Optimal default boundary in discrete time models

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Abstract

In this paper we solve the problem of determining the default time of a firm in such a way as to maximize its total value, which includes bankruptcy costs and tax benefits, with the condition that the value of equity must be nonnegative. By applying dynamic programming in discrete time, we find results which extends those of Merton (1974), and we give an application for the approximation of models driven by a Brownian motion or a Poisson process.

KEYWORDS. Credit and default risk, optimal stopping, dynamic programming.

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1 Introduction

The aim of this paper is to find the optimal time of default for a firm which has issued a coupon bond with a given maturity. The default time is optimal in the sense that it maximises the total value of the firm, given by its net value plus tax benefits of the coupons minus bankruptcy costs, with the constraint that equity, given by the difference between the total value of the firm minus the debt, must have a positive value at all times prior to default.

In the field of credit risk, valuation models can be divided into two categories which differ from one another in modelling default time (for a concise description of the two approaches, see [Ammann, 1999] and [Lando, 1997]). In the first category (*intensity based* models), the default time τ is typically modeled as a first jump time of a Poisson process; this captures the idea that the time of a default takes the bondholders by surprise (see [Jeanblanc and Rutkowski, 1999] for a survey and some bibliography). In the second category (models based on the *value of the firm*), the default time is determined by an underlying process V describing the value of the firm; default occurs when this process passes a certain boundary (like in a barrier option), and typically represents the fact that the firm cannot repay a debt with a given maturity, so the default time is of the form $\tau = \inf\{t \mid V_t \leq f(t)\}$. In this last category, the prices of credit derivatives as corporate bonds and swaps depend on the shape of the default boundary f (that can be a deterministic function as well as a stochastic process). One can see different choices of f in [Black and Cox, 1976], [Briys and de Varenne, 1997], [Johnson and Stulz, 1987], [Kim, Rawasmany and Sundaresan, 1993], [Leland, 1994], [Longstaff and Schwartz, 1995], [Leland and Toft, 1996], [Merton, 1974], [Nielsen, Saà-Requejo and Santa-Clara, 1993] (for a survey, see [Ammann, 1999]). The choice of this default boundary in the different models seems as arbitrary as the choice of a term structure of risk-free interest rates is, and is usually exogenously suggested by personal taste. The first example of boundary built in an endogenous way is in [Leland, 1994] and [Leland and Toft, 1996], where a constant boundary K is determined such that the total value (including tax benefits on the debt) of the firm is maximised, with the constraint that equity must have a positive value in all the times prior to default.

Our model generalises Leland-Toft's approach by searching for a default boundary f (not of the specific kind $f(t) \equiv K$) such that the total value of the firm is maximised, with the same constraint as Leland-Toft. In order to obtain explicit results, we choose to work in discrete time. From an economic point of view, this could reflect the fact that default can be decided by the stockholders only at particular times (that could be the maturities of the coupons of the debt). From a mathematical point of view, this can be the first step for a continuous time formulation: in fact the solution of a continuous time problem of this kind can be approximated by the solutions of discrete time problems (see [Kushner, 1977]).

We formulate this problem using the tools of stochastic control, so we write it as an optimal stopping problem with a constraint. While unconstrained optimal stopping can be treated by methods that are by now classical (see for example [Ŝirjaev, 1973]), we found that there are no specific references for our problem, so we developed an ad-hoc method of solution that makes use of the dynamic programming principle. We find that the optimal default time is the first time at which equity falls to zero, where equity turns out to be a deterministic function of the value of the firm. In the case of a debt with no coupons, we find nice parallels between our model and Merton's: namely, under assumptions more general than the ones in [Merton, 1974], we find that it is optimal for the firm to wait until the final maturity of the debt. We investigate in particular the case when the process V follows a binomial tree, and characterise the optimal default boundary in specific cases, mainly in view of using this model to approximate continuous time models.

The paper is organised as follows. In Section 2 we describe our model and present the default-dependent quantities relevant for our work. In Section 3 we formulate and solve the optimal stopping problem. In Section 4 we apply our results to the binomial model, and present how one can use a binomial model in order to approximate a lognormal model or a geometric Poisson model.

This paper is the "full version" of [Altieri and Vargiolu, 2001] (which does not contain the proof of the main theorem as well as other results). The authors wish to thank Stefan Jaschke, Nicole El Karoui, Monique Jeanblanc, Wolfgang Runggaldier and Jerzy Zabczyk, who gave useful contributions at various stages of the work.

2 The model

We consider a market in which the primary assets are a riskless asset B, called the *money market account*, and a risky asset V which represents the total value of the firm. We represent the value of the firm as a stochastic process $V = (V_t)_{t\in[0,N]}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $N \in \mathbb{N}$ is a given maturity.

We assume that the market is arbitrage free and complete. This is equivalent to assuming that there exists a unique equivalent martingale measure $\mathbb{Q}$ and every contingent claim can be priced via the expected value of its discounted final payoff under $\mathbb{Q}$. This situation occurs for example if V is a binomial process (i.e. the sample paths of V are pathwise constant and they can change assuming only two values conditional on the value of V on the preceding subinterval) or if V is a diffusion driven by a 1-dimensional Brownian motion. We will be interested in particular in some activities related to the firm (corporate debt, equity, tax benefits, bankruptcy costs, etc.), that will be viewed as a derivative asset of the value of the firm.

