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Abstract

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Integral Equations and Operator Theory

Analytic Dependence of Volume Potentials Corresponding to Parametric Families of Fundamental Solutions

Matteo Dalla Riva, Massimo Lanza de Cristoforis and Paolo Musolino

Abstract. We show that volume potentials associated to a parameter dependent analytic family of weakly singular kernels depend real-analytically upon the density function and on the parameter. Then we consider the special case in which the analytic family corresponds to a family of fundamental solutions of second order differential operators with constant coefficients.

Mathematics Subject Classification. Primary 26E05, 31B10; Secondary 35E05, 47H30.

Keywords. Volume potentials, second order differential operators with constant coefficients, domain perturbation, special nonlinear operators.

1. Introduction

The aim of this paper is to analyze the behavior of the volume potential corresponding to the fundamental solution of a parameter dependent second order differential operator upon variation of the density and of the parameter.

We first introduce our parameter dependent differential operators. We fix once for all a natural number

$$n \in \mathbb{N}\setminus\{0,1\}.$$

We denote by $N_{2,n}$ the set of multi-indexes $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2$. If $\mathbf{a} \equiv (a_\alpha)_{|\alpha|\leq 2} \in \mathbb{C}^{N_{2,n}}$, then we set

$$P[\mathbf{a}, x] \equiv \sum_{|\alpha|\leq 2} a_\alpha x^\alpha \quad \forall x \in \mathbb{R}^n.$$

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We also set

$$\mathcal{E} \equiv \left\{ \mathbf{a} \equiv (a_\alpha)_{|\alpha|\le 2} \in \mathbb{C}^{N_{2,n}} \colon \sum_{|\alpha|=2} a_\alpha \xi^\alpha \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\} \right\}.$$

Clearly, $\mathcal{E}$ coincides with the set of coefficients $\mathbf{a} \equiv (a_\alpha)_{|\alpha|\le 2}$ such that the complex coefficient partial differential operator

$$P[\mathbf{a}, D] \equiv \sum_{|\alpha|\le 2} a_\alpha D^\alpha$$

is elliptic. As is well known, $P[\mathbf{a}, D]$ has a fundamental solution for all $\mathbf{a} \in \mathcal{E}$. We are now interested into a parameter dependent family of fundamental solutions, and we want to consider the following assumptions

$$\begin{aligned}
&\text{Let } \mathcal{K} \text{ be a real Banach space. Let } \mathcal{O} \text{ be an open subset of } \mathcal{K}. \qquad (1.1)\\
&\text{Let } \mathbf{a}(\cdot) \text{ be a real analytic map from } \mathcal{O} \text{ to } \mathcal{E}.\\
&\text{Let } S(\cdot,\cdot) \text{ be a real analytic map from } (\mathbb{R}^n\setminus\{0\}) \times \mathcal{O} \text{ to } \mathbb{C} \text{ such that}\\
&S(\cdot,\kappa) \text{ is a fundamental solution of } P[\mathbf{a}(\kappa), D] \text{ for all } \kappa \in \mathcal{O}.
\end{aligned}$$

Next we fix an open bounded connected subset Ω of $\mathbb{R}^n$ of class C^1, and an open bounded subset Ω_1 of $\mathbb{R}^n$ such that

$$\mathrm{cl}\Omega_1 \subseteq \Omega.$$

Then we are interested into the dependence of the volume potential

$$\mathcal{P}_\kappa[\varphi] \equiv \int_\Omega S(x-y,\kappa)\varphi(y)\,dy \quad \forall x \in \mathrm{cl}\Omega_1, \tag{1.2}$$

upon φ and κ. Indeed, in the applications of volume potentials to perturbation problems for partial differential equations, one often needs to understand the dependence of the composition $\mathcal{P}_\kappa[\varphi] \circ \psi$ of $\mathcal{P}_\kappa[\varphi]$ with a function ψ in the subset $C^{m,\alpha}(\mathrm{cl}\Omega_\#, \Omega_1)$ upon the triple (κ, φ, ψ) (cf. Sect. 6.3). Here $\Omega_\#$ is a bounded open subset of $\mathbb{R}^n$ of class C^1, and $C^{m,\alpha}(\mathrm{cl}\Omega_\#, \Omega_1)$ denotes the set of functions from $\mathrm{cl}\Omega_\#$ to Ω_1 which belong to the Schauder space with exponents $m \in \mathbb{N}$ and $\alpha \in]0, 1[$.

As shown by Preciso [32,33], if we want that both ψ and $\mathcal{P}_\kappa[\varphi] \circ \psi$ belong to a Schauder space and that $\mathcal{P}_\kappa[\varphi] \circ \psi$ depends analytically on ψ, then a right choice for the space for $\mathcal{P}_\kappa[\varphi]$ is the Roumieu class $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1)$ built on the space of continuous functions on $\mathrm{cl}\Omega_1$ for some $\rho \in]0, +\infty[$ [see (2.1) below]. Thus it is natural to ask whether there exist ρ and $\rho_1 \in]0, +\infty[$ such that the map from $\mathcal{O} \times C^0_{\omega,\rho}(\mathrm{cl}\Omega)$ to $C^0_{\omega,\rho_1}(\mathrm{cl}\Omega_1)$ which takes (κ, φ) to $\mathcal{P}_\kappa[\varphi]$ is a real analytic map. We prove such analyticity in Theorem 5.1.

The dependence of integral operators associated to fundamental solutions of elliptic differential equations upon perturbations has long been investigated by several authors with the aim of applying those results to the study of boundary value problems.

For example, Fréchet differentiability results for the dependence of layer potentials for the Helmholtz equation upon the support of integration have been obtained by Potthast [29–31] in the framework of Schauder spaces, in order to analyze the domain derivative of the far field pattern for a scattering problem. In this context, we also mention the works by Haddar and Kress [11], Hettlich [13], Kirsch [17], and Kress and Päivärinta [18]. Instead, Fréchet differentiability properties of operators related to the inverse elastic scattering problem have been shown by Charalambopoulos [2]. Analogous results in the framework of Sobolev spaces on Lipschitz domains have been obtained by Costabel and Le Louër [3,4,26].

The authors of the present paper have developed a method based on potential theory to prove analyticity results for the solution of boundary value problems upon perturbations of the domain and of the data (cf. e.g., [20]). In order to exploit such a method, one has to study the dependence of layer and volume potentials upon perturbations. As a consequence, [24,25] have analyzed the layer potentials associated to the Laplace and Helmholtz equations. Then [6] has investigated the case of layer potentials corresponding to second order complex constant coefficient elliptic differentials operators, and [23] has considered a periodic analog.

The present paper extends such a technique to volume potentials in order to investigate perturbation results for the solutions of boundary value problems for non-homogeneous elliptic differential equations (cf. Sect. 6.3). The paper is organized as follows. In Sect. 2, we introduce some basic notation. In Sect. 3, we introduce some variant of some classical material on volume potentials in a form which is suitable to the developments of the present paper. In Sect. 4, we estimate the Roumieu norm of a volume potential corresponding to a general kernel in terms of a weighted norm of the kernel and of a norm of the density. Here the idea is to introduce a special weighted class of singular functions at the origin, which are analytic away from the origin (see Definition 4.1). In Sect. 5 we exploit the results of Sect. 3 to prove the analyticity Theorem 5.1 for volume potentials corresponding to a family of fundamental solutions. In Sect. 6, we present some concrete applications.

2. Notation

We denote the norm on a normed space $\mathcal{X}$ by $\|\cdot\|_{\mathcal{X}}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. We endow the space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x,y)\|_{\mathcal{X}\times\mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x,y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for $\mathbb{R}^n$. For standard definitions of Calculus in normed spaces, we refer to Deimling [8]. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\mathrm{cl}\mathbb{D}$ denotes the closure of $\mathbb{D}$, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$, and $\mathrm{diam}(\mathbb{D})$ denotes the diameter of $\mathbb{D}$. The symbol $|\cdot|$ denotes the Euclidean modulus in $\mathbb{R}^n$ or in $\mathbb{C}$. For all $R \in]0,+\infty[$, $x \in \mathbb{R}^n$, x_j denotes the jth coordinate of x, and $\mathbb{B}_n(x,R)$ denotes the ball $\{y \in \mathbb{R}^n : |x-y| < R\}$. Let Ω be an open subset of $\mathbb{R}^n$. The space of m times continuously differentiable complex-valued functions on Ω is denoted by $C^m(\Omega,\mathbb{C})$, or more simply by $C^m(\Omega)$. Let $f \in (C^m(\Omega))$. Then Df denotes the gradient of f. Let $\eta \equiv (\eta_1,\ldots,\eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \cdots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\mathrm{cl}\Omega$ is denoted $C^m(\mathrm{cl}\Omega)$.

The subspace of $C^m(\mathrm{cl}\Omega)$ whose functions have mth order derivatives that are Hölder continuous with exponent $\alpha \in]0,1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\Omega)$ (cf. e.g., Gilbarg and Trudinger [10]). Let $\mathbb{D} \subseteq \mathbb{C}^n$. Then $C^{m,\alpha}(\mathrm{cl}\Omega, \mathbb{D})$ denotes $\{f \in (C^{m,\alpha}(\mathrm{cl}\Omega))^n : f(\mathrm{cl}\Omega) \subseteq \mathbb{D}\}$. The subspace of $C^m(\mathrm{cl}\Omega)$ of those functions f such that $f_{|\mathrm{cl}(\Omega \cap \mathbb{B}_n(0,R))} \in C^{m,\alpha}(\mathrm{cl}(\Omega \cap \mathbb{B}_n(0,R)))$ for all $R \in]0,+\infty[$ is denoted $C^{m,\alpha}_{\mathrm{loc}}(\mathrm{cl}\Omega)$.

Now let Ω be a bounded open subset of $\mathbb{R}^n$. Then $C^m(\mathrm{cl}\Omega)$ and $C^{m,\alpha}(\mathrm{cl}\Omega)$ are endowed with their usual norm and are well known to be Banach spaces (cf. e.g., Troianiello [36, §1.2.1]). For the definition of a bounded open Lipschitz subset of $\mathbb{R}^n$, we refer for example to Nečas [28, §1.3]. We say that a bounded open subset Ω of $\mathbb{R}^n$ is of class C^m or of class $C^{m,\alpha}$, if it is a manifold with boundary imbedded in $\mathbb{R}^n$ of class C^m or $C^{m,\alpha}$, respectively (cf. e.g., Gilbarg and Trudinger [10, §6.2]). We denote by ν_Ω the outward unit normal to $\partial\Omega$. For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [10] and to Troianiello [36] (see also [24, §2]). We denote by $d\sigma$ the area element of a manifold imbedded in $\mathbb{R}^n$. We retain the standard notation for the Lebesgue spaces.

We note that throughout the paper 'analytic' means always 'real analytic'. For the definition and properties of analytic operators, we refer to Deimling [8, §15].

