

# Compactness estimates for Hamilton-Jacobi equations depending on space

Fabio Ancona\*, Piermarco Cannarsa†, Khai T. Nguyen‡

(Dedicated to Prof. Tai Ping Liu in the occasion of his 70<sup>th</sup> birthday)

April 8, 2015

## Abstract

We study quantitative estimates of compactness in  $\mathbf{W}_{loc}^{1,1}$  for the map  $S_t$ ,  $t > 0$  that associates to every given initial data  $u_0 \in \text{Lip}(\mathbb{R}^N)$  the corresponding solution  $S_t u_0$  of a Hamilton-Jacobi equation

$$u_t + H(x, \nabla_x u) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^N,$$

with a convex and coercive Hamiltonian  $H = H(x, p)$ . We provide upper and lower bounds of order  $1/\varepsilon^N$  on the the Kolmogorov  $\varepsilon$ -entropy in  $\mathbf{W}^{1,1}$  of the image through the map  $S_t$  of sets of bounded, compactly supported initial data. Quantitative estimates of compactness, as suggested by P.D. Lax [23], could provide a measure of the order of “resolution” and of “complexity” of a numerical method implemented for this equation. We establish these estimates deriving accurate a-priori bounds on the Lipschitz, semiconcavity and semiconvexity constant of a viscosity solution when the initial data is semiconvex. The derivation of a small time controllability result for the above Hamilton-Jacobi equation is also fundamental to establish the lower bounds on the  $\varepsilon$ -entropy.

## 1 Introduction

The theory of viscosity solutions to first-order Hamilton-Jacobi equations of the form

$$u_t(t, x) + H(x, \nabla_x u(t, x)) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^N, \quad (1)$$

provides existence, uniqueness and stability results. The concept of viscosity solution was introduced by M.G. Crandall and P.-L. Lions in [15] to cope with the lack of classical (smooth) solutions of the Cauchy problem for (1) globally defined in time. In fact, for such equations singularities in the gradient of the solution may arise in finite time, no matter how smooth the initial datum

$$u(0, \cdot) = u_0 \quad (2)$$

is assumed to be. We refer to [5] for a review of the notion of viscosity solution and the related theory for equation of type (1). In the case where the Hamiltonian  $H(x, p)$  is smooth in

---

\*Dipartimento di Matematica, Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy

†Dipartimento di Matematica, Università di Roma 'Tor Vergata', Via della Ricerca Scientifica 1, 00133 Roma, Italy

‡Department of Mathematics, Penn State University, University Park, Pa. 16802, U.S.A.

both variables and convex in the  $p$ -variable, the viscosity solution  $u(t, x)$  of (1)-(2), with initial datum  $u_0 : \mathbb{R}^N \rightarrow \mathbb{R}$  Lipschitz continuous, can be represented as the value function of a classical problem in the calculus of variation:

$$u(t, x) = \min_{\xi \in AC([0, t], \mathbb{R}^N)} \left\{ u_0(\xi(0)) + \int_0^t L(\xi(s), \dot{\xi}(s)) ds \mid \xi(t) = x \right\}, \quad (3)$$

where  $AC([0, t], \mathbb{R}^N)$  is the class of absolutely continuous functions from  $[0, t]$  to  $\mathbb{R}^n$  and  $L$  denotes the Legendre transform of  $H$  with respect to the second group of variables:

$$L(x, q) \doteq \sup_{p \in \mathbb{R}^N} \{ p \cdot q - H(x, p) \} \quad \forall x, q \in \mathbb{R}^N. \quad (4)$$

Under appropriate regularity assumptions on the map  $x \mapsto H(x, p)$ , this fact implies that  $u(t, x)$  is locally semiconcave in  $x$ , which in turn ensures that  $u(t, \cdot)$  is almost everywhere twice differentiable and that  $\nabla_x u(t, \cdot)$  has locally bounded variation ( $\nabla_x u(t, \cdot) \in \text{BV}_{loc}$ ), i.e. that the distributional Hessian  $D_x^2 u(t, \cdot)$  is a symmetric matrix of Radon measures.

There is a vast literature concerning the structure and the regularity of the gradient of a viscosity solution to (1), see for example [8, 9, 10, 11, 13, 19]. Instead, in this paper we are interested in analyzing the regularizing effect of the whole semigroup map

$$S_t : \text{Lip}_{loc}(\mathbb{R}^N) \rightarrow \text{Lip}_{loc}(\mathbb{R}^N), \quad t > 0$$

that associates to every initial data  $u_0 \in \text{Lip}(\mathbb{R}^N)$  the unique viscosity solution  $S_t u_0 \doteq u(t, \cdot)$  of the corresponding Cauchy problem (1)-(2), evaluated at time  $t$ . Namely, for  $\mathbf{W}^{1, \infty}$ -bounded subsets  $\mathcal{L}$  of  $\text{Lip}(\mathbb{R}^N)$  of the form

$$\mathcal{L}_{[R, M]} \doteq \left\{ u_0 \in \text{Lip}(\mathbb{R}^N) \mid \text{supp}(u_0) \subset [-R, R]^N, \text{Lip}[u_0] \leq M \right\}, \quad (5)$$

the semiconcavity constant of  $S_t u_0$ ,  $u_0 \in \mathcal{L}$ , on every bounded subset  $\Omega \subset \mathbb{R}^N$ , depends only on  $\Omega$ ,  $t$  and  $\mathcal{L}$ . Hence, thanks to the local uniform semiconcavity of  $S_t(\mathcal{L})$ , applying Helly's compactness theorem and a Poincaré inequality for BV-functions, one can show that the image set  $S_t(\mathcal{L})$  is compact with respect to the  $\mathbf{W}_{loc}^{1, 1}$ -topology. This property reflects the irreversibility feature of the equation (1) when the Hamiltonian  $H(x, p)$  is convex in the  $p$ -variable. Here, we are concerned with the compactifying effect of the map  $S_t$  when the space  $\text{Lip}(\mathbb{R}^N)$  is endowed with the  $\mathbf{W}_{loc}^{1, 1}$ -topology, rather than the classical  $\mathbf{L}^\infty$ -topology, having in mind the  $\mathbf{L}^1$ -stability theory and the  $\mathbf{L}^1$ -error estimates established for approximate solutions of Hamilton-Jacobi equations [24], which turn out to be sharper than the  $\mathbf{L}^\infty$  ones.

Inspired by a question posed by P.D. Lax [23] within the context of conservation laws, we employed in [1] the concept of Kolmogorov  $\varepsilon$ -entropy to provide a quantitative estimate of this regularizing effect of the semigroup map in the case where the Hamiltonian  $H = H(\nabla_x u)$  is a convex function depending only on the spatial gradient of the solution. We recall the notion of  $\varepsilon$ -entropy introduced by A. Kolmogorov [21]:

**Definition 1.** *Let  $(X, d)$  be a metric space and let  $K$  be a totally bounded subset of  $X$ . For  $\varepsilon > 0$ , let  $\mathcal{N}_\varepsilon(K|X)$  be the minimal number of sets in a cover of  $K$  by subsets of  $X$  having diameter no larger than  $2\varepsilon$ . Then the  $\varepsilon$ -entropy of  $K$  is defined as*

$$\mathcal{H}_\varepsilon(K|X) \doteq \log_2 \mathcal{N}_\varepsilon(K|X).$$

*Throughout the paper, we will call  $\varepsilon$ -cover a cover of  $K$  by subsets of  $X$  having diameter no larger than  $2\varepsilon$ .*

Actually, since in general  $S_t u_0$ ,  $u_0 \in \mathcal{L}$ , is not an element of  $\mathbf{W}^{1,1}(\mathbb{R}^N)$ , we have analyzed in [1] the Kolmogorov entropy of the translated set  $S_t(\mathcal{L}) - S_t 0$  which is a subset of  $\mathbf{W}^{1,1}(\mathbb{R}^N)$ . The main result of the present paper extends the estimates on the Kolmogorov entropy established in [1] to the semigroup map generated by (1), for Hamiltonians satisfying the **Standing Assumptions**:

**(H1)**  $H \in C^2(\mathbb{R}^N \times \mathbb{R}^N)$  is a coercive and convex map with respect to the second group of variables, i.e. it satisfies

$$\lim_{|p| \rightarrow \infty} \frac{H(x, p)}{|p|} = +\infty \quad \forall x \in \mathbb{R}^N, \quad (6)$$

$$D_p^2 H(x, p) > 0 \quad \forall x, p \in \mathbb{R}^N, \quad (7)$$

where  $D_p^2 H(x, p)$  denotes the Hessian of  $H$  with respect to the  $p$  variables and the inequality is understood in the sense that  $D_p^2 H(x, p)$  is a positive definite matrix.

**(H2)**  $H$  and its gradient satisfy the inequalities:

$$\begin{aligned} H(x, p) &\geq -c_1(1 + |x|) \\ \langle p, D_p H(x, p) \rangle - H(x, p) &\geq c_2 |D_p H(x, p)|^\alpha - c_3 \quad \forall x, p \in \mathbb{R}^N, \\ |D_x H(x, p)| &\leq c_4 |D_p H(x, p)|^\alpha + c_5 \end{aligned} \quad (8)$$

for some constants  $c_1, c_3, c_4, c_5 \geq 0$ ,  $c_2 > 0$  and  $\alpha > 1$ .

In fact, we shall provide upper bounds on the Kolmogorov entropy of  $S_t(\mathcal{L}_{[R,M]}) - S_t 0$  at any time  $t > 0$  and lower bounds for times  $t$  smaller than a quantity depending on  $R, M$ . Specifically, we prove the following

**Theorem 1.** *Let  $H : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$  be a function satisfying the assumptions **(H1)**-**(H2)** and  $\{S_t\}_{t \geq 0}$  be the semigroup of viscosity solutions generated by (1) on the domain  $\text{Lip}(\mathbb{R}^N)$ . Then, given  $R, M > 0$ , letting  $\mathcal{L}_{[R,M]}$  be the set defined in (5) the following hold.*

(i) *For any  $T > 0$  and for every  $\varepsilon > 0$  sufficiently small, one has*

$$\mathcal{H}_\varepsilon(S_T(\mathcal{L}_{[R,M]}) - S_T 0 \mid \mathbf{W}^{1,1}(\mathbb{R}^N)) \leq \Gamma_{[R,M,N,T]}^+ \cdot \frac{1}{\varepsilon^N}, \quad (9)$$

with

$$\Gamma_{[R,M,N,T]}^+ \doteq \omega_N^N \cdot \left(4N \cdot (1 + \mu_T + (\kappa_T + 1)l_T)\right)^{4N^2}, \quad (10)$$

where  $\omega_N$  denotes the Lebesgue measure of the unit ball of  $\mathbb{R}^N$  and  $l_T, \mu_T, \kappa_T$  are constants depending on  $R, M, N, T$  defined in (104), (105), (110).

(ii) *For any  $0 < T < \tau_{R,M}$  and for every  $\varepsilon > 0$  sufficiently small, one has*

$$\mathcal{H}_\varepsilon(S_T(\mathcal{L}_{[R,M]}) - S_T 0 \mid \mathbf{W}^{1,1}(\mathbb{R}^N)) \geq \Gamma_{[R,M,N,T]}^- \cdot \frac{1}{\varepsilon^N}, \quad (11)$$

with

$$\Gamma_{[R,M,N,T]}^- \doteq \frac{1}{8 \cdot \ln 2} \cdot \left(\frac{K_{R,M} \omega_N r_R^{N+1}}{48(N+1) 2^{N+1}}\right)^N \quad (12)$$

where  $\tau_{R,M}, r_R, K_{R,M}$  are constants depending on  $R, M, N$  defined as in Section 4.2.

Since the upper and lower bounds on the  $\varepsilon$ -entropy in  $\mathbf{W}^{1,1}$  of  $S_t(\mathcal{L}_{[R,M]}) - S_t 0$  are both of order  $1/\varepsilon^N$ , we deduce that, for Hamiltonians satisfying the assumptions **(H1)**, **(H2)**, such an  $\varepsilon$ -entropy is of the same size  $\approx 1/\varepsilon^N$  established in [1] for Hamiltonians not depending on the space variable. Entropy numbers play a central role in various areas of information theory and statistics as well as of ergodic and learning theory. In the present setting, this concept could provide a measure of the order of “resolution” and of the “complexity” of a numerical scheme, as suggested in [22, 23]. Roughly speaking, the order of magnitude of the  $\varepsilon$ -entropy should indicate the minimum number of operations that one should perform in order to obtain an approximate solution with a precision of order  $\varepsilon$  with respect to the considered topology.

**Remark 1.** Because of the assumption **(H1)**, for any given  $x, q \in \mathbb{R}^N$  there exists some point  $p_{x,q}$  where the supremum in (4) is attained (cfr. [12, Appendix A.2]). Thus, in particular, one finds that

$$L(x, 0) = -H(x, p_x), \quad (13)$$

for some  $p_x \in \mathbb{R}^n$ , and

$$\begin{aligned} q &= D_p H(x, p_{x,q}), \\ L(x, q) &= \langle p_{x,q}, q \rangle - H(x, p_{x,q}), \\ D_x L(x, q) &= -D_x H(x, p_{x,q}), \end{aligned} \quad (14)$$

for some  $p_{x,q} \in \mathbb{R}^N$ . Hence, relying on the inequalities of the assumption **(H2)**, one can show that

$$\begin{aligned} L(x, 0) &\leq c_1(1 + |x|) && \forall x \in \mathbb{R}^N, \\ L(x, q) &\geq c_2|q|^\alpha - c_3 && \forall x, q \in \mathbb{R}^N, \\ |D_x L(x, q)| &\leq c_4|q|^\alpha + c_5 && \forall x, q \in \mathbb{R}^N. \end{aligned}$$

These uniform bounds on the Legendre transform of  $H$  are fundamental to provide an estimate on the size of the support of the map  $x \mapsto S_t u_0(x) - S_t 0(x)$ , when  $u_0$  varies in a set  $\mathcal{L}_{[R,M]}$  as in (5), as well as to derive a-priori bounds on the minimizers for (3). The assumptions **(H1)**-**(H2)** are verified by a large class of Hamiltonians  $H(x, p)$  convex in the  $p$ -variable. For example, if we consider

$$H(x, p) = f(x)(1 + |p|^2)^m + g(x),$$

where  $m > 1/2$  and  $f, g \in C^2(\mathbb{R}^N)$  are such that

$$0 < f(x) \leq c_f, \quad -c_g(1 + |x|) \leq g(x) \leq c_g \quad \forall x \in \mathbb{R}^N,$$

for some  $c_f, c_g > 0$ , it is straightforward to verify that  $H$  satisfies **(H1)**-**(H2)** for  $\alpha = \frac{2m}{2m-1}$ . On the other hand, one can easily check that the assumption **(H2)** is certainly fulfilled, in particular, by the Hamiltonians that satisfy **(H1)** together with the (stronger) bounds

$$\begin{aligned} -c'_1(1 + |x|) &\leq H(x, 0) \leq c'_1 \\ |D_p H(x, p)|^\alpha &\leq c'_2(1 + |p|^2) \end{aligned} \quad \forall x, p \in \mathbb{R}^N, \quad (15)$$

for some constants  $c'_1, c'_2 > 0$  and  $\alpha > 1$ .

The key step of the proof of Theorem 1-(i) consists in deriving accurate estimates on the size of the support  $\omega_T(u_0) \doteq \text{Supp}(S_T u_0 - S_T 0)$  of the map  $x \mapsto S_T u_0(x) - S_T 0(x)$  and on the

semiconcavity constant of  $S_T u_0$  on  $\omega_T(u_0)$ , when  $u_0 \in \mathcal{L}_{[R,M]}$ . Notice that, since the Hamiltonian depends on the space variable, we cannot employ the explicit Hopf-Lax representation formula for the solutions of an Hamilton-Jacobi equation as in [1]. Instead, we shall obtain these estimates relying on the representation (3) of a solution to (1) as the value function of a Bolza problem in the calculus of variations and performing a careful analysis of the behaviour of the solution along the corresponding minimizers. Thanks to such a-priori bounds, one then recovers the upper estimate (9) invoking a similar estimate established in [1] for the Kolmogorov entropy of a class  $\mathcal{SC}_{[K]}$  of semiconcave functions with semiconcavity constant  $K$  defined on a bounded domain.

The proof of Theorem 1-(ii), as in [1], is based on a controllability type result for the elements of the class  $\mathcal{SC}_{[K]}$ . Namely, we show that, for times  $T > 0$  sufficiently small and for some constant  $K$  depending on  $R, M$ , every element of  $\mathcal{SC}_{[K]}$  which coincides with  $S_T 0$  outside a bounded domain can be obtained as the value  $u(T, \cdot)$  of a viscosity solution of (1) with initial data in  $\mathcal{L}_{[R,M]}$ . Notice that  $S_T 0$  is in general not a smooth function. Therefore, to establish such a controllability property one cannot expect to produce smooth solutions on the whole domain  $[0, T] \times \mathbb{R}^N$  that attain at time  $T$  the desired profile. However, we shall achieve this result relying on a fine analysis of the backward and forward optimizers of the problem of calculus of variation associated to a local smooth solution of (1) and performing accurate estimates on the semiconcavity and semiconvexity constants of a viscosity solution of (1). In turn, this result yields the lower bound (11) invoking the same type of estimates provided in [1] for the Kolmogorov entropy of  $\mathcal{SC}_{[K]}$ . It remains open the question whether a global (in time) controllability property for semiconcave functions hold for Hamilton-Jacobi equations with Hamiltonian depending on space (cfr. remark 10 in Section 4.2).