We also assume that default can take place only at discrete times $n = 0, 1, \ldots, N$. Since we are interested only in finding which is the optimal default time among the dates $n = 0, 1, \ldots, N$, we represent the dynamics of B and V as

$$B_{n+1} = B_n(1 + r) \ ,$$

where $r > 0$ is the deterministic risk-free interest rate, and as

$$V_{n+1} = V_n\omega_n \ , \quad n = 0, \ldots, N \ ,$$

where $(\omega_n)_{n\in\{0,1,\ldots,N\}}$ is a sequence of positive i.i.d. random variables on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$ and $(\mathcal{F}_n)_n$ is the filtration generated by V. Under these assumptions, V is a Markov chain with transition operator T defined by

$$Tf(v) = \mathbb{E}_{\mathbb{Q}}[f(V_{n+1}) \mid V_n = v]$$

(since the $(\omega_n)_n$ are i.i.d., the right hand side does not depend on n) for every $f : \mathbb{R}^+ \to \mathbb{R}$ measurable and such that the right hand side is well defined; besides, $(V_n/(1 + r)^n)_n$ is a $\mathbb{Q}$-martingale. We also introduce the following notation: for all $n \leq N$ and $v > 0$, we define the probability measure

$$\mathbb{Q}_{n,v}(A) = \mathbb{Q}\{A \mid V_n = v\} \ , \quad \forall A \in \mathcal{F} \ ,$$

and denote with $\mathbb{E}_{n,v}$ the expectation with respect to $\mathbb{Q}_{n,v}$ (for more details, see [Fleming and Soner, 1993]).

The firm is operated by its equityholders, and we assume for simplicity that their only choice is when to liquidate the firm. A liquidation policy is a $(\mathcal{F}_n)_n$-stopping time τ. As said before, we are assuming that τ takes values in the set $\{0, \ldots, N\}$. There is also the possibility that default does not take place before time N; when this happens, we will give τ the value $N + 1$. This is not a real date (because the terminal one is N) but it indicates that the firm has arrived at N without declaring bankruptcy.

Following [Leland, 1994] and [Leland and Toft, 1996], we consider three particular claims depending on the value of the firm and on default time. Since there exists a unique martingale measure $\mathbb{Q}$, the value of these contingent claims will be given by the discounted expectation under $\mathbb{Q}$ of the payments of the claim.

1. **Bankruptcy costs.** We assume that the costs connected with default are equal to αV_τ, where $\alpha \in (0,1)$ is a fixed fraction. These costs are equivalent to a claim that does not pay coupons and is worth αV_τ in the event of bankruptcy. Then the value of bankruptcy costs at time n is given by:

$$BC(v,n;\tau) = \mathbb{E}_{n,v}\left[\frac{\alpha V_\tau}{(1+r)^{\tau-n}}\mathbf{1}_{\{n\le\tau\le N\}}\right] .$$

2. **Debt.** We assume that debt issued by the firm pays coupon payments C_n at every date n prior to maturity or default and an amount P at the terminal date; in the event of default the bondholders receive the residual value of the firm minus the bankruptcy costs. Thus the value of the debt is:

$$D(v,n;\tau) = \mathbb{E}_{n,v}\left[\sum_{i=n}^{\tau-1}\frac{C_n}{(1+r)^{i-n}} + \frac{P}{(1+r)^{\tau-n}}\mathbf{1}_{\{\tau=N+1\}} + \frac{(1-\alpha)V_\tau}{(1+r)^{\tau-n}}\mathbf{1}_{\{n\le\tau\le N\}}\right] .$$

3. **Tax benefits.** The tax benefits associated with debt financing are proportional to the coupon C_n via a constant $\gamma \in (0,1)$. The value of tax benefits is:

$$TB(v,n;\tau) = \mathbb{E}_{n,v}\left[\sum_{i=n}^{\tau-1}\frac{\gamma C_n}{(1+r)^{i-n}}\right] .$$

The total value of the firm is given by:

$$W(v,n;\tau) = v + TB(v,n;\tau) - BC(v,n;\tau) .$$

By the Modigliani-Miller theorem, the value of equity is the total value of the firm minus the value of debt:

$$E(v,n;\tau) = W(v,n;\tau) - D(v,n;\tau) .$$

3 The optimal stopping problem

We now suppose that the firm can choose the time τ at which it declares bankruptcy and we assume that the firm does it in such a way as to maximize its total value with the condition that equity must be nonnegative. Hence the problem we want to solve is

$$W^*(v) = \max_{\tau\in\mathcal{T}_0} W(v,0,\tau) , \tag{1}$$

where $\mathcal{T}_n$ is the set of the $\{n,\dots,N+1\}$-valued stopping times such that $\{\tau > k\} \subseteq \{E(V_k,k;\tau) > 0\}$ for all $k = n,\dots,N$. This problem is not a classical optimal stopping problem (see for instance [Ŝirjaev, 1973]) because of the presence of a constraint for the stopping time. Therefore we present an ad-hoc method of solution that makes use of dynamic programming.