Next, we turn to introduce the Roumieu classes. For all bounded open subsets Ω of $\mathbb{R}^n$ and $\rho \in]0,+\infty[$, we set

$$C^0_{\omega,\rho}(\mathrm{cl}\Omega) \equiv \left\{ u \in C^\infty(\mathrm{cl}\Omega) \colon \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} < +\infty \right\}, \qquad (2.1)$$

and

$$\|u\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)} \equiv \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\mathrm{cl}\Omega)} \quad \forall u \in C^0_{\omega,\rho}(\mathrm{cl}\Omega).$$

As is well known, the Roumieu class $\left(C^0_{\omega,\rho}(\mathrm{cl}\Omega), \|\cdot\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)}\right)$ is a Banach space.

3. Preliminaries on Volume Potentials

We first introduce the following preliminary classical lemma. We denote by m_n the n-dimensional Lebesgue measure and by s_n the $(n-1)$-dimensional measure of $\partial\mathbb{B}_n(0,1)$.

Lemma 3.1. *Let $h \in L^1(\mathbb{R}^n)$. For each $\epsilon \in]0,+\infty[$ there exists $\delta \in]0,+\infty[$ such that*

$$\int_E |h|\, dx \le \epsilon,$$

for all measurable subsets E of $\mathbb{R}^n$ such that $m_n(E) \le \delta$.

For a proof, we refer to Folland [9, Cor. 3.6, p. 89]. Then we have the following elementary technical statement.

Lemma 3.2. *Let* $\lambda \in]0,n[$, $R \in]0,+\infty[$. *Let* $h \in C^0((\mathrm{cl}\mathbb{B}_n(0,R))\setminus\{0\})$. *Let*

$$\sup_{x\in(\mathrm{cl}\mathbb{B}_n(0,R))\setminus\{0\}} |h(x)|\,|x|^\lambda < +\infty.$$

Let $\rho \in]0,R[$. *For each* $\epsilon \in]0,+\infty[$ *there exists* $\delta \in]0,+\infty[$ *such that*

$$\int_E |h(x-y)|\,dy \le \epsilon,$$

for all measurable subsets E *of* $\mathrm{cl}\mathbb{B}_n(0,R-\rho)$ *such that* $m_n(E) \le \delta$ *and for all* $x \in \mathrm{cl}\mathbb{B}_n(0,\rho)$.

Proof. Let $\tilde{h}$ be the function from $\mathbb{R}^n$ to $\mathbb{R}$ defined by $\tilde{h}(x) \equiv h(x)$ if $x \in (\mathrm{cl}\mathbb{B}_n(0,R))\setminus\{0\}$, $\tilde{h}(x) \equiv 0$ if $x \in \mathbb{R}^n\setminus((\mathrm{cl}\mathbb{B}_n(0,R))\setminus\{0\})$. Then $\tilde{h} \in L^1(\mathbb{R}^n)$ and for each $\epsilon \in]0,+\infty[$, there exists $\delta \in]0,+\infty[$ such that

$$\int_F |h|\,dx = \int_F |\tilde{h}|\,dx \le \epsilon,$$

for all measurable subsets F of $\mathrm{cl}\mathbb{B}_n(0,R)$ such that $m_n(F) \le \delta$. Now if E is a measurable subset of $\mathbb{B}_n(0,R-\rho)$ and if $m_n(E) \le \delta$, and if $x \in \mathrm{cl}\mathbb{B}_n(0,\rho)$, then we have $m_n(x-E) = m_n(E) \le \delta$, $x - E \subseteq \mathrm{cl}\mathbb{B}_n(0,R)$ and accordingly,

$$\int_E |h(x-y)|\,dy = \int_{x-E} |h(y)|\,dy \le \epsilon. \qquad \square$$

Next we introduce the following class of singular functions in a punctured ball.

Definition 3.3. Let $\lambda \in]0,+\infty[$. Let $R \in]0,+\infty[$. Then we denote by $A^0_\lambda(R)$ the set of functions $h \in C^0((\mathrm{cl}\mathbb{B}_n(0,R))\setminus\{0\})$ such that

$$\sup_{x\in(\mathrm{cl}\mathbb{B}_n(0,R))\setminus\{0\}} |h(x)|\,|x|^\lambda < +\infty,$$

and we set

$$\|h\|_{A^0_\lambda(R)} \equiv \sup_{x\in(\mathrm{cl}\mathbb{B}_n(0,R))\setminus\{0\}} |h(x)|\,|x|^\lambda \quad \forall h \in A^0_\lambda(R).$$

One can readily verify that $(A^0_\lambda(R), \|\cdot\|_{A^0_\lambda(R)})$ is a Banach space. Then we prove the following.

Proposition 3.4. *Let* $\lambda \in]0,n[$. *Let* Ω *be a bounded open subset of* $\mathbb{R}^n$. *Then the following statements hold.*

(i) *If* $(h,\varphi) \in A^0_\lambda(\mathrm{diam}\,(\Omega)) \times L^\infty(\Omega)$ *and if* $x \in \mathrm{cl}\Omega$, *then the function from* Ω *to* $\mathbb{R}$ *which takes* $y \in \Omega$ *to* $h(x-y)\varphi(y)$ *is integrable.*

(ii) *If* $(h,\varphi) \in A^0_\lambda(\mathrm{diam}\,(\Omega)) \times L^\infty(\Omega)$, *then the function* $\mathcal{P}[h,\varphi]$ *from* $\mathrm{cl}\Omega$ *to* $\mathbb{R}$ *which takes* $x \in \mathrm{cl}\Omega$ *to*

$$\mathcal{P}[h,\varphi](x) \equiv \int_\Omega h(x-y)\varphi(y)\,dy,$$

is continuous.

(iii) $\mathcal{P}[h,\varphi]$ *is bounded and*

$$\|\mathcal{P}[h,\varphi]\|_{L^\infty(\Omega)} \leq s_n \frac{(\mathrm{diam}\,(\Omega))^{n-\lambda}}{n-\lambda}\|h\|_{A^0_\lambda(\mathrm{diam}\,(\Omega))}\|\varphi\|_{L^\infty(\Omega)}, \tag{3.1}$$

for all $(h,\varphi) \in A^0_\lambda(\mathrm{diam}\,(\Omega)) \times L^\infty(\Omega)$.

Proof. If $(h,\varphi) \in A^0_\lambda(\mathrm{diam}\,(\Omega)) \times L^\infty(\Omega)$, then we have

$$|h(x-y)\varphi(y)| \leq |h(x-y)|\|\varphi\|_{L^\infty(\Omega)} \quad \text{for a.a. } y \in \Omega,$$

for all $x \in \mathrm{cl}\Omega$. Then $h(x-\cdot)\varphi(\cdot)$ is integrable in Ω. Since $\Omega \subseteq \mathbb{B}_n(x, \mathrm{diam}\,(\Omega))$ for all $x \in \mathrm{cl}\Omega$, we have

$$\begin{aligned}\left|\int_\Omega h(x-y)\varphi(y)\,dy\right| &\leq \int_{\mathbb{B}_n(x,\mathrm{diam}\,(\Omega))} |h(x-y)|\,dy\|\varphi\|_{L^\infty(\Omega)} \\ &\leq \|h\|_{A^0_\lambda(\mathrm{diam}\,(\Omega))}\int_{\mathbb{B}_n(x,\mathrm{diam}\,(\Omega))}\frac{dy}{|x-y|^\lambda}\|\varphi\|_{L^\infty(\Omega)} \\ &= \|h\|_{A^0_\lambda(\mathrm{diam}\,(\Omega))} s_n \frac{(\mathrm{diam}\,(\Omega))^{n-\lambda}}{n-\lambda}\|\varphi\|_{L^\infty(\Omega)} \quad \forall x \in \mathrm{cl}\Omega.\end{aligned}$$

Hence, inequality (3.1) follows.

Next we show that $\mathcal{P}[h,\varphi]$ is continuous. Let $x_0 \in \mathrm{cl}\Omega$. Let $\epsilon \in]0,+\infty[$. By Lemma 3.2 with $\rho = \mathrm{diam}\,(\Omega)/2$, there exists $\delta \in]0, \mathrm{diam}\,(\Omega)/2[$ such that

$$\int_{\mathbb{B}_n(x_0,\delta)} |h(x-y)|\,dy = \int_{\mathbb{B}_n(0,\delta)} |h((x-x_0)-z)|\,dz \leq \epsilon/2$$

for all $x \in \mathbb{B}_n(x_0,\delta)$. Then we have

$$\begin{aligned}&|\mathcal{P}[h,\varphi](x) - \mathcal{P}[h,\varphi](x_0)| \\ &\leq \left|\int_{\Omega\setminus\mathbb{B}_n(x_0,\delta)} h(x-y)\varphi(y)\,dy - \int_{\Omega\setminus\mathbb{B}_n(x_0,\delta)} h(x_0-y)\varphi(y)\,dy\right| \\ &\quad + \int_{\mathbb{B}_n(x_0,\delta)} |h(x-y)|\,dy\|\varphi\|_{L^\infty(\Omega)} + \int_{\mathbb{B}_n(x_0,\delta)} |h(x_0-y)|\,dy\|\varphi\|_{L^\infty(\Omega)} \\ &\leq \left|\int_{\Omega\setminus\mathbb{B}_n(x_0,\delta)} h(x-y)\varphi(y)\,dy - \int_{\Omega\setminus\mathbb{B}_n(x_0,\delta)} h(x_0-y)\varphi(y)\,dy\right| + \epsilon\|\varphi\|_{L^\infty(\Omega)},\end{aligned}$$

for all $x \in \mathrm{cl}\Omega \cap \mathbb{B}_n(x_0,\delta)$. Since h is continuous in $\mathrm{cl}\mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\{0\}$, we have

$$\gamma \equiv \sup_{\xi \in \mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\mathbb{B}_n(0,\delta/2)} |h(\xi)| < \infty.$$

If $x \in \mathrm{cl}\Omega \cap \mathrm{cl}\mathbb{B}_n(x_0,\delta/2)$, we have $|x-y| \geq \delta/2$ for all $y \in \Omega\setminus\mathbb{B}_n(x_0,\delta)$. Then we have

$$|h(x-y) - h(x_0-y)|\,|\varphi(y)| \leq 2\gamma\|\varphi\|_{L^\infty(\Omega)}$$

for almost all $y \in \Omega\setminus\mathbb{B}_n(x_0,\delta)$ and for all $x \in \mathrm{cl}\Omega \cap \mathrm{cl}\mathbb{B}_n(x_0,\delta/2)$. Then the dominated convergence theorem implies that

$$\lim_{x\to x_0}\int_{\Omega\setminus\mathbb{B}_n(x_0,\delta)} [h(x-y) - h(x_0-y)]\varphi(y)\,dy = 0,$$

and we have

$$\limsup_{x\to x_0} |\mathcal{P}[h,\varphi](x) - \mathcal{P}[h,\varphi](x_0)| \le \|\varphi\|_{L^\infty(\Omega)}\epsilon.$$

Since $\epsilon \in]0,+\infty[$ has been chosen arbitrarily, we obtain

$$\lim_{x\to x_0} \mathcal{P}[h,\varphi](x) - \mathcal{P}[h,\varphi](x_0) = 0,$$

and accordingly, $\mathcal{P}[h,\varphi]$ is continuous at the point x_0. □

Next we introduce the following.

