds. If u is positive then $\lambda_1(-L,\Omega) = 0$. Thus, by Property A.4, u is the generalized principal eigenfunction of $-L$ in Ω. In particular, $Lu = 0$ and $u = 0$ (in the classical sense) on T. The result then follows from Theorem 3.1. Consider now the case $\lambda_1(-L,\Omega) > 0$. Set $f := Lu$ and let $\chi : \mathbb{R}^N \to [0,1]$ be a smooth function such that

$$\chi = 1 \quad \text{in } \Omega' + B_{\eta/4}, \qquad \chi = 0 \quad \text{outside } \Omega' + B_{\eta/2}.$$

Let $v \in W^{2,N}_{loc}(\Omega \cup T) \cap L^\infty(\Omega)$ be the solution of the problem

$$\begin{cases} Lv = f & \text{a.e. in } \Omega \\ v \overset{u_0}{=} \chi u & \text{on } \partial\Omega. \end{cases}$$

It is given by $v = w + \chi u$, where w is the unique bounded solution of

$$\begin{cases} Lw = f - L(\chi u) & \text{a.e. in } \Omega \\ w \overset{u_0}{=} 0 & \text{on } \partial\Omega, \end{cases}$$

provided by Property A.5 (note that $\chi u \in W^{2,N}(\Omega)$). We have $0 \le v \le u$ by the refined MP - which holds due to Property A.2. The refined Alexandrov-Bakelman-Pucci estimate - Property A.6 - yields

$$\sup_\Omega v \le \sup_T u + A\left(\|Lu\|_{L^N(\Omega)} + \sup_\Omega c^+ \sup_T u\right),$$

where A depends on Ω, $\lambda_1(-L,\Omega)$ and the coefficients of L. Applying Theorem 3.1 to $u - v$, we obtain

$$\sup_{\Omega'}(u-v) \le C' \inf_{\Omega^\delta}(u-v) \le C' \inf_{\Omega^\delta} u.$$

The result then follows by gathering the above inequalities. □

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