Theorem 1. *Given the problem (1) and defining successively the functions* $h_N, h_{N-1}, \ldots, h_0$, $\overline{W}_N, \overline{W}_{N-1}, \ldots, \overline{W}_0$ *as*

$$\begin{aligned}
h_N(v) &= (v - P - (1-\gamma)C_N)^+ \ , \\
\overline{W}_N(v) &= \begin{cases} v(1-\alpha) & \text{if } h_N(v) = 0 \ , \\ v + \gamma C_N & \text{if } h_N(v) > 0 \ , \end{cases} \\
h_k(v) &= \left(\frac{1}{1+r} T h_{k+1}(v) - (1-\gamma)C_k \right)^+ , \qquad k = N-1, \ldots, 0 \ , \\
\overline{W}_k(v) &= \begin{cases} \gamma C_k + \dfrac{1}{1+r} T \overline{W}_{k+1}(v) & \text{if } h_k(v) > 0 \ , \\ v(1-\alpha) & \text{if } h_k(v) = 0 \ , \end{cases} \qquad k = N-1, \ldots, 0 \ ,
\end{aligned}$$

an optimal stopping time is

$$\hat{\tau} = \begin{cases} \inf\{ j \mid V_j \leq v_j^* \} \ , \\ N+1 \quad \text{if the above set is empty} \ , \end{cases} \tag{2}$$

where v_j^* *is the maximal solution of the equation*

$$h_j(v_j^*) = 0 \tag{3}$$

and we have:

$$\begin{aligned}
\overline{W}_0(v) &= W^*(v) = W(v, 0; \hat{\tau}) \ , \\
E(v, 0; \hat{\tau}) &= h_0(v) \quad \text{if } h_0(v) > 0 \ .
\end{aligned}$$

Proof. As announced, we prove this theorem using dynamic programming. For this purpose we consider for $v > 0$ and $n = 0, 1, \ldots, N$ the family of problems:

$$J_n(v) = \max_{\tau \in \mathcal{T}_n} W(v, n; \tau). \tag{4}$$

We will prove that the theorem is true for each optimisation problem defined above. More precisely, we will prove that, in the notation of the theorem, an optimal stopping time is given by

$$\hat{\tau}_n = \begin{cases} \inf\{ j \geq n \mid V_j \leq v_j^* \} \ , \\ N+1 \quad \text{if the above set is empty} \ , \end{cases} \tag{5}$$

and we have

$$\begin{aligned}
\overline{W}_n(v) &= J_n(v) = W(v, n; \hat{\tau}_n) \ , \qquad (6) \\
E(v, n; \hat{\tau}_n) &= h_n(v) \quad \text{if } h_n(v) > 0 \ .
\end{aligned}$$

We proceed by induction starting to prove the theorem for $n = N$. Every $\tau \in \mathcal{T}_N$ can take only the values N or $N+1$. Since $\{\tau = N\} \in \mathcal{F}_N$, we have that $\mathbb{Q}_{N,v}\{\tau = N\}$ can take only the values 0 or 1 (strictly speaking, it is a measurable function of v); this means that τ is constant (conditionally on V_N), and takes either the value N or the value $N+1$. If $\tau = N$, then the condition $\emptyset = \{\tau > N\} \subseteq \{E(v, N; \tau) > 0\}$ is always satisfied, so the stopping time $\tau = N$ is admissible. If $\tau = N+1$, then the condition $E(v, N; N+1) > 0$ has to hold $\mathbb{Q}_{N,v}$-almost surely, and we have

$$E(v, N; N+1) = v - P - (1-\gamma)C_N > 0 \,,$$

that is $h_N(v) > 0$. If this is true, then

$$J_N(v) = \max(W(v, N; N), W(v, N; N+1)) = \max(v(1-\alpha), v + \gamma C_N) = v + \gamma C_N$$

If the condition $h_N(v) > 0$ does not hold, we have that $h_N(v) = 0$ and

$$J_N(v) = \max(W(v, N; N)) = v(1-\alpha) \,,$$

so we have that $J_N = \overline{W}_N$. Moreover, the maximal solution of the equation $h_N(v) = 0$ is $v_N^* = P + (1-\gamma)C_N$; this means that the stopping time $\hat{\tau}_N$ is optimal, and $J_N(v) = W(v, N; \hat{\tau}_N)$. Hence the proof is complete for $n = N$.

Now let us suppose that the theorem is true for the step $n+1$, and we prove it for the step n. We assume that at each step k, $n < k < N$, Equation (6) holds and the optimal stopping time τ_k is given by Equation (5). Since $\{\tau = n\} \in \mathcal{F}_n$, by an argument analogous to step $n = N$, either $\tau = n$ or $\tau \in \mathcal{T}_{n+1}$ (that is, $\tau > n$ $\mathbb{Q}_{n,v}$-a.s.). As before, if $\tau = n$, then the condition $\emptyset = \{\tau > n\} \subseteq \{E(v, N; \tau) > 0\}$ is always satisfied, so the stopping time $\tau = n$ is admissible. If $\tau \in \mathcal{T}_{n+1}$, then the optimum is achieved in $\hat{\tau}_{n+1}$ (which for brevity we indicate with $\hat{\tau}$ from now on), and the condition $E(v, n; \hat{\tau}) > 0$ has to hold $\mathbb{Q}_{n,v}$-almost surely (in fact we know by the previous steps that $\{\hat{\tau} > k\} \subseteq \{E(V_k, k; \hat{\tau}) > 0\}$ for all $k = n, \ldots, N$). By applying the martingale stopping theorem to $(V_n/(1+r)^n)_n$ for the stopping time $\hat{\tau} \wedge N$, we obtain

$$\begin{aligned} E(v, n; \hat{\tau}) &= v - \mathbb{E}_{n,v}\left[\frac{V_{\hat{\tau}}}{(1+r)^{\hat{\tau}-n}}\mathbf{1}_{\{n \le \hat{\tau} \le N\}} + \sum_{i=n}^{\hat{\tau}-1} \frac{(1-\gamma)C_i}{(1+r)^{i-n}} + \frac{P}{(1+r)^{N-n}}\mathbf{1}_{\{\hat{\tau} = N+1\}}\right] = \\ &= \mathbb{E}_{n,v}\left[\frac{V_N}{(1+r)^{N-n}}\mathbf{1}_{\{\hat{\tau} = N+1\}} - \sum_{i=n}^{N} \frac{(1-\gamma)C_i}{(1+r)^{i-n}}\mathbf{1}_{\{\hat{\tau} > i\}} - \frac{P}{(1+r)^{N-n}}\mathbf{1}_{\{\hat{\tau} = N+1\}}\right] \,. \end{aligned}$$