The paper is organized as follows. In Section 2, we collect preliminary results and definitions concerning semiconcave functions and Hamilton-Jacobi equations, as well as the quantitative compactness estimates on classes of semiconcave functions established in [1]. In Section 3 we derive local a-priori bounds on the Lipschitz and semiconcavity constant of a viscosity solution to (1), which then yield the upper bound stated in Theorem 1-(i). In the first part of Section 4 we provide local a-priori bounds on the semiconvexity constant of a viscosity solution to (1) when the initial data is semiconvex. Next, we establish a local controllability result for a class of semiconcave functions, which allows us to obtain the lower bound stated in Theorem 1-(ii).

## 2 Notation and preliminaries

Let  $N \geq 1$  be an integer. Throughout the paper we shall denote by:

- $|\cdot|$  the Euclidean norm in  $\mathbb{R}^N$ ,
- $\langle \cdot, \cdot \rangle$  the Euclidean inner product in  $\mathbb{R}^N$ ,
- $[x, y]$  the segment joining two points  $x, y \in \mathbb{R}^N$ ,
- $B(x_0, r)$  the open ball of  $\mathbb{R}^N$  with radius  $r > 0$  and centered at  $x_0$ ,
- $\#(S)$  the number of elements of any finite set  $S$ ,
- $\text{Vol}(D)$  the Lebesgue measure of a measurable set  $D \subset \mathbb{R}^N$ ,
- $\omega_N := \text{Vol}(B(0, 1)) = \frac{\pi^{N/2}}{\Gamma(N/2+1)}$  the Lebesgue measure of the unit ball of  $\mathbb{R}^N$ ,
- $\|A\|$  the usual operator norm of the  $N \times N$  matrix  $A$ ,

- $\text{Lip}(\mathbb{R}^N)$  the space of all Lipschitz continuous functions  $f : \mathbb{R}^N \rightarrow \mathbb{R}$ , and by  $\text{Lip}[f]$  the Lipschitz seminorm of  $f$ , while  $\text{Lip}[f; \Omega]$  denotes the Lipschitz seminorm of the restriction of  $f$  to a domain  $\Omega \subset \mathbb{R}^N$ ,
- $\text{supp}(u)$  the support of  $u \in \text{Lip}(\mathbb{R}^N)$ , that is, the closure of  $\{x \in \mathbb{R}^N \mid u(x) \neq 0\}$ ,
- $AC([a, b], \mathbb{R}^N)$ , with  $[a, b]$  interval of  $\mathbb{R}$ , the space of all absolutely continuous functions from  $[a, b]$  to  $\mathbb{R}^N$ ,
- $\mathbf{L}^1(D)$ , with  $D \subset \mathbb{R}^N$  a measurable set, the Lebesgue space of all (equivalence classes of) summable functions on  $D$ , equipped with the usual norm  $\|\cdot\|_{\mathbf{L}^1(D)}$ ,
- $\mathbf{L}^\infty(D)$ , with  $D \subset \mathbb{R}^N$  a measurable set, the space of all essentially bounded functions on  $D$ , and by  $\|u\|_{\mathbf{L}^\infty(D)}$  the essential supremum of a function  $u \in \mathbf{L}^\infty(D)$  (we shall use the same symbol in case  $u$  is vector-valued),
- $\mathbf{W}^{1,1}(\Omega)$ , with  $\Omega$  a convex domain in  $\mathbb{R}^N$ , the Sobolev space of functions with summable first order distributional derivatives, and by  $\|\cdot\|_{\mathbf{W}^{1,1}(\Omega)}$  its norm,
- $\mathbf{W}_0^{1,1}(\Omega)$ , with  $\Omega$  a convex domain in  $\mathbb{R}^N$ , the Sobolev space of functions  $F \in \mathbf{W}^{1,1}(\Omega)$  with zero trace on the boundary  $\partial\Omega$ ,
- $BV(\Omega, \mathbb{R}^N)$ , with  $\Omega$  a domain in  $\mathbb{R}^N$ , the space of all vector-valued functions  $F : \Omega \rightarrow \mathbb{R}^N$  of bounded variation (that is, all  $F \in \mathbf{L}^1(\Omega, \mathbb{R}^N)$  such that the first partial derivatives of  $F$  in the sense of distributions are measures with finite total variation in  $\Omega$ ).

Moreover  $[a] := \max\{z \in \mathbb{Z} \mid z \leq a\}$  denotes the integer part of  $a$ .

## 2.1 Generalized gradients and semiconcave functions

We shall adopt the notation  $Du$  for the distributional gradient of a continuous function  $u$ . A notion of generalized differentials that specially fits viscosity solutions is recalled in the following

**Definition 2.** Let  $u : \Omega \rightarrow \mathbb{R}$ , with  $\Omega \subseteq \mathbb{R}^N$  open. For every  $x \in \Omega$ , the sets

$$\begin{aligned} D^+u(x) &:= \left\{ p \in \mathbb{R}^N \mid \limsup_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}, \\ D^-u(x) &:= \left\{ p \in \mathbb{R}^N \mid \liminf_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\}, \end{aligned} \tag{16}$$

are called, respectively, the  $D$ -superdifferential and the  $D$ -subdifferential of  $u$  at  $x$ . Moreover,

$$D^*u(x) := \left\{ p = \lim_{k \rightarrow \infty} \nabla u(x_k) \mid f \text{ is differentiable at } x_k \text{ and } x_k \rightarrow x \right\}, \tag{17}$$

is called the set of reachable gradients of  $u$  at  $x$ .

From definition (16) it follows that there holds

$$D^-u(x) = -D^+(-u)(x) \quad \forall x \in \Omega. \tag{18}$$

**Remark 2.** When  $u$  is locally Lipschitz in  $\Omega$ ,  $D^*u(x)$  is a nonempty compact set for every  $x \in \Omega$ . Moreover, if  $L$  is a Lipschitz constant for  $u$  on a neighborhood of  $x$ , then we have that

$$|p| \leq L \quad \forall p \in D^*u(x).$$

In this case, the convex hull  $\text{co } D^*u(x)$  gives Clarke's generalized gradient,  $\partial u(x)$ , see [14]. Consequently, there also holds

$$|p| \leq L \quad \forall p \in \partial u(x). \quad (19)$$

On the other hand, if  $u$  is semiconvex then one has  $\partial u(x) = D^-u(x)$ .

We collect below some basic definitions and properties of semiconcave functions in  $\mathbb{R}^N$  that will be used in the paper. We refer the reader to [12] for a comprehensive introduction to the corresponding theory.

**Definition 3.** A continuous function  $u : \Omega \rightarrow \mathbb{R}$ , with  $\Omega \subset \mathbb{R}^N$ , is called semiconcave if there exists  $K > 0$  such that

$$u(x+h) + u(x-h) - 2u(x) \leq K|h|^2, \quad (20)$$

for all  $x, h \in \mathbb{R}^N$  such that  $[x-h, x+h] \subset \Omega$ . When this property holds true, we also say that  $u$  is semiconcave in  $\Omega$  with constant  $K$ , and call  $K$  a semiconcavity constant for  $u$ .

- We say that  $u$  is semiconvex with constant  $K$  if  $-u$  is semiconcave with constant  $K$ .
- We say that  $u : \Omega \rightarrow \mathbb{R}$ , with  $\Omega \subset \mathbb{R}^N$  open, is locally semiconcave (or locally semiconvex) if  $u$  is semiconcave (semiconvex) in every compact set  $A \subset \subset \Omega$ .

**Remark 3.** The notion of semiconcavity introduced here is the most commonly used in the literature, often denoted as *linear semiconcavity*. A more general definition of semiconcavity can be found in [12]. It is easy to see that a function  $u$  is semiconcave (semiconvex) in  $\Omega$  with constant  $K > 0$  if and only if the function

$$\tilde{u}(x) \doteq u(x) - \frac{K}{2}|x|^2 \quad \left( \tilde{u}(x) \doteq u(x) + \frac{K}{2}|x|^2 \right), \quad x \in \Omega$$

is concave (convex).

Semiconcave functions and their superdifferential enjoy the properties stated in the following (see [12, Theorem 2.31, Proposition 3.3.1, Proposition 3.3.4, Theorem 3.3.6, Proposition 3.3.10])

**Theorem 2.** Let  $\Omega \subseteq \mathbb{R}^N$  be open and  $u : \Omega \rightarrow \mathbb{R}$  be semiconcave with semiconcavity constant  $K$ . Then, the following properties hold true.

- (i)  $u$  is Lipschitz continuous and almost everywhere differentiable.
- (ii) The superdifferential  $D^+u(x)$  is a compact, convex, nonempty set for all  $x \in \Omega$ .
- (iii)  $D^+u(x) = \text{co } D^*u(x)$  for all  $x \in \Omega$ , where  $\text{co}$  stands for the convex hull.
- (iv)  $D^+u(x)$  is a singleton if and only if  $u$  is differentiable at  $x$ .
- (v) If  $D^+u(x)$  is a singleton for every  $x \in \Omega$ , then  $u \in C^1(\Omega, \mathbb{R})$ .
- (vi)  $p \in D^+u(x)$  if and only if

$$u(x+h) - u(x) - \langle p, h \rangle \leq \frac{K}{2}|h|^2, \quad (21)$$

for all  $x, h \in \mathbb{R}^N$  such that  $[x, x+h] \subset \Omega$ .

**Proposition 1.** Let  $\Omega \subseteq \mathbb{R}^N$  be open convex and  $u : \Omega \rightarrow \mathbb{R}$  be semiconcave with constant  $K$ . Then, for every  $x, y \in \Omega$ , and for any  $p_x \in D^+u(x)$ ,  $p_y \in D^+u(y)$ , there holds

$$\langle p_y - p_x, y - x \rangle \leq K |y - x|^2.$$

**Remark 4.** Relying on the properties of the generalized gradients one can show that if a function  $u : \Omega \rightarrow \mathbb{R}$  ( $\Omega \subseteq \mathbb{R}^N$  open and convex) is both semiconcave and semiconvex in  $\Omega$  then  $u \in C^{1,1}(\Omega, \mathbb{R})$  (see [12, Corollary 3.3.8]).

## 2.2 Upper and lower bounds on the $\varepsilon$ -entropy for a class of semiconcave functions

We report here the estimates obtained in [1] on the  $\varepsilon$ -entropy in  $\mathbf{W}^{1,1}$  of classes of semiconcave functions on  $\mathbb{R}^N$ .

Given any  $R, M, K > 0$ , and any  $\psi \in \text{Lip}([-R, R]^N)$ , consider the classes of functions

$$\mathcal{L}_{[R,M]}^\psi \doteq \left\{ u_0 \in \text{Lip}([-R, R]^N) \mid u_0 = \psi \text{ on } \partial[-R, R]^N, \text{Lip}[u_0] \leq M \right\}, \quad (22)$$

and

$$\mathcal{SC}_{[R,M,K]}^\psi \doteq \left\{ u_0 \in \mathcal{L}_{[R,M]}^\psi \mid u_0 \text{ is semiconcave on } [-R, R]^N \text{ with semiconcavity constant } K \right\}. \quad (23)$$

**Theorem 3.** Given any  $R, M, K > 0$  and a semiconcave function  $\psi \in \text{Lip}([-R, R]^N)$  having Lipschitz constant  $M$  and semiconcavity constant  $K$ , with the above notations the followings hold:

(i) for every  $0 < \varepsilon < \frac{MR^2}{5} (\min\{R, M\})^N$ , one has

$$\mathcal{H}_\varepsilon \left( \mathcal{SC}_{[R,M,K]}^\psi \mid \mathbf{W}^{1,1}([-R, R]^N) \right) \leq \gamma_{[R,M,K,N]}^+ \cdot \frac{1}{\varepsilon^N} \quad (24)$$

where

$$\gamma_{[R,M,K,N]}^+ \doteq \omega_N^N \cdot \left( 4N \cdot (1 + M + (K + 1)R) \right)^{4N^2}. \quad (25)$$

(ii) for every  $0 < \varepsilon < \frac{\omega_N R^N}{N 2^{N+9}} \min\{M, K\}$ , one has

$$\mathcal{H}_\varepsilon \left( \mathcal{SC}_{[R,2M,2K]}^\psi \mid \mathbf{W}^{1,1}([-R, R]^N) \right) \geq \gamma_{[R,K,N]}^- \cdot \frac{1}{\varepsilon^N}, \quad (26)$$

where

$$\gamma_{[R,K,N]}^- \doteq \frac{1}{8 \cdot \ln 2} \cdot \left( \frac{K \omega_N R^{N+1}}{48(N+1) 2^{N+1}} \right)^N. \quad (27)$$

*Proof.* The estimates stated in Theorem 3 were established in [1, Proposition 8, Proposition 10] for the class of functions

$$\mathcal{SC}_{[R,M,K]} \doteq \left\{ u_0 \in \mathcal{L}_{[R,M]} \mid u_0 \text{ is semiconcave with semiconcavity constant } K \right\} \quad (28)$$

(with  $\mathcal{L}_{[R,M]}$  as in (5)), which consists of the extensions to  $\mathbb{R}^N$  of the elements in  $\mathcal{SC}_{[R,M,K]}^0$ . However, with the same arguments of the proof of [1, Proposition 8] one obtains the upper bound (24) for the class of function in (22) and for a general  $\psi \in \text{Lip}([-R, R]^N)$ . In fact,



as in [1], we may define the map  $\mathcal{T}_K : \mathbf{L}^1([-R, R]^N) \rightarrow \mathbf{L}^1([-R, R]^N)$ , that associates to any  $f \in \mathbf{L}^1([-R, R]^N)$  the function

$$\mathcal{T}_K f(x) \doteq f(x) + \frac{K}{2}|x|^2, \quad (29)$$

and then consider the class of concave functions

$$\mathcal{C}_{[R,M,K]}^\psi \doteq \left\{ f \in \mathbf{W}^{1,1}([-R, R]^N) \mid \mathcal{T}_K f \in \mathcal{SC}_{[R,M,K]}^\psi \right\}. \quad (30)$$

Since  $\mathcal{T}_K : \mathcal{C}_{[R,M,K]}^\psi \rightarrow \mathcal{SC}_{[R,M,K]}^\psi$  is a surjective isometry, it is sufficient to provide an upper bound on the  $\varepsilon$ -entropy of the set  $\mathcal{C}_{[R,M,K]}^\psi$ . Then, observing that for every  $f_1, f_2 \in \mathcal{C}_{[R,M,K]}^\psi$  there holds  $f_1 - f_2 \in \mathbf{W}_0^{1,1}([-R, R]^N)$ , and applying the Poincaré inequality for trace-zero  $\mathbf{W}^{1,1}$  functions, we produce as shown in [1] an  $\varepsilon$ -covering of  $\mathcal{C}_{[R,M,K]}^\psi$  in  $\mathbf{W}^{1,1}$  with a cardinality of order  $2^{\frac{\gamma_{[R,M,K,N]}^+}{\varepsilon^N}}$ . This yields the upper bound (24). Similarly, one can recover the lower bound (26) as follows. For  $\varepsilon$  sufficiently small, it was shown in the proof of [1, Proposition 10] that there exists a class of  $C^1$  semiconcave functions  $\mathcal{U}_\varepsilon \subset \mathcal{SC}_{[R,M,K]}^\psi$  for which

$$\mathcal{H}_\varepsilon(\mathcal{U}_\varepsilon \mid \mathbf{W}^{1,1}(\mathbb{R}^N)) \geq \gamma_{[R,K,N]}^- \cdot \frac{1}{\varepsilon^N}. \quad (31)$$

On the other hand, since  $\psi$  is a semiconcave function with Lipschitz constant  $M$  and semiconcavity constant  $K$ , by definition (23) it follows that the restrictions to  $[-R, R]^N$  of the maps in  $\mathcal{U}_\varepsilon + \psi$  are all elements of  $\mathcal{SC}_{[R,2M,2K]}^\psi$ . Thus, one has

$$\begin{aligned} \mathcal{H}_\varepsilon(\mathcal{SC}_{[R,2M,2K]}^\psi \mid \mathbf{W}^{1,1}([-R, R]^N)) &\geq \mathcal{H}_\varepsilon(\mathcal{U}_\varepsilon + \psi \mid \mathbf{W}^{1,1}([-R, R]^N)) \\ &= \mathcal{H}_\varepsilon(\mathcal{U}_\varepsilon \mid \mathbf{W}^{1,1}([-R, R]^N)) \\ &= \mathcal{H}_\varepsilon(\mathcal{U}_\varepsilon \mid \mathbf{W}^{1,1}(\mathbb{R}^N)), \end{aligned} \quad (32)$$

which, together with (31), yields (26).  $\square$

**Remark 5.** If we assume that  $\psi \in C^1([-R, R]^N)$ , the same lower bound (26) of Theorem 3-(ii) holds for the class of  $C^1$  elements of the set  $\mathcal{SC}_{[R,M,K]}^\psi$ :

$$\mathcal{SC}_{[R,M,K]}^{\psi,1} \doteq \left\{ u_0 \in \mathcal{SC}_{[R,M,K]}^\psi \cap C^1([-R, R]^N) \mid Du_0 = D\psi \text{ on } \partial[-R, R]^N \right\}. \quad (33)$$

### 2.3 Hamilton Jacobi equation

Consider the Hamilton-Jacobi equation (1), and observe that the assumptions **(H1)**-**(H2)** imply that the Legendre transform  $L(x, q)$  of  $p \rightarrow H(x, p)$  defined in (4) enjoy similar properties as  $H$  (cfr. Remark 1 in the Introduction and [12, Appendix A.2]):

**(L1)**  $L \in C^2(\mathbb{R}^N \times \mathbb{R}^N)$  is a convex and coercive map with respect to the second group of variables, i.e.,

$$0 < D_q^2 L(x, q) \quad \forall x, q \in \mathbb{R}^N, \quad (34)$$

$$\lim_{|q| \rightarrow \infty} \frac{L(x, q)}{|q|} = +\infty \quad \forall x \in \mathbb{R}^N. \quad (35)$$

**(L2)** There exist constants  $c_1, c_3, c_4, c_5 \geq 0$ ,  $c_2 > 0$  and  $\alpha > 1$  so that

$$L(x, 0) \leq c_1(1 + |x|) \quad \forall x \in \mathbb{R}^N, \quad (36)$$

$$L(x, q) \geq c_2|q|^\alpha - c_3 \quad \forall x, q \in \mathbb{R}^N, \quad (37)$$

$$|D_x L(x, q)| \leq c_4|q|^\alpha + c_5 \quad \forall x, q \in \mathbb{R}^N. \quad (38)$$

As we mentioned in the introduction, since solutions of the Cauchy problem for (1) may develop singularities in the gradient in finite time, even with smooth initial data, a concept of generalized solution, the *viscosity solution*, was introduced in [15] (see also [16]). We recall here the:

**Definition 4.** We say that a continuous function  $u : [0, T] \times \mathbb{R}^N$  is a viscosity solution of (1) if:

- (i)  $u$  is a viscosity subsolution of (1), i.e., for every point  $(t_0, x_0) \in ]0, T[ \times \mathbb{R}^N$  and test function  $v \in C^1((0, +\infty) \times \mathbb{R}^N)$  such that  $u - v$  has a local maximum at  $(t_0, x_0)$ , it holds

$$v_t(t_0, x_0) + H(x_0, \nabla_x v(t_0, x_0)) \leq 0,$$

- (ii)  $u$  is a viscosity supersolution of (1), i.e., for every point  $(t_0, x_0) \in ]0, T[ \times \mathbb{R}^N$  and test function  $v \in C^1((0, +\infty) \times \mathbb{R}^N)$  such that  $u - v$  has a local minimum at  $(t_0, x_0)$ , it holds

$$v_t(t_0, x_0) + H(x_0, \nabla_x v(t_0, x_0)) \geq 0.$$

In addition, we say that  $u$  is a viscosity solution of the Cauchy problem (1)-(2) if condition (2) is satisfied in the classical sense.