Definition 3.5. Let $\lambda \in]0,+\infty[$. Let $R \in]0,+\infty[$. Then we denote by $A^1_\lambda(R)$ the set of functions $h \in C^1((\mathrm{cl}\mathbb{B}_n(0,R))\setminus\{0\})$ such that

$$h \in A^0_\lambda(R), \quad \frac{\partial h}{\partial x_j} \in A^0_{\lambda+1}(R) \quad \forall j \in \{1,\dots,n\},$$

and we set

$$\|h\|_{A^1_\lambda(R)} \equiv \|h\|_{A^0_\lambda(R)} + \sum_{j=1}^n \left\|\frac{\partial h}{\partial x_j}\right\|_{A^0_{\lambda+1}(R)} \qquad \forall h \in A^1_\lambda(R).$$

One can easily verify that $\left(A^1_\lambda(R), \|\cdot\|_{A^1_\lambda(R)}\right)$ is a Banach space. In the following proposition we consider the function $\mathcal{P}[h,\varphi]$ with (h,φ) in $A^1_\lambda(\mathrm{diam}\,(\Omega)) \times L^\infty(\Omega)$.

Proposition 3.6. *Let $\lambda \in]0,n-1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$. Then the following statements hold.*

(i) *If $(h,\varphi) \in A^1_\lambda(\mathrm{diam}\,(\Omega)) \times L^\infty(\Omega)$ and if $x \in \mathrm{cl}\Omega$, then the functions from Ω to $\mathbb{R}$ which take $y \in \Omega$ to $h(x-y)\varphi(y)$ and to $\frac{\partial h}{\partial x_j}(x-y)\varphi(y)$ for $j \in \{1,\dots,n\}$ are integrable.*

(ii) *If $(h,\varphi) \in A^1_\lambda(\mathrm{diam}\,(\Omega)) \times L^\infty(\Omega)$, then $\mathcal{P}[h,\varphi] \in C^1(\mathrm{cl}\Omega)$ and*

$$\frac{\partial}{\partial x_j}\mathcal{P}[h,\varphi] = \mathcal{P}[\frac{\partial h}{\partial x_j},\varphi] \quad \text{in } \mathrm{cl}\Omega. \tag{3.2}$$

Proof. Statement (i) is an immediate consequence of Proposition 3.4 applied to h, $\frac{\partial h}{\partial x_j}$.

We now consider statement (ii). By Proposition 3.4 (ii), $\mathcal{P}[h,\varphi]$ and $\mathcal{P}[\frac{\partial h}{\partial x_j},\varphi]$ are continuous in $\mathrm{cl}\Omega$ for all $j \in \{1,\dots,n\}$. Thus it suffices to show that $\frac{\partial}{\partial x_j}\mathcal{P}[h,\varphi]$ exists in Ω and that (3.2) holds in Ω. We proceed by a standard argument. Let $g \in C^\infty(\mathbb{R})$ be such that

$$g(t) = 0 \quad \forall t \in]-\infty,1], \quad g(t) = 1 \quad \forall t \in [2,+\infty[.$$

Then we set

$$g_\delta(t) = g(t/\delta) \quad \forall t \in \mathbb{R},$$

and

$$u_\delta(x) \equiv \int_\Omega g_\delta(|x-y|)h(x-y)\varphi(y)\,dy \quad \forall x \in \mathrm{cl}\Omega,$$

for all $\delta \in]0,+\infty[$. We also observe that the function which takes $(x,y) \in \mathrm{cl}\Omega \times \mathrm{cl}\Omega$ to $g_\delta(|x-y|)h(x-y)$ is of class C^1. We now show that $u_\delta \in C^1(\mathrm{cl}\Omega)$, by applying the classical theorem of differentiation for integrals depending on a parameter. Clearly,

$$|g_\delta(|x-y|)h(x-y)\varphi(y)| \le \|g\|_{L^\infty(\mathbb{R})}\left(\sup_{\mathrm{cl}\mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\mathbb{B}_n(0,\delta)}|h|\right)|\varphi(y)|, \tag{3.3}$$

for all $x \in \mathrm{cl}\Omega$ and for almost all $y \in \Omega$. Since $\varphi \in L^1(\Omega)$, inequality (3.3) and the continuity theorem for integrals depending on a parameter imply that u_δ is continuous in $\mathrm{cl}\Omega$. Then we have

$$\begin{aligned}&\left|\frac{\partial}{\partial x_j}\{g_\delta(|x-y|)h(x-y)\varphi(y)\}\right| \\ &\quad \le \left|g'_\delta(|x-y|)\frac{x_j-y_j}{|x-y|}h(x-y)\varphi(y)\right| + \left|g_\delta(|x-y|)\frac{\partial h}{\partial x_j}(x-y)\varphi(y)\right|,\end{aligned} \tag{3.4}$$

for all $x \in \mathrm{cl}\Omega$ and for almost all $y \in \Omega$. The functions h and $\frac{\partial h}{\partial x_j}$ are continuous in $\mathrm{cl}\mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\{0\}$. Hence, h and $\frac{\partial h}{\partial x_j}$ are bounded in $\mathrm{cl}\mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\mathbb{B}_n(0,\delta)$. Then the right hand side of (3.4) is less than or equal to

$$\begin{aligned}&\frac{1}{\delta}\|g'\|_{L^\infty(\mathbb{R})}\left(\sup_{\mathrm{cl}\mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\mathbb{B}_n(0,\delta)}|h|\right)|\varphi(y)| \\ &+\|g\|_{L^\infty(\mathbb{R})}\left(\sup_{\mathrm{cl}\mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\mathbb{B}_n(0,\delta)}\left|\frac{\partial h}{\partial x_j}\right|\right)|\varphi(y)|,\end{aligned} \tag{3.5}$$

for all $x \in \mathrm{cl}\Omega$ and for almost all $y \in \Omega$. Since $\varphi \in L^1(\Omega)$, inequalities (3.4), (3.5) and the differentiability theorem for integrals depending on a parameter imply that

$$\frac{\partial u_\delta}{\partial x_j}(x) = \int_\Omega \frac{\partial}{\partial x_j}\left[g_\delta(|x-y|)h(x-y)\right]\varphi(y)\,dy \quad \forall x \in \Omega,$$

and that $\frac{\partial u_\delta}{\partial x_j}$ has a continuous extension to $\mathrm{cl}\Omega$. Hence, $u_\delta \in C^1(\mathrm{cl}\Omega)$. In order to prove that $\mathcal{P}[h,\varphi]$ belongs to $C^1(\mathrm{cl}\Omega)$, it suffices to show that

$$\lim_{\delta\to 0} u_\delta = \mathcal{P}[h,\varphi] \text{ uniformly in } \mathrm{cl}\Omega, \tag{3.6}$$

$$\lim_{\delta\to 0}\frac{\partial u_\delta}{\partial x_j} = \mathcal{P}\left[\frac{\partial h}{\partial x_j},\varphi\right] \text{ uniformly in } \mathrm{cl}\Omega, \tag{3.7}$$

for all $j \in \{1,\ldots,n\}$. We first consider (3.6). Since $1-g_\delta(|x-y|)=0$ for $|x-y| \ge 2\delta$, we have

$$
\begin{aligned}
&|\mathcal{P}[h,\varphi](x)-u_\delta(x)| \\
&\quad = \left|\int_{\mathbb{B}_n(x,2\delta)\cap\Omega}(1-g_\delta(|x-y|))h(x-y)\varphi(y)\,dy\right| \\
&\quad \le (1+\|g\|_{L^\infty(\mathbb{R})})\|\varphi\|_{L^\infty(\Omega)}\int_{\mathbb{B}_n(x,2\delta)\cap\mathbb{B}_n(x,\mathrm{diam}\,(\Omega))}|h(x-y)|\,dy \\
&\quad = (1+\|g\|_{L^\infty(\mathbb{R})})\|\varphi\|_{L^\infty(\Omega)}\int_{\mathbb{B}_n(0,2\delta)\cap\mathbb{B}_n(0,\mathrm{diam}\,(\Omega))}|h(y)|\,dy \\
&\quad = (1+\|g\|_{L^\infty(\mathbb{R})})\|\varphi\|_{L^\infty(\Omega)} \\
&\qquad \times\left(\sup_{x\in\mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\{0\}}|h(x)|\,|x|^\lambda\right)\int_{\mathbb{B}_n(0,2\delta)}|x|^{-\lambda}\,dx \\
&\quad = (1+\|g\|_{L^\infty(\mathbb{R})})\|\varphi\|_{L^\infty(\Omega)}\|h\|_{A^0_\lambda(\mathrm{diam}\,(\Omega))}s_n\frac{(2\delta)^{n-\lambda}}{n-\lambda},
\end{aligned}
$$

for all $x\in \mathrm{cl}\Omega$ and for all $\delta\in]0,\mathrm{diam}\,(\Omega)/2]$. Hence, (3.6) holds.

We now turn to prove (3.7). Since the support of g'_δ is contained in $[\delta,2\delta]$, the same argument we have exploited to prove (3.6) implies that

$$
\begin{aligned}
&\left|\mathcal{P}\left[\frac{\partial h}{\partial x_j},\varphi\right](x)-\frac{\partial u_\delta}{\partial x_j}(x)\right| \\
&\quad \le \left|\int_\Omega(1-g_\delta(|x-y|))\frac{\partial h}{\partial x_j}(x-y)\varphi(y)\,dy\right| \\
&\qquad +\left|\int_\Omega\frac{1}{\delta}g'\left(\frac{|x-y|}{\delta}\right)\frac{x_j-y_j}{|x-y|}h(x-y)\varphi(y)\,dy\right| \\
&\quad \le (1+\|g\|_{L^\infty(\mathbb{R})})\|\varphi\|_{L^\infty(\Omega)} \\
&\qquad \times\left(\sup_{x\in\mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\{0\}}\left|\frac{\partial h}{\partial x_j}(x)\right|\,|x|^{\lambda+1}\right)s_n\frac{(2\delta)^{n-\lambda-1}}{n-\lambda-1} \\
&\qquad +\frac{1}{\delta}\|g'\|_{L^\infty(\mathbb{R})}\int_{\mathbb{B}_n(x,2\delta)\setminus\mathbb{B}_n(x,\delta)}|h(x-y)|\,dy\|\varphi\|_{L^\infty(\Omega)} \\
&\quad \le s_n(1+\|g\|_{L^\infty(\mathbb{R})}+\|g'\|_{L^\infty(\mathbb{R})})\|\varphi\|_{L^\infty(\Omega)} \\
&\qquad \times\Bigg\{\left(\sup_{x\in\mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\{0\}}\left|\frac{\partial h}{\partial x_j}(x)\right|\,|x|^{\lambda+1}\right)\frac{(2\delta)^{n-\lambda-1}}{n-\lambda-1} \\
&\qquad +\frac{1}{\delta}\left(\sup_{x\in\mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\{0\}}|h(x)|\,|x|^\lambda\right)\frac{(2\delta)^{n-\lambda}}{n-\lambda}\Bigg\}\quad\forall x\in\Omega,
\end{aligned}
$$

for all $\delta\in]0,\mathrm{diam}\,(\Omega)/2[$. Since $n-\lambda-1>0$, the limiting relation (3.7) follows. □

Then we present the following variant of the classical formula for the partial derivatives of a volume potential with a differentiable density.