Since $\{\hat{\tau} > i, \hat{\tau} > i-1, \ldots, \hat{\tau} > n\} = \{h_i(V_i) > 0, \hat{\tau} > i-1, \ldots, \hat{\tau} > n\}$ for all $i = n, \ldots, N$, we have

$$\begin{aligned} E(v, n; \hat{\tau}) = \mathbb{E}_{n,v}\Bigg[&\frac{V_N - P}{(1+r)^{N-n}}\mathbf{1}_{\{h_N(V_N) > 0\}}\mathbf{1}_{\{\hat{\tau} > N-1\}} \cdots \mathbf{1}_{\{\hat{\tau} > n\}} - \\ &- \sum_{i=n}^{N} \frac{(1-\gamma)C_i}{(1+r)^{i-n}}\mathbf{1}_{\{\hat{\tau} > i\}}\mathbf{1}_{\{\hat{\tau} > i-1\}} \cdots \mathbf{1}_{\{\hat{\tau} > n\}}\Bigg] = \\ = \mathbb{E}_{n,v}\Bigg[&\mathbb{E}_{N-1, V_{N-1}}\left[\frac{V_N - P - (1-\gamma)C_N}{(1+r)^{N-n}}\mathbf{1}_{\{h_N(V_N) > 0\}}\right]\mathbf{1}_{\{\hat{\tau} > N-1\}} \cdots \mathbf{1}_{\{\hat{\tau} > n\}} - \end{aligned}$$

$$
\begin{aligned}
& -\sum_{i=n}^{N-1} \frac{(1-\gamma)C_i}{(1+r)^{i-n}} \mathbf{1}_{\{\hat\tau>i\}} \mathbf{1}_{\{\hat\tau>i-1\}} \cdots \mathbf{1}_{\{\hat\tau>n\}} \Bigg] = \\
= \ & \mathbb{E}_{n,v} \left[\mathbb{E}_{N-2,V_{N-2}} \left[\left(\frac{Th_N(V_{N-1})}{(1+r)^{N-n}} - \frac{(1-\gamma)C_{N-1}}{(1+r)^{N-n+1}} \right) \mathbf{1}_{\{h_{N-1}(V_{N-1})>0\}} \right] \mathbf{1}_{\{\hat\tau>N-2\}} \cdots \mathbf{1}_{\{\hat\tau>n\}} - \right. \\
& \left. -\sum_{i=n}^{N-2} \frac{(1-\gamma)C_i}{(1+r)^{i-n}} \mathbf{1}_{\{\hat\tau>i\}} \mathbf{1}_{\{\hat\tau>i-1\}} \cdots \mathbf{1}_{\{\hat\tau>n\}} \right] ,
\end{aligned}
$$

and by induction we come to

$$
E(v,n;\hat\tau) \ = \ \mathbb{E}_{n,v} \left[\left(\frac{Th_{n+1}(v)}{1+r} - (1-\gamma)C_n \right) \mathbf{1}_{\{h_n(v)>0\}} \right] = h_n(v) > 0 \, .
$$

If this condition holds, then

$$
J_n(v) = \max \left(W(v,n;n), \max_{\tau \in \mathcal{T}_{n+1}} W(v,n;\tau) \right) = \max(v(1-\alpha), W(v,n;\hat\tau_{n+1})) \, ,
$$

and we have:

$$
\begin{aligned}
W(v,n;\hat\tau_{n+1}) \ &= \ v + \mathbb{E}_{n,v} \left[\sum_{i=n}^{\hat\tau_{n+1}-1} \frac{\gamma C_i}{(1+r)^{i-n}} - \frac{\alpha V_{\hat\tau_{n+1}}}{(1+r)^{\hat\tau_{n+1}-n}} \mathbf{1}_{\{n \le \hat\tau_{n+1} \le N\}} \right] = \\
&= \ \gamma C_n + \mathbb{E}_{n,v} \left[\frac{V_{n+1}}{1+r} + \sum_{i=n+1}^{\hat\tau_{n+1}-1} \frac{\gamma C_i}{(1+r)^{i-n}} - \frac{\alpha V_{\hat\tau_{n+1}}}{(1+r)^{\hat\tau_{n+1}-n}} \mathbf{1}_{\{n+1 \le \hat\tau_{n+1} \le N\}} \right] = \\
&= \ \gamma C_n + \frac{1}{1+r} \mathbb{E}_{n,v}[W(V_{n+1}, n+1; \hat\tau_{n+1})] = \\
&= \ \gamma C_n + \frac{T\overline{W}_{n+1}(v)}{1+r} \, .
\end{aligned}
$$