**Remark 6.** An alternative equivalent definition of viscosity solution is expressed in terms of the sub and superdifferential of the function (see [16]). Relying on this definition, and because of Theorem 2-(iv), one immediately see that every  $C^1$  solution of (1) is also a viscosity solution of (1). On the other hand, if  $u$  is a viscosity solution of (1), then  $u$  satisfies the equation at every point of differentiability. Moreover, one can show that if  $u : [0, T] \times \Omega$ ,  $\Omega \subset \mathbb{R}^N$ , is a viscosity solution of (1) and we know that  $u(t, \cdot)$  is both semiconcave and semiconvex in  $\Omega$  for all  $t \in ]0, T[$ , then  $u$  is a continuously differentiable classical solution of (1) on  $]0, T[ \times \Omega$  (e.g. see [1, Proposition 3]).

Under the assumption **(L1)**, for every  $u_0 \in \text{Lip}(\mathbb{R}^N)$ , the value function defined in (3) in connection with the Bolza problem of calculus of variation with running cost  $L$  and initial cost  $u_0$ :

$$\min_{\xi \in AC([0, t], \mathbb{R}^N)} \left\{ u_0(\xi(0)) + \int_0^t L(\xi(s), \dot{\xi}(s)) ds \mid \xi(t) = x \right\} \quad (CV)_{t,x}$$

provides the (unique) viscosity solution of the Hamilton-Jacobi equation (1) with initial data  $u(0, \cdot) = u_0$  (see [12, Section 6.4]). We recall below some properties of viscosity solutions of interest in this paper which follow from the representation formula (3) (cfr. [12, Sections 1.2, 6.3, 6.4]).

**Theorem 4.** Assume that the Legendre transform  $L$  of  $H$  in (4) satisfies the assumptions **(L1)**-**(L2)** and, given  $u_0 \in \text{Lip}(\mathbb{R}^N)$ , let  $u$  be the viscosity solution of (1)-(2) on  $[0, +\infty[ \times \mathbb{R}^N$ , defined by (3). Then, the following holds true.

(i) **Dynamic programming principle:** for all  $x \in \mathbb{R}^N$  and  $s \in [0, t[$ ,  $t > 0$ , we have

$$u(t, x) = \min_{\xi \in AC([s, t], \mathbb{R}^N)} \left\{ u(s, \xi(s)) + \int_s^t L(\xi(\tau), \dot{\xi}(\tau)) d\tau \mid \xi(t) = x \right\}. \quad (39)$$

Moreover, if  $\xi^*$  is a minimizer for  $(CV)_{t,x}$ , the restriction of  $\xi^*$  to  $[s, t]$  is also a minimizer in (39).

(ii) **Euler-Lagrange equation:** for all  $x \in \mathbb{R}^N$  and  $t > 0$ , if  $\xi^*$  is a minimizer for  $(CV)_{t,x}$ ,  $\xi^*$  is a Caratheodory solution of the equation

$$\frac{d}{ds} D_q L(\xi(s), \dot{\xi}(s)) = D_x L(\xi(s), \dot{\xi}(s)) \quad (40)$$

on  $[0, t]$ , i.e.  $\xi^*$  satisfies (40) for almost every  $s \in [0, t]$ . Moreover, one has

$$D_q L(\xi^*(0), \dot{\xi}^*(0)) \in \partial u_0(\xi^*(0)), \quad (41)$$

where  $\partial u$  denotes Clarke's generalized gradient.

(iii) **Generalized backward characteristics:** for all  $x \in \mathbb{R}^N$  and  $t > 0$ , if  $\xi^*$  is a minimizer for  $(CV)_{t,x}$ , there exists  $p^* \in AC([0, t], \mathbb{R}^N)$  (called the dual or co-state arc associated with  $\xi^*$ ) so that  $(\xi^*, p^*)$  provides the solution of the system

$$\begin{cases} \dot{\xi} = D_p H(\xi, p), \\ \dot{p} = -D_x H(\xi, p), \end{cases} \quad (42)$$

on  $[0, t]$ , with terminal condition

$$\begin{cases} \xi(t) = x, \\ p(t) \in D_x^+ u(t, x). \end{cases} \quad (43)$$

Moreover,  $u(s, \cdot)$  is differentiable at  $\xi^*(s)$  for any  $s \in ]0, t[$  and one has

$$p^*(s) \in D_x u(s, \xi^*(s)) \quad \forall s \in ]0, t[, \quad (44)$$

$$p^*(0) \in \partial u_0(\xi^*(0)). \quad (45)$$

By the above observations, the family of nonlinear operators

$$S_t : \text{Lip}_{loc}(\mathbb{R}^N) \rightarrow \text{Lip}_{loc}(\mathbb{R}^N), \quad u_0 \mapsto S_t u_0, \quad t \geq 0,$$

defined by

$$S_t u_0(x) \doteq \min_{\xi \in AC([0, t], \mathbb{R}^N)} \left\{ u_0(\xi(0)) + \int_0^t L(\xi(s), \dot{\xi}(s)) ds \mid \xi(t) = x \right\} \quad (46)$$

enjoy the following properties:

- (i) for every  $u_0 \in \text{Lip}(\mathbb{R}^N)$ ,  $u(t, x) \doteq S_t u_0(x)$  provides the unique viscosity solution of the Cauchy problem (1)-(2);
- (ii) (semigroup property)

$$S_{t+s} u_0 = S_t S_s u_0, \quad \forall t, s \geq 0, \quad \forall u_0 \in \text{Lip}(\mathbb{R}^N).$$

### 3 Upper compactness estimates

#### 3.1 A-priori bounds on the value function

Let  $H : \mathbb{R}^N \rightarrow \mathbb{R}$  be a function satisfying the assumptions **(H1)**-**(H2)** and let  $L$  be the corresponding Legendre transform in (4). We establish here an a-priori bound on the support of  $S_t u_0 - S_t 0$  in terms of the support of  $u_0$ , and we collect some a-priori local bounds on the semiconcavity constant and on the gradient of the value function  $S_t u_0$  in (46). In particular, given  $R, M > 0$ , we shall derive such properties in connection with the set of initial data introduced in (5):

$$\mathcal{L}_{[R,M]} = \left\{ u_0 \in \text{Lip}(\mathbb{R}^N) \mid \text{supp}(u_0) \subset [-R, R]^N, \text{Lip}[u_0] \leq M \right\}.$$

**Lemma 1.** *Assume that the Legendre transform  $L$  of  $H$  satisfies the assumptions **(L1)**-**(L2)** and, given  $u_0 \in \text{Lip}(\mathbb{R}^N) \cap \mathbf{L}^\infty(\mathbb{R}^N)$ , let  $u$  be the viscosity solution of (1)-(2) on  $[0, +\infty[ \times \mathbb{R}^N$ , defined by (46). Then, given  $(t, x) \in ]0, \infty[ \times \mathbb{R}^N$ , and letting  $\xi^*$  be a minimizer for  $(CV)_{t,x}$ , one has*

$$|x - \xi^*(\tau)| \leq l_1(\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, t) + \frac{|x|}{2} \quad \forall \tau \in [0, t], \quad (47)$$

where

$$l_1(\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, t) \doteq \frac{\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} + (c_1 + c_3)t}{1 + c_1 t} + t \left( \frac{2(1 + c_1 t)}{c_2} \right)^{\frac{1}{\alpha-1}}. \quad (48)$$

Moreover, if  $x \in [-l, l]^N$ ,  $l > 0$ , then there holds

$$|\dot{\xi}^*(\tau)| \leq \chi_1(\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, \text{Lip}[u_0], l, t) \quad \forall \tau \in [0, t], \quad (49)$$

for some constant  $\chi_1(\|u_0\|_{\mathbf{L}^\infty}, \text{Lip}[u_0], l, t) > 0$  depending on  $\|u_0\|_{\mathbf{L}^\infty}$ ,  $\text{Lip}[u_0]$ ,  $l$ ,  $t$ .

*Proof.*

1. Given  $(t, x) \in [0, \infty[ \times \mathbb{R}^N$ , let  $\xi^*$  be a minimizer for  $(CV)_{t,x}$ . First observe that since

$$z \geq \left( \frac{2(1 + c_1 t)}{c_2} \right)^{\frac{1}{\alpha-1}} \implies \frac{c_2}{2(1 + c_1 t)} z^\alpha \geq z,$$

we deduce

$$\begin{aligned} |x - \xi^*(\tau)| &= |\xi^*(t) - \xi^*(\tau)| \leq \int_0^t |\dot{\xi}^*(s)| ds \\ &\leq \frac{c_2}{2(1 + c_1 t)} \int_0^t |\dot{\xi}^*(s)|^\alpha ds + t \left( \frac{2(1 + c_1 t)}{c_2} \right)^{\frac{1}{\alpha-1}} \quad \forall \tau \in [0, t], x \in \mathbb{R}^N. \end{aligned} \quad (50)$$

Towards an estimation of  $\int_0^t |\dot{\xi}^*(s)|^\alpha ds$ , relying on (37) we derive

$$c_2 \int_0^t |\dot{\xi}^*(s)|^\alpha ds \leq \int_0^t L(\xi^*(s), \dot{\xi}^*(s)) ds + c_3 t. \quad (51)$$

On the other hand, by definition (46) and thanks to (36), we have

$$\begin{aligned} u(t, x) &\geq -\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} + \int_0^t L(\xi^*(s), \dot{\xi}^*(s)) ds \\ u(t, x) &\leq \|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} + t \cdot L(x, 0) \\ &\leq \|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} + t \cdot c_1(1 + |x|), \end{aligned}$$

which, together, yield

$$\int_0^t L(\xi^*(s), \dot{\xi}^*(s)) ds \leq 2\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} + t \cdot c_1(1 + |x|). \quad (52)$$

Thus, combining (51), (52), we find

$$c_2 \int_0^t |\dot{\xi}^*(s)|^\alpha ds \leq 2\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} + (c_1 + c_3)t + c_1 \cdot |x|t. \quad (53)$$

Hence, from (50), (53), we deduce

$$|x - \xi^*(\tau)| \leq \frac{\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} + (c_1 + c_3)t}{1 + c_1t} + t \left( \frac{2(1 + c_1t)}{c_2} \right)^{\frac{1}{\alpha-1}} + \frac{|x|}{2} \quad \forall \tau \in [0, t], \quad (54)$$

which proves (47).

**2.** Towards a bound on  $|\dot{\xi}^*|$ , observe that, if  $x \in [-l, l]^N$ , because of (47) we find

$$\begin{aligned} |\xi^*(\tau)| &\leq |\xi^*(\tau) - x| + |x| \\ &\leq l_1(\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, t) + \frac{3|x|}{2} \quad \forall \tau \in [0, t]. \\ &\leq l_1(\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, t) + \frac{3\sqrt{N} \cdot l}{2} \end{aligned} \quad (55)$$

On the other hand, thanks to the convexity of  $L(x, q)$  with respect to  $q$ , relying on (36)-(38), (40), (41), (53), (55), and recalling Remark 2, we derive

$$\begin{aligned} c_2 |\dot{\xi}^*(\tau)|^\alpha &\leq c_3 + L(\xi^*(\tau), \dot{\xi}^*(\tau)) \\ &\leq c_3 + L(\xi^*(\tau), 0) + \langle D_q L(\xi^*(\tau), \dot{\xi}^*(\tau)), \dot{\xi}^*(\tau) \rangle \\ &\leq c_1 + c_3 + c_1 |\xi^*(\tau)| + \left\langle \int_0^\tau \frac{d}{ds} D_q L(\xi^*(s), \dot{\xi}^*(s)) ds + D_q L(\xi^*(0), \dot{\xi}^*(0)), \dot{\xi}^*(\tau) \right\rangle \\ &\leq c_1 \cdot \left(1 + l_1(\|u_0\|_{\mathbf{L}^\infty}, t)\right) + \frac{3\sqrt{N}c_1 \cdot l}{2} + c_3 + \left( \int_0^\tau |D_x L(\xi^*(s), \dot{\xi}^*(s))| ds + \text{Lip}[u_0] \right) \cdot |\dot{\xi}^*(\tau)| \\ &\leq c_1 \cdot \left(1 + l_1(\|u_0\|_{\mathbf{L}^\infty}, t)\right) + \frac{3\sqrt{N}c_1 \cdot l}{2} + c_3 + \left( c_4 \int_0^t |\dot{\xi}^*(s)|^\alpha ds + c_5 t + \text{Lip}[u_0] \right) \cdot |\dot{\xi}^*(\tau)| \\ &\leq c_1 \cdot \left(1 + l_1(\|u_0\|_{\mathbf{L}^\infty}, t)\right) + \frac{3\sqrt{N}c_1 \cdot l}{2} + c_3 + \\ &\quad + \left( \frac{2c_4 \|u_0\|_{\mathbf{L}^\infty}}{c_2} + \frac{(c_1 + c_3)c_4}{c_2} t + \frac{c_1 c_4 \cdot l t}{c_2} + c_5 t + \text{Lip}[u_0] \right) \cdot |\dot{\xi}^*(\tau)| \quad \forall \tau \in [0, t]. \end{aligned} \quad (56)$$

Then, setting

$$\begin{aligned} b_1 &\doteq b_1(l, t) = \frac{c_1}{c_2} \left(1 + \frac{3\sqrt{N}l}{2}\right) + \frac{c_1 + c_3}{c_2} + \frac{c_1}{c_2} \cdot l_1(\|u_0\|_{\mathbf{L}^\infty}, t), \\ b_2 &\doteq b_2(\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, \text{Lip}[u_0], l, t) = \frac{(c_1 + c_3 + c_1 \cdot l)c_4 + c_2 c_5}{c_2^2} t + \frac{2c_4 \|u_0\|_{\mathbf{L}^\infty}}{c_2^2} + \frac{\text{Lip}[u_0]}{c_2}, \end{aligned} \quad (57)$$

we obtain from (56) the estimate (49) with

$$\chi_1(\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, \text{Lip}[u_0], l, t) \doteq \max \{b_1, (1 + b_2)^{\frac{1}{\alpha-1}}\}. \quad (58)$$

□

**Remark 7.** By the proof of Lemma 1 one deduces that the following further properties hold

- (i) Given any  $l > 0$ , there exist constants  $\tau_1(l), c_6(l) > 0$  such that, for every  $u_0 \in \text{Lip}(\mathbb{R}^N) \cap \mathbf{L}^\infty(\mathbb{R}^N)$  with  $\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \leq c_6(l)$ , the following hold:

- letting  $\xi^*$  be any minimizer for  $(CV)_{t,x}$ ,  $t \leq \tau_1(l)$ ,  $x \in [-l, l]^N$ , one has

$$|\xi^*(s)| \leq 2l \quad \forall s \in [0, t]; \quad (59)$$

- letting  $\xi^*$  be any maximizer for

$$\max_{\xi \in AC([t, \tau_1(l)], \mathbb{R}^N)} \left\{ u_0(\xi(\tau_1(l))) - \int_t^{\tau_1(l)} L(\xi(s), \dot{\xi}(s)) ds \mid \xi(t) = x \right\}, \quad (CV)^{t,x}$$

with  $t \leq \tau_1(l)$ ,  $\xi^*(\tau_1(l)) \in [-l, l]^N$ , one has

$$|\xi^*(s)| \leq 2l \quad \forall s \in [t, \tau_1(l)]. \quad (60)$$

- (ii) Given any  $l > 0$ , there exist constants  $\tau_2(l), c_7(l) > 0$  such that, for every  $u_0 \in \text{Lip}(\mathbb{R}^N) \cap \mathbf{L}^\infty(\mathbb{R}^N)$  with  $\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \leq c_7(l)$ , letting  $\xi^*$  be any minimizer for  $(CV)_{t,x}$ ,  $t \leq \tau_2(l)$ ,  $x \in \mathbb{R}^N \setminus ]-l, l[^N$ , one has

$$|\xi^*(s)| > \frac{|x|}{3} \quad \forall s \in [0, t]. \quad (61)$$

Moreover, by definition (46), relying on (36), (37) and on (49), one can show that:

- (iii) Given any  $l > 0$ , there exists a constant  $c_8(l) > 0$  depending on  $l$  such that there holds

$$|S_t 0(x)| \leq c_8(l) \cdot t \quad \forall x \in [-l, l]^N, t > 0. \quad (62)$$

**Corollary 1.** Let  $H : \mathbb{R}^N \rightarrow \mathbb{R}$  be a function satisfying the assumptions **(H1)**-**(H2)** and  $\{S_t : \text{Lip}(\mathbb{R}^N) \rightarrow \text{Lip}(\mathbb{R}^N)\}_{t \geq 0}$  be the semigroup of viscosity solutions generated by (1). Then, letting  $\mathcal{L}_{[R, M]}$ ,  $R, M > 0$ , be the set in (5), for any  $u_0 \in \mathcal{L}_{[R, M]}$ ,  $t > 0$ , one has

$$\text{supp}(S_t u_0 - S_t 0) \subseteq [-l_2(R, M, t), l_2(R, M, t)]^N, \quad (63)$$

where

$$l_2(R, M, t) \doteq 2 \left( \frac{M\sqrt{N} \cdot R + (c_1 + c_3)t}{1 + c_1 t} + t \left( \frac{2(1 + c_1 t)}{c_2} \right)^{\frac{1}{\alpha-1}} \right) + 2\sqrt{N} \cdot R. \quad (64)$$

*Proof.* As observed in Section 2.3, the Legendre transform  $L$  of  $H$  satisfies the assumptions **(L1)**-**(L2)**. Thus, given  $(t, x) \in [0, \infty[ \times \mathbb{R}^n$ , and  $u_0 \in \mathcal{L}_{[R, M]}$ , we can apply Lemma 1 for the viscosity solutions to (1) with initial data  $u_0$  and 0. Notice that, recalling (5),  $u_0 \in \mathcal{L}_{[R, M]}$  implies

$$\text{supp}(u_0) \subset [-R, R]^N \quad (65)$$

and

$$|u_0(x)| \leq M\sqrt{N} \cdot R \quad \forall x \in \mathbb{R}^N. \quad (66)$$

Then, employing the notation in (48), set

$$l_1(u_0, t) \doteq l_1(M\sqrt{N} \cdot R, t). \quad (67)$$

Hence, relying on (47) and recalling (46) we find

$$S_t u_0(x) \doteq \min_{\xi \in AC([0, t], \mathbb{R}^N)} \left\{ u_0(\xi(0)) + \int_0^t L(\xi(s), \dot{\xi}(s)) ds \mid \xi(t) = x, \right. \\ \left. |x - \xi(0)| \leq l_1(u_0, t) + \frac{|x|}{2} \right\}, \quad (68)$$

$$S_t 0(x) \doteq \min_{\xi \in AC([0, t], \mathbb{R}^N)} \left\{ \int_0^t L(\xi(s), \dot{\xi}(s)) ds \mid \xi(t) = x, \right. \\ \left. |x - \xi(0)| \leq l_1(u_0, t) + \frac{|x|}{2} \right\}.$$

Observe now that, for every given  $\xi \in AC([0, t], \mathbb{R}^N)$ , if

$$|x - \xi(0)| \leq l_1(u_0, t) + \frac{|x|}{2}, \quad |x| \geq 2l_1(u_0, t) + 2\sqrt{N} \cdot R, \quad (69)$$

it follows

$$\begin{aligned} |\xi(0)| &\geq |x| - |x - \xi(0)| \\ &\geq \frac{|x|}{2} - l_1(u_0, t) \\ &\geq \sqrt{N} \cdot R. \end{aligned}$$

Therefore, by (65) we deduce that for all  $x \in \mathbb{R}^N$  satisfying (69), one has

$$u_0(\xi(0)) = 0,$$

which, in turn, because of (68) implies

$$S_t u_0(x) = S_t 0(x) \quad \forall x \in \mathbb{R}^N \setminus B(0, 2l_1(u_0, t) + 2\sqrt{N} \cdot R). \quad (70)$$

This, proves (63) since, comparing (48), (64), (67), we have

$$l_2(R, M, t) = 2l_1(u_0, t) + 2\sqrt{N} \cdot R. \quad (71)$$

□

**Proposition 2.** *In the same setting of Corollary 1, for every  $u_0 \in \text{Lip}(\mathbb{R}^N) \cap \mathbf{L}^\infty(\mathbb{R}^N)$  and for any  $l, T > 0$ , the following properties hold true:*

(i)  $S_T u_0$  is Lipschitz continuous on  $[-l, l]^N$  with a Lipschitz constant  $\mu(\|u_0\|_{\mathbf{L}^\infty}, \text{Lip}[u_0], l, T)$  depending on  $\|u_0\|_{\mathbf{L}^\infty}, \text{Lip}[u_0], l, T$ .

(ii)  $S_T(u_0)$  is semiconcave on  $]-l, l[^N$  with a semiconcavity constant  $\kappa(\|u_0\|_{\mathbf{L}^\infty}, \text{Lip}[u_0], l, T)$  depending on  $\|u_0\|_{\mathbf{L}^\infty}, \text{Lip}[u_0], l, T$ .

*Proof.*

1. Given any  $x \in [-l, l]^N$ , let  $\xi^* \in AC([0, T], \mathbb{R}^N)$  be a minimizer for  $(CV)_{T,x}$ , so that one has

$$\xi^*(T) = x, \quad S_T u_0(x) = u_0(\xi^*(0)) + \int_0^T L(\xi^*(s), \dot{\xi}^*(s)) ds. \quad (72)$$

As observed in the proof of Corollary 1, the Legendre transform  $L$  of  $H$  satisfies the assumptions **(L1)**-**(L2)** and thus we can apply Lemma 1. Then, relying on (55), (49), we find

$$\begin{aligned} |\xi^*(\tau)| &\leq r_T \doteq l_1(\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, T) + \frac{3\sqrt{N} \cdot l}{2}, & \forall \tau \in [0, T]. \\ |\dot{\xi}^*(\tau)| &\leq q_T \doteq \chi_1(\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, \text{Lip}[u_0], l, T), \end{aligned} \quad (73)$$

Next, set

$$\beta_{1,T} \doteq \sup \left\{ \max \{ |D_x L(x, q)|, |D_q L(x, q)| \} \mid |x| \leq r_T + 2\sqrt{N} \cdot l, |q| \leq q_T \right\}, \quad (74)$$

and, given any  $y \in [-l, l]^N$ , consider the map  $\xi \in AC([0, T], \mathbb{R}^N)$  defined by

$$\xi(\tau) = \xi^*(\tau) + y - x \quad \forall \tau \in [0, T]. \quad (75)$$

Notice that, by (72), (73), one has

$$\begin{aligned} \xi(T) &= y, \\ |\xi(\tau)| &\leq r_T + 2\sqrt{N} \cdot l, & |\dot{\xi}(\tau)| &\leq q_T \quad \forall \tau \in [0, T]. \end{aligned} \quad (76)$$

Thus, recalling (46) and because of (72), (73), (76), we obtain

$$\begin{aligned} S_T u_0(y) - S_T u_0(x) &\leq u_0(\xi(0)) - u_0(\xi^*(0)) + \int_0^T \left( L(\xi(s), \dot{\xi}(s)) - L(\xi^*(s), \dot{\xi}^*(s)) \right) ds \\ &\leq \text{Lip}[u_0] \cdot |\xi(0) - \xi^*(0)| + \int_0^T \beta_{1,T} \left( |\xi(s) - \xi^*(s)| + |\dot{\xi}(s) - \dot{\xi}^*(s)| \right) ds \\ &\leq (\text{Lip}[u_0] + 2\beta_{1,T} T) \cdot |y - x|. \end{aligned} \quad (77)$$

Performing a similar computation interchanging the role of  $x$  and  $y$ , considering a minimizer  $\xi^*$  for  $(CV)_{T,y}$  and a map  $\xi \in AC([0, T], \mathbb{R}^N)$  as in (75), we find

$$S_T u_0(x) - S_T u_0(y) \leq (\text{Lip}[u_0] + 2\beta_{1,T} T) \cdot |y - x|. \quad (78)$$



Thus, (77)-(78) together yield the Lipschitz continuity of  $S_T u_0$  on  $[-l, l]^N$  with a Lipschitz constant

$$\mu(\|u_0\|_{\mathbf{L}^\infty}, \text{Lip}[u_0], l, T) \doteq \text{Lip}[u_0] + 2\beta_{1,T} T, \quad (79)$$

proving (i).

**2.** Given any  $x, x-h, x+h \in [-l, l]^N$ , let  $\xi^* \in AC([0, T], \mathbb{R}^N)$  be a minimizer for  $(CV)_{T,x}$ , and consider the maps  $\xi^+, \xi^- \in AC([0, T], \mathbb{R}^N)$  defined by

$$\xi^\pm(\tau) = \xi^*(\tau) \pm \frac{\tau}{T} \cdot h \quad \forall \tau \in [0, T]. \quad (80)$$

Notice that, by (72), (73), one has

$$\begin{aligned} \xi^\pm(0) &= \xi^*(0), & \xi^\pm(T) &= x \pm h, \\ |\xi^\pm(\tau)| &\leq r_T + 2\sqrt{N} \cdot l, & |\dot{\xi}^\pm(\tau)| &\leq q_T + 2\sqrt{N} \cdot \frac{l}{T} \quad \forall \tau \in [0, T]. \end{aligned} \quad (81)$$

Then, setting

$$\beta_{2,T} \doteq \sup \left\{ \max \left\{ |D_{xx}^2 L(x, q)|, |D_{qx}^2 L(x, q)|, |D_{qq}^2 L(x, q)| \right\} \mid \begin{aligned} |x| &\leq r_T + 2\sqrt{N} \cdot l, \\ |q| &\leq q_T + 2\sqrt{N} \cdot \frac{l}{T} \end{aligned} \right\}, \quad (82)$$

recalling (46) and relying on (72), (73), (81), we have

$$\begin{aligned} S_T u_0(x+h) + S_T u_0(x-h) - 2S_T u_0(x) &\leq \\ &\leq \int_0^T L(\xi^+(s), \dot{\xi}^+(s)) ds + \int_0^T L(\xi^-(s), \dot{\xi}^-(s)) ds - 2 \int_0^T L(\xi^*(s), \dot{\xi}^*(s)) ds \\ &= \int_0^T \left( L\left(\xi^*(s) + \frac{s}{T}h, \dot{\xi}^*(s)\right) + L\left(\xi^*(s) - \frac{s}{T}h, \dot{\xi}^*(s)\right) - 2L(\xi^*(s), \dot{\xi}^*(s)) \right) ds + \\ &\quad + \int_0^T \left( L(\xi^+(s), \dot{\xi}^+(s)) - L(\xi^+(s), \dot{\xi}^*(s)) \right) ds + \int_0^T \left( L(\xi^-(s), \dot{\xi}^-(s)) - L(\xi^-(s), \dot{\xi}^*(s)) \right) ds \\ &\leq \beta_{2,T} \cdot \int_0^T \left(\frac{s}{T}\right)^2 |h|^2 ds + \\ &\quad + \int_0^T \left( \int_0^1 \left( \left\langle D_q L\left(\xi^+(s), \dot{\xi}^*(s) + \lambda \frac{h}{T}\right), \frac{h}{T} \right\rangle + \left\langle D_q L\left(\xi^-(s), \dot{\xi}^*(s) - \lambda \frac{h}{T}\right), \frac{-h}{T} \right\rangle \right) d\lambda \right) ds \\ &\leq \frac{\beta_{2,T}}{3} T \cdot |h|^2 + \frac{|h|}{T} \cdot \int_0^T \left( \int_0^1 \left| D_q L\left(\xi^+(s), \dot{\xi}^*(s) + \lambda \frac{h}{T}\right) - D_q L\left(\xi^-(s), \dot{\xi}^*(s) - \lambda \frac{h}{T}\right) \right| d\lambda \right) ds \\ &\leq \frac{\beta_{2,T}}{3} T \cdot |h|^2 + \beta_{2,T} \frac{|h|}{T} \cdot \int_0^T \left( \frac{2s}{T} |h| + \frac{2}{T} |h| \right) ds \\ &\leq \beta_{2,T} \left( \frac{T}{3} + 1 + \frac{1}{T} \right) \cdot |h|^2. \end{aligned} \quad (83)$$

Thus,  $S_T u_0$  is semiconcave on  $]-l, l[^N$  with a semiconcavity constant

$$\kappa(\|u_0\|_{\mathbf{L}^\infty}, \text{Lip}[u_0], l, T) \doteq \beta_{2,T} \left( 1 + \frac{3+T^2}{3T} \right), \quad (84)$$

proving (ii).  $\square$

**Remark 8.** By the proof of Lemma 1 and Proposition 2 it follows that, under the same hypotheses of Corollary 1, given any  $c, l, m > 0$  there exists a constant  $\tau_3(c, l, m) > 0$  depending on  $c, l$  and  $m$ , such that the following holds. For every  $u_0 \in \text{Lip}(\mathbb{R}^N) \cap \mathbf{L}^\infty(\mathbb{R}^N)$  with  $\|u_0\|_{\mathbf{L}^\infty} \leq c$  and  $\text{Lip}[u_0] \leq m$ , the map  $S_t u_0$  is Lipschitz continuous on  $[-l, l]^N$  with  $\text{Lip}[S_t u_0; [-l, l]^N] \leq 2 \cdot m$ , for all  $t \leq \tau_3(c, l, m)$ .

### 3.2 Continuity of the semigroup map $S_t$ , $t > 0$

It was shown in [1] that, for every fixed  $t \geq 0$ , the map  $S_t : \text{Lip}(\mathbb{R}^N) \rightarrow \text{Lip}_{loc}(\mathbb{R}^N)$  is continuous when the space  $\text{Lip}(\mathbb{R}^N)$  is endowed with the  $\mathbf{W}_{loc}^{1,1}$ -topology and  $S_t$  is restricted to sets of functions with uniform Lipschitz constant on bounded domains. The proof of this property was obtained in [1] exploiting the Hopf-Lax representation formula of solutions valid for Hamiltonians depending only on the gradient of the solution. We shall extend here this result to the case of Hamiltonians possibly depending also on the space variable providing a direct proof of this property that relies only on the a-priori bounds on the solutions established in Section 3.1.

**Proposition 3.** *Let  $u, u^\nu \in \text{Lip}(\mathbb{R}^N)$  ( $\nu \in \mathbb{N}$ ) be such that*

$$\|u^\nu\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \leq C, \quad \forall \nu, \quad \text{for some } C > 0, \quad (85)$$

$$u^\nu \xrightarrow{\nu \rightarrow \infty} u \quad \text{in} \quad \mathbf{W}_{loc}^{1,1}(\mathbb{R}^N). \quad (86)$$

Moreover, assume that for every bounded domain  $\Omega \subset \mathbb{R}^N$ , there exists some constant  $M_\Omega > 0$  such that

$$\text{Lip}[u^\nu; \Omega] \leq M_\Omega \quad \forall \nu. \quad (87)$$

Then, for every fixed  $t \geq 0$ , one has

$$S_t u^\nu \xrightarrow{\nu \rightarrow \infty} S_t u \quad \text{in} \quad \mathbf{W}_{loc}^{1,1}(\mathbb{R}^N). \quad (88)$$

*Proof.*

1. In order to establish the proposition it will be sufficient to show that, given any bounded domain  $\Omega \subset \mathbb{R}^N$ , for any fixed  $t > 0$ , there holds

$$S_t u^\nu \xrightarrow{\nu \rightarrow \infty} S_t u \quad \text{in} \quad \mathbf{W}^{1,1}(\Omega). \quad (89)$$

Consider the set

$$\Omega_\tau \doteq \left\{ x \in \mathbb{R}^N \mid d(x, \Omega) \leq l_1(C, \tau) + 2 \cdot \left( \sup_{y \in \Omega} |y| + \text{diam}(\Omega) \right) \right\}, \quad \tau \in [0, t], \quad (90)$$

where  $l_1(C, \tau)$  is defined as in (48) with  $C$  in place of  $\|u_0\|_{\mathbf{L}^\infty}$ . Observe that, because of (85), (87), applying Lemma 1 we deduce that, for any  $x \in \Omega$ , letting  $\xi^*$  be a minimizer for  $(CV)_{t,x}$ , one has

$$\begin{aligned} \xi^*(\tau) + (y - x) &\in \Omega_{t-\tau} \quad \forall y \in \Omega, \quad \tau \in [0, t], \\ |\dot{\xi}^*(\tau)| &\leq \chi_t \quad \forall \tau \in [0, t], \end{aligned} \quad (91)$$

for some constant  $\chi_t$  depending on  $C, M_{\Omega_t}, \Omega_t, t$ . Then, with the same arguments of the proof of Proposition 2 we deduce that there exist constants  $\mu_t, \kappa_t > 0$  so that:

(i)  $S_\tau u^\nu$  is Lipschitz continuous on  $\Omega_{t-\tau}$  with  $\text{Lip}[S_\tau u^\nu; \Omega_{t-\tau}] \leq \mu_t$  for all  $\nu \in \mathbb{N}, \tau \in [0, t]$ ;

(ii)  $S_t u^\nu$  is semiconcave on  $\Omega$  with semiconcavity constant  $\kappa_t$  for all  $\nu \in \mathbb{N}$ .