Lemma 3.7. *Let $\lambda \in]0, n-1[$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. If $(h, \varphi) \in A^1_\lambda(\mathrm{diam}\,(\Omega)) \times C^1(\mathrm{cl}\Omega)$ and $j \in \{1, \dots, n\}$, then*

$$\frac{\partial}{\partial x_j}\mathcal{P}[h,\varphi](x) = \mathcal{P}\left[h, \frac{\partial \varphi}{\partial x_j}\right](x) - \int_{\partial\Omega} h(x-y)\varphi(y)(\nu_\Omega)_j(y)\, d\sigma_y \quad \forall x \in \Omega. \tag{3.8}$$

Proof. By the previous proposition and by standard computations, we have

$$\begin{aligned}\frac{\partial}{\partial x_j}\mathcal{P}[h,\varphi](x) &= \int_\Omega \frac{\partial h}{\partial x_j}(x-y)\varphi(y)\, dy \\ &= -\int_\Omega \frac{\partial}{\partial y_j}(h(x-y))\varphi(y)\, dy \\ &= -\int_\Omega \frac{\partial}{\partial y_j}\left(h(x-y)\varphi(y)\right)\, dy \\ &\quad + \int_\Omega h(x-y)\frac{\partial \varphi}{\partial y_j}(y)\, dy \quad \forall x \in \mathrm{cl}\Omega. \end{aligned} \tag{3.9}$$

Next we fix $x \in \Omega$ and we take $\epsilon_x \in]0, \mathrm{dist}\,(x, \partial\Omega)[$. Then $\mathrm{cl}\mathbb{B}_n(x,\epsilon) \subseteq \Omega$ and the set

$$\Omega_\epsilon \equiv \Omega \setminus \mathrm{cl}\mathbb{B}_n(x,\epsilon)$$

is of Lipschitz class for all $\epsilon \in]0, \epsilon_x[$. By the divergence theorem, we have

$$\begin{aligned}&\int_\Omega \frac{\partial}{\partial y_j}\left(h(x-y)\varphi(y)\right)\, dy \\ &= \int_{\Omega_\epsilon} \frac{\partial}{\partial y_j}\left(h(x-y)\varphi(y)\right)\, dy + \int_{\mathbb{B}_n(x,\epsilon)} \frac{\partial}{\partial y_j}\left(h(x-y)\varphi(y)\right)\, dy \\ &= \int_{\partial\Omega} h(x-y)\varphi(y)(\nu_\Omega)_j(y)\, d\sigma_y + \int_{\partial\mathbb{B}_n(x,\epsilon)} h(x-y)\varphi(y)\frac{x_j - y_j}{|x-y|}\, d\sigma_y \\ &\quad - \int_{\mathbb{B}_n(x,\epsilon)} \frac{\partial h}{\partial x_j}(x-y)\varphi(y)\, dy + \int_{\mathbb{B}_n(x,\epsilon)} h(x-y)\frac{\partial \varphi}{\partial y_j}(y)\, dy \end{aligned} \tag{3.10}$$

for all $\epsilon \in]0, \epsilon_x[$. Next we note that

$$\begin{aligned}&\left|\int_{\partial\mathbb{B}_n(x,\epsilon)} h(x-y)\varphi(y)\frac{x_j - y_j}{|x-y|}\, d\sigma_y\right| \\ &\quad \le \left(\sup_{x \in \mathbb{B}_n(0,\mathrm{diam}\,(\Omega))\setminus\{0\}} |h(x)|\,|x|^\lambda\right) \|\varphi\|_{L^\infty(\Omega)} \int_{\partial\mathbb{B}_n(0,\epsilon)} |y|^{-\lambda}\, d\sigma_y \\ &\quad = \|h\|_{A^0_\lambda(\mathrm{diam}\,(\Omega))} \|\varphi\|_{L^\infty(\Omega)} s_n \epsilon^{n-1-\lambda} \end{aligned} \tag{3.11}$$

for all $\epsilon \in]0, \epsilon_x[$. By Proposition 3.6 (i), the functions $\frac{\partial h}{\partial x_j}(x - \cdot)\varphi(\cdot)$ and $h(x - \cdot)\frac{\partial \varphi}{\partial x_j}(\cdot)$ are integrable in Ω, and accordingly

$$\begin{aligned}\lim_{\epsilon\to 0} \int_{\mathbb{B}_n(x,\epsilon)} \frac{\partial h}{\partial x_j}(x-y)\varphi(y)\, dy &= 0, \\ \lim_{\epsilon\to 0} \int_{\mathbb{B}_n(x,\epsilon)} h(x-y)\frac{\partial \varphi}{\partial y_j}(y)\, dy &= 0. \end{aligned} \tag{3.12}$$

By (3.11) and (3.12), we can take the limit as ϵ tends to 0 in (3.10) and deduce that

$$\int_{\Omega} \frac{\partial}{\partial y_j} \left(h(x-y)\varphi(y)\right) \, dy = \int_{\partial\Omega} h(x-y)\varphi(y)(\nu_{\Omega})_j(y) \, d\sigma_y.$$

Then equality (3.9) implies that formula (3.8) holds. □

4. Volume Potentials Corresponding to General Kernels in Roumieu Classes

In order to estimate the Roumieu norm of a volume potential in terms of a norm of the kernel and of a norm of the density, we introduce the following class of functions which are singular at the origin and analytic away from the origin.

Definition 4.1. Let δ_1, $\delta_2 \in]0,+\infty[$ with $\delta_1 < \delta_2$. Let $\lambda, \rho \in]0,+\infty[$. Then we set

$$H^{\lambda,\rho}(\delta_1,\delta_2) \equiv \left\{h \in A^1_\lambda(\delta_2)\colon h_{|\mathrm{cl}\mathbb{B}_n(0,\delta_2)\setminus\mathbb{B}_n(0,\delta_1)} \in C^0_{\omega,\rho}((\mathrm{cl}\mathbb{B}_n(0,\delta_2))\setminus\mathbb{B}_n(0,\delta_1))\right\},$$

and we set

$$\|h\|_{H^{\lambda,\rho}(\delta_1,\delta_2)} \equiv \|h\|_{A^1_\lambda(\delta_2)} + \|h\|_{C^0_{\omega,\rho}(\mathrm{cl}\mathbb{B}_n(0,\delta_2)\setminus\mathbb{B}_n(0,\delta_1))} \quad \forall h \in H^{\lambda,\rho}(\delta_1,\delta_2).$$

One can readily verify that $(H^{\lambda,\rho}(\delta_1,\delta_2), \|\cdot\|_{H^{\lambda,\rho}(\delta_1,\delta_2)})$ is a Banach space. Then we can prove the following.

Proposition 4.2. *Let $\rho \in]0,+\infty[$, $\lambda \in]0,n-1[$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let Ω_1 be a nonempty open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega_1 \subseteq \Omega$. Let*

$$\delta^* \equiv \mathrm{diam}(\Omega), \quad \delta_* \equiv \min\left\{|t-s|\colon t \in \mathrm{cl}\Omega_1, s \in \partial\Omega\right\}. \tag{4.1}$$

Then the restriction of $\mathcal{P}[h,\varphi]$ to $\mathrm{cl}\Omega_1$ belongs to $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1)$ for all $(h,\varphi) \in H^{\lambda,\rho}(\delta_*,\delta^*) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega)$.

Moreover, the map from $H^{\lambda,\rho}(\delta_*,\delta^*) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega)$ to $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1)$ which takes (h,φ) to $\mathcal{P}[h,\varphi]_{|\mathrm{cl}\Omega_1}$ is bilinear and continuous.

Proof. We first prove that if $m \in \mathbb{N}\setminus\{0\}$ and if $(h,\varphi) \in H^{\lambda,\rho}(\delta_*,\delta^*) \times C^m(\mathrm{cl}\Omega)$, then $\mathcal{P}[h,\varphi]_{|\mathrm{cl}\Omega_1} \in C^m(\mathrm{cl}\Omega_1)$ and

$$\begin{aligned}\partial^\beta\mathcal{P}[h,\varphi](x) &= \mathcal{P}[h,\partial^\beta\varphi](x) \\ &\quad - \sum_{k=1}^{n}\sum_{l_k=0}^{\beta_k-1} \partial^{\beta_n}_{x_n}\dots\partial^{\beta_{k+1}}_{x_{k+1}}\partial^{l_k}_{x_k}\left\{\int_{\partial\Omega} h(x-y)\right. \\ &\quad \left.\times\,(\partial^{\beta_k-1-l_k}_{y_k}\partial^{\beta_{k-1}}_{y_{k-1}}\dots\partial^{\beta_1}_{y_1}\varphi(y))(\nu_\Omega)_k(y)\,d\sigma_y\right\},\end{aligned} \tag{4.2}$$

for all $x \in \mathrm{cl}\Omega_1$ and for all $\beta \in \mathbb{N}^n$ such that $|\beta| \le m$, where we understand that $\sum_{l_k=0}^{\beta_k-1}$ is omitted if $\beta_k = 0$. We proceed by induction on m. If $m=1$, then the statement follows by Lemma 3.7. Next we assume that the statement holds for m and we prove it for $m+1$. Let $(h,\varphi) \in H^{\lambda,\rho}(\delta_*,\delta^*) \times C^{m+1}(\mathrm{cl}\Omega)$. By the inductive assumption, we have $\mathcal{P}[h, \frac{\partial \varphi}{\partial x_j}]_{|\mathrm{cl}\Omega_1} \in C^m(\mathrm{cl}\Omega_1)$ for all $j \in \{1,\ldots,n\}$. Since $h_{|\mathrm{cl}\mathbb{B}_n(0,\delta^*)\setminus\mathbb{B}_n(0,\delta_*)} \in C^m(\mathrm{cl}\mathbb{B}_n(0,\delta^*)\setminus\mathbb{B}_n(0,\delta_*))$ and φ, $(\nu_\Omega)_j \in C^0(\partial\Omega)$, the classical differentiability theorem for integrals depending on a parameter implies that the second term in the right hand side of formula (3.8) defines a function of class $C^m(\mathrm{cl}\Omega_1)$. Then formula (3.8) implies that $\frac{\partial}{\partial x_j}\mathcal{P}[h,\varphi]_{|\mathrm{cl}\Omega_1}$ belongs to $C^m(\mathrm{cl}\Omega_1)$. Hence, $\mathcal{P}[h,\varphi]_{|\mathrm{cl}\Omega_1} \in C^{m+1}(\mathrm{cl}\Omega_1)$. Next we prove the formula for the derivatives by following the lines of the corresponding argument of [20, p. 856]. We first prove the formula for $\partial^\beta = \partial^{\beta_j}_{x_j}$ by finite induction on the length of β_j. Then we prove the formula for $\partial^\beta = \partial^{\beta_1}_{x_1}\ldots\partial^{\beta_j}_{x_j}$ by finite induction on $j \in \{1,\ldots,n\}$. As a consequence, the formula holds for $|\beta| \leq m+1$. For the details, we refer to [20, p. 856].