Since $\hat\tau_{n+1}$ is bounded by $N+1$, by applying the martingale stopping theorem again to $(V_n/(1+r)^n)_n$ for the stopping time $\hat\tau_{n+1}$, we obtain

$$
\begin{aligned}
W(v,n;\hat\tau_{n+1}) \ &= \ v - \alpha \mathbb{E}_{n,v} \left[\frac{V_{\hat\tau_{n+1}}}{(1+r)^{\hat\tau_{n+1}-n}} \mathbf{1}_{\{n < \hat\tau_{n+1} \le N\}} \right] + \mathbb{E}_{n,v} \left[\sum_{i=n}^{\hat\tau_{n+1}-1} \frac{\gamma C_i}{(1+r)^{i-n}} \right] \ge \\
&\ge \ v - \alpha \mathbb{E}_{n,v} \left[\frac{V_{\hat\tau_{n+1}}}{(1+r)^{\hat\tau_{n+1}-n}} \mathbf{1}_{\{n < \hat\tau_{n+1}\}} \right] = v - \alpha v \, ,
\end{aligned}
$$

so

$$
J_n(v) = \gamma C_n + \frac{T\overline{W}_{n+1}(v)}{1+r} \, .
$$

If the condition $h_N(v) > 0$ does not hold, we have that $h_N(v) = 0$ and

$$
J_n(v) = \max(W(v,n;n)) = v(1-\alpha) \, ,
$$

so we have that $J_n = \overline{W}_n$. Moreover, $h_n(v) = 0$ is equivalent to $v < v_n^*$, so the stopping time $\hat\tau_n$ is optimal, and $J_N(v) = W(v,N;\hat\tau_N)$. Hence the proof is complete also for step n. By induction, the proof is complete. □

Corollary 1. *If $C_i = 0$, $i = 1, \ldots, N$, and the random variables ω_n have support equal to $\mathbb{R}^+$, then the optimal stopping time is:*

$$\hat{\tau} = \begin{cases} N & \text{if } V_N \leq P \; , \\ \\ N+1 & \text{if } V_N > P \; , \end{cases}$$

and we have:

$$\begin{aligned} W_n(v) &= v - \mathbb{E}_{n,v}\left[\frac{\alpha V_N}{(1+r)^{N-n}}\mathbf{1}_{\{V_N \leq P\}}\right] \; , \\ D_n(v) &= \mathbb{E}_{n,v}\left[\frac{(1-\alpha) V_N}{(1+r)^{N-n}}\mathbf{1}_{\{V_N \leq P\}} + \frac{P}{(1+r)^{N-n}}\mathbf{1}_{\{V_N > P\}}\right] \; , \\ BC_n(v) &= \mathbb{E}_{n,v}\left[\frac{\alpha V_N}{(1+r)^{N-n}}\mathbf{1}_{\{V_N \leq P\}}\right] \; , \\ TB_n(v) &= 0 \; , \\ E_n(v) &= v - \mathbb{E}_{n,v}\left[\frac{V_N}{(1+r)^{N-n}}\mathbf{1}_{\{V_N \leq P\}} + \frac{P}{(1+r)^{N-n}}\mathbf{1}_{\{V_N > P\}}\right] \; . \end{aligned}$$

Proof. By Theorem 1 we have

$$\begin{aligned} h_N(v) &= (v-P)^+ \; , \\ h_k(v) &= \left(\frac{1}{1+r} T h_{k+1}(v)\right)^+ \; , \quad k = N-1, \ldots, 0 \; , \end{aligned}$$

and the optimal stopping time is

$$\hat{\tau} = \begin{cases} \inf\{j \mid h_j(V_j) = 0\} \; , \\ \\ N+1 \text{ if the above set is empty} \; . \end{cases}$$

We prove by induction that the functions Th_{k+1} for $k = N-1, \ldots, 0$, are all strictly positive. In fact, if we assume that $h_{k+1} \geq 0$ and h_{k+1} is different from the zero function we have:

$$Th_{k+1}(v) \;=\; \int_0^{+\infty} h_{k+1}(y)\; p(v, dy) > 0 \quad \forall v \in \mathbb{R} \; ,$$

where p is the transition kernel of the Markov chain $\{V_n\}_{n \in \{0,\ldots,N\}}$ under $\mathbb{Q}$. Since $h_N \geq 0$ and h_N is different from the zero function, we deduce that

$$h_k(v) = \frac{1}{1+r} T h_{k+1}(v) > 0 \quad \forall k = N-1, \ldots, 0 \; ,$$

so $h_j(V_j) > 0$ for all $j = 0, \ldots, N-1$, and the only nontrivial condition to stop is $h_N(V_N) = 0$. Thus the optimal stopping time is

$$\hat{\tau} = \begin{cases} N & \text{if } \; V_N \leq P \; , \\ N+1 & \text{if } \; V_N > P \; . \end{cases}$$

□

This corollary draws a nice parallel between our model and the seminal one by Merton [Merton, 1974]. In fact if there are no intermediate coupons then the optimal rule for the firm to default is to wait until the maturity and then to default if and only if its net value is worth less than the principal, as in Merton's model. Moreover, if we take $\omega_n = \exp((r - 1/2\sigma^2)T/N + \sigma W_N)$, where $W_n \sim N(0, T/N)$, then we obtain exactly the same quantities for D and E as Merton.

Remark 1. If the random variables ω_n have a support smaller than $\mathbb{R}^+$ (such as in the case when the possible values are a finite number as in the binomial model which we will discuss in the next section), then the results of Corollary 1 are still valid if we take the admissible default times in the set $\tilde{\mathcal{T}}_n$ of the $\{n, \ldots, N+1\}$-valued stopping times such that $\{\tau > k\} \subseteq \{E(V_k, k; \tau) \geq 0\}$ for all $k = n, \ldots, N$. This corresponds to allowing the equity to be exactly zero in some periods of time. This could be quite unpleasant to the intuition, but it could be justified by the fact that in our model the primary asset is not the equity but the value of the firm.

4 The binomial model

In this section we apply our results, collected in Theorem 1, to the case of the binomial model. The purpose is to use a binomial model to approximate a lognormal model or a geometric Poisson model, in order to obtain (for example) results similar to the ones in [Leland, 1994] and [Leland and Toft, 1996]; a sketch of how this can be done is in the Subsections 4.1 and 4.2 (a rigorous analysis can be obtained from the authors on request).