Thanks to the uniform bound in (i), in order to prove that

$$\nabla S_t u^\nu \xrightarrow{\nu \rightarrow \infty} \nabla S_t u \quad \text{in} \quad \mathbf{L}^1(\Omega), \quad (92)$$

it will be sufficient to show that

$$\nabla S_t u^\nu(x) \xrightarrow{\nu \rightarrow \infty} \nabla S_t u(x) \quad \text{for a.e. } x \in \Omega. \quad (93)$$

On the other hand, notice that because of (87), we have

$$|u^\nu(x) - u^\nu(y)| \leq M_{\Omega_t} \cdot |x - y| \quad \forall x, y \in \overline{\Omega_t}, \quad \forall \nu. \quad (94)$$

Hence, relying on (85), (94), and applying the Ascoli-Arzelà compactness theorem we find that

$$u^\nu \xrightarrow{\nu \rightarrow \infty} u \quad \text{uniformly on } \Omega_t. \quad (95)$$

Observe now that, by (91) and because of (i), the value of a solution to (1) with initial datum  $u^\nu$  depends at any point  $(t, x)$ ,  $x \in \Omega$  only by the values of the Hamiltonian  $H$  on the bounded domain

$$\left( \bigcup_{\tau \in [0, t]} \Omega_\tau \right) \times B(0, \mu_t). \quad (96)$$

Since the restriction of  $H(x, p)$  to such a domain is uniformly Lipschitz continuous in both variables, we may invoke the well-known contraction property of the semigroup map  $S_t$  with respect to the uniform convergence on compacta, which holds for Hamiltonians  $H$  enjoying this property (e.g. see [12, Theorem 5.2.12]). Thus, we deduce from (95) that

$$S_t u^\nu \xrightarrow{\nu \rightarrow \infty} S_t u \quad \text{uniformly on } \Omega, \quad (97)$$

which, in turn, implies

$$S_t u^\nu \xrightarrow{\nu \rightarrow \infty} S_t u \quad \text{in} \quad \mathbf{L}^1(\Omega). \quad (98)$$

**2.** Towards a proof of (93), let  $\Omega'$  be a subset of  $\Omega$ , with  $\text{meas}(\Omega') = \text{meas}(\Omega)$ , where all  $S_t u^\nu$ ,  $\nu \in \mathbb{N}$ , and  $S_t u$  are differentiable. Then, invoking properties (iv), (vi) stated in Theorem 2, and relying on property (ii) above, we infer that, at every  $x \in \Omega'$ , there holds

$$S_t u^\nu(x + h) - S_t u^\nu(x) - \langle \nabla S_t u^\nu(x), h \rangle \leq \frac{\kappa_t}{2} \cdot |h|^2 \quad (99)$$

for all  $h \in \mathbb{R}^N$  such that  $[x, x + h] \subset \Omega$ . Since  $\nabla S_t u^\nu(x)$  are uniformly bounded by property (i) above, let  $p \in \mathbb{R}^N$  be any accumulation point i.e. such that

$$\nabla S_t u^{\nu_k}(x) \xrightarrow{k \rightarrow \infty} p, \quad (100)$$

for some subsequence  $\{S_t u^{\nu_k}\}_k$ . Then, taking the limit in (99) of  $S_t u^{\nu_k}$ ,  $\nabla S_t u^{\nu_k}$ , as  $k \rightarrow \infty$ , and using (98), (100), we obtain

$$S_t u(x + h) - S_t u(x) - \langle p, h \rangle \leq \frac{\kappa_t}{2} \cdot |h|^2 \quad (101)$$

for all  $h \in \mathbb{R}^N$  such that  $[x, x + h] \subset \Omega$ . Recalling Definition 2 and by Theorem 2-(iv), this inequality implies that  $p \in D^+ S_t u(x) = \{\nabla S_t u(x)\}$ . Since  $p$  is an arbitrary accumulation point of  $\{\nabla S_t u^\nu(x)\}_\nu$ , it follows that

$$\nabla S_t u^\nu(x) \xrightarrow{\nu \rightarrow \infty} \nabla S_t u(x) \quad \forall x \in \Omega', \quad (102)$$

which proves (93) and hence (92). In turn, (92) together with (98) yields (89), completing the proof of the proposition.  $\square$

### 3.3 Conclusion of the proof of Theorem 1-(i)

Given  $R, M, T > 0$ , consider the set of initial data  $\mathcal{L}_{[R,M]}$  introduced in (5). Then, invoking Corollary 1-(ii) we know that, for every  $u_0 \in \mathcal{L}_{[R,M]}$ , there holds

$$S_T u_0 = S_T 0 \quad \text{on } \mathbb{R}^N \setminus ]-l_T, l_T[^N, \quad (103)$$

with

$$l_T = 2 \left( \frac{M\sqrt{N} \cdot R + (c_1 + c_3)T}{1 + c_1 T} + T \left( \frac{2(1 + c_1 T)}{c_2} \right)^{\frac{1}{\alpha-1}} \right) + 2\sqrt{N} \cdot R. \quad (104)$$

On the other hand, relying on (66) and applying Proposition 2-(i) we find that  $S_T u_0$  is a Lipschitz continuous map with Lipschitz constant

$$\mu_T \doteq M + 2\beta_{1,T} T, \quad (105)$$

where

$$\beta_{1,T} \doteq \sup \left\{ \max \{ |D_x L(x, q)|, |D_q L(x, q)| \} \mid |x| \leq \tilde{r}_T, |q| \leq \tilde{q}_{1,T} \right\}, \quad (106)$$

and

$$\begin{aligned} \tilde{r}_T &= l_1(M\sqrt{N} \cdot R, T) + \left( \frac{3\sqrt{N}}{2} + 2\sqrt{N} \right) \cdot l_T \\ &= \frac{M\sqrt{N} \cdot R + (c_1 + c_3)T}{1 + c_1 T} + T \left( \frac{2(1 + c_1 T)}{c_2} \right)^{\frac{1}{\alpha-1}} + \frac{7\sqrt{N} \cdot l_T}{2} \\ &= (1 + 7\sqrt{N}) \left( \frac{M\sqrt{N} \cdot R + (c_1 + c_3)T}{1 + c_1 T} + T \left( \frac{2(1 + c_1 T)}{c_2} \right)^{\frac{1}{\alpha-1}} \right) + 7N \cdot R, \end{aligned} \quad (107)$$

while

$$\begin{aligned} \tilde{q}_{1,T} &\doteq \chi_1(M\sqrt{N} \cdot R, M, l_T, T) + 2\sqrt{N} \cdot l_T \\ &= \max \left\{ b_1, (1 + b_2)^{\frac{1}{\alpha-1}} \right\} + 4\sqrt{N} \left( \frac{M\sqrt{N} \cdot R + (c_1 + c_3)T}{1 + c_1 T} + T \left( \frac{2(1 + c_1 T)}{c_2} \right)^{\frac{1}{\alpha-1}} \right) + 4N \cdot R, \end{aligned} \quad (108)$$

with

$$\begin{aligned} b_1 &= \frac{c_1}{c_2} \left( 1 + \frac{3\sqrt{N} l_T}{2} \right) + \frac{c_1 + c_3}{c_2} + \frac{c_1}{c_2} \cdot l_1(M\sqrt{N} \cdot R, T) \\ &= \frac{c_1}{c_2} \left( 1 + (1 + 3\sqrt{N}) \left( \frac{M\sqrt{N} \cdot R + (c_1 + c_3)T}{1 + c_1 T} + T \left( \frac{2(1 + c_1 T)}{c_2} \right)^{\frac{1}{\alpha-1}} \right) + 3N \cdot R \right) + \frac{c_1 + c_3}{c_2}, \end{aligned}$$

$$\begin{aligned} b_2 &= \frac{1}{c_2^2} \left( (c_1 c_4 + c_3 c_4 + c_2 c_5) T + c_1 c_4 \cdot l_T \cdot T + (c_2 + 2c_4 \sqrt{N} \cdot R) M \right) \\ &= \frac{1}{c_2^2} \left[ (c_1 c_4 + c_3 c_4 + c_2 c_5) T + (c_2 + 2c_4 \sqrt{N} \cdot R) M \right] + \\ &\quad + \frac{2c_1 c_4}{c_2^2} \left[ \left( \frac{M\sqrt{N} \cdot R + (c_1 + c_3)T}{1 + c_1 T} + T \left( \frac{2(1 + c_1 T)}{c_2} \right)^{\frac{1}{\alpha-1}} \right) T + \sqrt{N} \cdot R \cdot T \right]. \end{aligned} \quad (109)$$

Moreover, by Proposition 2-(ii), using again (66) we deduce that  $S_T u_0$  is a semiconcave map with semiconcavity constant

$$\kappa_T \doteq \beta_{2,T} \left( 1 + \frac{3 + T^2}{3T} \right), \quad (110)$$

where

$$\beta_{2,T} \doteq \sup \left\{ \max \left\{ |D_{xx}^2 L(x, q)|, |D_{qx}^2 L(x, q)|, |D_{qq}^2 L(x, q)| \right\} \mid |x| \leq \tilde{r}_T, \quad |q| \leq \tilde{q}_{2,T} \right\}, \quad (111)$$

with  $\tilde{r}_T$  defined in (107) and

$$\begin{aligned} \tilde{q}_{2,T} &\doteq \chi_1(M\sqrt{N} \cdot R, M, l_T, T) + 2\sqrt{N} \cdot \frac{l_T}{T} \\ &= \max \left\{ b_1, (1 + b_2)^{\frac{1}{\alpha-1}} \right\} + \frac{4\sqrt{N}}{T} \left( \frac{M\sqrt{N} \cdot R + (c_1 + c_3)T}{1 + c_1 T} + T \left( \frac{2(1 + c_1 T)}{c_2} \right)^{\frac{1}{\alpha-1}} \right) + \frac{4N \cdot R}{T}, \end{aligned} \quad (112)$$

$b_1, b_2$  being the constants in (109). Therefore, recalling definition (23) we find that

$$S_T u_0 \upharpoonright_{[-l_T, l_T]^N} \in \mathcal{SC}_{[l_T, \mu_T, \kappa_T]}^{S_T 0} \quad \forall u_0 \in \mathcal{L}_{[R, M]}, \quad (113)$$

where  $S_T u_0 \upharpoonright_{[-l_T, l_T]^N}$  denotes the restriction of  $S_T u_0$  to the  $N$ -dimensional cube  $[-l_T, l_T]^N$ . On the other hand, by (103) one has

$$\text{supp}(\varphi) \subset [-l_T, l_T]^N \quad \forall \varphi \in S_T(\mathcal{L}_{[R, M]}) - S_T 0. \quad (114)$$

Moreover, by Proposition 2-(ii) also  $S_T 0$  is a semiconcave map with semiconcavity constant  $\kappa_T$ . Hence, applying Theorem 3-(i), we deduce that, for  $\varepsilon$  sufficiently small, there holds

$$\begin{aligned} \mathcal{H}_\varepsilon \left( S_T(\mathcal{L}_{[R, M]}) - S_T 0 \mid \mathbf{W}^{1,1}(\mathbb{R}^N) \right) &= \mathcal{H}_\varepsilon \left( S_T(\mathcal{L}_{[R, M]}) - S_T 0 \mid \mathbf{W}^{1,1}([-l_T, l_T]^N) \right) \\ &= \mathcal{H}_\varepsilon \left( S_T(\mathcal{L}_{[R, M]}) \mid \mathbf{W}^{1,1}([-l_T, l_T]^N) \right) \\ &\leq \mathcal{H}_\varepsilon \left( \mathcal{SC}_{[l_T, \mu_T, \kappa_T]}^{S_T 0} \mid \mathbf{W}^{1,1}([-l_T, l_T]^N) \right) \\ &\leq \gamma_{[l_T, \mu_T, \kappa_T, N]}^+ \cdot \frac{1}{\varepsilon^N}, \end{aligned} \quad (115)$$

where

$$\gamma_{[l_T, \mu_T, \kappa_T, N]}^+ = \omega_N^N \cdot \left( 4N \cdot (1 + \mu_T + (\kappa_T + 1)l_T) \right)^{4N^2}, \quad (116)$$

with the constants  $l_T, \mu_T, \kappa_T$  defined in (104), (105), (110). This completes the proof of the upper bound (9).  $\square$

## 4 Lower compactness estimates

### 4.1 Controllability of a class of semiconcave functions

The proof of Theorem 1-(ii) is based on a local controllability result for the class of semiconcave functions introduced in (23). Towards this goal, we will first show that a solution of (1) with a semiconvex initial condition preserves the semiconvexity for a time interval that depends on the semiconvexity constant of the initial condition. We shall obtain this property exploiting the

representation of a solution to (1) as the value function of the Bolza problem in the calculus of variations with running cost  $L$  and initial cost  $u_0$ :

$$\min_{\xi \in AC([0,t], \mathbb{R}^N)} \left\{ u_0(\xi(0)) + \int_0^t L(\xi(s), \dot{\xi}(s)) ds \mid \xi(t) = x \right\}. \quad (CV)_{t,x}$$

**Proposition 4.** *Let  $H : \mathbb{R}^N \rightarrow \mathbb{R}$  be a function satisfying the assumptions **(H1)**-**(H2)** and  $\{S_t : \text{Lip}_{loc}(\mathbb{R}^N) \rightarrow \text{Lip}_{loc}(\mathbb{R}^N)\}_{t \geq 0}$  be the semigroup of viscosity solutions generated by (1). Then, given any  $c, l > 0$  there exists a constant  $\tau_4(c, l) > 0$  depending on  $c, l$ , such that the following holds. Given any  $T < \tau_4(c, l)$ , there exists  $K_T > 0$  such that, for every semiconvex map  $u_0 \in \text{Lip}_{loc}(\mathbb{R}^N) \cap \mathbf{L}^\infty(\mathbb{R}^N)$  with semiconvexity constant  $K \leq K_T$  and with  $\|u_0\|_{\mathbf{L}^\infty} \leq c$ , the following hold true.*

(i)  $x \mapsto S_t u_0(x)$  is semiconvex on  $[-l, l]^N$  for all  $t \in [0, T]$ .

(ii)  $(t, x) \mapsto S_t u_0(x)$  is a  $C^1$  classical solution of (1) on  $]0, T] \times [-l, l]^N$ .

*Proof.* Recall that, as observed in Section 2.3, the Legendre transform  $L$  of  $H$  satisfies the assumptions **(L1)**-**(L2)**.

1. Let  $u_0 \in \text{Lip}_{loc}(\mathbb{R}^N) \cap \mathbf{L}^\infty(\mathbb{R}^N)$  be such that  $\|u_0\|_{\mathbf{L}^\infty} \leq c$ . Fix  $T > 0$ , and observe that letting  $\xi^*$  be any minimizer for  $(CV)_{t,x}$ ,  $t \leq T$ ,  $x \in [-l, l]^N$ , thanks to Lemma 1 one has

$$|\xi^*(s)| \leq r_T, \quad |\dot{\xi}^*(s)| \leq \chi_T \quad \forall s \in [0, t], \quad (117)$$

for some constants  $r_T = r_T(c, l)$ ,  $\chi_T = \chi_T(c, \text{Lip}[u_0; [-r_t, r_t]^N], l)$  depending on  $c$ ,  $\text{Lip}[u_0; [-r_T, r_T]^N]$ ,  $l$  and  $T$ . Then, with the same arguments of the proof of Proposition 2 we deduce that there exist constants  $\mu_T, C_T > 0$  depending on  $c$ ,  $\text{Lip}[u_0; [-r_T, r_T]^N]$ ,  $l$  and  $T$ , so that:

(a)  $S_t u_0$  is Lipschitz continuous on

$$\Xi_t \doteq \left\{ \xi^*(t) \mid \xi^* \text{ is a minimizer for } (CV)_{T,x}, x \in [-l, l]^N \right\}, \quad (118)$$

for all  $t \in [0, T]$ , with Lipschitz constant  $\mu_T$ ;

(b)  $S_t u_0$  is semiconcave on  $\Xi_t$  for all  $t \in ]0, T]$ , with semiconcavity constant  $C_T$ .

Hence, in order to prove the proposition, we only have to show that there exists  $\tau_4 > 0$  so that, for any  $T < \tau_4$  there holds

$$S_t u_0(x+h) + S_t u_0(x-h) - 2S_t u_0(x) \geq k_T \cdot |h|^2 \quad \forall t \leq T, \quad (119)$$

for all  $x, x-h, x+h \in [-l, l]^N$ , and for some constant  $k_T > 0$ . In fact, it follows from (119) that property (i) is verified for all  $T < \tau_4$ . On the other hand, once we know that  $S_t u_0$  is both semiconcave and semiconvex on  $[-l, l]^N$ , for  $t \in ]0, T]$ , invoking Remark 6 we immediately deduce that also the property (ii) holds.