If $(h,\varphi) \in H^{\lambda,\rho}(\delta_*,\delta^*) \times C^\infty(\mathrm{cl}\Omega)$, then by applying the above statement for all $m \in \mathbb{N}\setminus\{0\}$ we deduce that $\mathcal{P}[h,\varphi]_{|\mathrm{cl}\Omega_1}$ belongs to $C^\infty(\mathrm{cl}\Omega_1)$ and that formula (4.1) holds for all order derivatives.

We now assume that $(h,\varphi) \in H^{\lambda,\rho}(\delta_*,\delta^*) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega)$ and we turn to estimate the supnorm in $\mathrm{cl}\Omega_1$ of the double summation in the right hand side of (4.2), which we denote by I. To do so, we abbreviate by $I(k,l_k)$ the (k,l_k)th term in the sum I, and we estimate the supremum of $I(k,l_k)$ in $\mathrm{cl}\Omega_1$. We can clearly assume that $\beta_k > 0$. Then we have

$$\begin{aligned}\sup_{\mathrm{cl}\Omega_1}|I(k,l_k)| &= \sup_{x\in\mathrm{cl}\Omega_1}\Bigg|\partial^{\beta_n}_{x_n}\ldots\partial^{\beta_{k+1}}_{x_{k+1}}\partial^{l_k}_{x_k}\Bigg\{\int_{\partial\Omega} h(x-y)\\ &\qquad \times\partial^{\beta_k-1-l_k}_{y_k}\partial^{\beta_{k-1}}_{y_{k-1}}\ldots\partial^{\beta_1}_{y_1}\varphi(y)(\nu_\Omega)_k(y)\,d\sigma_y\Bigg\}\Bigg|\\ &\leq \int_{\partial\Omega}\sup_{\xi\in A}\left|\partial^{\beta_n}_{\xi_n}\ldots\partial^{\beta_{k+1}}_{\xi_{k+1}}\partial^{l_k}_{\xi_k}h(\xi)\right|\left|\partial^{\beta_k-1-l_k}_{y_k}\partial^{\beta_{k-1}}_{y_{k-1}}\ldots\partial^{\beta_1}_{y_1}\varphi(y)\right|\,d\sigma_y,\end{aligned}$$

where $A \equiv \{x-y\colon x\in\mathrm{cl}\Omega_1,\ y\in\partial\Omega\}$. Since $h \in H^{\lambda,\rho}(\delta_*,\delta^*)$, we have

$$\sup_{\xi\in A}\left|\partial^{\beta_n}_{\xi_n}\ldots\partial^{\beta_{k+1}}_{\xi_{k+1}}\partial^{l_k}_{\xi_k}h(\xi)\right| \leq \|h\|_{H^{\lambda,\rho}(\delta_*,\delta^*)}\frac{(\beta_n+\cdots+\beta_{k+1}+l_k)!}{\rho^{\beta_n+\cdots+\beta_{k+1}+l_k}}.$$

Moreover,

$$\left|\partial^{\beta_k-1-l_k}_{y_k}\partial^{\beta_{k-1}}_{y_{k-1}}\ldots\partial^{\beta_1}_{y_1}\varphi(y)\right| \leq \|\varphi\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)}\frac{(\beta_1+\cdots+\beta_{k-1}+\beta_k-1-l_k)!}{\rho^{\beta_1+\cdots+\beta_{k-1}+\beta_k-1-l_k}},$$

for all $y \in \mathrm{cl}\Omega$. Then we have

$$\begin{aligned}\sup_{\mathrm{cl}\Omega_1}|I(k,l_k)| &\leq m_{n-1}(\partial\Omega)\|h\|_{H^{\lambda,\rho}(\delta_*,\delta^*)}\|\varphi\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)}\\ &\qquad\times\frac{(\beta_n+\cdots+\beta_{k+1}+l_k)!(\beta_1+\cdots+\beta_{k-1}+\beta_k-1-l_k)!}{\rho^{|\beta|-1}},\end{aligned} \tag{4.3}$$

where $m_{n-1}(\partial\Omega)$ denotes the $(n-1)$ dimensional Lebesgue measure of $\partial\Omega$. Next we note that

$$m_1!m_2! \leq (m_1+m_2)!,$$

for all m_1, $m_2 \in \mathbb{N}$. Indeed,

$$1 \le \binom{m_1+m_2}{m_1} = \frac{(m_1+m_2)!}{m_1!m_2!}.$$

Then (4.3) implies that

$$\sup_{\mathrm{cl}\Omega_1} |I(k,l_k)| \le m_{n-1}(\partial\Omega)\|h\|_{H^{\lambda,\rho}(\delta_*,\delta^*)}\|\varphi\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)}\frac{(|\beta|-1)!}{\rho^{|\beta|-1}}.$$

Hence,

$$\sup_{\mathrm{cl}\Omega_1} |I| \le n\rho m_{n-1}(\partial\Omega)\|h\|_{H^{\lambda,\rho}(\delta_*,\delta^*)}\|\varphi\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)}\frac{|\beta|!}{\rho^{|\beta|}}. \tag{4.4}$$

By Proposition 3.4 (iii), we have

$$\|\mathcal{P}[h,\partial^\beta\varphi]\|_{L^\infty(\Omega)} \le s_n\frac{(\mathrm{diam}\,(\Omega))^{n-\lambda}}{n-\lambda}\|h\|_{A^0_\lambda(\delta^*)}\|\partial^\beta\varphi\|_{L^\infty(\Omega)}. \tag{4.5}$$

Then equality (4.2) and inequalities (4.4) and (4.5) imply that there exists $C \in]0,+\infty[$ such that

$$\|\partial^\beta\mathcal{P}[h,\varphi]_{|\Omega_1}\|_{L^\infty(\Omega_1)} \le C\|h\|_{H^{\lambda,\rho}(\delta_*,\delta^*)}\|\varphi\|_{C^0_{\omega,\rho}(\mathrm{cl}\Omega)}\frac{|\beta|!}{\rho^{|\beta|}} \quad \forall\beta\in\mathbb{N}^n,$$

for all $(h,\varphi) \in H^{\lambda,\rho}(\delta_*,\delta^*) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega)$. □

Proposition 4.2 can be applied in case h is replaced by a fundamental solution of a second order elliptic operator. As shown in John [15], if S is a fundamental solution of a second order elliptic operator and if $\delta \in]0,+\infty[$, then

$$\sup_{x\in\mathbb{B}_n(0,\delta)\setminus\{0\}} |S(x)|\,|x|^{n-2} < +\infty, \qquad \sup_{x\in\mathbb{B}_n(0,\delta)\setminus\{0\}} \left|\frac{\partial S}{\partial x_j}(x)\right| |x|^{n-1} < +\infty,$$

for all $j \in \{1,\dots,n\}$, if $n-2>0$, and

$$\sup_{x\in\mathbb{B}_n(0,\delta)\setminus\{0\}} |S(x)|\,|x|^{1/2} < +\infty, \qquad \sup_{x\in\mathbb{B}_n(0,\delta)\setminus\{0\}} \left|\frac{\partial S}{\partial x_j}(x)\right| |x|^{3/2} < +\infty,$$

for all $j \in \{1,\dots,n\}$, if $n-2=0$. Moreover, S is analytic in $\mathbb{R}^n\setminus\{0\}$, and the classical Cauchy inequalities for the derivatives of S on a compact set imply that $S \in C^0_{\omega,\rho}(\mathrm{cl}\mathbb{B}_n(0,\delta_2)\setminus\mathbb{B}_n(0,\delta_1))$ for all δ_1, $\delta_2 \in]0,+\infty[$ such that $\delta_1 < \delta_2$ and for $\rho \in]0,+\infty[$ sufficiently small (cf. e.g., John [14, p. 65]). Hence,

$$S \in H^{\max\{n-2,\frac{1}{2}\},\rho}(\delta_1,\delta_2).$$

Thus if we plan to apply Proposition 4.2 with h replaced by a fundamental solution of a second order elliptic operator, we can choose $\lambda = \max\{n-2,\frac{1}{2}\}$.

5. A Real Analyticity Result for Volume Potentials Corresponding to Analytic Families of Fundamental Solutions

We now exploit Proposition 4.2 of the previous section in order to analyze the analytic dependence of the volume potentials of (1.2) upon (κ, φ) both under the assumption (1.1) and under the following assumption.

$$\begin{aligned}&\text{Let } \kappa_0 \in \mathcal{O}. \text{ Let } \delta_1, \delta_2 \in]0, +\infty[,\ \delta_1 < \delta_2. \text{ Then}\\ &\text{there exist } \rho \in]0, +\infty[\text{ and an open neighborhood } V_{\kappa_0} \text{ of } \kappa_0 \text{ in } \mathcal{O}\\ &\text{such that the map from } V_{\kappa_0} \text{ to } H^{\max\{n-2,\frac{1}{2}\},\rho}(\delta_1, \delta_2), \text{ which takes}\\ &\kappa \text{ to } S(\cdot, \kappa)_{|\mathrm{cl}\mathbb{B}_n(0,\delta_2)\setminus\{0\}} \text{ is real analytic.}\end{aligned} \tag{5.1}$$

Then we are ready to deduce the validity of the following.

Theorem 5.1. *Let $n \in \mathbb{N}\setminus\{0, 1\}$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let Ω_1 be an open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega_1 \subseteq \Omega$. Let assumption (1.1) hold. Let assumption (5.1) hold with $\delta_1 = \delta_*$, $\delta_2 = \delta^*$ [see (4.1)]. Then the map from $V_{\kappa_0} \times C^0_{\omega,\rho}(\mathrm{cl}\Omega)$ to $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1)$ which takes (κ, φ) to $\mathcal{P}_\kappa[\varphi]_{|\mathrm{cl}\Omega_1}$ is real analytic [see (1.2)].*

Proof. Let $\delta_1 \equiv \delta_*$, $\delta_2 \equiv \delta^*$ be as in (4.1). Let ρ, V_{κ_0} be as in (5.1). Then assumption (1.1) implies that the map from V_{κ_0} to $H^{\max\{n-2,\frac{1}{2}\},\rho}(\delta_*, \delta^*)$ which takes κ to $S(\cdot, \kappa)_{|(\mathrm{cl}\mathbb{B}_n(0,\delta_2))\setminus\{0\}}$ is real analytic.