Let us suppose that the ω_n are random variables which take only the two values $1 + u$ and $1 + d$ with probability p and $1 - p$ $(0 < p < 1)$, respectively, under the equivalent martingale measure $\mathbb{Q}$. We want the discounted value of V to be a $\mathbb{Q}$-martingale, so

$$p = \frac{r - d}{u - d}\ , \qquad 1 - p = \frac{u - r}{u - d}\ ,$$

with $-1 < d < r < u$. The transition operator T this time is such that

$$T\varphi(x) = p\varphi(x(1 + u)) + (1 - p)\varphi(x(1 + d))\ .$$

We first apply Theorem 1 to the case $C_n \equiv 0$ for all $n = 0, \ldots, N$. The functions h_n, $n = 0, \ldots, N$ turn out to be pathwise affine increasing functions, and it is easy to determine the optimal default boundary v_n^*, $n = 0, \ldots, N$.

Theorem 2. *If $C_n \equiv 0$ for all $n = 0, \ldots, N$, then the optimal default boundary is given by*

$$v_n^* = \frac{P}{(1 + u)^{N-n}}\ , \quad n = 0, \ldots, N\ ,$$

and the optimal stopping time is given by Equation (2).

Proof. We have $h_N(v) = (v - P)^+$. Since $C_n = 0$ and T sends nonnegative functions into nonnegative functions, then

$$h_{N-n}(v) = \frac{T^n h_N(v)}{(1 + r)^n}\ .$$

Now we need to find the maximal solution v_n of the equation $Th_{n+1}(v) = 0$, that becomes $T^{N-n}h_N(v) = 0$. We claim that $v_n = P/(1+u)^{N-n}$ for all $n = 0, \ldots, N$. This is true for $n = N$, because $h_N = 0$ on $(0, P]$ and $h_N > 0$ on $(P, +\infty)$. We now proceed by induction, assuming that it is true for n. This means that $h_n = 0$ on $(0, P/(1+u)^{N-n}]$ and $h_n > 0$ on $(P/(1+u)^{N-n}, +\infty)$. Then for $v \in (0, P/(1+u)^{N-n+1}]$:

$$h_{n-1}(v) = \frac{Th_n(v)}{1+r} = ph_n(v(1+u)) + (1-p)h_n(v(1+d)) = 0 \ ,$$

since $v(1+u), v(1+d) \in (0, P/(1+u)^{N-n}]$. For $v \in (P/(1+u)^{N-n+1}, +\infty)$,

$$h_{n-1}(v) = \frac{Th_n(v)}{1+r} = ph_n(v(1+u)) + (1-p)h_n(v(1+d)) > 0 \ ,$$

since $v(1+u) \in (P/(1+u)^{N-n}, +\infty)$. The result follows from Theorem 1. □

Now we apply Theorem 1 to the case $C_n \neq 0$. The functions h_n, $n = 0, \ldots, N$, turn out to be pathwise affine increasing functions also now, but in general it is more difficult to find explicitly the optimal default boundary v_n^*, $n = 0, \ldots, N$. However we find that if the conditions

$$\sum_{i=0}^{N-n} \frac{(1-\gamma)C_i}{(1+r)^i} + \frac{P}{(1+r)^{N-n}} \leq \frac{(1-\gamma)C_n(1+d)(1+r)}{(r-d)} \ , \quad n = 0, \ldots, N \ , \tag{7}$$

hold, then it is possible to obtain explicitly v_n^*, $n = 0, \ldots, N$. It is not straightforward to understand how to interpret the conditions in Equation (7): in fact, while in the left hand side we have the discounted value of the debt to be paid net of the tax benefits, in the right hand side we have the present coupon multiplied by a constant which has not a clear interpretation. However, one can see some situations where the conditions in Equation (7) hold in the case of a constant coupon in Corollary 6.

Theorem 3. *If the condition in Equation (7) holds, then the optimal default boundary is given by*

$$v_n^* = \sum_{i=0}^{N-n} \frac{(1-\gamma)C_i}{(1+r)^i} + \frac{P}{(1+r)^{N-n}} \ , \quad n = 0, \ldots, N \ , \tag{8}$$

and the optimal stopping time is given by Equation (2).

Proof. We have that $h_N(v) = (v - P - (1-\gamma)C_N)^+$, and we claim that

$$h_n(v) = (v - v_n^*)^+ \ ,$$

where v_n^* is given by Equation (8), $n = 0, \ldots, N$. This is true for $n = N$. Assume now that it is true for n. Then

$$\frac{Th_n(v)}{1+r} = \begin{cases} 0 & \text{if } v < \dfrac{v_n^*}{1+u} \ , \\[2ex] \dfrac{p}{1+r}\,(v(1+u) - v_n^*) & \text{if } \dfrac{v_n^*}{1+u} < v < \dfrac{v_n^*}{1+d} \ , \\[2ex] v - \dfrac{v_n^*}{1+r} & \text{if } v \geq \dfrac{v_n^*}{1+d} \ . \end{cases}$$

If

$$\frac{Th_n}{1+r}\left(\frac{v_n^*}{1+d}\right) \le (1-\gamma)C_n \ , \tag{9}$$

then it is immediate to see that the function $Th_n(\cdot)/(1+r) - (1-\gamma)C_n$ is greater than or equal to zero for $v > v_n^*/(1+r) + (1-\gamma)C_n = v_{n-1}^*$ and less than or equal to zero for $v < v_{n-1}^*$. This means that $h_{n-1}(v) = (v - v_{n-1}^*)^+$, and the inductive step is complete. By Theorem 1, the result follows.