2. Towards a proof of (119), given any  $x, x-h, x+h \in [-l, l]^N$ , let  $\xi^\pm$  be a minimizer for  $(CV)_{t, x \pm h}$ ,  $t \leq T$ , and consider the map  $\xi_x \in AC([0, t], \mathbb{R}^N)$  defined by

$$\xi_x(s) = \frac{\xi^+(s) + \xi^-(s)}{2} \quad \forall s \in [0, t]. \quad (120)$$

Then, one has

$$\xi^\pm(t) = x \pm h, \quad \xi_x(t) = x, \quad (121)$$

and

$$\begin{aligned} S_t u_0(x \pm h) &= u_0(\xi^\pm(0)) + \int_0^t L(\xi^\pm(s), \dot{\xi}^\pm(s)) ds, \\ S_t u_0(x) &\leq u_0(\xi_x(0)) + \int_0^t L(\xi_x(s), \dot{\xi}_x(s)) ds. \end{aligned} \quad (122)$$

Moreover, recalling Theorem 4-(iii), and because of property (a) above, there exist dual arcs  $p^\pm \in AC([0, t], \mathbb{R}^N)$  so that  $(\xi^\pm, p^\pm)$  provides the solution of

$$\begin{cases} \dot{\xi} = D_p H(\xi, p), \\ \dot{p} = -D_x H(\xi, p), \end{cases} \quad (123)$$

on  $[0, t]$ , with terminal condition

$$\begin{cases} \xi^\pm(t) = x \pm h, \\ p(t) \in D^+ S_t u_0(x \pm h), \end{cases} \quad (124)$$

that satisfy

$$|p^\pm(s)| = |DS_s u_0(\xi^\pm(s))| \leq \mu_T \quad \forall s \in ]0, t[ \quad (125)$$

and

$$p^\pm(0) \in \partial u_0(\xi^\pm(0)). \quad (126)$$

Then, relying on (117), (125), and setting

$$\beta_{3,T} \doteq \sup \left\{ \max \{ |D_{px}^2 H(x, p)|, |D_{pp}^2 H(x, p)| \} \mid |x| \leq r_T, |p| \leq \mu_T \right\}, \quad (127)$$

we find

$$\begin{aligned} |\dot{\xi}^+(s) - \dot{\xi}^-(s)| &= \left| D_p H(\xi^+(s), p^+(s)) - D_p H(\xi^-(s), p^-(s)) \right| \\ &\leq \beta_{3,T} \left( |\xi^+(s) - \xi^-(s)| + |p^+(s) - p^-(s)| \right) \quad \forall s \in ]0, t[. \end{aligned} \quad (128)$$

Next, notice that, since  $u_0$  is semiconvex with constant  $K$ , we have

$$u_0(x+h) + u_0(x-h) - 2u_0(x) \geq -K \cdot |h|^2 \quad \forall x, h \in \mathbb{R}^N. \quad (129)$$

On the other hand, thanks to the convexity of  $L(x, q)$  with respect to  $q$ , it follows

$$\int_0^t L(\xi_x(s), \dot{\xi}^+(s)) ds + \int_0^t L(\xi_x(s), \dot{\xi}^-(s)) ds - 2 \int_0^t L(\xi_x(s), \dot{\xi}_x(s)) ds \geq 0. \quad (130)$$

Hence, setting

$$\beta_{4,T} \doteq \sup \left\{ \max \{ |D_{xx}^2 L(x, q)|, |D_{xq}^2 L(x, q)| \} \mid |x| \leq r_T, |q| \leq \chi_T \right\}, \quad (131)$$

and relying on (117), (128), (130), we derive

$$\begin{aligned}
& \int_0^t L(\xi^+(s), \dot{\xi}^+(s)) ds + \int_0^t L(\xi^-(s), \dot{\xi}^-(s)) ds - 2 \int_0^t L(\xi_x(s), \dot{\xi}_x(s)) ds \\
&= \int_0^t L(\xi_x(s), \dot{\xi}^+(s)) ds + \int_0^t L(\xi_x(s), \dot{\xi}^-(s)) ds - 2 \int_0^t L(\xi_x(s), \dot{\xi}_x(s)) ds + \\
&\quad + \int_0^t \left( L(\xi^+(s), \dot{\xi}^+(s)) - L(\xi_x(s), \dot{\xi}^+(s)) \right) ds + \int_0^t \left( L(\xi^-(s), \dot{\xi}^-(s)) - L(\xi_x(s), \dot{\xi}^-(s)) \right) ds \\
&\geq -\beta_{4,T} \int_0^t \left( \frac{|\xi^+(s) - \xi^-(s)|^2}{2} + |\xi^+(s) - \xi^-(s)| |\dot{\xi}^+(s) - \dot{\xi}^-(s)| \right) ds \\
&\geq -\frac{\beta_{4,T}(1 + 3\beta_{3,T})}{2} \int_0^t \left( |\xi^+(s) - \xi^-(s)|^2 + |p^+(s) - p^-(s)|^2 \right) ds.
\end{aligned} \tag{132}$$

Therefore, (122), (129), (132), together yield

$$\begin{aligned}
& S_t u_0(x+h) + S_t u_0(x-h) - 2S_t u_0(x) \geq \\
&\quad \geq u_0(x+h) + u_0(x-h) - 2u_0(x) + \\
&\quad\quad + \int_0^t L(\xi^+(s), \dot{\xi}^+(s)) ds + \int_0^t L(\xi^-(s), \dot{\xi}^-(s)) ds - 2 \int_0^t L(\xi_x(s), \dot{\xi}_x(s)) ds \\
&\quad \geq -\frac{\beta_{4,T}(1 + 3\beta_{3,T})}{2} \int_0^t \left( |\xi^+(s) - \xi^-(s)|^2 + |p^+(s) - p^-(s)|^2 \right) ds \quad \forall t \leq T.
\end{aligned} \tag{133}$$

**3.** In order to recover the estimate (119) from (133) we need to provide an upper bound on  $\int_0^t |\xi^+(s) - \xi^-(s)|^2 ds$  and  $\int_0^t |p^+(s) - p^-(s)|^2 ds$ . To this end observe first that, by the same computations at point 2., because of (121), (128), for  $0 \leq t' < t$  we find

$$\begin{aligned}
|\xi^+(t') - \xi^-(t')|^2 &= |\xi^+(t) - \xi^-(t)|^2 - \frac{1}{2} \int_{t'}^t \frac{d}{ds} |\xi^+(s) - \xi^-(s)|^2 ds \\
&= 4|h|^2 - \int_{t'}^t \left\langle D_p H(\xi^+(s), p^+(s)) - D_p H(\xi^-(s), p^-(s)), \xi^+(s) - \xi^-(s) \right\rangle ds \\
&\leq 4|h|^2 + \int_{t'}^t \left| D_p H(\xi^+(s), p^+(s)) - D_p H(\xi^-(s), p^-(s)) \right| \cdot |\xi^+(s) - \xi^-(s)| ds \\
&\leq 4|h|^2 + \int_{t'}^t \beta_{3,T} \left( |\xi^+(s) - \xi^-(s)| + |p^+(s) - p^-(s)| \right) \cdot |\xi^+(s) - \xi^-(s)| ds \\
&\leq 4|h|^2 + \frac{\beta_{3,T}}{2} \int_0^t |p^+(s) - p^-(s)|^2 ds + \frac{3\beta_{3,T}}{2} \int_{t'}^t |\xi^+(s) - \xi^-(s)|^2 ds.
\end{aligned} \tag{134}$$

By a Gronwall type inequality, (134) implies

$$\int_0^t |\xi^+(s) - \xi^-(s)|^2 ds \leq \frac{8}{3\beta_{3,T}} \left( e^{\frac{3\beta_{3,T}}{2} t} - 1 \right) |h|^2 + \frac{1}{3} \left( e^{\frac{3\beta_{3,T}}{2} t} - 1 \right) \int_0^t |p^+(s) - p^-(s)|^2 ds. \tag{135}$$



Towards an estimate of the second term in (135), observe that, by (34), there holds

$$\langle q, D_{pp}^2 H(x, p) q \rangle \geq \beta_{5,T} |q|^2 \quad \forall x \in B(0, r_T), p \in B(0, \mu_T), q \in \mathbb{R}^N, \quad (136)$$

with

$$\beta_{5,T} \doteq \inf \left\{ \langle q, D_{pp}^2 H(x, p) q \rangle \mid |x| \leq r_T, |p| \leq \mu_T, |q| = 1 \right\} > 0. \quad (137)$$

Hence, recalling that  $(\xi^\pm, p^\pm)$  are solutions of (123), defining the averaged matrices

$$\begin{aligned} \tilde{H}_{px}(s) &\doteq \tilde{H}_{xp}(s) \doteq \int_0^1 D^2 H_{px}(\lambda \xi^+(s) + (1-\lambda)\xi^-(s), \lambda p^+(s) + (1-\lambda)p^-(s)) d\lambda, \\ \tilde{H}_{pp}(s) &\doteq \int_0^1 D^2 H_{pp}(\lambda \xi^+(s) + (1-\lambda)\xi^-(s), \lambda p^+(s) + (1-\lambda)p^-(s)) d\lambda, \\ \tilde{H}_{xx}(s) &\doteq \int_0^1 D^2 H_{xx}(\lambda \xi^+(s) + (1-\lambda)\xi^-(s), \lambda p^+(s) + (1-\lambda)p^-(s)) d\lambda, \end{aligned} \quad (138)$$

and relying on (117), (125), (136), we get

$$\begin{aligned} &\frac{d}{ds} \langle p^+(s) - p^-(s), \xi^+(s) - \xi^-(s) \rangle \\ &= - \left\langle D_x H(\xi^+(s), p^+(s)) - D_x H(\xi^-(s), p^-(s)), \xi^+(s) - \xi^-(s) \right\rangle + \\ &\quad + \left\langle p^+(s) - p^-(s), D_p H(\xi^+(s), p^+(s)) - D_p H(\xi^-(s), p^-(s)) \right\rangle \\ &= - \left\langle \tilde{H}_{xx}(s)(\xi^+(s) - \xi^-(s)) + \tilde{H}_{xp}(s)(p^+(s) - p^-(s)), \xi^+(s) - \xi^-(s) \right\rangle + \\ &\quad + \left\langle p^+(s) - p^-(s), \tilde{H}_{px}(s)(\xi^+(s) - \xi^-(s)) + \tilde{H}_{pp}(s)(p^+(s) - p^-(s)) \right\rangle \\ &= - \left\langle \tilde{H}_{xx}(s)(\xi^+(s) - \xi^-(s)), \xi^+(s) - \xi^-(s) \right\rangle + \left\langle p^+(s) - p^-(s), \tilde{H}_{pp}(s)(p^+(s) - p^-(s)) \right\rangle \\ &\geq -\beta_{3,T} |\xi^+(s) - \xi^-(s)|^2 + \beta_{5,T} |p^+(s) - p^-(s)|^2. \end{aligned} \quad (139)$$

Observe now that  $u_0$  is semiconvex with constant  $K$  and  $S_t u_0$  is semiconcave on the domain  $\Xi_t$  in (118) with semiconcavity constant  $C_T$ . Hence, invoking Proposition 1, recalling Remark 2, and relying on (18), (121), (124), (126) we have

$$\begin{aligned} &\left\langle p^+(t) - p^-(t), \xi^+(t) - \xi^-(t) \right\rangle - \left\langle p^+(0) - p^-(0), \xi^+(0) - \xi^-(0) \right\rangle \\ &\leq C_T |\xi^+(t) - \xi^-(t)|^2 + K |\xi^+(0) - \xi^-(0)|^2 \\ &\leq 4C_T |h|^2 + K |\xi^+(0) - \xi^-(0)|^2. \end{aligned} \quad (140)$$

On the other hand, by the same computations in (134) we find

$$\begin{aligned} |\xi^+(0) - \xi^-(0)|^2 &\leq 4|h|^2 + \int_0^t \beta_{3,T} \left( |\xi^+(s) - \xi^-(s)| + |p^+(s) - p^-(s)| \right) \cdot |\xi^+(s) - \xi^-(s)| ds \\ &\leq 4|h|^2 + \frac{\epsilon \cdot \beta_{3,T}}{2} \int_0^t |p^+(s) - p^-(s)|^2 ds + \left(1 + \frac{1}{2\epsilon}\right) \beta_{3,T} \int_0^t |\xi^+(s) - \xi^-(s)|^2 ds. \end{aligned} \quad (141)$$

Thus, (140), (141), together yield

$$\begin{aligned}
& \left\langle p^+(t) - p^-(t), \xi^+(t) - \xi^-(t) \right\rangle - \left\langle p^+(0) - p^-(0), \xi^+(0) - \xi^-(0) \right\rangle \\
& \leq 4(C_T + K)|h|^2 + \frac{\beta_{5,T}}{2} \int_0^t |p^+(s) - p^-(s)|^2 ds + \\
& \quad + \left(1 + \frac{K\beta_{3,T}}{2\beta_{5,T}}\right) K\beta_{3,T} \int_0^t |\xi^+(s) - \xi^-(s)|^2 ds.
\end{aligned} \tag{142}$$

Then, relying on (139), (142), we derive

$$\begin{aligned}
\beta_{5,T} \int_0^t |p^+(s) - p^-(s)|^2 ds & \leq \beta_{3,T} \int_0^t |\xi^+(s) - \xi^-(s)|^2 ds \\
& \quad + \int_0^t \frac{d}{ds} \left\langle p^+(s) - p^-(s), \xi^+(s) - \xi^-(s) \right\rangle ds \\
& \leq 4(C_T + K)|h|^2 + \frac{\beta_{5,T}}{2} \int_0^t |p^+(s) - p^-(s)|^2 ds + \\
& \quad + \left( \left(1 + \frac{K\beta_{3,T}}{2\beta_{5,T}}\right) K + 1 \right) \beta_{3,T} \int_0^t |\xi^+(s) - \xi^-(s)|^2 ds,
\end{aligned} \tag{143}$$

which implies

$$\int_0^t |p^+(s) - p^-(s)|^2 ds \leq \frac{8}{\beta_{5,T}} (C_T + K)|h|^2 + \left( \left(1 + \frac{K\beta_{3,T}}{2\beta_{5,T}}\right) K + 1 \right) \frac{8\beta_{3,T}}{\beta_{5,T}} \int_0^t |\xi^+(s) - \xi^-(s)|^2 ds. \tag{144}$$

4. Combining together (135) and (144), we find

$$\begin{aligned}
& \int_0^t \left( |\xi^+(s) - \xi^-(s)|^2 + |p^+(s) - p^-(s)|^2 \right) ds \leq \\
& \quad \left( \frac{8}{3\beta_{3,T}} + \frac{8}{3\beta_{5,T}} (C_T + K) \right) \left( e^{\frac{3\beta_{3,T}}{2}t} - 1 \right) |h|^2 + \\
& \quad + \frac{8\beta_{3,T}}{3\beta_{5,T}} \left( e^{\frac{3\beta_{3,T}}{2}t} - 1 \right) \left( \left(1 + \frac{K\beta_{3,T}}{2\beta_{5,T}}\right) K + 1 \right) \int_0^t \left( |\xi^+(s) - \xi^-(s)|^2 + |p^+(s) - p^-(s)|^2 \right) ds.
\end{aligned} \tag{145}$$

Observe now that, setting

$$\tau_4 \doteq \sup \left\{ \inf \left\{ \tau, \frac{2}{3\beta_{3,\tau}} \ln \left( 1 + \frac{3\beta_{5,\tau}}{8\beta_{3,\tau}} \right) \right\} \mid \tau > 0 \right\}, \tag{146}$$

for any  $T < \tau_4$  we can find  $\mathcal{T}_T \geq T$  and  $K_T > 0$  such that, for all  $K \leq K_T$ , one has

$$T \leq \frac{2}{3\beta_{3,\mathcal{T}_T}} \ln \left( 1 + \frac{3\beta_{5,\mathcal{T}_T}^2}{4\beta_{3,\mathcal{T}_T} (2\beta_{5,\mathcal{T}_T} + 2\beta_{5,\mathcal{T}_T}K + \beta_{3,\mathcal{T}_T}K^2)} \right). \tag{147}$$

Since

$$t \leq \frac{2}{3\beta_{3,\tau_T}} \ln \left( 1 + \frac{3\beta_{5,\tau_T}^2}{4\beta_{3,\tau_T}(2\beta_{5,\tau_T} + 2\beta_{5,\tau_T}K + \beta_{3,\tau_T}K^2)} \right) \quad (148)$$

implies

$$\frac{8\beta_{3,\tau_T}}{3\beta_{5,\tau_T}} \left( e^{\frac{3\beta_{3,\tau_T}}{2}t} - 1 \right) \left( \left( 1 + \frac{K\beta_{3,\tau_T}}{2\beta_{5,\tau_T}} \right) K + 1 \right) \leq \frac{1}{2}, \quad (149)$$

we infer from (145) that, for any  $t \leq T$ , with  $T$  satisfying (147), and for  $K \leq K_T$ , there holds

$$\int_0^t \left( |\xi^+(s) - \xi^-(s)|^2 + |p^+(s) - p^-(s)|^2 \right) ds \leq \frac{16(C_T + K)}{3} \left( \frac{1}{\beta_{3,\tau_T}} + \frac{1}{\beta_{5,\tau_T}} \right) \left( e^{\frac{3\beta_{3,\tau_T}}{2}T} - 1 \right) |h|^2. \quad (150)$$

Because of (133), we deduce from (150) that, for any  $T < \tau_4$ , with  $\tau_4$  as in (146), and for  $K \leq K_T$ , there holds

$$\begin{aligned} S_t u_0(x+h) + S_t u_0(x-h) - 2S_t u_0(x) &\geq \\ &\geq -\frac{8\beta_{4,\tau_T}(1+3\beta_{3,\tau_T})(C_T+K)}{3} \left( \frac{1}{\beta_{3,\tau_T}} + \frac{1}{\beta_{5,\tau_T}} \right) \left( e^{\frac{3\beta_{3,\tau_T}}{2}T} - 1 \right) |h|^2 \quad \forall t \leq T, \end{aligned} \quad (151)$$

which proves (119) and thus concludes the proof of the proposition.  $\square$

**Remark 9.** By the proof of Proposition 4 it follows that, under the same hypotheses of the proposition, given any  $c, l, K > 0$  there exists a constant  $\tau_5(c, l, K) > 0$  depending on  $c, l$  and  $K$ , such that the following holds. Given any  $T \leq \tau_5(c, l, K)$ , for every semiconvex map  $u_0 \in \text{Lip}_{loc}(\mathbb{R}^N) \cap \mathbf{L}^\infty(\mathbb{R}^N)$  with semiconvexity constant  $K' \leq K$  and with  $\|u_0\|_{\mathbf{L}^\infty} \leq c$ , the statements (i)-(ii) of Proposition 4 hold.