By Proposition 4.2, the map from $H^{\max\{n-2,\frac{1}{2}\},\rho}(\delta_*, \delta^*) \times C^0_{\omega,\rho}(\mathrm{cl}\Omega)$ to $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1)$ which takes (h, φ) to $\mathcal{P}[h, \varphi]_{|\mathrm{cl}\Omega_1}$ is bilinear and continuous. Since a composition of real analytic maps is real analytic, the map from $V_{\kappa_0} \times C^0_{\omega,\rho}(\mathrm{cl}\Omega)$ to $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1)$ which takes (κ, φ) to

$$\mathcal{P}_\kappa[\varphi]_{|\mathrm{cl}\Omega_1} = \mathcal{P}[S(\cdot, \kappa), \varphi]_{|\mathrm{cl}\Omega_1}$$

is real analytic. □

If the Banach space $\mathcal{K}$ of assumption (1.1) coincides with $\mathbb{R}^{n_1}$ for some $n_1 \in \mathbb{N}\setminus\{0\}$, then the condition in (5.1) can be relaxed and replaced by the following.

$$\begin{aligned}&\text{Let } \kappa_0 \in \mathcal{O}. \text{ Let } \delta_2 \in]0, +\infty[. \text{ Then there exists an open}\\ &\text{neighborhood } V_{\kappa_0} \text{ of } \kappa_0 \text{ in } \mathcal{O} \text{ such that the map from } V_{\kappa_0}\\ &\text{to } A^1_{\max\{n-2,\frac{1}{2}\}}(\delta_2) \text{ which takes } \kappa \text{ to } S(\cdot, \kappa)_{|\mathrm{cl}\mathbb{B}_n(0,\delta_2)\setminus\{0\}}\\ &\text{is real analytic.}\end{aligned} \tag{5.2}$$

Indeed, in such a case, the real analyticity of the map which takes κ to $S(\cdot, \kappa)_{|\mathrm{cl}\mathbb{B}_n(0,\delta_2)\setminus\mathbb{B}_n(0,\delta_1)}$ from V_{κ_0} to $C^0_{\omega,\rho}((\mathrm{cl}\mathbb{B}_n(0, \delta_2))\setminus\mathbb{B}_n(0, \delta_1))$, for some $\rho \in]0, +\infty[$, is guaranteed by Proposition A.1 of the Appendix as long as V_{κ_0} is bounded. Then we have the following.

Theorem 5.2. *Let $n \in \mathbb{N}\setminus\{0, 1\}$, $n_1 \in \mathbb{N}\setminus\{0\}$. Let assumption (1.1) hold with $\mathcal{K} = \mathbb{R}^{n_1}$. Let $\kappa_0 \in \mathcal{O}$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Assume that condition (5.2) holds with $\delta_2 = \mathrm{diam}(\Omega)$, and that V_{κ_0} is bounded, and that $\mathrm{cl}V_{\kappa_0} \subseteq \mathcal{O}$. Let Ω_1 be an open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega_1 \subseteq$*

Journal: 20 Article No.: 2236 □ TYPESET □ DISK □ LE □ CP Disp.:2015/4/21 Pages: 23

Ω. Then there exists $\rho \in]0,+\infty[$ such that the map from $V_{\kappa_0} \times C^0_{\omega,\rho}(\mathrm{cl}\Omega)$ to $C^0_{\omega,\rho}(\mathrm{cl}\Omega_1)$ which takes (κ,φ) to $\mathcal{P}_\kappa[\varphi]_{|\mathrm{cl}\Omega_1}$ is real analytic [see (1.2)].

Proof. Let δ_*, δ^* be as in (4.1). Let $W \equiv \mathbb{B}_n(0,\delta^*)\backslash\mathrm{cl}\mathbb{B}_n(0,\delta_*)$. Since S is real analytic on $\mathcal{O}\times(\mathbb{R}^n\backslash\{0\})$ and $\mathrm{cl}(V_{\kappa_0}\times W)$ is a compact subset of $\mathcal{O}\times(\mathbb{R}^n\backslash\{0\})$, there exists $\rho_1 \in]0,+\infty[$ such that $S_{|\mathrm{cl}(V_{\kappa_0}\times W)} \in C^0_{\omega,\rho_1}(\mathrm{cl}(V_{\kappa_0}\times W))$. Let $\rho \in]0,\rho_1[$. Then by Proposition A.1 of the Appendix, the map from V_{κ_0} to $C^0_{\omega,\rho}(\mathrm{cl}W)$ which takes κ to $S(\cdot,\kappa)_{|\mathrm{cl}W}$ is real analytic. Then by taking $\delta_1 = \delta_*$, $\delta_2 = \delta^*$, our assumptions imply that condition (5.1) holds, and thus Theorem 5.1 implies the validity of the statement. □

6. Applications

6.1. A Family of Fundamental Solutions for Second Order Elliptic Partial Differential Operators

In the following Theorem 6.1 we introduce a family of fundamental solutions for second order elliptic partial differential operators. For the construction of such a family we refer the reader to [7, Thm. 5.5], where the case of quaternion coefficient partial differential operators is considered (see also [5] for the case of real coefficients). Then the validity of Theorem 6.1 can be deduced by the embedding of $\mathbb{C}$ in the quaternion algebra $\mathbb{H}$, by the basic multiplication rules of the quaternion units, and by standard properties of real analytic functions.

Theorem 6.1. *Let $n \in \mathbb{N}\backslash\{0,1\}$. There exist a real analytic function A from $\partial\mathbb{B}_n(0,1)\times\mathbb{R}\times\mathcal{E}$ to $\mathbb{C}$, and two real analytic functions B and C from $\mathbb{R}^n\times\mathcal{E}$ to $\mathbb{C}$ such that the function $E(\cdot,\mathbf{a})$ from $\mathbb{R}^n\backslash\{0\}$ to $\mathbb{C}$, defined by*

$$E(x,\mathbf{a}) \equiv |x|^{2-n}A(x/|x|,|x|,\mathbf{a}) + B(x,\mathbf{a})\log|x| + C(x,\mathbf{a}) \quad \forall x \in \mathbb{R}^n\backslash\{0\},$$

is a fundamental solution of $P[\mathbf{a},D]$ for all $\mathbf{a}\in\mathcal{E}$. Moreover, the functions B and C are identically equal to 0 if n is odd.

Then one can verify that the function $S \equiv E$ of Theorem 6.1 satisfies condition (1.1) with $\mathcal{K} = \mathbb{C}^{N_{2,n}}$, and $\mathcal{O} = \mathcal{E}$, and $\mathbf{a}(\cdot)$ equal to the identity function from $\mathcal{E}$ to itself. We now show that $S \equiv E$ satisfies also the condition in (5.2). To do so we prove the following.

Proposition 6.2. *Let $n \in \mathbb{N}\backslash\{0,1\}$. Let $\mathbf{a}_0 \in \mathcal{E}$. Let $V_{\mathbf{a_0}}$ be an open bounded neighborhood of $\mathbf{a}_0$ in $\mathcal{E}$ such that $\mathrm{cl}V_{\mathbf{a_0}} \subseteq \mathcal{E}$. Let $\delta_2 \in]0,+\infty[$. Then the map from $V_{\mathbf{a_0}}$ to $A^1_{\max\{n-2,\frac{1}{2}\}}(\delta_2)$ which takes $\mathbf{a}$ to $E(\cdot,\mathbf{a})_{|(\mathrm{cl}\mathbb{B}_n(0,\delta_2))\backslash\{0\}}$ is real analytic.*

Proof. Let A, B, and C be as in Theorem 6.1. Then there exist an open neighborhood $W_{\partial\mathbb{B}_n(0,1)}$ of $\partial\mathbb{B}_n(0,1)$ in $\mathbb{R}^n$ and a real analytic function $\tilde{A}$ from $W_{\partial\mathbb{B}_n(0,1)}\times\mathbb{R}\times\mathcal{E}$ such that $\tilde{A}_{|\partial\mathbb{B}_n(0,1)\times\mathbb{R}\times\mathcal{E}} = A$ (cf. [7, §4]). Let $V_{\partial\mathbb{B}_n(0,1)}$ be an open bounded neighborhood of $\partial\mathbb{B}_n(0,1)$ with $\mathrm{cl}V_{\partial\mathbb{B}_n(0,1)} \subseteq W_{\partial\mathbb{B}_n(0,1)}$. By the classical Cauchy inequalities for the derivatives of analytic functions, there exists $\rho' \in]0,+\infty[$ such that $\tilde{A}_{|\mathrm{cl}V_{\partial\mathbb{B}_n(0,1)}\times[-\delta_2,\delta_2]\times\mathrm{cl}V_{\mathbf{a_0}}} \in C^0_{\omega,\rho'}(\mathrm{cl}V_{\partial\mathbb{B}_n(0,1)}\times[-\delta_2,\delta_2]\times\mathrm{cl}V_{\mathbf{a_0}})$ (cf. e.g., John [14, p. 65]). Let $\rho \in]0,\rho'[$.

Then Proposition A.1 of the Appendix implies that the map from $V_{\mathbf{a_0}}$ to $C^0_{\omega,\rho}(\mathrm{cl}V_{\partial\mathbb{B}_n(0,1)}\times[-\delta_2,\delta_2])$ which takes $\mathbf{a}$ to $\tilde{A}(\cdot,\cdot,\mathbf{a})_{|\mathrm{cl}V_{\partial\mathbb{B}_n(0,1)}\times[-\delta_2,\delta_2]}$ is real analytic. Then we observe that the map from $C^0_{\omega,\rho}(\mathrm{cl}V_{\partial\mathbb{B}_n(0,1)}\times[-\delta_2,\delta_2])$ to $A^1_{\max\{n-2,\frac{1}{2}\}}(\delta_2)$ which takes a function F to the function $|x|^{2-n}F(x/|x|,|x|)$ of $x\in(\mathrm{cl}\mathbb{B}_n(0,\delta_2))\setminus\{0\}$ is linear and continuous. As a consequence, we conclude that the map from $V_{\mathbf{a_0}}$ to $A^1_{\max\{n-2,\frac{1}{2}\}}(\delta_2)$ which takes $\mathbf{a}$ to the function $|x|^{2-n}A(x/|x|,|x|,\mathbf{a})=|x|^{2-n}\tilde{A}(x/|x|,|x|,\mathbf{a})$ of $x\in(\mathrm{cl}\mathbb{B}_n(0,\delta_2))\setminus\{0\}$ is real analytic. Similarly one shows that the maps from $V_{\mathbf{a_0}}$ to $A^1_{\max\{n-2,\frac{1}{2}\}}(\delta_2)$ which take $\mathbf{a}$ to the function $B(x,\mathbf{a})\log|x|$ of $x\in(\mathrm{cl}\mathbb{B}_n(0,\delta_2))\setminus\{0\}$ and to the function $C(x,\mathbf{a})$ of $x\in(\mathrm{cl}\mathbb{B}_n(0,\delta_2))\setminus\{0\}$ are real analytic. Now the validity of the proposition follows by Theorem 6.1 and by standard calculus in Banach spaces. □

6.2. Families of Fundamental Solutions for the Helmholtz Operator

We now consider two specific families of fundamental solutions for the Helmholtz operator $\Delta+\lambda$ with $\lambda\in\mathbb{C}\setminus\{0\}$. Such families have been exploited in [22] to study a singularly perturbed Neumann eigenvalue problem for the Laplace operator. As we shall see, the first family consists of functions which can be extended to entire holomorphic functions of the variable $\lambda\in\mathbb{C}$ when the spatial variable x is fixed. Instead, the second family consists of fundamental solutions which satisfy a Bohr-Sommerfeld outgoing radiation condition corresponding to a suitable choice of a square root of λ.