It remains to prove that condition (7) implies (9). Equation (9) is equivalent to

$$\frac{v_n^*}{1+d} - \frac{v_n^*}{1+r} \le (1-\gamma)C_n \ .$$

So we have:

$$v_n^*\left(\frac{1}{1+d} - \frac{1}{1+r}\right) = v_n^*\frac{r-d}{(1+d)(1+r)} \le (1-\gamma)C_n \ .$$

Using Equation (8), we easily see that this is equivalent to Equation (7). □

If the coupons are constant during the life of the bond, then Equation (7) can be verified in simple ways, as the corollary below shows.

Corollary 2. *If $C_n \equiv C$ for all $n = 0, \ldots, N$, and one of the assumptions*

i) $\dfrac{r-d}{1+2d+rd}P \le (1-\gamma)C \le rP$ *and* $d \notin \left(-\dfrac{1}{2+r}, 0\right]$,

ii) $(1-\gamma)C \ge rP$ *and* $d \notin \left(-\dfrac{1}{(1+r)^{N+2}-1}, 0\right]$,

iii) $(1-\gamma)C \ge \dfrac{r(r-d)}{d(1+r)^{N+2}+r-d}P$ *and* $d \in \left(-\dfrac{1}{(1+r)^{N+2}-1}, 0\right]$,

holds, then the optimal default boundary is given by Equation (8), and the optimal stopping time is given by Equation (2).

Proof. We start by noting that the default boundary $(v_n^*)_n$ given by Equation (8) is decreasing (resp. constant, increasing) with n if the net coupon $(1-\gamma)C$ is greater (resp. equal, less) than the interest on a single period rP.

For case i), it is sufficient to check Equation (7) only for $n = N$:

$$(1-\gamma)C + P \le \frac{(1-\gamma)C(1+d)(1+r)}{(r-d)} \ .$$

After some algebra, we arrive at

$$(1-\gamma)C \ge \frac{r-d}{1+2d+rd}P \ .$$

In order for case i) to be verified, we have to check that

$$r \ge \frac{r-d}{1+2d+rd} \ .$$

This is equivalent to

$$\frac{d(1+r)^2}{1+2d+rd} \geq 0 \ ,$$

which is verified if $d \geq 0$ or if $d < -1/(2+r)$, so the result follows for case i).

For case ii), it is sufficient to check Equation (7) only for $n = 0$:

$$\sum_{i=0}^{N} \frac{(1-\gamma)C}{(1+r)^i} + \frac{P}{(1+r)^N} \leq \frac{(1-\gamma)C(1+d)(1+r)}{(r-d)} \ .$$

We can see, after some algebra, that this is equivalent to

$$(1-\gamma)C \geq \frac{r(r-d)}{d(1+r)^{N+2}+r-d} P \ . \tag{10}$$

We can easily see that

$$rP \geq \frac{r(r-d)}{d(1+r)^{N+2}+r-d} P$$

is equivalent to

$$\frac{d(1+r)^{N+2}}{d(1+r)^{N+2}+r-d} \geq 0 \ ,$$

which is satisfied if and only if $d \notin (-1/((1+r)^{N+2}-1), 0]$, so if the assumptions in ii) are satisfied, the result follows. Conversely, if $d \in (-1/((1+r)^{N+2}-1), 0]$, then Equation (10) has to hold, and the result follows also for case iii). $\square$

The cases in Theorems 2 and 3 can be regarded as extreme cases: in fact, the optimal default boundary $(v_n^*)_n$ lies always between the two boundaries found in these cases.

Theorem 4. *The optimal default boundary $(v_n^*)_n$ satisfies the following bound:*

$$\frac{P}{(1+u)^{N-n}} \leq v_n^* \leq \sum_{i=0}^{N-n} \frac{(1-\gamma)C_i}{(1+r)^i} + \frac{P}{(1+r)^{N-n}} \ , \quad n = 0, \ldots, N \ ,$$

and the optimal stopping time is given by Equation (2).

Proof. We have the following bound on h_N:

$$v - P - (1-\gamma)C_N \leq h_N(v) \leq (v-P)^+ \ .$$

Since the operator T is linear and monotone, this implies:

$$v - \frac{P + (1-\gamma)C_N}{1+r} - (1-\gamma)C_{N-1} \leq \frac{Th_N(v)}{1+r} - (1-\gamma)C_{N-1} \leq \frac{T(v-P)^+}{1+r} \ .$$

By taking the positive parts of the two last quantities, this implies that

$$v - \frac{P + (1-\gamma)C_N}{1+r} - (1-\gamma)C_{N-1} \leq h_{N-1}(v) \leq \frac{T(v-P)^+}{1+r} \ .$$

By induction we have that

$$v - \sum_{i=0}^{n} \frac{(1-\gamma)C_i}{(1+r)^i} - \frac{P}{(1+r)^n} \le h_{N-n}(v) \le \frac{T^n(v-P)^+}{(1+r)^n} \,.$$

By taking positive parts we have that

$$\left(v - \sum_{i=0}^{n} \frac{(1-\gamma)C_i}{(1+r)^i} - \frac{P}{(1+r)^n}\right)^+ \le h_{N-n}(v) \le \frac{T^n(v-P)^+}{(1+r)^n} \,.$$

The left hand side is equal to zero for

$$v \le \sum_{i=0}^{n} \frac{(1-\gamma)C_i}{(1+r)^i} + \frac{P}{(1+r)^n} \,,$$

while by Theorem 2 the right hand side is equal to zero for $v \le P/(1+u)^n$. This ends the proof. □

We notice that the default boundary $(v_n^*)_n$ in general depends on time. In particular, it is bounded by two boundaries that evolve in an exponential way with respect to time.