**Proposition 5.** *In the same setting of Proposition 4, given any  $R > 0$ , there exists  $r_1(R) > 0$  depending on  $R$ , and, for every  $r \leq r_1(R)$ ,  $M > 0$ , there exist  $m_1(r, M) > 0$ ,  $k_1(r, m) > 0$ ,  $\tau_6(r, m) > 0$  depending on  $r \leq r_1(R)$ ,  $m \leq m_1(r, M)$  and  $M$ , so that the following holds. Letting  $\mathcal{L}_{[R,M]}$ ,  $\mathcal{SC}_{[r,m,K]}^{S_T^{0,1}}$  be the sets defined in (5), (33), for any  $\psi \in \mathcal{SC}_{[r,2m,2K]}^{S_T^{0,1}}$ , with  $r \leq r_1(R)$ ,  $m \leq m_1(r, M)$ ,  $\tau \leq \tau_6(r, m)$ ,  $K = k_1(r, m)$ , there exists  $u_0 \in \mathcal{L}_{[R,M]}$  such that*

$$S_\tau u_0(x) = \begin{cases} \psi(x) & \text{if } x \in [-r, r]^N, \\ S_\tau 0(x) & \text{if } x \in \mathbb{R}^N \setminus [-r, r]^N. \end{cases} \quad (152)$$

*Proof.*

1. Given  $R, M > 0$ , fix  $r, m > 0$  such that

$$r \leq r_1(R), \quad m \leq \min \left\{ \frac{M}{10}, \frac{1}{20 \cdot \sqrt{N} \cdot r} \cdot \min \left\{ c_6(36 \cdot r), c_7(6 \cdot r) \right\} \right\}, \quad (153)$$

where  $c_6(36 \cdot r), c_7(6 \cdot r) > 0$  are constants enjoying the properties stated in Remark 7. Moreover, choose positive constants  $\tau_1(36 \cdot r), \tau_2(6 \cdot r), \tau_3(c_6(36 \cdot r), 72 \cdot r, m), c_8(72 \cdot r)$ , according with Remark 7 and Remark 8, so that:

- a) For every  $\tau \leq \tau_1(36 \cdot r)$ ,  $u_0 \in \text{Lip}(\mathbb{R}^N) \cap \mathbf{L}^\infty(\mathbb{R}^N)$  with  $\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \leq c_6(36 \cdot r)$ , the following holds:

– letting  $\xi^*$  be any minimizer for  $(CV)_{t,x}$ ,  $t \leq \tau$ , one has

$$\begin{aligned} x \in [-r, r]^N &\implies |\xi^*(s)| \leq 2 \cdot r \\ x \in [-36 \cdot r, 36 \cdot r]^N &\implies |\xi^*(s)| \leq 72 \cdot r \end{aligned} \quad \forall s \in [0, t]; \quad (154)$$

– letting  $\xi^*$  be any maximizer for

$$\max_{\xi \in AC([t, \tau], \mathbb{R}^N)} \left\{ u_0(\xi(\tau)) - \int_t^\tau L(\xi(s), \dot{\xi}(s)) ds \mid \xi(t) = x \right\}, \quad (CV)^{t,x}$$

with  $t \leq \tau$ , one has

$$\begin{aligned} \xi^*(\tau) \in [-r, r]^N &\implies |\xi^*(s)| \leq 2 \cdot r \\ \xi^*(\tau) \in [-6 \cdot r, 6 \cdot r]^N &\implies |\xi^*(s)| \leq 12 \cdot r \quad \forall s \in [t, \tau]. \\ \xi^*(\tau) \in [-36 \cdot r, 36 \cdot r]^N &\implies |\xi^*(s)| \leq 72 \cdot r \end{aligned} \quad (155)$$

b) For every  $\tau \leq \tau_2(6 \cdot r)$ ,  $u_0 \in \text{Lip}(\mathbb{R}^N) \cap \mathbf{L}^\infty(\mathbb{R}^N)$  with  $\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \leq c_7(6 \cdot r)$ , the following holds:

– letting  $\xi^*$  be any minimizer for  $(CV)_{t,x}$ ,  $t \leq \tau$ ,  $x \in \mathbb{R}^N \setminus ]-6 \cdot r, 6 \cdot r[^N$ , one has

$$|\xi^*(s)| > \frac{|x|}{3} \quad \forall s \in [0, t]. \quad (156)$$

– letting  $\xi^*$  be any maximizer for  $(CV)^{t,x}$ ,  $t \leq \tau$ , with  $\xi^*(\tau) \in \mathbb{R}^N \setminus ]-6 \cdot r, 6 \cdot r[^N$ , one has

$$|\xi^*(s)| > \frac{|\xi^*(\tau)|}{3} \quad \forall s \in [t, \tau]. \quad (157)$$

c) Setting

$$I_r \doteq [-72 \cdot r, 72 \cdot r], \quad (158)$$

there holds

$$|S_t 0(x)| \leq c_8(72 \cdot r) \cdot t \quad \forall x \in I_r^N, t > 0. \quad (159)$$

d) For every  $\tau \leq \tau_3(c_6(36 \cdot r), 72 \cdot r, m)$ , and for any  $u_0 \in \text{Lip}(\mathbb{R}^N) \cap \mathbf{L}^\infty(\mathbb{R}^N)$  with  $\|u_0\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \leq c_6(36 \cdot r)$ , one has

$$\begin{aligned} \text{Lip}[u_0] \leq m &\implies \text{Lip}[S_\tau u_0; I_r^N] \leq 2 \cdot m, \\ \text{Lip}[u_0] \leq 5m &\implies \text{Lip}[S_\tau u_0; I_r^N] \leq 10 \cdot m. \end{aligned} \quad (160)$$

Next, set

$$\bar{\tau}(r, m) \doteq \min \left\{ \tau_1(36 \cdot r), \tau_2(6 \cdot r), \tau_3(c_6(36 \cdot r), 72 \cdot r, m), \frac{\min\{c_6(36 \cdot r), c_7(6 \cdot r)\} - 8m \cdot \sqrt{N} \cdot r}{2c_8(72 \cdot r)} \right\}, \quad (161)$$

and let  $\bar{k}(r, m) > 0$  be a semiconcavity constant for  $S_\tau 0$ ,  $\tau \leq \bar{\tau}(r, m)$ , on  $I_r^N$ . Then, fix  $K \geq \bar{k}(r, m)$ , and take

$$\tau \leq \bar{\tau}'(r, m) \doteq \min \left\{ \bar{\tau}(r, m), \tau_5(c_6(36 \cdot r), 72 \cdot r, 3K) \right\}, \quad (162)$$

where  $\tau_5(c_6(36 \cdot r), 72 \cdot r, 3K)$  is a constant with the property stated in Remark 9. Observe that, by property (d) above, one has  $\text{Lip}[S_\tau 0; I_r^N] \leq 2 \cdot m$ . Moreover, since the zero map is semiconvex with semiconvexity constant any  $K > 0$ , by virtue of Remark 9 we deduce that  $S_t 0(x)$  provides a  $C^1$  classical solution of (1) on  $[0, \tau] \times I_r^N$ . This, in particular, implies by (164) that  $S_\tau 0$  is a  $C^1$  map on  $I_r^N$ . Then, given

$$\psi \in \mathcal{SC}_{[r, 4m, 2K]}^{S_\tau 0, 1}, \quad (163)$$

we can define a  $C^1$  semiconcave map  $\tilde{\psi}^\tau : \mathbb{R}^N \rightarrow \mathbb{R}$  with the properties:

$$(i) \quad \begin{aligned} \tilde{\psi}^\tau(x) &= \psi(x) & \forall x \in [-r, r]^N, \\ \tilde{\psi}^\tau(x) &= S_\tau 0(x) & \forall x \in I_r^N \setminus [-r, r]^N; \end{aligned} \quad (164)$$

(ii)  $\tilde{\psi}^\tau$  has Lipschitz constant  $5m$  and semiconcavity constant  $3K$  on  $\mathbb{R}^N$ ;

$$(iii) \quad \|\tilde{\psi}^\tau\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \leq 2 \max \left\{ \|\psi\|_{\mathbf{L}^\infty([-r, r]^N)}, \|S_\tau 0\|_{\mathbf{L}^\infty(I_r^N)} \right\}. \quad (165)$$

Recall that, by definitions (22), (23), (33),  $\psi$  is a Lipschitz continuous map on  $[-r, r]^N$  with Lipschitz constant  $4m$ , and that  $\psi = S_\tau 0$ ,  $D\psi = DS_\tau 0$  on  $\partial[-r, r]^N$ . Hence, because of (159), (161), one finds

$$\|S_\tau 0\|_{\mathbf{L}^\infty(I_r^N)} \leq c_8(72 \cdot r) \cdot \bar{\tau} \leq \frac{\min\{c_6(36 \cdot r), c_7(6 \cdot r)\}}{2},$$

and

$$\begin{aligned} \|\psi\|_{\mathbf{L}^\infty([-r, r]^N)} &\leq \|S_\tau 0\|_{\mathbf{L}^\infty([-r, r]^N)} + 4m \cdot \sqrt{N} \cdot r, \\ &\leq c_8(72 \cdot r) \cdot \bar{\tau} + 4m \cdot \sqrt{N} \cdot r \\ &\leq \frac{\min\{c_6(36 \cdot r), c_7(6 \cdot r)\}}{2}, \end{aligned}$$

which, in turn, together with (165), yields

$$\|\tilde{\psi}^\tau\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \leq \min \{c_6(36 \cdot r), c_7(6 \cdot r)\}. \quad (166)$$

**2.** We will show that, for  $\tau$  satisfying (162), with  $\bar{\tau}(r, m)$  as in (161), the map  $\tilde{\psi}^\tau$  defined above can be obtained as the value at time  $\tau$  of a classical solution to (1) by reversing the direction of time and constructing a backward solution to (1) that starts at time  $\tau$  from  $\tilde{\psi}^\tau$ . Namely, set

$$w_0^\tau(x) \doteq -\tilde{\psi}^\tau(-x) \quad \forall x \in \mathbb{R}^N, \quad (167)$$

and consider the viscosity solution  $w^\tau(t, x)$  of

$$w_t(t, x) + H(-x, \nabla_x w(t, x)) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^N, \quad (168)$$

with initial datum

$$w(0, \cdot) = w_0^\tau. \quad (169)$$

Notice that the Hamiltonian  $\hat{H}(x, p) \doteq H(-x, p)$  satisfy the assumptions **(H1)**-**(H2)** as does  $H(x, p)$ . Moreover, by (167) and because of (ii),  $w_0^\tau$  is semiconvex with semiconvexity

constant  $3K$ . Thus, invoking Remark 9 and thanks to (162), (166), we know that the function  $w(t, x) \doteq w^\tau(t, x)$  is a  $C^1$  classical solution of (1) on  $[0, \tau] \times I_r^N$ , with  $I_r$  as in (158). Furthermore, by properties (d) and (ii) above, and by virtue of (161), (162), we deduce that

$$\text{Lip}[w(\tau, \cdot); I_r^N] \leq 10m. \quad (170)$$

Next, observe that by a direct computation the function

$$u(t, x) \doteq -w(\tau - t, -x) \quad (171)$$

is also a  $C^1$  classical solution of (1) on  $[0, \tau] \times I_r^N$ . Moreover, because of (167), (169), (171), one has

$$u(\tau, x) = -w_0^\tau(-x) = \tilde{\psi}^\tau(x) \quad \forall x \in \mathbb{R}^N. \quad (172)$$

Then, for every  $y \in [-18 \cdot r, 18 \cdot r]^N$ , consider the pair  $(\xi^y, p^y) \in (AC([0, \tau], \mathbb{R}^N))^2$  that satisfies the Hamiltonian system

$$\begin{cases} \dot{\xi} = D_p H(\xi, p), \\ \dot{p} = -D_x H(\xi, p), \end{cases} \quad (173)$$

on  $[0, \tau]$ , with terminal condition

$$\begin{cases} \xi^y(\tau) = y, \\ p^y(\tau) = D\tilde{\psi}^\tau(y). \end{cases} \quad (174)$$

Notice that, for any  $t \in [0, \tau]$ ,  $y \in [-36 \cdot r, 36 \cdot r]^N$ , the restriction of  $\xi^y$  to  $[t, \tau]$  provides the (unique) optimal solution for the backward maximization problem

$$\max_{\xi \in AC([t, \tau], \mathbb{R}^N)} \left\{ \tilde{\psi}^\tau(\xi(\tau)) - \int_t^\tau L(\xi(s), \dot{\xi}(s)) ds \mid \xi(t) = \xi^y(t) \right\} \quad (CV)^{t, \xi^y(t)}$$

(cfr. [6]). Moreover, since by (161), (162) we have chosen  $\tau \leq \min\{\tau_1(36 \cdot r), \tau_2(6 \cdot r)\}$ , and because of (166), relying on properties (a), (b) at point 1, and recalling (158) we find

$$\begin{aligned} y \in [-r, r]^N &\implies \xi^y(s) \in B(0, 2 \cdot r) \\ y \in [-6 \cdot r, 6 \cdot r]^N &\implies \xi^y(s) \in B(0, 12 \cdot r) \\ y \in \partial[-6 \cdot r, 6 \cdot r]^N &\implies \xi^y(s) \in \mathbb{R}^N \setminus B(0, 2 \cdot r) \\ y \in [-36 \cdot r, 36 \cdot r]^N &\implies \xi^y(s) \in I_r \end{aligned} \quad \forall s \in [0, \tau]. \quad (175)$$

In particular, if  $y \in [-36 \cdot r, 36 \cdot r]^N \setminus ]-r, r[^N$ , because of (164) the pair  $(\xi^y, p^y)$  satisfies the Hamiltonian system (173) with terminal condition

$$\begin{cases} \xi^y(\tau) = y, \\ p^y(\tau) = DS_\tau 0(y), \end{cases} \quad (176)$$

and the restriction of  $\xi^y$  to  $[t, \tau]$  provides the (unique) optimal solution for the backward maximization problem

$$\max_{\xi \in AC([t, \tau], \mathbb{R}^N)} \left\{ S_\tau 0(\xi(\tau)) - \int_t^\tau L(\xi(s), \dot{\xi}(s)) ds \mid \xi(t) = \xi^y(t) \right\}. \quad (CV)^{t, \xi^y(t)}$$

Hence, since  $u(t, x)$  and  $S_t 0(x)$  are both  $C^1$  classical solutions of (1) on  $[0, \tau] \times I_r^N$ , we deduce

$$\begin{aligned} u(t, \xi^y(t)) &= \tilde{\psi}^\tau(y) - \int_t^\tau L(\xi^y(s), \dot{\xi}^y(s)) ds \\ &= S_\tau 0(y) - \int_t^\tau L(\xi^y(s), \dot{\xi}^y(s)) ds \\ &= S_t 0(\xi^y(t)), \end{aligned} \tag{177}$$

for every  $t \in [0, \tau]$ ,  $y \in [-36 \cdot r, 36 \cdot r]^N \setminus ]-r, r[^N$ . This, in particular, implies that

$$u(0, \xi^y(0)) = S_0 0(\xi^y(0)) = 0 \quad \forall y \in [-36 \cdot r, 36 \cdot r]^N \setminus ]-r, r[^N. \tag{178}$$

Moreover, for every  $y \in [-36 \cdot r, 36 \cdot r]^N$ , the first component  $\xi^y$  of the solution to (173)-(174) provides the minimizer for  $(CV)_{\tau, y}$ , with initial cost  $u(0, \cdot)$ .