We start by introducing the holomorphic family, which we denote by $S^\sharp_{h,n}$. Here the subscript h stands for 'holomorphic'. To do so, we need the following notation. We denote by $J^\sharp_\nu$ the function from $\mathbb{C}$ to $\mathbb{C}$ defined by

$$J^\sharp_\nu(z)\equiv\sum_{j=0}^{\infty}\frac{(-1)^j z^j(1/2)^{2j}(1/2)^\nu}{\Gamma(j+1)\Gamma(j+\nu+1)}\quad\forall z\in\mathbb{C},$$

if $\nu\in\mathbb{C}\setminus\{-j\colon j\in\mathbb{N}\setminus\{0\}\}$, and by

$$J^\sharp_\nu(z)\equiv\sum_{j=-\nu}^{\infty}\frac{(-1)^j z^j(1/2)^{2j}(1/2)^\nu}{\Gamma(j+1)\Gamma(j+\nu+1)}\quad\forall z\in\mathbb{C},$$

if $\nu\in\{-j\colon j\in\mathbb{N}\setminus\{0\}\}$. Then $J^\sharp_\nu(z)$ is well known to be an entire function of $z\in\mathbb{C}$ for all fixed $\nu\in\mathbb{C}$ and $z^\nu J^\sharp_\nu(z^2)$ is the Bessel function of the first kind of index ν. Moreover, if $\nu\in\mathbb{N}$, then we set

$$N^\sharp_\nu(z)\equiv-\frac{2^\nu}{\pi}\sum_{0\le j\le\nu-1}\frac{(\nu-j-1)!}{j!}z^j(1/2)^{2j}$$
$$-\frac{z^\nu}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j z^j(1/2)^{2j}(1/2)^\nu}{j!(\nu+j)!}\left(2\sum_{0<l\le j}\frac{1}{l}+\sum_{j<l\le j+\nu}\frac{1}{l}\right)\quad\forall z\in\mathbb{C}.$$

As one can see, also $N^\sharp_\nu(z)$ is an entire holomorphic function of the variable $z\in\mathbb{C}$ for all $\nu\in\mathbb{N}$, and $\frac{2}{\pi}(\log z-\log 2+\gamma)J_\nu(z)-z^\nu N^\sharp_\nu(z^2)$ coincides with the Bessel function of the second kind and index ν for all $z\in\mathbb{C}\setminus]-\infty,0]$. Here log is the principal branch of the logarithm and γ is the Euler-Mascheroni constant. Then we have the following proposition (for a proof, see e.g., [22]).

Proposition 6.3. *Let* $n \in \mathbb{N}\setminus\{0,1\}$. *Let*

$$b_n \equiv \begin{cases} \pi^{1-(n/2)}2^{-1-(n/2)} & \text{if } n \text{ is even,} \\ (-1)^{\frac{n-1}{2}}\pi^{1-(n/2)}2^{-1-(n/2)} & \text{if } n \text{ is odd.} \end{cases} \tag{6.1}$$

Let $S^\sharp_{h,n}(\cdot,\cdot)$ *be the map from* $(\mathbb{R}^n\setminus\{0\})\times\mathbb{C}$ *to* $\mathbb{C}$ *defined by*

$$S^\sharp_{h,n}(x,\lambda) \equiv \begin{cases} b_n\Big\{\frac{2}{\pi}\lambda^{\frac{n-2}{2}}J^\sharp_{\frac{n-2}{2}}(\lambda|x|^2)\log|x| \\ \quad +|x|^{2-n}N^\sharp_{\frac{n-2}{2}}(\lambda|x|^2)\Big\} & \text{if } n \text{ is even,} \\ b_n|x|^{2-n}J^\sharp_{-\frac{n-2}{2}}(\lambda|x|^2) & \text{if } n \text{ is odd,} \end{cases}$$

for all $(x,\lambda) \in (\mathbb{R}^n\setminus\{0\})\times\mathbb{C}$. *Then* $S^\sharp_{h,n}(\cdot,\lambda)$ *is a fundamental solution of* $\Delta+\lambda$ *for all* $\lambda\in\mathbb{C}$. *Moreover, the function* $S^\sharp_{h,n}(x,\cdot)$ *is holomorphic in* $\mathbb{C}$ *for all fixed* $x\in\mathbb{R}^n\setminus\{0\}$.

Now, one readily verifies that the function from $(\mathbb{R}^n\setminus\{0\})\times\mathbb{C}$ to $\mathbb{C}$ which takes (x,λ) to $S^\sharp_{h,n}(x,\lambda)$ is real analytic. Accordingly, the function $S\equiv S^\sharp_{h,n}$ satisfies condition (1.1) with $\mathcal{K}=\mathcal{O}=\mathbb{C}$, and $\mathbf{a}(\cdot)\equiv(a_\alpha(\cdot))_{|\alpha|\leq 2}$ defined by

$$a_\alpha(\lambda) \equiv \begin{cases} 1 & \text{if } \alpha = 2e_j \text{ with } j\in\{1,\dots,n\}, \\ 0 & \text{if } |\alpha|=1 \text{ or if } \alpha=e_j+e_k \text{ with } j,k\in\{1,\dots,n\},\ j\neq k, \\ \lambda & \text{if } |\alpha|=0 \end{cases} \tag{6.2}$$

for all $\lambda\in\mathbb{C}$. Here $\{e_1,\dots,e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. We now show that $S^\sharp_{h,n}$ verifies also the condition in (5.2). To do so we prove the following.

Proposition 6.4. *Let* $n\in\mathbb{N}\setminus\{0,1\}$. *Let* $\lambda_0\in\mathbb{C}$. *Let* V_{λ_0} *be an open bounded neighborhood of* λ_0 *in* $\mathbb{C}$. *Let* $\delta_2\in]0,+\infty[$. *Then the map from* V_{λ_0} *to the space* $A^1_{\max\{n-2,\frac{1}{2}\}}(\delta_2)$ *which takes* λ *to* $S^\sharp_{h,n}(\cdot,\lambda)_{|(\mathrm{cl}\mathbb{B}_n(0,\delta_2))\setminus\{0\}}$ *is real analytic.*

Proof. Assume that n is even. Then, by the classical Cauchy inequalities for real analytic functions, there exists $\rho'\in]0,+\infty[$ such that the function from $\mathrm{cl}V_{\lambda_0}\times\mathrm{cl}\mathbb{B}_n(0,\delta_2)$ to $\mathbb{C}$ which takes (λ,x) to $\lambda^{\frac{n-2}{2}}J^\sharp_{\frac{n-2}{2}}(\lambda|x|^2)$ belongs to $C^0_{\omega,\rho'}(\mathrm{cl}V_{\lambda_0}\times\mathrm{cl}\mathbb{B}_n(0,\delta_2))$ (cf. e.g., John [14, p. 65]). Then let $\rho\in]0,\rho'[$. By Proposition A.1 of the Appendix, the map from V_{λ_0} to $C^0_{\omega,\rho}(\mathrm{cl}\mathbb{B}_n(0,\delta_2))$ which takes λ to the function $\lambda^{\frac{n-2}{2}}J^\sharp_{\frac{n-2}{2}}(\lambda|x|^2)$ of $x\in\mathrm{cl}\mathbb{B}_n(0,\delta_2)$ is real analytic. We also observe that the map from $C^0_{\omega,\rho}(\mathrm{cl}\mathbb{B}_n(0,\delta_2))$ to $A^1_{\max\{n-2,\frac{1}{2}\}}(\delta_2)$ which takes a function F to the function $F(x)\log|x|$ of $x\in(\mathrm{cl}\mathbb{B}_n(0,\delta_2))\setminus\{0\}$ is linear and continuous. Hence we conclude that the map from V_{λ_0} to $A^1_{\max\{n-2,\frac{1}{2}\}}(\delta_2)$ which takes λ to the function $\lambda^{\frac{n-2}{2}}J^\sharp_{\frac{n-2}{2}}(\lambda|x|^2)\log|x|$ of $x\in(\mathrm{cl}\mathbb{B}_n(0,\delta_2))\setminus\{0\}$ is real analytic. Similarly one can show that the map from V_{λ_0} to $A^1_{\max\{n-2,\frac{1}{2}\}}(\delta_2)$ which takes λ to the function $|x|^{2-n}N^\sharp_{\frac{n-2}{2}}(\lambda|x|^2)$ of $x \in (\mathrm{cl}\mathbb{B}_n(0,\delta_2))\setminus\{0\}$ is real analytic. Now the validity of the proposition for n even follows by standard calculus in Banach spaces. The proof for n odd is similar and is accordingly omitted. □

We now turn to consider the family of fundamental solutions $S^\sharp_{r,n}(\cdot,\lambda)$, where the subscript r stands for 'radiation'. As well known in scattering theory, if $\lambda \in \mathbb{C}\setminus]-\infty,0]$ and $\operatorname{Im}\lambda \geq 0$, then a function $u \in C^1(\mathbb{R}^n\setminus\{0\})$ is said to satisfy the outgoing $(e^{\frac{1}{2}\log\lambda})$-radiation condition if we have

$$\lim_{x\to\infty}|x|^{\frac{n-1}{2}}\left(Du(x)\frac{x}{|x|} - ie^{\frac{1}{2}\log\lambda}u(x)\right) = 0.$$

Then we have the following (for a proof we refer the reader to [22]).

Proposition 6.5. *Let $n \in \mathbb{N}\setminus\{0,1\}$. Let γ_n be the function from $\mathbb{C}$ to $\mathbb{C}$ defined by setting*

$$\gamma_n(z) \equiv \begin{cases} [-i + \frac{2}{\pi}(z - \log 2 + \gamma)]b_n & \text{if } n \text{ is even,} \\ -e^{-i\frac{n-2}{2}\pi}b_n & \text{if } n \text{ is odd,} \end{cases}$$

for all $z \in \mathbb{C}$, with b_n as in (6.1). Let

$$S^\sharp_{r,n}(x,\lambda) \equiv S^\sharp_{h,n}(x,\lambda) + \gamma_n(2^{-1}\log\lambda)e^{\frac{n-2}{2}\log\lambda}J^\sharp_{\frac{n-2}{2}}(\lambda|x|^2) \quad \forall x \in \mathbb{R}^n\setminus\{0\},$$

for all $\lambda \in \mathbb{C}\setminus]-\infty,0]$. Then $S^\sharp_{r,n}(\cdot,\lambda)$ is a fundamental solution of $\Delta+\lambda$ for all $\lambda \in \mathbb{C}\setminus]-\infty,0]$, and satisfies the the outgoing $(e^{\frac{1}{2}\log\lambda})$-radiation condition for all $\lambda \in \mathbb{C}\setminus]-\infty,0]$ with $\operatorname{Im}\lambda \geq 0$.