Now we present two ideas of how we can use our results in order to find the optimal default boundary when the process V is a geometric Brownian motion or a geometric Poisson process.

4.1 Approximation of a geometric Brownian motion

As anticipated at the beginning of Section 4, we now want to use our results in order to find the optimal default boundary when the process V is a geometric Brownian motion: let us assume that in the interval $[0,T]$ the value of the firm V is given by

$$\tilde{V}_t = V_0 \exp\left(\left(R - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$$

where R is the instantaneous risk-free interest rate, σ its volatility, and W a standard Brownian motion. Suppose that the firm issues a debt with instantaneous coupon C and final payment P, as in [Leland and Toft, 1996]. In order to approximate V, we proceed as in [Hull] by discretising the interval $[0,T]$ in N subintervals of length T/N, and by letting

$$C_N = \frac{C}{N}\,, \quad r_N = e^{\frac{RT}{N}} - 1\,, \quad u_N = e^{\sigma\sqrt{\frac{T}{N}}} - 1\,, \quad d_N = e^{-\sigma\sqrt{\frac{T}{N}}} - 1\,.$$

This results in a discrete time model as in Section 2, with V following a binomial tree as in Section 4. Then the process $(V_{[tN/T]T/N})_t$ (where $[x]$ denotes the greatest integer smaller than or equal to x) approximates $\tilde{V}$ (in particular for $N \to \infty$ we have convergence in law). Since $d_N \nearrow 0$, the only hope to apply Corollary 2 is to check condition $iii)$. We have that

$$\frac{r_N(r_N - d_N)}{d_N(1+r_N)^{N+2} + r_N - d_N} = \frac{\left(e^{\frac{RT}{N}} - 1\right)\left(e^{\frac{RT}{N}} - e^{-\sigma\sqrt{\frac{T}{N}}}\right)}{\left(e^{-\sigma\sqrt{\frac{T}{N}}} - 1\right)e^{RT\frac{N+2}{N}} + e^{\frac{RT}{N}} - e^{-\sigma\sqrt{\frac{T}{N}}}}\,,$$

so we must impose that the quantity above is less or equal than $(1-\gamma)C/N$. For $N \to \infty$, this corresponds to

$$(1-\gamma)C \geq -\frac{RT}{e^{RT}-1}P \ .$$

Since $C > 0$, this is always verified, so the shape of the functions h_n remains stable. The conclusion is that, for N sufficiently large, the optimal default boundary is given explicitly by Equation (8).

4.2 Approximation of a geometric Poisson process

We now want to use our results in order to find the optimal default boundary when the process V is a geometric Poisson process with drift: let us assume that in the interval $[0,T]$ the value of the firm V is given by

$$\tilde{V}_t = V_0 e^{(R+\gamma\lambda)t}(1-\gamma)^{\tilde{N}_t}$$

where R is the instantaneous risk-free interest rate, $\gamma \in (0,1)$ is the height of a multiplicative downward jump and $\tilde{N}$ is a standard Poisson process with intensity λ. Thus the value of the firm grows with an instantaneous rate $R+\gamma\lambda$ and can jump downwards losing a proportion γ of its value at stochastic times, namely the jump times of the Poisson process $\tilde{N}$. As before, assume that the firm issues a debt with instantaneous coupon C and final payment P. In order to approximate V, we proceed in a similar way as in [Hull] by discretising the interval $[0,T]$ in N subintervals of length T/N, and by letting

$$C_N = \frac{C}{N} \ , \quad r_N = e^{\frac{RT}{N}} - 1 \ , \quad u_N = e^{(R+\gamma\lambda)\frac{T}{N}} - 1 \ , \quad d_N = -\lambda \ .$$

This results in a discrete time model as in Section 2, with V following a binomial tree as in Section 4. Then the process $(V_{[tN/T]T/N})_t$ (where $[x]$ denotes the greatest integer smaller than or equal to x) approximates $\tilde{V}$ (in particular for $N \to \infty$ we have convergence in law). Here we always have $d_N < 0$, so the only hope to apply Corollary 2 is to check condition iii). We have that

$$\frac{r_N(r_N - d_N)}{d_N(1+r_N)^{N+2} + r_N - d_N} = \frac{\left(e^{\frac{RT}{N}} - 1\right)\left(e^{\frac{RT}{N}} - 1 + \gamma\right)}{-\gamma e^{RT\frac{N+2}{N}} + e^{\frac{RT}{N}} - 1 + \gamma} \ ,$$

so we must impose that the quantity above is less or equal than $(1-\gamma)C/N$. For $N \to \infty$, this corresponds to

$$(1-\gamma)C \geq -\frac{RT}{e^{RT}-1}P \ .$$

Since $C > 0$, this is always verified, so the shape of the functions h_n remains stable, provided that

$$d_N \in \left(-\frac{1}{(1+r_N)^{N+2}-1}, 0\right] \ .$$

This happens if and only if $\gamma < 1/(e^{RT\frac{N+2}{N}} - 1)$. For $N \to \infty$, this corresponds to $\gamma < 1/(e^{RT} - 1)$. The conclusion is that, for N sufficiently large, the optimal default boundary is given explicitly by Equation (8) if $\gamma < 1/(e^{RT} - 1)$. Conversely, if this condition is not satisfied, then the function h_n becomes complex as n gets far from N.

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