**3.** Fix  $\tau > 0$  satisfying (161), (162) and let  $u(t, x)$  be the map defined in (171). For every  $y \in ]-6 \cdot r, 6 \cdot r[^N$ , let  $\xi^y$  be the first component of the solution to (173)-(174). By the regularity of  $\tilde{\psi}^\tau$  it follows that the sets

$$\begin{aligned} \Omega_1 &\doteq \left\{ \xi^y(0) \mid y \in ]-r, r[^N \right\}, \\ \Omega_2 &\doteq \left\{ \xi^y(0) \mid y \in ]-6 \cdot r, 6 \cdot r[^N \right\}, \end{aligned} \tag{179}$$

are open domains of  $\mathbb{R}^N$ . Observe that, by (158), (175), and by the definition of  $\Omega_i$ , we have

$$\bar{\Omega}_1 \subset B(0, 2 \cdot r), \quad \bar{\Omega}_2 \subset B(0, 12 \cdot r) \subset I_r^N. \tag{180}$$

Moreover, notice that for every  $x \in B(0, 2 \cdot r)$ , the first component  $\xi_x$  of the solution to (173) on  $[0, \tau]$ , with initial condition

$$\begin{cases} \xi_x(0) = x, \\ p_x(0) = D_x u(0, x), \end{cases} \tag{181}$$

provides the minimizer for  $(CV)_{\tau, \xi_x(\tau)}$ , with initial cost  $u(0, \cdot)$ . In turn, because of property (b) at point 1, this implies that  $\xi_x(\tau) \in ]-6 \cdot r, 6 \cdot r[^N$ . On the other hand, setting  $y \doteq \xi_x(\tau)$ , by the above notation we have  $\xi_x = \xi^y$ ,  $x = \xi^y(0)$ . Therefore, from the definition (179) it follows in particular that

$$B(0, 2 \cdot r) \subset \Omega_2. \tag{182}$$

Furthermore, because of (178), one has

$$u(0, x) = 0 \quad \forall x \in \Omega_2 \setminus \Omega_1. \tag{183}$$

Hence, by virtue of (180), and recalling (170), (171), we deduce

$$|u(0, x)| \leq 20m \cdot r \quad \forall x \in \Omega_1. \tag{184}$$

Then, define the function

$$u_0^\sharp(x) \doteq \begin{cases} u(0, x) & \text{if } x \in \Omega_1 \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega_1 \end{cases} \tag{185}$$

and notice that, by (180), (183),  $u_0^\sharp$  is a continuous map, while (170), (171), (180), (185) imply

$$\text{supp}(u_0^\sharp) \subset [-12 \cdot r, 12 \cdot r]^N, \quad \text{Lip}[u_0^\sharp] \leq 10m. \tag{186}$$

Therefore, recalling definition (5), and because of (153), there holds

$$u_0^\sharp \in \mathcal{L}_{[12 \cdot r, 10m]} \subset \mathcal{L}_{[R, M]}. \quad (187)$$

We claim that

$$S_\tau u_0^\sharp(x) = \begin{cases} \psi(x) & \text{if } x \in [-r, r]^N, \\ S_\tau 0(x) & \text{if } x \in \mathbb{R}^N \setminus [-r, r]^N. \end{cases} \quad (188)$$

In fact, for every  $x \in \mathbb{R}^N$  let  $\xi_x^\sharp, \xi_x^*$  be any minimizer for  $(CV)_{\tau, x}$  with initial cost  $u_0^\sharp, u(0, \cdot)$ , respectively, so that one has

$$\begin{aligned} \xi_x^\sharp(\tau) = x, \quad S_\tau u_0^\sharp(x) &= u_0^\sharp(\xi_x^\sharp(0)) + \int_0^\tau L(\xi_x^\sharp(s), \dot{\xi}_x^\sharp(s)) ds, \\ \xi_x^*(\tau) = x, \quad u(\tau, x) &= u(0, \xi_x^*(0)) + \int_0^\tau L(\xi_x^*(s), \dot{\xi}_x^*(s)) ds. \end{aligned} \quad (189)$$

Observe that, because of (153), (184), (185), one has

$$\|u_0^\sharp\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \leq \min \{c_6(36 \cdot r), c_7(6 \cdot r)\}. \quad (190)$$

Then, by the choice of  $\tau$  in (161), (162), relying on properties (a), (b) at point 1 and on (180), (182), we deduce that

$$\begin{aligned} x \in [-r, r]^N &\implies \xi_x^\sharp(0) \in \Omega_2 \\ x \in \mathbb{R}^N \setminus [-36 \cdot r, 36 \cdot r]^N &\implies \xi_x^\sharp(0) \in \mathbb{R}^N \setminus \Omega_2. \end{aligned} \quad (191)$$

**4.** In order to establish (188), we shall distinguish three cases.

Case 1:  $x \in [-r, r]^N$ .

By definition of  $\Omega_1$  we have  $\xi_x^*(0) \in \Omega_1$  for all  $x \in [-r, r]^N$ . Therefore, because of (185), there holds  $u(0, \xi_x^*(0)) = u^\sharp(\xi_x^*(0))$ , while (183), (185), (191) imply  $u^\sharp(\xi_x^\sharp(0)) = u(0, \xi_x^\sharp(0))$ . Hence, by (189), we deduce

$$\begin{aligned} u(\tau, x) &= u^\sharp(\xi_x^*(0)) + \int_0^\tau L(\xi_x^*(s), \dot{\xi}_x^*(s)) ds \\ &\geq S_\tau u_0^\sharp(x) \\ &= u_0^\sharp(\xi_x^\sharp(0)) + \int_0^\tau L(\xi_x^\sharp(s), \dot{\xi}_x^\sharp(s)) ds \\ &= u(0, \xi_x^\sharp(0)) + \int_0^\tau L(\xi_x^\sharp(s), \dot{\xi}_x^\sharp(s)) ds \\ &\geq u(\tau, x), \end{aligned} \quad (192)$$

which proves (188), recalling (164), (167), (169), (171).

Case 2:  $x \in [-36 \cdot r, 36 \cdot r]^N \setminus [-r, r]^N$ .

By the observations at point 3 and because of (178), (185), we know that  $u(0, \xi_x^*(0)) = 0 = u^\sharp(\xi_x^*(0))$  for all  $x \in [-36 \cdot r, 36 \cdot r]^N \setminus [-r, r]^N$ . Moreover, by (158), (164), (167), (169), (171), one has

$$u(\tau, x) = S_\tau 0(x) \quad \forall x \in [-36 \cdot r, 36 \cdot r]^N \setminus [-r, r]^N. \quad (193)$$



Thus, if  $u_0^\sharp(\xi_x^\sharp(0)) \geq u(0, \xi_x^\sharp(0))$ , by (189) we derive

$$\begin{aligned}
u(\tau, x) &= \int_0^\tau L(\xi_x^*(s), \dot{\xi}_x^*(s)) ds \\
&\geq S_\tau u_0^\sharp(x) \\
&= u_0^\sharp(\xi_x^\sharp(0)) + \int_0^\tau L(\xi_x^\sharp(s), \dot{\xi}_x^\sharp(s)) ds \\
&\geq u(0, \xi_x^\sharp(0)) + \int_0^\tau L(\xi_x^\sharp(s), \dot{\xi}_x^\sharp(s)) ds \\
&\geq u(\tau, x).
\end{aligned} \tag{194}$$

Otherwise, if  $u_0^\sharp(\xi_x^\sharp(0)) < u(0, \xi_x^\sharp(0))$ , by (185) it must be  $u_0^\sharp(\xi_x^\sharp(0)) = 0$ . Hence, relying on (189), (193), we get

$$\begin{aligned}
S_\tau 0(x) &= \int_0^\tau L(\xi_x^*(s), \dot{\xi}_x^*(s)) ds \\
&\geq S_\tau u_0^\sharp(x) \\
&= \int_0^\tau L(\xi_x^\sharp(s), \dot{\xi}_x^\sharp(s)) ds \\
&\geq S_\tau 0(x).
\end{aligned} \tag{195}$$

Together (193), (194), (195), yield (188).

Case 3:  $x \in \mathbb{R}^N \setminus [-36 \cdot r, 36 \cdot r]^N$ .

By (185), (191) we know that  $u_0^\sharp(\xi_x^\sharp(0)) = 0$  for all  $x \in \mathbb{R}^N \setminus [-36 \cdot r, 36 \cdot r]^N$ . Moreover, letting  $\xi_x^o$  be any minimizer for  $(CV)_{\tau, x}$  with initial cost zero, relying on properties (b) at point 1 and on (180), (185), we deduce that  $\xi_x^o(0) \in \mathbb{R}^N \setminus \Omega_1$  and hence also  $u_0^\sharp(\xi_x^o(0)) = 0$  for all  $x \in \mathbb{R}^N \setminus [-36 \cdot r, 36 \cdot r]^N$ . Then, using (189), we derive

$$\begin{aligned}
S_\tau 0(x) &= \int_0^\tau L(\xi_x^o(s), \dot{\xi}_x^o(s)) ds \\
&\geq S_\tau u_0^\sharp(x) \\
&= \int_0^\tau L(\xi_x^\sharp(s), \dot{\xi}_x^\sharp(s)) ds \\
&\geq S_\tau 0(x),
\end{aligned} \tag{196}$$

which proves (188).

This completes the proof of the proposition taking

$$\begin{aligned}
r_1(R) &\doteq \frac{R}{12}, \\
m_1(r, M) &\doteq 2 \min \left\{ \frac{M}{10}, \frac{1}{120 \cdot \sqrt{N} \cdot r} \cdot \min \left\{ c_6(36 \cdot r), c_7(6 \cdot r) \right\} \right\},
\end{aligned} \tag{197}$$

$$\tau_6(r, m) \doteq \bar{\tau}'(r, m/2),$$

with  $\bar{\tau}'$  as in (161), (162), and letting  $k_1(r, m)$  be a semiconcavity constant for  $S_\tau 0$ ,  $\tau \leq \tau_6(r, m)$  on  $I_r^N$ .  $\square$

**Remark 10.** By Proposition 5, for every  $r \leq r_1(R)$ ,  $m \leq m_1(r, M)$ ,  $T \leq \tau_6(r, m)$  and  $K = k_1(r, m)$ , one has

$$\widetilde{\mathcal{SC}}_{[r, 2m, 2K]}^{S_T^{0,1}} \subset S_T(\mathcal{L}_{[R, M]}), \quad (198)$$

where  $\mathcal{L}_{[R, M]}$  is the set defined in (5) while

$$\widetilde{\mathcal{SC}}_{[r, 2m, 2K]}^{S_T^{0,1}} \doteq \left\{ u_0 \in \mathcal{L}_{[r, 2m]} \mid u_0 \upharpoonright_{[-r, r]^N} \in \mathcal{SC}_{[r, 2m, 2K]}^{S_T^{0,1}}, \quad u_0 \upharpoonright_{\mathbb{R}^N \setminus [-r, r]^N} = S_T 0 \right\}, \quad (199)$$

with  $\mathcal{SC}_{[r, m, K]}^{S_T^{0,1}}$  defined as in (33).

It remains an interesting open problem to analyze the global in time exact controllability of (1). Namely, one would like to determine whether there exist constants  $r_1(R, M)$ ,  $m_1(r, M)$  so that, for every time  $T > 0$  and for any  $r \leq r_1(R, M)$ ,  $m \leq m_1(r, M)$ , there holds (198) for some  $K = k_1(r, m, T)$ .

## 4.2 Conclusion of the proof of Theorem 1-(ii)

Given  $R, M > 0$ , let  $r_1(R)$ ,  $m_1(r, M)$ ,  $k_1(r, m)$  and  $\tau_6(r, m)$  be the constants provided by Proposition 5 and set  $r_R \doteq r_1(R)$ ,  $m_{R, M} \doteq m_1(r_R, M)$ ,  $K_{R, M} \doteq k_1(r_R, m_{R, M})$ ,  $\tau_{R, M} \doteq \tau_6(r_R, m_{R, M})$ , so that, as observed in Remark 10, there holds

$$\widetilde{\mathcal{SC}}_{[r_R, 2m_{R, M}, 2K_{R, M}]}^{S_T^{0,1}} \subset S_T(\mathcal{L}_{[R, M]}) \quad \forall T \leq \tau_{R, M}. \quad (200)$$

On the other hand, by definition (199), one has

$$\text{supp}(\varphi) \subset [-r_R, r_R]^N \quad \forall \varphi \in \widetilde{\mathcal{SC}}_{[r_R, 2m_{R, M}, 2K_{R, M}]}^{S_T^{0,1}} - S_T 0. \quad (201)$$

Therefore, relying on (200), (201), and applying Theorem 3-(ii), we deduce that, for  $T \leq \tau_{R, M}$  and  $\varepsilon$  sufficiently small, there holds

$$\begin{aligned} \mathcal{H}_\varepsilon \left( S_T(\mathcal{L}_{[R, M]}) - S_T 0 \mid \mathbf{W}^{1,1}(\mathbb{R}^N) \right) &\geq \mathcal{H}_\varepsilon \left( \widetilde{\mathcal{SC}}_{[r_R, 2m_{R, M}, 2K_{R, M}]}^{S_T^{0,1}} - S_T 0 \mid \mathbf{W}^{1,1}(\mathbb{R}^N) \right) \\ &= \mathcal{H}_\varepsilon \left( \widetilde{\mathcal{SC}}_{[r_R, 2m_{R, M}, 2K_{R, M}]}^{S_T^{0,1}} - S_T 0 \mid \mathbf{W}^{1,1}([-r_R, r_R]^N) \right) \\ &= \mathcal{H}_\varepsilon \left( \widetilde{\mathcal{SC}}_{[r_R, 2m_{R, M}, 2K_{R, M}]}^{S_T^{0,1}} \mid \mathbf{W}^{1,1}([-r_R, r_R]^N) \right) \\ &\geq \gamma_{[l_T, \mu_T, \kappa_T, N]}^- \cdot \frac{1}{\varepsilon^N}, \end{aligned} \quad (202)$$

where

$$\gamma_{[r_R, K_{R, M}, N]}^- = \frac{1}{8 \cdot \ln 2} \cdot \left( \frac{K_{R, M} \omega_N r_R^{N+1}}{48(N+1) 2^{N+1}} \right)^N. \quad (203)$$

This completes the proof of the lower bound (11). □

## Acknowledgements

This work was partially supported by the National Group for Mathematical Analysis and Probability (GNAMPA) of the Istituto Nazionale di Alta Matematica “Francesco Severi” (INdAM) and by the INdAM-CNRS European Research Group (GDRE) on Control of Partial Differential Equations (CONEDP). Fabio Ancona was partially supported by the Miur-Prin 2012 Project “Nonlinear Hyperbolic Partial Differential Equations, Dispersive and Transport Equations: theoretical and applicative aspects” and by the University of Padova grant “PRAT 2013 - Traffic Flow on Networks: Analysis and Control”.

## References

- [1] F. Ancona, P. Cannarsa, Khai T. Nguyen, Quantitative compactness estimates for Hamilton-Jacobi equations, preprint 2014 (<http://arxiv.org/abs/1403.4556>).
- [2] F. Ancona, O. Glass and K. T. Nguyen, Lower compactness estimates for scalar balance laws, *Comm. Pure Appl. Math* 65 (2012), no. 9, 1303-1329.
- [3] F. Ancona, O. Glass and K. T. Nguyen, On compactness estimates for hyperbolic systems of conservation laws, preprint (2014) (<http://arxiv.org/abs/1403.5070>), to appear in *Ann. Inst. H. Poincaré Anal. Non Linéaire*.
- [4] F. Ancona, O. Glass and K. T. Nguyen, On quantitative compactness estimates for hyperbolic conservation laws, *Hyperbolic problems: theory, numerics and applications. Proceedings of the 14th International Conference on Hyperbolic Problems (HYP2012)*, AIMS, Springfield, MO, 2014, 249-257.
- [5] M. Bardi and I. Capuzzo Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi equations*. Birkhäuser, Boston (1997).
- [6] E.N. Barron, P.Cannarsa, R. Jensen and C. Sinestrari, Regularity of Hamilton-Jacobi equations when forward is backward, *Indiana Univ. Math. J.* 48, no. 2 (1999), 385-409.
- [7] P. L. Bartlett, S. R. Kulkarni and S.E. Posner, Covering numbers for real-valued function classes. *IEEE Trans. Inform. Theory* 43 (1997), no. 5, 1721-1724.
- [8] S. Bianchini and D. Tonon, SBV Regularity for Hamilton-Jacobi equations with Hamiltonian depending on  $(t, x)$ , *Siam J. Math. Anal.* 44, no. 3, 2179-2203.
- [9] S. Bianchini, C. DeLellis and R.Roby, SBV Regularity for Hamilton-Jacobi equations in  $\mathbb{R}^n$ , *Arch. Ration. Mech. Anal.* 200 (2011), 1003-1021.
- [10] P.Cannarsa and H.Frankowska, From pointwise to local regularity for solutions of Hamilton-Jacobi equations. *Calc. Var. Partial Differ. Equ.* 49 (2014), 1061-1074.
- [11] P.Cannarsa, A. Mennucci and C. Sinestrari, Regularity results for solutions of a class of Hamilton-Jacobi equations. *Arch. Ration. Mech. Anal.* 140 (1997), 197-223.
- [12] P.Cannarsa and C.Sinestrari, Semiconcave functions, Hamilton-Jacobi equations, and optimal control. *Progress in Nonlinear Differential Equations and their Applications*, 58. Birkhäuser Boston, 2004.

- [13] P. Cannarsa and H. Soner, On the singularities of viscosity solutions to Hamilton-Jacobi equations. *Indiana Univ. Math. J.* 36 (1987), 501-524.
- [14] F.H. Clarke, *Optimization and nonsmooth analysis*, Wiley, New York, 1983.
- [15] M.G. Crandall and P.-L. Lions, Viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* 277 (1983), no. 1, 1-42.
- [16] M.G. Crandall, L.C. Evans and P.-L. Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* 282 (1984), 487-502.
- [17] C. De Lellis and F. Golse, A Quantitative Compactness Estimate for Scalar Conservation Laws, *Comm. Pure Appl. Math.* 58 (2005), no. 7, 989-998.
- [18] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 319, AMS, Providence, 1991.
- [19] W. Fleming, The Cauchy problem for a nonlinear first order partial differential equations, *J. Differential Equations*, 5 (1969), 515-530.
- [20] W. Hoeffding, Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* 58 (1963), 13-30.
- [21] A.N. Kolmogorov and V.M Tikhomirov,  $\varepsilon$ -Entropy and  $\varepsilon$ -capacity of sets in functional spaces. *Uspekhi Mat. Nauk* 14 (1959), 3-86.
- [22] P. D. Lax, Accuracy and resolution in the computation of solutions of linear and nonlinear equations. *Recent advances in numerical analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1978)*. *Publ. Math. Res. Center Univ. Wisconsin*, 107-117. Academic Press, New York, 1978.
- [23] P.D. Lax, *Course on hyperbolic systems of conservation laws.. XXVII Scuola Estiva di Fis. Mat., Ravello, 2002.*
- [24] C.-T. Lin and E. Tadmor,  $L^1$ -Stability and error estimates for approximate Hamilton-Jacobi equations, *Numer. Math.* 87 (2001), no. 4, 701735.