Then one verifies that the function from $(\mathbb{R}^n\setminus\{0\}) \times (\mathbb{C}\setminus]-\infty,0])$ to $\mathbb{C}$ which takes (x,λ) to $S^\sharp_{r,n}(x,\lambda)$ is real analytic.

Accordingly, the function $S \equiv S^\sharp_{r,n}$ satisfies condition (1.1) with $\mathcal{K} = \mathbb{C}$, and $\mathcal{O} = \mathbb{C}\setminus]-\infty,0]$, and $\mathbf{a}(\cdot) \equiv (a_\alpha(\cdot))_{|\alpha|\leq 2}$ with a_α as in (6.2). Moreover, the following Proposition 6.6 implies that $S \equiv S^\sharp_{r,n}$ satisfies also the condition in (5.2). Its proof is similar to the one of Proposition 6.4 and is accordingly omitted.

Proposition 6.6. *Let $n \in \mathbb{N}\setminus\{0,1\}$. Let $\lambda_0 \in \mathbb{C}\setminus]-\infty,0]$. Let V_{λ_0} be an open bounded neighborhood of λ_0 in $\mathbb{C}\setminus]-\infty,0]$. Let $\delta_2 \in]0,+\infty[$. Then the map from V_{λ_0} to $A^1_{\max\{n-2,\frac{1}{2}\}}(\delta_2)$ which takes λ to $S^\sharp_{r,n}(\cdot,\lambda)_{|(\mathrm{cl}\mathbb{B}_n(0,\delta_2))\setminus\{0\}}$ is real analytic.*

6.3. An Application to Domain Perturbation Problems

The study of the dependence of the solution of a boundary value problem upon regular and singular perturbations of the domain has long been investigated by several authors and with many different approaches. So for example, we mention Burenkov and Lamberti [1], Henry [12], Keldysh [16], Maz'ya et al. [27], Sokolowski and Zolésio [35], and Ward and Keller [37]. We now briefly outline an application of the results of the previous sections to an operator which appears when dealing with the investigation of the dependence of the solution of a boundary value problem upon perturbation of the coefficients of the differential operator, of the domain, and of the data.

So let assumption (1.1) hold and let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Suppose we are interested in studying the dependence of the solution of a certain boundary value problem for the partial differential equation

$$P[\mathbf{a}(\kappa), D]u = \varphi \quad \text{in } \psi(\Omega_\#)\,, \tag{6.3}$$

upon κ, φ, and ψ, where $\Omega_\#$ is a bounded open Lipschitz subset of $\mathbb{R}^n$, $\kappa \in \mathcal{O}$, φ is a sufficiently regular function defined in $\mathrm{cl}\Omega$, and ψ a certain diffeomorphism of class $C^{m,\alpha}$ from $\mathrm{cl}\Omega_\#$ onto $\psi(\mathrm{cl}\Omega_\#) \subseteq \Omega$. The set $\Omega_\#$ represents a 'base domain' which is perturbed by means of the diffeomorphism ψ. In order to investigate the dependence of the solution on the triple (κ, φ, ψ), one may need to convert the boundary value problem for the non-homogeneous equation (6.3) defined on the varying domain $\psi(\Omega_\#)$ into a boundary value problem for an homogeneous equation defined on the fixed domain $\Omega_\#$. Thus, as in [20], one may find useful to consider the composition $\mathcal{P}_k[\varphi] \circ \psi$ of the volume potential $\mathcal{P}_k[\varphi]$ with the diffeomorphism ψ, and study the regularity of the map which takes the triple (κ, φ, ψ) to $\mathcal{P}_k[\varphi] \circ \psi$. As observed in the introduction, a convenient choice of the function space for φ in order to ensure the real analyticity of such operator with the Schauder class $C^{m,\alpha}$ as target space is a Roumieu class.

Then, in the following proposition, by combining Theorem 5.1 and Proposition A.2 of the Appendix, we deduce under suitable assumptions the analyticity of the operator which takes the triple (κ, φ, ψ) to the composite function $\mathcal{P}_k[\varphi] \circ \psi$.

Proposition 6.7. *Let $n \in \mathbb{N}\setminus\{0, 1\}$. Let $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in]0, 1[$. Let assumption (1.1) hold. Let Ω, $\Omega_\#$ be bounded open Lipschitz subsets of $\mathbb{R}^n$. Let Ω_1 be an open subset of $\mathbb{R}^n$ such that $\mathrm{cl}\Omega_1 \subseteq \Omega$. Let assumption (5.1) hold with $\delta_1 = \delta_*$, $\delta_2 = \delta^*$ [see (4.1)]. Then the map from $V_{\kappa_0} \times C^0_{\omega,\rho}(\mathrm{cl}\Omega) \times C^{m,\alpha}(\mathrm{cl}\Omega_\#, \Omega_1)$ to $C^{m,\alpha}(\mathrm{cl}\Omega_\#)$ which takes (κ, φ, ψ) to $\mathcal{P}_\kappa[\varphi] \circ \psi$ is real analytic [see (1.2)].*

Appendix A.

We introduce in this appendix some technical results which we exploit in the paper.

Proposition A.1. *Let n_1, $n_2 \in \mathbb{N}\setminus\{0\}$. Let V, W be bounded open subsets of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Let $\rho' \in]0, +\infty[$. Let $H \in C^0_{\omega,\rho'}(\mathrm{cl}(V \times W))$. Then $H(x, \cdot) \in C^0_{\omega,\rho'}(\mathrm{cl}W)$ for all $x \in \mathrm{cl}V$. Moreover, if $\rho \in]0, \rho'[$ then the map from V to $C^0_{\omega,\rho}(\mathrm{cl}W)$ which takes x to $H(x, \cdot)$ is real analytic and*

$$\|\partial_x^\alpha H(x, \cdot)\|_{C^0_{\omega,\rho}(\mathrm{cl}W)} \leq \|H\|_{C^0_{\omega,\rho'}(\mathrm{cl}(V\times W))} |\alpha|!/(\rho' - \rho)^{|\alpha|} \quad \forall x \in \mathrm{cl}V, \tag{A.1}$$

for all $\alpha \in \mathbb{N}^{n_1}$.

Proof. By the membership of H in $C^0_{\omega,\rho'}(\mathrm{cl}(V \times W))$ we have

$$|\partial_x^\alpha \partial_y^\beta H(x, y)| \leq \|H\|_{C^0_{\omega,\rho'}(\mathrm{cl}(V\times W))} (|\alpha| + |\beta|)!/\rho'^{|\alpha|+|\beta|} \tag{A.2}$$

for all $x \in \mathrm{cl}V$, $y \in \mathrm{cl}W$, $\alpha \in \mathbb{N}^{n_1}$, and $\beta \in \mathbb{N}^{n_2}$. Then by taking $\alpha = (0, \dots, 0)$ we deduce that $H(x, \cdot) \in C^0_{\omega,\rho'}(\mathrm{cl}W)$ for all $x \in \mathrm{cl}V$. Now let $\rho \in]0, \rho'[$ and observe that

$$\frac{(|\alpha| + |\beta|)!}{\rho'^{|\alpha|+|\beta|}} = \binom{|\alpha| + |\beta|}{|\beta|} \left(\frac{\rho' - \rho}{\rho'}\right)^{|\alpha|} \left(\frac{\rho}{\rho'}\right)^{|\beta|} \frac{|\alpha|!}{(\rho' - \rho)^{|\alpha|}} \frac{|\beta|!}{\rho^{|\beta|}}$$

and

$$\begin{aligned} &\binom{|\alpha| + |\beta|}{|\beta|} \left(\frac{\rho' - \rho}{\rho'}\right)^{|\alpha|} \left(\frac{\rho}{\rho'}\right)^{|\beta|} \\ &\leq \sum_{j=0}^{|\alpha|+|\beta|} \binom{|\alpha| + |\beta|}{j} \left(\frac{\rho' - \rho}{\rho'}\right)^{|\alpha|+|\beta|-j} \left(\frac{\rho}{\rho'}\right)^{j} \\ &= \left(\frac{\rho' - \rho}{\rho'} + \frac{\rho}{\rho'}\right)^{|\alpha|+|\beta|} = 1. \end{aligned}$$

Thus inequality (A.2) implies that

$$|\partial_x^\alpha \partial_y^\beta H(x, y)| \leq \|H\|_{C^0_{\omega,\rho'}(\mathrm{cl}(V \times W))} \frac{|\alpha|!}{(\rho' - \rho)^{|\alpha|}} \frac{|\beta|!}{\rho^{|\beta|}}$$

and the validity of (A.1) follows by the definition of $\| \cdot \|_{C^0_{\omega,\rho}(\mathrm{cl}W)}$. Now the real analyticity of the map from V to $C^0_{\omega,\rho}(\mathrm{cl}W)$ which takes x to $H(x, \cdot)$ can be deduced by inequality (A.1) and by the classical Cauchy inequalities for real analytic maps in Banach spaces (cf. e.g., Prodi and Ambrosetti [34, Thm. 10.5]). □

Then we introduce the following slight variant of Preciso [32, Prop. 4.2.16, p. 51], Preciso [33, Prop. 1.1, p. 101] on the real analyticity of a composition operator. See also [19, Prop. 2.17, Rem. 2.19] and the slight variant of the argument of Preciso of the proof of [21, Prop. 9, p. 214]. Indeed, bounded open connected Lipschitz subsets of the Euclidean space are easily seen to be Whitney regular as requested by the statement of Preciso.

Proposition A.2. *Let $h, k \in \mathbb{N} \setminus \{0\}$, $m \in \mathbb{N}$. Let $\alpha \in]0, 1]$, $\rho > 0$. Let Ω, Ω' be bounded open subsets of $\mathbb{R}^h$, $\mathbb{R}^k$, respectively. Let Ω' be a Lipschitz subset. Then the operator T defined by*

$$T[\zeta, \psi] \equiv \zeta \circ \psi$$

for all $(\zeta, \psi) \in C^0_{\omega,\rho}(\mathrm{cl}\Omega) \times C^{m,\alpha}(\mathrm{cl}\Omega', \Omega)$ is real analytic from the open subset $C^0_{\omega,\rho}(\mathrm{cl}\Omega) \times C^{m,\alpha}(\mathrm{cl}\Omega', \Omega)$ of $C^0_{\omega,\rho}(\mathrm{cl}\Omega) \times C^{m,\alpha}(\mathrm{cl}\Omega', \mathbb{R}^h)$ to $C^{m,\alpha}(\mathrm{cl}\Omega')$.

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